PSEUDO-RIEMANNIAN SYMMETRIES ON HEISENBERG GROUPS

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Abstract. The notion of Γ-symmetric space is a natural generalization of the classical notion of symmetric space based on $\mathbb{Z}_2$-grading on Lie algebras. We consider homogeneous spaces G/H such that the Lie algebra $\mathfrak{g}$ of G admits a Γ-grading where Γ is a finite abelian group. In this work we study Riemannian metrics and Lorentzian metrics on the Heisenberg group $\mathbb{H}_3$ adapted to the symmetries of a Γ-symmetric structure on $\mathbb{H}_3$. We prove that the classification of Riemannian and Lorentzian $\mathbb{Z}_2^2$-symmetric metrics on $\mathbb{H}_3$ corresponds to the classification of its left-invariant Riemannian and Lorentzian metrics, up to isometry. We study also the $\mathbb{Z}_2^k$-symmetric structures on G/H when G is the $(2p+1)$-dimensional Heisenberg group. This gives examples of non-Riemannian symmetric spaces. When $k \geq 1$, we show that there exists a family of flat and torsion free affine connections adapted to the $\mathbb{Z}_2^k$-symmetric structures.

1. Introduction

A symmetric space can be considered as a reductive homogeneous space G/H on which acts an abelian subgroup Γ of the automorphisms group of G with Γ isomorphic to $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and H the subgroup of G composed of the fixed points of the automorphisms belonging to Γ. If we suppose that the Lie groups G and H are connected and that G is simply connected, it is equivalent to provide G/H with a symmetric structure or to provide the Lie algebra $\mathfrak{g}$ of G with a $\mathbb{Z}_2$-graduation $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $[\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_{i+j(mod\ 2)}$. Riemannian symmetric spaces form an interesting class of symmetric spaces. But there are symmetric spaces which are not Riemannian symmetric. We describe examples when G is the Heisenberg group. Nevertheless, a symmetric space is always provided with an affine connection ∇ which is torsion free and has a curvature tensor satisfying $\nabla R = 0$. When the symmetric space is Riemannian, this connection is the Levi-Civita connection of the metric. A natural generalization of the notion of symmetric space can be obtained by considering that the subgroup Γ is abelian, finite and not necessarily isomorphic to $\mathbb{Z}_2$. When Γ is cyclic isomorphic to $\mathbb{Z}_k$ it corresponds to the generalized symmetric spaces of [2, 13, 15]. These structures are also characterized by $\mathbb{Z}_k$-graduations of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ of $\mathfrak{g}$. We get another interesting case when $\Gamma = \mathbb{Z}_2^k$ because the characteristic graduation is defined on $\mathfrak{g}$ and not on $\mathfrak{g}_{\mathbb{C}}$. When $\mathfrak{g}$ is simple the $\mathbb{Z}_2^2$-graduations of $\mathfrak{g}$ have been classified as well as the $\mathbb{Z}_2^2$-symmetric spaces G/H when G is simple connected ([1, 12]). All these spaces are Riemannian (see [16]). But, in this paper, we provide some examples of non Riemannian symmetric spaces studying symmetric spaces G/H when G is the Heisenberg group $\mathbb{H}_{2p+1}$. We study also, for $k > 1$, $\mathbb{Z}_2^k$-symmetric structures on these homogeneous spaces showing, in particular, that these spaces are Riemannian and affine. But contrary to the symmetric

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case, there exist on these spaces affine connections different from the canonical (or the Levi-Civita) connection and more adapted to the symmetries of G/H that the canonical one. We describe these connections and we prove that there exists connections adapted to the $\mathbb{Z}_2^k$ symmetries which are flat and torsion free.

2. $\mathbb{Z}_2^k$-symmetric spaces

2.1. **Recall on symmetric and Riemannian symmetric spaces.** A symmetric space is a triple (G, H, σ) where G is a connected Lie group, H a closed subgroup of G and σ an involutive automorphism of G such that $G_e^\sigma \subset H \subset G^\sigma$ where $G^\sigma = \{x \in G,\ \sigma(x) = x\}$, G_e^σ the identity component of G^σ. If (G, H, σ) is a symmetric space, to each point $\overline{x}$ of the homogeneous manifold $M = G/H$ corresponds an involutive diffeomorphism $\sigma_{\overline{x}}$ which has $\overline{x}$ as an isolated fixed point. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H. The automorphism $\sigma \in Aut(G)$ induces an involutive automorphism of $\mathfrak{g}$, denoted by σ again, such that $\mathfrak{h}$ consists of all elements of $\mathfrak{g}$ which are left fixed by σ. We deduce that the Lie algebra $\mathfrak{g}$ is $\mathbb{Z}_2$-graded: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathfrak{m} = \{X \in \mathfrak{g},\ \sigma(X) = -X\}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. If we assume that G is simply connected and H connected, then the $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ defines a symmetric space stucture (G, H, σ). Thus, under these hypothesis, it is equivalent to speak about $\mathbb{Z}_2$-grading of Lie algebras or symmetric spaces.

An important class of symmetric spaces consists of Riemannian symmetric spaces. A Riemannian symmetric space is a Riemannian manifold M whose curvature tensor field associated with the Levi-Civita connection is parallel. In this case the geodesic symmetry at a point $u \in M$ attached to the Levi-Civita connection is an isometry and, if we fix u, it defines an involutive automorphism σ of the largest group of isometries G of M which acts transitively on M. We deduce that M is an homogeneous manifold $M = G/H$ and the triple (G, H, σ) is a symmetric space. Let us note that, in this case, H is compact. When $H \cap Z(G) = \{e\}$, this last condition is equivalent to $ad_{\mathfrak{g}}(H)$ compact. Here $Z(G)$ denotes the center of G. Conversely, if (G, H, σ) is a symmetric space such that the image $ad_{\mathfrak{g}}(H)$ of H under the adjoint representation of G is a compact subgroup of $Gl(\mathfrak{g})$, then $\mathfrak{g}$ admits an $ad_{\mathfrak{g}}(H)$-invariant inner product and $\mathfrak{h}$ and $\mathfrak{m}$ are orthogonal with respect to it. This inner product restricted to $\mathfrak{m}$ induces an G-invariant Riemannian metric on G/H and G/H is a Riemannian symmetric space. For example, if H is compact, $ad_{\mathfrak{g}}(H)$ is also compact and (G, H, σ) is a Riemannian symmetric space. Assume now that H is connected, then $ad_{\mathfrak{g}}(H)$ is compact if and only if the connected Lie group associated with the linear algebra $ad_{\mathfrak{g}}(\mathfrak{h}) = \{adX,\ X \in \mathfrak{h}\}$ is compact. In this case, $\mathfrak{g}$ admits an $ad_{\mathfrak{g}}(\mathfrak{h})$-invariant inner product φ, that is, $\varphi([X, Y], Z) + \varphi(Y, [X, Z]) = 0$ for all $X \in \mathfrak{h}$ and $Y, Z \in \mathfrak{g}$ such that $\varphi(\mathfrak{h}, \mathfrak{m}) = 0$. An interesting particular case is the following. Assume that $\mathfrak{g}$ is $\mathbb{Z}_2$-graded and that this grading is effective that is $\mathfrak{h}$ doesn't contain non trivial ideal of $\mathfrak{g}$. If $ad_{\mathfrak{g}}(\mathfrak{h})$ is irreducible on $\mathfrak{m}$, then $\mathfrak{g}$ is simple, or a sum $\mathfrak{g}_1 + \mathfrak{g}_1$ with $\mathfrak{g}_1$ simple or $\mathfrak{m}$ abelian. In the first case, the Killing-Cartan form K of $\mathfrak{g}$ induces a negative or positive defined bilinear form on $\mathfrak{m}$. It follows a classification of $\mathbb{Z}_2$-graded Lie algebras when $\mathfrak{g}$ is simple or semi-simple.

Many results on the problem of classifications concern more particularly the simple Lie algebras. For solvable or nilpotent Lie algebras, it is an open problem. A first approach is to study induced grading on Borel or parabolic subalgebras of simple Lie algebras. In this work we describe Γ-grading of the Heisenberg algebras. Two reasons for this study

- Heisenberg algebras are nilradical of some Borel subalgebras.
- The Riemannian and Lorentzian geometries on the 3-dimensional Heisenberg group have been studied recently by many authors.

Thus it is interesting to study the Riemannian and Lorentzian symmetries with the natural symmetries associated with a Γ-symmetric structure on the Heisenberg group. In this paper we prove that these geometries are entirely determinated by Riemannian and Lorentzian structures adapted to $\mathbb{Z}_2^2$-symmetric structures.

2.2. **Γ-symmetric spaces.** Let Γ be a finite abelian group.

Definition 1. *A Γ-symmetric space is a triple $(G,H,\tilde{\Gamma})$ where G is a connected Lie group, H a closed subgroup of G and $\tilde{\Gamma}$ a finite abelian subgroup of the group $\mathrm{Aut}(G)$ of automorphisms of G isomorphic to Γ such that $G_e^\Gamma \subset H \subset G^\Gamma$ where $G^\Gamma = \{x \in G,\ \sigma(x)=x\ \ \forall \sigma \in \tilde{\Gamma}\}$, G_e^Γ the identity component of G^Γ.*

If Γ is isomorphic to $\mathbb{Z}_2$ then we find the notion of symmetric spaces again. If Γ is isomorphic to $\mathbb{Z}_k$ with $k \geq 3$, then Γ is a cyclic group generated by an automorphism of order k. The corresponding spaces are called generalized symmetric spaces and have been studied by A.J. Ledger, M. Obata [15], A. Gray, J. A. Wolf, [9] and O. Kowalski [13]. The general notion of Γ-symmetric spaces was introduced by R. Lutz [14] and was algebraically reconsidered by Y. Bahturin and M. Goze [1].
An equivalent and useful definition is the following:

Definition 2. *Let Γ be a finite abelian group. A Γ-symmetric space is an homogeneous space G/H such that there exists an injective homomorphism $\rho : \Gamma \to Aut(G)$ where $Aut(G)$ is the group of automorphisms of the Lie group G, the subgroup H satisfies $G_e^\Gamma \subset H \subset G^\Gamma$ where $G^\Gamma = \{x \in G/\rho(\gamma)(x) = x, \forall \gamma \in \Gamma\}$ and G_e^Γ is the connected identity component of G^Γ of G.*

In [1], one proves that, if G and H are connected, then the triple $(G,H,\tilde{\Gamma})$ is a Γ-symmetric space if and only if the complexified Lie algebra $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$ of $\mathfrak{g}$ is Γ-graded: $\mathfrak{g}_\mathbb{C} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ where $\mathfrak{g}_\epsilon = \mathfrak{h}$ is the Lie algebra of H with ϵ the unit of Γ. In this case, we have the relations $[\mathfrak{g}_\gamma, \mathfrak{g}_{\gamma'}] \subset \mathfrak{g}_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \Gamma$.

In fact, the derivative of an automorphism σ of G belonging to $\tilde{\Gamma}$ is an automorphism of $\mathfrak{g}$, still denoted σ. So if γ runs over $\tilde{\Gamma}$, we obtain a subgroup $\hat{\Gamma}$ of the group of automorphisms of $\mathfrak{g}$ which is isomorphic to Γ. The elements of $\hat{\Gamma}$ are automorphisms of $\mathfrak{g}$ of finite order, pairwise commuting and the Γ-grading corresponds to the spectral decomposition of $\mathfrak{g}_\mathbb{C}$ associated with the abelian finite group $\hat{\Gamma}$. Conversely, if we have a Γ-grading of $\mathfrak{g}_\mathbb{C}$, and if we denote by $\check{\Gamma}$ the dual group of Γ, that is, the group of characters, thus $\check{\Gamma}$ is a finite abelian group isomorphic to Γ. Any element $\chi \in \check{\Gamma}$ can be considered as an automorphism of $\mathfrak{g}_\mathbb{C}$ by $\chi(X) = \chi(\gamma)X$ for any homogeneous vector $X \in \mathfrak{g}_\gamma$. Thus $\check{\Gamma}$ is an abelian subgroup of $Aut(\mathfrak{g}_\mathbb{C})$ isomorphic to Γ and the Γ-grading of $\mathfrak{g}$ corresponds to the spectral decomposition associated with $\check{\Gamma}$ considered as an abelian finite subgroup of $Aut(\mathfrak{g}_\mathbb{C})$. Then, if we assume that G is also simply connected, we have a one-to-one correspondence between the set of Γ-symmetric stuctures and the Γ-gradings of $\mathfrak{g}$.

In [14], it is shown that for any $\bar{x} \in M = G/H$, there exists a subgroup $\Gamma_{\bar{x}}$ of the group $\mathcal{D}\mathrm{iff}(M)$ of diffeomorphisms of M, isomorphic to Γ, such that $\bar{x}$ is the unique point of M satisfying $\sigma(\bar{x}) = \bar{x}$ for any $\sigma \in \Gamma_{\bar{x}}$. By extension, the elements of $\Gamma_{\bar{x}}$ are also called symmetries of M.

2.3. **$\mathbb{Z}_2^k$-symmetric spaces.** Assume that $\Gamma = \mathbb{Z}_2^k$. In this case any element of $\hat{\Gamma}$ is an involutive automorphism of $\mathfrak{g}$ and the eigenvalues are real. Since the elements of $\hat{\Gamma}$ are pairwise commuting, we define a spectral decomposition of $\mathfrak{g}$ itself. This implies a $\mathbb{Z}_2^k$-grading defined on $\mathfrak{g}$: $\mathfrak{g} = \bigoplus\limits_{\gamma \in \Gamma} \mathfrak{g}_\gamma$. For example, if $k=2$, then $\Gamma = \{a,b,c,\epsilon\}$ where ϵ is the identity, with $a^2 = b^2 = c^2 = \epsilon, \quad ab = c, \quad bc = a, \quad ca = b.$ and $\hat{\Gamma}$ contains 4 elements, $\sigma_a, \sigma_b, \sigma_c$ and the identity Id. These maps are involutive and satisfy $\sigma_a \circ \sigma_b = \sigma_c, \quad \sigma_b \circ \sigma_c = \sigma_a, \quad \sigma_c \circ \sigma_a = \sigma_b.$

Each one of these linear maps is diagonalizable, and because they are pairwise commuting, we can diagonalize all these maps simultaneously. Let $\mathfrak{g}_a = \{X \in \mathfrak{g}, \sigma_a(X) = X, \sigma_b(X) = -X\}$, $\mathfrak{g}_b = \{X \in \mathfrak{g}, \sigma_a(X) = -X, \sigma_b(X) = X\}$, $\mathfrak{g}_c = \{X \in \mathfrak{g}, \sigma_a(X) = -X, \sigma_b(X) = -X\}$ and $\mathfrak{g}_\epsilon = \{X \in \mathfrak{g}, \sigma_a(X) = X, \sigma_b(X) = X\}$ be the root spaces. We have $\mathfrak{g} = \mathfrak{g}_\epsilon \oplus \mathfrak{g}_a \oplus \mathfrak{g}_b \oplus \mathfrak{g}_c$.

Let us return to the general case $\Gamma = \mathbb{Z}_2^k$. If G is connected and simply connected and H connected, then the Γ-grading of $\mathfrak{g}$ determine a structure of Γ-symmetric space on the triple $(G, H, \tilde{\Gamma})$. We will say also that the homogeneous space G/H is a $\mathbb{Z}_2^k$-symmetric space.

Proposition 3. *Any $\mathbb{Z}_2^k$-symmetric homogeneous space G/H is reductive.*

Proof. In fact if $\mathfrak{g} = \bigoplus_{\gamma \in \mathbb{Z}_2^k} \mathfrak{g}_\gamma$ is the associated decomposition of $\mathfrak{g}$, thus putting $\mathfrak{m} = \bigoplus_{\gamma \in \Gamma, \gamma \neq \epsilon} \mathfrak{g}_\gamma$, we have $\mathfrak{g} = \mathfrak{g}_\epsilon \oplus \mathfrak{m}$ with $[\mathfrak{g}_\epsilon, \mathfrak{g}_\epsilon] \subset \mathfrak{g}_\epsilon$ and $[\mathfrak{g}_\epsilon, \mathfrak{m}] \subset \mathfrak{m}$. The decomposition $\mathfrak{g} = \mathfrak{g}_\epsilon \oplus \mathfrak{m}$ is reductive.

In general $[\mathfrak{m}, \mathfrak{m}]$ is not a subset of $\mathfrak{g}_\epsilon$, except if $k = 1$.

Two $\mathbb{Z}_2^k$-gradings $\mathfrak{g} = \bigoplus_{\gamma \in \mathbb{Z}_2^k} \mathfrak{g}_\gamma$ and $\mathfrak{g} = \bigoplus_{\gamma' \in \mathbb{Z}_2^k} \mathfrak{g}'_{\gamma'}$ of $\mathfrak{g}$ are called equivalent if there exist an automorphism π of $\mathfrak{g}$ and an automorphism ω of $\mathbb{Z}_2^k$ such that $\mathfrak{g}'_{\gamma'} = \pi(\mathfrak{g}_{\omega(\gamma)})$ for any $\gamma' \in \mathbb{Z}_2^k$. If we consider only connected and simply connected groups G, and connected subgroups H, then the classification of $\mathbb{Z}_2^k$-symmetric spaces is equivalent to the classification, up to equivalence, to $\mathbb{Z}_2^k$-gradings on Lie algebras. For example, the $\mathbb{Z}_2^2$-grading of classical simple complex Lie algebras are classified in [1]. This classification is completed for exceptional simple algebras in [12].

2.4. **Riemannian and pseudo-Riemannian $\mathbb{Z}_2^k$-symmetric spaces.** Let $(G, H, \mathbb{Z}_2^k)$ be a $\mathbb{Z}_2^k$-symmetric space with G and H connected. The homogeneous space $M = G/H$ is reductive. Then there exists a one-to-one correspondence between the G-invariant pseudo-Riemannian metrics g on M and the non-degenerated symmetric bilinear form B on $\mathfrak{m}$ satisfying $B([Z, X], Y) + B(X, [Z, Y]) = 0$ for all $X, Y \in \mathfrak{m}$ and $Z \in \mathfrak{g}_\epsilon$.

Definition 4. [8] *A $\mathbb{Z}_2^k$-symmetric space $M = G/H$ with $Ad_G(H)$ compact, is called Riemannian $\mathbb{Z}_2^k$-symmetric if M is provided with a G-invariant Riemannian metric g whose associated bilinear form B satisfies*

1. $B(\mathfrak{g}_\gamma, \mathfrak{g}_{\gamma'}) = 0$ *if* $\gamma \neq \gamma' \neq \epsilon \neq \gamma$
2. *The restriction of B to $\mathfrak{m} = \oplus_{\gamma \neq \epsilon} \mathfrak{g}_\gamma$ is positive definite.*

In this case the linear automorphisms which belong to $\hat{\Gamma}$ are linear isometries. Some examples are described in [16].

Proposition 5. *Let $(G, H, \mathbb{Z}_2^k)$ be a Riemannian $\mathbb{Z}_2^k$-symmetric space, G and H supposed to be connected. Then H is compact.*

Proof. In fact, H coincides with the identity component of the isotropy group which is compact.

Example: $\mathbb{Z}_2^k$-symmetric nilpotent spaces. Let $(G, H, \mathbb{Z}_2^k)$ be a $\mathbb{Z}_2^k$-symmetric space with G nilpotent. Such a space will be called a $\mathbb{Z}_2^k$-symmetric nilpotent space. If $k = 1$, we cannot have on G/H a Riemannian symmetric metric except if G is abelian. But, if $k \geq 2$, there exist Riemannian $\mathbb{Z}_2^k$-symmetric nilpotent spaces. For example, let G be the 3-dimensional Heisenberg Lie group. Its Lie algebra $\mathfrak{h}_3$ admits a basis $\{X_1, X_2, X_3\}$ with $[X_1, X_2] = X_3$. We have a $\mathbb{Z}_2^2$-grading of $\mathfrak{h}_3$:

$$\mathfrak{h}_3 = \{0\} \oplus \mathbb{R}\{X_1\} \oplus \mathbb{R}\{X_2\} \oplus \mathbb{R}\{X_3\}$$

and the metric $g = \omega_1^2 + \omega_2^2 + \omega_3^2$ defines a structure of Riemannian $\mathbb{Z}_2^k$-symmetric nilpotent space on $H_3/\{e\} = H_3$ where $\{\omega_1, \omega_2, \omega_3\}$ is the dual basis of $\{X_1, X_2, X_3\}$. We will develop this calculus in the next sections.

A Lorentzian metric on a n-dimensional differential manifold M is a smooth field of non-degenerate quadratic forms of signature $(n-1, 1)$. We say that a homogeneous space $(M = G/H, g)$ provided with a Lorentzian metric g is *Lorentzian* if the canonical action of G on M preserves the metric. If M is reductive and if $\mathfrak{g} = \mathfrak{g}_\epsilon \oplus \mathfrak{m}$, the Lorentzian metric is determinate by the $ad\mathfrak{g}_\epsilon$-invariant non-degenerate bilinear form B with signature $(n-1, 1)$.

Definition 6. *Let $(G, H, \mathbb{Z}_2^k)$ be a $\mathbb{Z}_2^k$-symmetric space. It is called Lorentzian if there exists on the homogeneous space $M = G/H$ a Lorentzian metric g such that one of the two conditions is satisfied:*

1. *The homogeneous non trivial components $\mathfrak{g}_\gamma$ of the $\mathbb{Z}_2^k$-graded Lie algebra $\mathfrak{g}$ are orthogonal and non-degenerate with respect to the induced bilinear form B.*
2. *One non trivial component $\mathfrak{g}_{\lambda_0}$ is degenerate, the other components are orthogonal and non-degenerate.*

Let us note that, in this case, H is not necessarily compact. Some examples of Lorentzian $\mathbb{Z}_2^k$-symmetric nilpotent spaces are described in the next sections.

3. Affine structures on $\mathbb{Z}_2^k$-symmetric spaces

Let $(G, H, \mathbb{Z}_2^k)$ be a $\mathbb{Z}_2^k$-symmetric space. Since the homogeneous space G/H is reductive, from [11], Chapter X, we deduce that $M = G/H$ admits two G-invariant canonical connections denoted by ∇ and $\overline{\nabla}$. The *first canonical connection*, ∇, satisfies

$$\begin{cases} R(X,Y) = -\mathrm{ad}([X,Y]_{\mathfrak{h}}), \qquad T(X,Y) = -[X,Y]_{\mathfrak{m}}, \quad \forall X, Y \in \mathfrak{m} \\ \nabla T = 0 \\ \nabla R = 0 \end{cases}$$

where T and R are the torsion and the curvature tensors of ∇. The tensor T is trivial if and only if $[X, Y]_{\mathfrak{m}} = 0$ for all $X, Y \in \mathfrak{m}$. This means that $[X, Y] \in \mathfrak{h}$ that is $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. If the grading of $\mathfrak{g}$ is given by $\mathbb{Z}_2^k$ with $k > 1$, then $[\mathfrak{m}, \mathfrak{m}]$ is not a subset of $\mathfrak{h}$ and then the torsion T need not to vanish. In this case the other connection $\overline{\nabla}$ is given by $\overline{\nabla}_X Y = \nabla_X Y - T(X, Y)$. This is an affine invariant torsion free connection on G/H which has the same geodesics as ∇. This connection is called the *second canonical connection* or the *torsion-free canonical connection.*

Remark. Actually, there is another way of writing the canonical affine connection of a Γ-symmetric space, without any reference to Lie algebras. This is done by an intrinsic construction of Γ-symmetric spaces proposed by Lutz in [14].

3.1. Associated affine connection.

Any symmetric space G/H is an affine symmetric space, that is, it is provided with an affine connection ∇ whose torsion tensor T and curvature tensor R satisfy

$$T = 0, \qquad \nabla R = 0$$

where

$$\begin{aligned} \nabla R(X_1, X_2, X_3, Y) = \; & \nabla(Y, R(X_1, X_2, X_3)) - R(\nabla(Y, X_1), X_2, X_3) \\ & - R(X_1, \nabla(Y, X_2), X_3) - R(X_1, X_2, \nabla(Y, X_3)) \end{aligned}$$

for any vector fields X_1, X_2, X_3, Y on G/H. It is the only affine connection which is invariant by the symmetries of G/H. This means that the two canonical connections, which are defined on an homogeneous reductive space, coincide if the reductive space is symmetric.

For example, if G/H is a Riemannian symmetric space, this connection ∇ coincides with the Levi-Civita connection associated with the Riemannian metric.

Let us return to the general case. Let us assume that G/H is a reductive homogeneous space, and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the reductive decomposition of $\mathfrak{g}$. Any connection on G/H is given by a linear map $\bigwedge : \mathfrak{m} \rightarrow gl(\mathfrak{m})$ satisfying $\bigwedge[X,Y] = [\bigwedge(X), \lambda(Y)]$ for all $X \in \mathfrak{m}$ and $Y \in \mathfrak{h}$, where λ is the linear isotropy representation of $\mathfrak{h}$. The corresponding torsion and curvature tensors are given by: $T(X,Y) = \bigwedge(X)(Y) - \bigwedge(Y)(X) - [X,Y]_{\mathfrak{m}}$ and $R(X,Y) = [\bigwedge(X), \bigwedge(Y)] - \bigwedge[X,Y] - \lambda([X,Y]_{\mathfrak{h}})$ for any $X, Y \in \mathfrak{m}$.

Let $(G, H, \mathbb{Z}_2^k)$ be a $\mathbb{Z}_2^k$-symmetric space. We have recalled that, when $k = 1$, the homogeneous space G/H is an affine symmetric space. But, as soon as $k > 1$, in general the two canonical connections do not coincide and the torsion tensor of the first one is not trivial. We can consider connections adapted to the $\mathbb{Z}_2^k$-symmetric structures.

Definition 7. *Let ∇ be an affine connection on the $\mathbb{Z}_2^k$-symmetric space G/H defined by the linear map $\bigwedge : \mathfrak{m} \rightarrow gl(\mathfrak{m})$. Then this connection is called adapted to the $\mathbb{Z}_2^k$-symmetric structure, if $\bigwedge(X_\gamma)(\mathfrak{g}_{\gamma'}) \subset \mathfrak{g}_{\gamma\gamma'}$ for any $\gamma, \gamma' \in \mathbb{Z}_2^k$, $\gamma, \gamma' \neq \epsilon$. The connection is called homogeneous if any homogeneous component $\mathfrak{g}_\gamma$ of $\mathfrak{m}$ is invariant by $\bigwedge$.*

Examples

(1) If $k = 1$, the affine canonical connection is adapted and homogeneous.

(2) Let us consider the 5-dimensional nilpotent Lie algebra, $\mathfrak{l}_5$ whose Lie brackets are given in a basis $\{X_1, \cdots, X_5\}$ by $[X_1, X_i] = X_{i+1}, i = 2, 3, 4$. This algebra admits a $\mathbb{Z}_2$-grading $\mathfrak{l}_5 = \mathbb{R}\{X_3, X_5\} \oplus \mathbb{R}\{X_1, X_2, X_4\}$. Thus $\bigwedge(X_1), \bigwedge(X_2), \bigwedge(X_4)$ are matrices of order 3. If we assume that the torsion T is zero, we obtain

$$\bigwedge(X_1) = \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & d & \frac{a}{2} \end{pmatrix}, \quad \bigwedge(X_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e & 0 \\ d & f & \frac{a}{2} \end{pmatrix}, \quad \bigwedge(X_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{a}{2} & 0 & 0 \end{pmatrix}.$$

The linear isotropy representation of H whose Lie algebra is $\mathfrak{h}$ is given by taking the differential of the map $\mathfrak{l}_5/H \rightarrow \mathfrak{l}_5/H$ corresponding to the left multiplication $\overline{x} \rightarrow h\overline{x}$ with $\overline{x} = xH_4$. We obtain

$$\lambda(X_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda(X_5) = (0).$$

We deduce that the curvature is always non zero.

4. The $\mathbb{Z}_2^k$-symmetric spaces $(\mathbb{H}_3, H, \mathbb{Z}_2^k)$

We denote by $\mathbb{H}_3$ the 3-dimensional Heisenberg group, that is the linear group of dimension 3 consisting of matrices

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \qquad a, b, c \in \mathbb{R}.$$

Its Lie algebra, $\mathfrak{h}_3$ is the real Lie algebra whose elements are matrices

$$\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{with} \qquad x, y, z \in \mathbb{R}.$$

The elements of $\mathfrak{h}_3$, X_1, X_2, X_3, corresponding to $(x,y,z)=(1,0,0),(0,1,0)$ and $(0,0,1)$ form a basis of $\mathfrak{h}_3$ and the Lie brackets are given in this basis by $[X_1,X_2]=X_3$, $[X_1,X_3]=[X_2,X_3]=0$.

4.1. **Description of $Aut(\mathfrak{h}_3)$.** Denote by $Aut(\mathfrak{h}_3)$ the group of automorphisms $\mathfrak{h}_3$. Every $\tau\in Aut(\mathfrak{h}_3)$ admits in the basis $\{X_1,X_2,X_3\}$ the following matricial representation:

$$\begin{pmatrix}\alpha_1 & \alpha_2 & 0\\ \alpha_3 & \alpha_4 & 0\\ \alpha_5 & \alpha_6 & \Delta\end{pmatrix} \quad \text{with} \quad \Delta=\alpha_1\alpha_4-\alpha_2\alpha_3\neq 0. \tag{1}$$

We will denote by $\tau(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6)$ any element of $Aut(\mathfrak{h}_3)$ in this representation. Let Γ be a finite abelian subgroup of $Aut(\mathfrak{h}_3)$. It admits a cyclic decomposition. If Γ contains a component of the cyclic decomposition which is isomorphic to $\mathbb{Z}_k$, then there exists an automorphism τ satisfying $\tau^k=Id$. The aim of this section is to determinate the cyclic decomposition of any finite abelian subgroup Γ.

• **Subgroups of $Aut(\mathfrak{h}_3)$ isomorphic to $\mathbb{Z}_2$**

Let $\tau\in Aut(\mathfrak{h}_3)$ satisfying $\tau^2=Id$. If we consider the matricial representation (1) of τ, we obtain:

$$\begin{pmatrix}\alpha_1^2+\alpha_2\alpha_3 & \alpha_1\alpha_2+\alpha_2\alpha_4 & 0\\ \alpha_1\alpha_3+\alpha_3\alpha_4 & \alpha_2\alpha_3+\alpha_4^2 & 0\\ \alpha_1\alpha_5+\alpha_3\alpha_6+\Delta\alpha_5 & \alpha_2\alpha_5+\alpha_4\alpha_6+\Delta\alpha_6 & \Delta^2\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

Proposition 8. *Any involutive automorphism τ of $Aut(\mathfrak{h}_3)$ is equal to one of the following automorphisms*

$$Id,\qquad \tau_1(\alpha_3,\alpha_6)=\begin{pmatrix}-1&0&0\\ \alpha_3 & 1 & 0\\ \dfrac{\alpha_3\alpha_6}{2} & \alpha_6 & -1\end{pmatrix},\qquad \tau_2(\alpha_3,\alpha_5)=\begin{pmatrix}1&0&0\\ \alpha_3&-1&0\\ \alpha_5&0&-1\end{pmatrix},$$

$$\tau_3(\alpha_1,\alpha_2\neq 0,\alpha_6)=\begin{pmatrix}\alpha_1 & \alpha_2 & 0\\ \dfrac{1-\alpha_1^2}{\alpha_2} & -\alpha_1 & 0\\ \dfrac{(1+\alpha_1)\alpha_6}{\alpha_2} & \alpha_6 & -1\end{pmatrix},\qquad \tau_4(\alpha_5,\alpha_6)=\begin{pmatrix}-1&0&0\\0&-1&0\\ \alpha_5&\alpha_6&1\end{pmatrix}.$$

Corollary 9. *Any subgroup of $Aut(\mathfrak{h}_3)$ isomorphic to $\mathbb{Z}_2$ is one of the following:*

$$\begin{array}{ll}\Gamma_1(\alpha_3,\alpha_6)=\{Id,\tau_1(\alpha_3,\alpha_6)\}, & \Gamma_2(\alpha_3,\alpha_5)=\{Id,\tau_2(\alpha_3,\alpha_5)\},\\ \Gamma_3(\alpha_1,\alpha_2,\alpha_6)=\{Id,\tau_3(\alpha_1,\alpha_2,\alpha_6),\ \alpha_2\neq 0\}, & \Gamma_4(\alpha_5,\alpha_6)=\{Id,\tau_4(\alpha_5,\alpha_6)\}.\end{array}$$

• **Subgroups of $Aut(\mathfrak{h}_3)$ isomorphic to $\mathbb{Z}_k,\ k\geq 3$.** If $\tau=\tau(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6)\in Aut(\mathfrak{h}_3)$ satisfies $\tau^k=Id$, then $\Delta=\alpha_1\alpha_4-\alpha_2\alpha_3=1$ and its minimal polynomial has 3 simple roots and it is of degree 3. More precisely, it is written

$$m_\tau(x)=(x-1)(x-\mu_k)(x-\overline{\mu_k})$$

where μ_k is a root of order k of 1. Since we can assume that τ is a generator of a cyclic subgroup of $Aut(\mathfrak{h}_3)$ isomorphic to $\mathbb{Z}_k$, the root μ_k is a primitive root of 1. There exists m relatively prime with k such that $\mu_k=exp\left(\dfrac{2mi\pi}{k}\right)$. We have $\alpha_1+\alpha_4=\mu_k+\overline{\mu_k}$ and $\alpha_1+\alpha_4=2\cos\dfrac{2m\pi}{k}$. Thus

$$\alpha_1=\cos\frac{2m\pi}{k}-\sqrt{\cos^2\frac{2m\pi}{k}-1-\alpha_2\alpha_3},$$

$$\alpha_4 = \cos\frac{2m\pi}{k} + \sqrt{\cos^2\frac{2m\pi}{k} - 1 - \alpha_2\alpha_3}$$

or

$$\alpha_1 = \cos\frac{2m\pi}{k} + \sqrt{\cos^2\frac{2m\pi}{k} - 1 - \alpha_2\alpha_3},$$
$$\alpha_4 = \cos\frac{2m\pi}{k} - \sqrt{\cos^2\frac{2m\pi}{k} - 1 - \alpha_2\alpha_3}.$$

If τ' and τ'' denote the automorphisms corresponding to these solutions, we have, for a good choice of the parameters α_i, $\tau' \circ \tau'' = Id$ and $\tau'' = (\tau')^{k-1}$. Thus these automorphisms generate the same subgroup of $Aut(\mathfrak{h}_3)$. Moreover, with same considerations, we can choose $m = 1$. Thus we have determinate the automorphism $\tau_5(\alpha_2, \alpha_3, \alpha_5, \alpha_6)$ whose matrix is

$$\begin{pmatrix} \cos\frac{2\pi}{k} + \sqrt{\cos^2\frac{2\pi}{k} - 1 - \alpha_2\alpha_3} & \alpha_2 & 0 \\ \alpha_3 & \cos\frac{2\pi}{k} - \sqrt{\cos^2\frac{2\pi}{k} - 1 - \alpha_2\alpha_3} & 0 \\ \alpha_5 & \alpha_6 & 1 \end{pmatrix}$$

Proposition 10. *Any abelian subgroup of $Aut(\mathfrak{h}_3)$ isomorphic to $\mathbb{Z}_k$,$k \geq 3$, is equal to*

$$\Gamma_{6,k}(\alpha_2, \alpha_3, \alpha_5, \alpha_6) = \left\{ Id, \tau_6(\alpha_2, \alpha_3, \alpha_5, \alpha_6), \cdots, \tau_6^{k-1}, \quad \alpha_2\alpha_3 \leq -1 + \cos^2\frac{2\pi}{k} \right\}.$$

General case. Suppose now that the cyclic decomposition of a finite abelian subgroup Γ of $Aut(\mathfrak{h}_3)$ is isomorphic to $\mathbb{Z}_2^{k_2} \times \mathbb{Z}_3^{k_3} \times \cdots \times \mathbb{Z}_p^{k_p}$ with $k_i \geq 0$.

Lemma 11. *Let Γ be an abelian finite subgroup of $Aut(\mathfrak{h}_3)$ with a cyclic decomposition isomorphic to $\mathbb{Z}_2^{k_2} \times \mathbb{Z}_3^{k_3} \times \cdots \times \mathbb{Z}_p^{k_p}$. Then*

1. *If there is $i \geq 3$ such that $k_i \neq 0$, then $k_2 \leq 1$.*
2. *If $k_2 \geq 2$, then Γ is isomorphic to $\mathbb{Z}_2^{k_2}$.*

Proof. Assume that there is $i \geq 3$ such that $k_i \geq 1$. If $k_2 \geq 1$, there exist two automorphisms τ and τ' satisfying $\tau'^i = \tau^2 = Id$ and $\tau' \circ \tau = \tau \circ \tau'$. Thus τ' and τ can be reduced simultaneously in the diagonal form and admit a common basis of eigenvectors. Since for any $\sigma \in Aut(\mathfrak{h}_3)$ we have $\sigma(X_3) = \Delta X_3$, X_3 is an eigenvector for τ' and τ associated to the eigenvalue 1 for τ' and ± 1 for τ. As the two other eigenvalues of τ' are complex conjugate numbers, the corresponding eigenvectors are complex conjugate. This implies that the eigenvalues of τ distinguished of $\Delta = \pm 1$ are equal and from Proposition 8, $\tau = \tau_4(\alpha_5, \alpha_6)$. If we assume that $k_2 \geq 2$, there exist τ and τ'' not equal and belonging to $\mathbb{Z}_2^{k_2}$. Thus we have $\tau = \tau_4(\alpha_5, \alpha_6)$ and $\tau'' = \tau_4(\alpha_5', \alpha_6')$. But $\tau_4(\alpha_5, \alpha_6) \circ \tau_4(\alpha_5', \alpha_6') = \tau_4(\alpha_5', \alpha_6') \circ \tau_4(\alpha_5, \alpha_6)$ if and only if $\alpha_5 = \alpha_5'$, $\alpha_6 = \alpha_6'$ and $\tau = \tau''$, this contradicts the hypothesis. $\square$

From this lemma, we have to determine, in a first step, the subgroups Γ of $Aut(\mathfrak{h}_3)$ isomorphic a $(\mathbb{Z}_2)^k$ with $k \geq 2$.

• Any involutive automorphism τ commuting with $\tau_1(\alpha_3, \alpha_6)$ with $\tau \neq \tau_1(\alpha_3, \alpha_6)$ is equal to $\tau_2(-\alpha_3, \alpha_5)$ or $\tau_4(\alpha_5, -\alpha_6)$ and we have $\tau_1(\alpha_3, \alpha_6) \circ \tau_2(-\alpha_3, \alpha_5) = \tau_4\left(-\frac{\alpha_3\alpha_6}{2} - \alpha_5, -\alpha_6\right)$ and $\left[\tau_2(-\alpha_3, \alpha_5), \tau_4\left(-\frac{\alpha_3\alpha_6}{2} - \alpha_5, -\alpha_6\right)\right] = 0$. Thus

$$\Gamma_7(\alpha_3, \alpha_5, \alpha_6) = \left\{ Id, \tau_1(\alpha_3, \alpha_6), \tau_2(-\alpha_3, \alpha_5), \tau_4\left(-\frac{\alpha_3\alpha_6}{2} - \alpha_5, -\alpha_6\right) \right\}$$

is a subgroup of $Aut(\mathfrak{h}_3)$ isomorphic to $\mathbb{Z}_2^2$. Moreover it is the only subgroup of $Aut(\mathfrak{h}_3)$ of type $(\mathbb{Z}_2)^k$, $k \geq 2$, containing an automorphism of type $\tau_1(\alpha_3, \alpha_6)$.

• A direct computation shows that any abelian subgroup Γ containing $\tau_2(\alpha_3,\alpha_5)$ is either isomorphic to $\mathbb{Z}_2$ or equal to Γ_7.
• Assume that $\tau_3(\alpha_1,\alpha_3,\alpha_6)\in\Gamma$. The automorphisms $\tau_3(-\alpha_1,-\alpha_2,\alpha_6')$ and $\tau_4(\alpha_5,\alpha_6')$ commute with $\tau_3(\alpha_1,\alpha_3,\alpha_6)$. Since

$$\tau_3(\alpha_1,\alpha_2,\alpha_6)\circ\tau_3(-\alpha_1,-\alpha_2,\alpha_6')=\tau_4\left(\frac{\alpha_6'(1-\alpha_1)-\alpha_6(1+\alpha_1)}{\alpha_2},-\alpha_6-\alpha_6'\right)$$

we obtain the following subgroup, denoted $\Gamma_8(\alpha_1,\alpha_2,\alpha_6,\alpha_6')$:

$$\left\{Id,\tau_3(\alpha_1,\alpha_2,\alpha_6),\tau_3(-\alpha_1,-\alpha_2,\alpha_6'),\tau_4\left(\frac{\alpha_6'(1-\alpha_1)-\alpha_6(1+\alpha_1)}{\alpha_2},-\alpha_6-\alpha_6'\right)\right\}$$

which is isomorphic to $\mathbb{Z}_2^2$.
• We suppose that $\tau_4(\alpha_5,\alpha_6)\in\Gamma$. If Γ is not isomorphic to $\mathbb{Z}_2$, then Γ is one of the groups Γ_7,Γ_8.

Theorem 12. *Any finite abelian subgroup Γ of $Aut(\mathfrak{h}_3)$ isomorphic to $(\mathbb{Z}_2)^k$ is one of the following*

(1) $k=1$, $\Gamma=\Gamma_1(\alpha_3,\alpha_6),\ \Gamma_2(\alpha_3,\alpha_5),\ \Gamma_3(\alpha_1,\alpha_2,\alpha_6),\ \alpha_2\neq 0,\ \Gamma_4(\alpha_5,\alpha_6)$,
(2) $k=2$, $\Gamma=\Gamma_7(\alpha_3,\alpha_5,\alpha_6),\ \Gamma_8(\alpha_1,\alpha_2,\alpha_6,\alpha_6')$.

Assume now that Γ is isomorphic to $\mathbb{Z}_3^{k_3}$ with $k_3\geq 2$. If $\tau\in\Gamma_5$, its matricial representation is

$$\begin{pmatrix}\dfrac{-1-\sqrt{-3-4\alpha_2\alpha_3}}{2} & \alpha_2 & 0\\ \alpha_3 & \dfrac{-1+\sqrt{-3-4\alpha_2\alpha_3}}{2} & 0\\ \alpha_5 & \alpha_6 & 1\end{pmatrix}.$$

To simplify, we put $\lambda=\dfrac{-1-\sqrt{-3-4\alpha_2\alpha_3}}{2}$. The eigenvalues of τ are $1,j,j^2$ and the corresponding eigenvectors $X_3,V,\overline{V}$ with

$$V=\left(1,-\frac{\lambda-j}{\alpha_2},-\frac{\alpha_5}{1-j}+\frac{\alpha_6(\lambda-j)}{\alpha_2(1-j)}\right)$$

if $\alpha_2\neq 0$. If τ' is an automorphism of order 3 commuting with τ, then $\tau'V=jV$ or j^2V. But the two first components of $\tau'(V)$ are $\lambda'-\dfrac{\beta_2}{\alpha_2}(\lambda-j)$, $\beta_3-\frac{\lambda'(\lambda-j)}{\alpha_2}$ where β_i and λ' are the corresponding coefficients of the matrix of τ'. This implies $\alpha_2\lambda'-\beta_2(\lambda-j)=\alpha_2 j$ or $\alpha_2 j^2$. Considering the real and complex parts of this equation, we obtain

$$\begin{cases}\alpha_2\lambda'-\beta_2\lambda=0,\\ \beta_2 j=\alpha_2 j \qquad\text{or}\qquad \alpha_2 j^2.\end{cases}$$

As $\alpha_2\neq 0$, we obtain $\alpha_2=\beta_2$ and $\lambda=\lambda'$. Let us compare the second component of $\tau'(V)$. We obtain $\beta_3\alpha_2-\lambda'(\lambda-j)=-(\lambda-j)j$ or $-(\lambda-j)j^2$. As $\lambda=\lambda'$, we have in the first case $2\lambda j=j^2$ and in the second case $2\lambda j=j^3=1$. In any case, this is impossible. Thus $\alpha_2=0$ and, from Section 2.2, $\tau=Id$. This implies that $k_3=1$ or 0.

Theorem 13. *Let Γ be a finite abelian subgroup of $Aut(\mathfrak{h}_3)$. Thus Γ is isomorphic to one of the following group*

(1) $\mathbb{Z}_2\times\mathbb{Z}_2$,
(2) $\mathbb{Z}_2^{k_2}\times\mathbb{Z}_3^{k_3}\times\cdots\times\mathbb{Z}_p^{k_p}$ *with $k_i=0$ or 1 for $i=2,\cdots,p$.*

To prove the second part, we show as in the case $i = 3$ that $k_i = 1$ as soon as $k_i \neq 0$.

Remark. We have determined the finite abelian subgroups of $Aut(\mathfrak{h}_3)$. There are non-abelian finite subgroups with elements of order at most 3. Take for example the subgroup generated by

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} -\frac{1}{2} & \alpha & 0 \\ -\frac{3}{4\alpha} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \alpha \neq 0.$$

The relations on the generators are $\sigma_1^2 = Id$, $\sigma_2^3 = Id$, $\sigma_1\sigma_2\sigma_1 = \sigma_2^2$. Thus the group generated by σ_1 and σ_2 is isomorphic to the symmetric group Σ_3 of degree 3.

4.2. **Description of the $\mathbb{Z}_2$ and $\mathbb{Z}_2^2$-gradings of $\mathfrak{h}_3$.** Let Γ be a finite abelian subgroup of $Aut(\mathfrak{h}_3)$ isomorphic to $\mathbb{Z}_2^k$ ($k = 1$ or 2).

• If $\Gamma = \mathbb{Z}_2$, we have obtained $\Gamma = \Gamma_i$, $i = 1, 2, 3, 4$. Up to equivalence of gradings, the $\mathbb{Z}_2$-grading of $\mathfrak{h}_3$ are:

$$\mathfrak{h}_3 = \mathbb{R}\{X_2\} \bigoplus \mathbb{R}\{X_1, X_3\} \quad \text{and} \quad \mathfrak{h}_3 = \mathbb{R}\{X_1\} \bigoplus \mathbb{R}\{X_2, X_3\}.$$

• If $\Gamma = \mathbb{Z}_2^2$ then $\Gamma = \Gamma_7$ or $\Gamma = \Gamma_8$.

Lemma 14. *There is an automorphism $\sigma \in Aut(\mathfrak{h}_3)$ such that $\sigma^{-1}\Gamma_7\sigma = \Gamma_8$.*

The proof is a simple computation. There also exists $\sigma \in Aut(\mathfrak{h}_3)$ such that

$$\begin{cases} \sigma^{-1}\tau_1(\alpha_3, \alpha_6)\sigma = \tau_1(0, 0), \\ \sigma^{-1}\tau_2(-\alpha_3, \alpha_5)\sigma = \tau_2(0, 0). \end{cases}$$

We deduce:

Proposition 15. *Every $\mathbb{Z}_2^2$-grading on $\mathfrak{h}_3$ is equivalent to the grading defined by $\Gamma_7(0, 0, 0) = \{Id, \tau_1(0, 0), \tau_2(0, 0), \tau_4(0, 0)\}$.*

This grading corresponds to $\mathfrak{h}_3 = \{0\} \oplus \mathbb{R}\{X_1\} \oplus \mathbb{R}\{X_2\} \oplus \mathbb{R}\{X_3\}$.

4.3. **Non existence of Riemannian symmetric structures on $\mathbb{H}_3/H$.** Consider the symmetric space $\mathbb{H}_3/H_1$ associated with the grading

$$\mathfrak{h}_3 = \mathbb{R}\{X_2\} \bigoplus \mathbb{R}\{X_1, X_3\}.$$

Let $\{\omega_1, \omega_2, \omega_3\}$ be the dual basis of $\{X_1, X_2, X_3\}$. Any pseudo-Riemannian metric on the symmetric space $\mathbb{H}_3/H_1$ where H_1 is a one-dimensional connected Lie group whose Lie algebra $\mathfrak{g}_0 = \mathbb{R}\{X_2\}$ is given by a non-degenerate $ad\mathfrak{g}_0$-invariant bilinear form $B = a\omega_1^2 + b\omega_1 \wedge \omega_3 + c\omega_3^2$ on $\mathfrak{g}_1 = \mathbb{R}\{X_1, X_3\}$. This implies $B([X_2, X_1], X_3) = -B(X_3, X_3) = -c = 0$. But we have also $B([X_2, X_1], X_1) + B(X_1, [X_2, X_1]) = -2B(X_3, X_1) = -2b = 0$. We deduce

Proposition 16. *The nilpotent symmetric space $\mathbb{H}_3/H$ associated to the grading $\mathfrak{h}_3 = \mathbb{R}\{X_2\} \bigoplus \mathbb{R}\{X_1, X_3\}$ doesn't admit any pseudo-Riemannian symmetric metric.*

Consider now the symmetric space $\mathbb{H}_3/H_2$ associated with the grading $\mathfrak{h}_3 = \mathbb{R}\{X_3\} \bigoplus \mathbb{R}\{X_1, X_2\}$. Then H_2 is the Lie subgroup whose Lie algebra is $\mathbb{R}\{X_3\}$ and the bilinear form $B = a\omega_1^2 + b\omega_1 \wedge \omega_2 + c\omega_2^2$ on $\mathfrak{g}_1 = \mathbb{R}\{X_1, X_2\}$ is adX_3-invariant because $adX_3 = 0$. But Ad_G is an homomorphism of G onto the group of inner automorphisms of $\mathfrak{g}$ with kernel the center of G, we deduce that $Ad_G(H)$ is compact in this case and any non-degenerate bilinear form B on $\mathfrak{g}_1$ defines a Riemannian or a Lorentzian structure on the symmetric space $\mathbb{H}_3/H_2$.

Proposition 17. *The nilpotent symmetric space $\mathbb{H}_3/H_2$ associated to the grading $\mathfrak{h}_3 = \mathbb{R}\{X_3\} \bigoplus \mathbb{R}\{X_1, X_2\}$ admits a structure of Riemannian symmetric space. It admits also a structure of Lorentzian symmetric space.*

4.4. **Riemannian $\mathbb{Z}_2^2$-symmetric structures on $\mathbb{H}_3$.** Consider on $\mathbb{H}_3$ a $\mathbb{Z}_2^2$-symmetric structure. It is determined, up to equivalence, by the $\mathbb{Z}_2^2$-grading of $\mathfrak{h}_3$

$$\mathfrak{h}_3 = \{0\} \oplus \mathbb{R}\{X_1\} \oplus \mathbb{R}\{X_2\} \oplus \mathbb{R}\{X_3\}.$$

Since every automorphism of $\mathfrak{h}_3$ is an isometry of any invariant Riemannian metric on $\mathbb{H}_3$, we deduce

Theorem 18. *Any Riemannian $\mathbb{Z}_2^2$-symmetric structure on $\mathbb{H}_3$ is isometric to the Riemannian structure associated with the grading $\mathfrak{h}_3 = \{0\} \oplus \mathbb{R}\{X_1\} \oplus \mathbb{R}\{X_2\} \oplus \mathbb{R}\{X_3\}$ and the Riemannian $\mathbb{Z}_2^2$-symmetric metric is written $g = \omega_1^2 + \omega_2^2 + \lambda^2\omega_3^2$ with $\lambda \neq 0$, where $\{\omega_1, \omega_2, \omega_3\}$ is the dual basis of $\{X_1, X_2, X_3\}$.*

Proof. Indeed, since the components of the grading are orthogonal, the Riemannian metric g, which coincides with the form B satisfies $g = \alpha_1\omega_1^2 + \alpha_2\omega_2^2 + \alpha_3\omega_3^2$ with $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_3 > 0$.

According to [6], we reduce the coefficients to $\alpha_1 = \alpha_2 = 1$. □

Remark. According to [7] and [10], this metric is naturally reductive for any λ.

Corollary 19. *A Riemannian tensor g on $\mathbb{H}_3$ determines a Riemannian $\mathbb{Z}_2^2$-symmetric structure over $\mathbb{H}_3$ if and only if it is a left-invariant metric on $\mathbb{H}_3$.*

This is a consequence of the previous theorem and of the classification of left-invariant metrics on Heisenberg groups ([6]).

4.5. **Lorentzian $\mathbb{Z}_2^2$-symmetric structures on $\mathbb{H}_3$.** We say that an homogeneous space $(M = G/H, g)$ is *Lorentzian* if the canonical action of G on M preserves a Lorentzian metric (i.e. a smooth field of non-degenerate quadratic forms of signature $(n-1, 1)$) (see [3]).

Proposition 20 ([5])**.** *Modulo an automorphism and a multiplicative constant, there exists on $\mathbb{H}_3$ one left-invariant metric assigning a strictly positive length on the center of $\mathfrak{h}_3$.*

The Lie algebra $\mathfrak{h}_3$ is generated by the central vector X_3 and X_1 and X_2 such that $[X_1, X_2] = X_3$. The automorphisms of the Lie algebra preserve the center and then send the element X_3 on λX_3, with $\lambda \in \mathbb{R}^*$. Such an automorphism acts on the plane generated by X_1 and X_2 as an automorphism of determinant λ.
It is shown in [17] and [18] that, modulo an automorphism of $\mathfrak{h}_3$, there are three classes of invariant Lorentzian metrics on $\mathbb{H}_3$, corresponding to the cases where $||X_3||$ is negative, positive or zero.
We propose to look at the Lorentzian metrics that are associated with the $\mathbb{Z}_2^2$-symmetric structures over $\mathbb{H}_3$. If $\mathfrak{g}$ is the Heisenberg algebra equipped with a $\mathbb{Z}_2^2$-grading, then by automorphism, we can reduce to the case where $\Gamma = \Gamma_7$. In this case, the grading of $\mathfrak{h}_3$ is given by:

$$\mathfrak{h}_3 = \mathfrak{g}_0 + \mathfrak{g}_{+-} + \mathfrak{g}_{-+} + \mathfrak{g}_{--}$$

with $\mathfrak{g}_0 = \{0\}$, and

$$\mathfrak{g}_{+-} = \mathbb{R}\left\{X_2 - \frac{\alpha_6}{2}X_3\right\}, \qquad \mathfrak{g}_{-+} = \mathbb{R}\left\{X_1 - \frac{\alpha_3}{2}X_2 + \frac{\alpha_5}{2}X_3\right\}, \quad \mathfrak{g}_{--} = \mathbb{R}\left\{X_3\right\}.$$

Assume $Y_1 = X_1 - \frac{\alpha_3}{2}X_2 + \frac{\alpha_5}{2}X_3$, $Y_2 = X_2 - \frac{\alpha_6}{2}X_3$, $Y_3 = X_3$. The dual basis is

$$\vartheta_1 = \omega_1 \qquad \vartheta_2 = \omega_2 + \frac{\alpha_3}{2}\omega_1 \qquad \vartheta_3 = \omega_3 - \frac{\alpha_6}{2}\omega_2 - \left(\frac{\alpha_3\alpha_6}{4} + \frac{\alpha_5}{2}\right)\omega_1$$

where $\{\omega_1, \omega_2, \omega_3\}$ is the dual basis of the base $\{X_1, X_2, X_3\}$.

Case I The components $\mathfrak{g}_{+-}$, $\mathfrak{g}_{-+}$, $\mathfrak{g}_{--}$ are non-degenerate. The quadratic form induced on $\mathfrak{h}_3$ therefore writes

$$g = \lambda_1\omega_1^2 + \lambda_2\left(\omega_2 + \frac{\alpha_3}{2}\omega_1\right)^2 + \lambda_3\left(\omega_3 - \frac{\alpha_6}{2}\omega_2 - \left(\frac{\alpha_5}{2} + \frac{\alpha_3\alpha_6}{4}\right)\omega_1\right)^2$$

with $\lambda_1, \lambda_2, \lambda_3 \neq 0$. The change of basis associated with the matrix

$$\begin{pmatrix} 1 & 0 & \\ \frac{\alpha_3}{2} & 1 & 0 \\ -\frac{\alpha_5}{2} - \frac{\alpha_3\alpha_6}{4} & -\frac{\alpha_6}{2} & 1 \end{pmatrix}$$

is an automorphism. Thus g is isometric to $g = \lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2$. Since the signature is $(2,1)$ one of the λ_i is negative and the two others positive.

Proposition 21. *Every Lorentzian $\mathbb{Z}_2^2$-symmetric metric g on $\mathbb{H}_3$ such that the components of the grading of $\mathfrak{h}_3$ are non-degenerate, is reduced to one of these two forms: $g = -\omega_1^2 + \omega_2^2 + \lambda^2\omega_3^2$ or $g = \omega_1^2 + \omega_2^2 - \lambda^2\omega_3^2$*

Case II Suppose that a component is degenerate. When this component is $\mathbb{R}\{X_2 + \frac{\alpha_6}{2}X_3\}$ or $\mathbb{R}\{X_1 - \frac{\alpha_3}{2}X_2 + \frac{\alpha_5}{2}X_3\}$ then, by automorphism, it reduces to the above case.
Suppose then that the component containing the center is degenerate.
Thus the quadratic form induced on $\mathfrak{h}_3$ is written

$$g = \omega_1^2 + \left[\omega_3 - \frac{\alpha_6}{2}\omega_2 - \frac{2\alpha_5 + \alpha_3\alpha_6}{4}\omega_1\right]^2 - \left[\omega_2 - \omega_3 + \frac{\alpha_6}{2}\omega_2 + \frac{2\alpha_5 + \alpha_3\alpha_6}{4}\omega_1\right]^2.$$

The change of basis associated with the matrix

$$\begin{pmatrix} 1 & 0 & \\ \frac{\alpha_3}{2} & 1 & 0 \\ -\frac{\alpha_5}{2} - \frac{\alpha_3\alpha_6}{4} & -\frac{\alpha_6}{2} & 1 \end{pmatrix}$$

is given by an automorphism. Thus g is isomorphic to $g = \omega_1^2 + \omega_3^2 - (\omega_2 - \omega_3)^2$.

Proposition 22. *Every Lorentzian $\mathbb{Z}_2^2$-symmetric metric g on $\mathbb{H}_3$ such that the component of the grading of $\mathfrak{h}_3$ containing the center is degenerate, is reduced to the form $g = \omega_1^2 + \omega_3^2 - (\omega_2 - \omega_3)^2$.*

Corollary 23. *A Lorentzian tensor g on $\mathbb{H}_3$ determines a Lorentzian $\mathbb{Z}_2^2$-symmetric structure over $\mathbb{H}_3$ if and only if it is a left-invariant Lorentzian metric on $\mathbb{H}_3$.*

The classification, up to isometry, of left-invariant Lorentzian metrics on $\mathbb{H}_3$ is described in [4] and in [18]. It corresponds to the previous classification of Lorentzian $\mathbb{Z}_2^2$-symmetric metrics.

5. $\mathbb{Z}_2^k$-symmetric spaces based on $\mathbb{H}_{2p+1}$

5.1. $\mathbb{Z}_2^k$-gradings of $\mathfrak{h}_{2p+1}$. Let σ be an involutive automorphism of the $(2p+1)$-dimensional Heisenberg algebra $\mathfrak{h}_{2p+1}$. Let $\{X_1, \cdots, X_{2p+1}\}$ be a basis of $\mathfrak{h}_{2p+1}$ whose structure constants are given by

$$[X_1, X_2] = \cdots = [X_{2p-1}, X_{2p}] = X_{2p+1}.$$

Since the center $\mathbb{R}\{X_{2p+1}\}$ is invariant by σ, it is contained in an homogeneous component of the grading $\mathfrak{h}_{2p+1} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ associated with σ. But for any $X \in \mathfrak{h}_{2p+1}$, $X \neq 0$, there exists $Y \neq 0$ such that $[X, Y] = aX_{2p+1}$ with $a \neq 0$. We deduce that any $\mathbb{Z}_2$-grading is equivalent to one of the following:

(1) If $X_{2p+1} \in \mathfrak{g}_0$, then
- $\mathfrak{h}_{2p+1} = \mathbb{R}\{X_{2p+1}\} \oplus \mathbb{R}\{X_1, X_2, \cdots, X_{2p}\}$
- $\mathfrak{h}_{2p+1} = \mathbb{R}\{X_1, X_2, X_3, \cdots, X_{2k}, X_{2p+1}\} \oplus \mathbb{R}\{X_{2k+1}, X_{2k+2}, \cdots, X_{2p}\}$

(2) If $X_{2p+1} \in \mathfrak{g}_1$, then
- $\mathfrak{h}_{2p+1} = \mathbb{R}\{X_1, X_3, \cdots, X_{2p-1}\} \oplus \mathbb{R}\{X_2, X_4, \cdots, X_{2p}, X_{2p+1}\}$
- $\mathfrak{h}_{2p+1} = \mathbb{R}\{X_2, X_4, \cdots, X_{2p}\} \oplus \mathbb{R}\{X_1, X_3, \cdots, X_{2p-1}, X_{2p+1}\}$.

Let $\mathfrak{h}_{2p+1} = \bigoplus_{\gamma \in \mathbb{Z}_2^k} \mathfrak{g}_\gamma$ be a $\mathbb{Z}_2^k$-grading of the Heisenberg algebra. The support of this grading is the subset $\{\gamma \in \mathbb{Z}_2^k,\ \mathfrak{g}_\gamma \neq 0\}$. We will say that this grading is irreducible if the subgroup of $\mathbb{Z}_2^k$ generated by its support is the full group $\mathbb{Z}_2^k$.

Lemma 24. *If $\mathfrak{h}_{2p+1}$ admits an irreducible $\mathbb{Z}_2^k$-grading, then $k = 1$ or $k = 2$.*

In fact, this is a consequence of the previous classification of the $\mathbb{Z}_2$-gradings of $\mathfrak{h}_{2p+1}$. We deduce also that any $\mathbb{Z}_2^2$-grading is equivalent to

$$\mathfrak{h}_{2p+1} = \{0\} \oplus \mathbb{R}\{X_{2p+1}\} \oplus \mathbb{R}\{X_1, X_3, \cdots, X_{2p-1}\} \oplus \mathbb{R}\{X_2, X_4, \cdots, X_{2p}\}.$$

5.2. **Pseudo-Riemannian symmetric spaces $\mathbb{H}_{2p+1}/H$.** We consider the symmetric spaces $\mathbb{H}_{2p+1}/H$ corresponding to the previous symmetric decomposition of $\mathfrak{h}_{2p+1}$, where H is a connected Lie subgroup of $\mathbb{H}_{2p+1}$ whose Lie algebra is $\mathfrak{g}_0$.
• With the $\mathbb{Z}_2$-grading $\mathfrak{h}_{2p+1} = \mathbb{R}\{X_{2p+1}\} \oplus \mathbb{R}\{X_1, X_2, \cdots, X_{2p}\}$. Since $ad(X_{2p+1})$ is zero any non-degenerate bilinear form on $\mathfrak{g}_1$ defines a symmetric pseudo-Riemannian metric on $\mathbb{H}_{2p+1}/H$ where H is a connected one-dimensional Lie Group.
• Consider the $\mathbb{Z}_2$-grading

$$\mathfrak{h}_{2p+1} = \mathbb{R}\{X_1, X_2, X_3, \cdots, X_{2k}, X_{2p+1}\} \oplus \mathbb{R}\{X_{2k+1}, X_{2k+2}, \cdots, X_{2p}\}$$

In this case, H is a Lie subgroup isomorphic to $\mathbb{H}_{2k+1}$. Since we have $[\mathfrak{g}_0, \mathfrak{g}_1] = 0$, any non-degenerate bilinear form on $\mathfrak{g}_1$ defines a symmetric pseudo-Riemannian metric on $\mathbb{H}_{2p+1}/\mathbb{H}_{2k+1}$. $\blacksquare$
• We consider the $\mathbb{Z}_2$-gradings

$$\begin{aligned} &\mathfrak{h}_{2p+1} = \mathbb{R}\{X_1, X_3, \cdots, X_{2p-1}\} \oplus \mathbb{R}\{X_2, X_4, \cdots, X_{2p}, X_{2p+1}\} \\ \text{or}\quad &\mathfrak{h}_{2p+1} = \mathbb{R}\{X_2, X_4, \cdots, X_{2p}\} \oplus \mathbb{R}\{X_1, X_3, \cdots, X_{2p-1}, X_{2p+1}\}. \end{aligned}$$

In this case, any bilinear form on $\mathfrak{g}_1$ which is $ad(\mathfrak{g}_0)$-invariant is degenerate. In fact, if B is such a form, we have

$$B([X_{2k+1}, X_{2k+2}], X_1) = B(X_{2p+1}, X_{2p+1}) = 0$$

and for any $k = 0, \cdots, p-1$ and $s \neq k+1$

$$B([X_{2k+1}, X_{2k+2}], X_{2s}) = B(X_{2p+1}, X_{2s}) = 0,$$

and X_{2p+1} is in the kernel of B. We have the same proof for the second grading.

Proposition 25. *The symmetric spaces $\mathbb{H}_{2p+1}/H$ corresponding to the $\mathbb{Z}_2$-grading of $\mathfrak{h}_{2p+1}$:*
- $\mathfrak{h}_{2p+1} = \mathbb{R}\{X_1, X_3, \cdots, X_{2p-1}\} \oplus \mathbb{R}\{X_2, X_4, \cdots, X_{2p}, X_{2p+1}\}$
- $\mathfrak{h}_{2p+1} = \mathbb{R}\{X_2, X_4, \cdots, X_{2p}\} \oplus \mathbb{R}\{X_1, X_3, \cdots, X_{2p-1}, X_{2p+1}\}$

are not pseudo-Riemannian symmetric spaces.

5.3. **Riemannian $\mathbb{Z}_2^2$-symmetric spaces $\mathbb{H}_{2p+1}/H$.** Let us consider the $\mathbb{Z}_2^2$-grading of the Heisenberg algebra

$$\mathfrak{h}_{2p+1} = \{0\} \oplus \mathbb{R}\{X_{2p+1}\} \oplus \mathbb{R}\{X_1, X_3, \cdots, X_{2p-1}\} \oplus \mathbb{R}\{X_2, X_4, \cdots, X_{2p}\}.$$

Since $\mathfrak{g}_0 = \{0\}$, then H is reduced to the identity and the $\mathbb{Z}_2^2$-symmetric space $\mathbb{H}_{2p+1}/H$ is isomorphic to $\mathbb{H}_{2p+1}$. The reductive decomposition $\mathfrak{h}_{2p+1} = \mathfrak{g}_0 \oplus \mathfrak{m}$ is reduced to $\mathfrak{m}$. Since $\mathfrak{g}_0 = \{0\}$, any bilinear definite positive form on $\mathfrak{m}$ for which the homogeneous components

$\mathbb{R}\{X_{2p+1}\}$, $\mathbb{R}\{X_1, X_3, \cdots, X_{2p-1}\}$ and $\mathbb{R}\{X_1, X_4, \cdots, X_{2p}\}$ are pairwise orthogonal defines a Riemannian $\mathbb{Z}_2^2$-symmetric structure on $\mathbb{H}_{2p+1}$.

The Levi-Civita connection associated with this Riemannian metric is an affine connection. In case of Riemannian symmetric space, the Levi-Civita connection associated with the Riemannian symmetric metric is torsion-free and the curvature tensor R satisfies $\nabla R = 0$, where ∇ is the covariant derivative of this connection, and correspond to the canonical connection defined in [11] which defines the natural affine structure on a symmetric space. This is not the case for Riemannian $\mathbb{Z}_2^2$-symmetric spaces. In the next section, we define a class of affine connections adapted to the $\mathbb{Z}_2^2$-symmetric structures, and we prove, in case of the Riemannian $\mathbb{Z}_2^2$-symmetric space $\mathbb{H}_{2p+1}/H$, that there exist adapted connections with torsion and curvature-free.

5.4. **Adapted affine connections on the $\mathbb{Z}_2^2$-symmetric spaces $\mathbb{H}_{2p+1}/H$.** Let G/H be a $\mathbb{Z}_2^k$-symmetric space. Since G/H is a reductive homogeneous space, that is $\mathfrak{g}$ admits a decomposition $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{m}$ with $[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0$ and $[\mathfrak{g}_0, \mathfrak{m}] \subset \mathfrak{m}$, any connection is given by a linear map

$$\bigwedge : \mathfrak{m} \to gl(\mathfrak{m})$$

satisfying

$$\bigwedge [X, Y] = \left[\bigwedge(X), \lambda(Y)\right]$$

for all $X \in \mathfrak{m}$ and $Y \in \mathfrak{g}_0$, where λ is the linear isotropy representation of $\mathfrak{g}_0$. The corresponding torsion and curvature tensors are given by:

$$\begin{array}{rl} & T(X,Y) = \bigwedge(X)(Y) - \bigwedge(Y)(X) - [X,Y]_{\mathfrak{m}} \\ \text{and} & R(X,Y) = \left[\bigwedge(X), \bigwedge(Y)\right] - \bigwedge[X,Y] - \lambda([X,Y]_{\mathfrak{g}_0}) \end{array}$$

for any $X, Y \in \mathfrak{m}$.

Definition 26. *Consider the affine connection on the $\mathbb{Z}_2^k$-symmetric space G/H defined by the linear map*

$$\bigwedge : \mathfrak{m} \to gl(\mathfrak{m}).$$

Then this connection is called adapted to the $\mathbb{Z}_2^k$-symmetric structure, if any

$$\bigwedge(X_\gamma)(\mathfrak{g}_{\gamma'}) \subset \mathfrak{g}_{\gamma\gamma'}$$

for any $\gamma, \gamma' \in \mathbb{Z}_2^k$, $\gamma, \gamma \neq 0$. The connection is called homogeneous if any homogeneous component $\mathfrak{g}_\gamma$ of $\mathfrak{m}$ is invariant by $\bigwedge$.

Now we consider the case where $G/H = \mathbb{H}_{2p+1}/H$ is the $\mathbb{Z}_2^2$-symmetric space defined by the grading

$$\mathfrak{h}_{2p+1} = \{0\} \oplus \mathbb{R}\{X_{2p+1}\} \oplus \mathbb{R}\{X_1, X_3, \cdots, X_{2p-1}\} \oplus \mathbb{R}\{X_2, X_4, \cdots, X_{2p}\}.$$

We have seen that H is reduced to the identity and $\mathbb{H}_{2p+1}/H$ is isomorphic to $\mathbb{H}_{2p+1}$. Consider an adapted connection and let $\bigwedge$ be the associated linear map. Since the connection is adapted to the $\mathbb{Z}_2^2$-symmetric structure, $\bigwedge$ satisfies:

$$\left\{ \begin{array}{lll} \bigwedge(X_{2k+1})(X_{2l+1}) = \bigwedge(X_{2s})(X_{2t}) = 0, & k,l = 0, \cdots, p-1, & s,t = 1, \cdots, p, \\ \bigwedge(X_{2k+1})(X_{2s}) = C_s^{2k+1} X_{2p+1}, & s = 1, \cdots, p, & k = 0, \cdots, p-1, \\ \bigwedge(X_{2s})(X_{2k+1}) = C_k^{2s} X_{2p+1}, & s = 1, \cdots, p, & k = 0, \cdots, p-1, \\ \bigwedge(X_{2k+1})(X_{2p+1}) = \sum_{s=1}^p a_{2k+1}^s X_{2s}, & k = 0, \cdots, p-1, & \\ \bigwedge(X_{2s})(X_{2p+1}) = \sum_{k=0}^p a_{2s}^k X_{2k+1}, & s = 1, \cdots, p. & \end{array} \right.$$

Theorem 27. *Any adapted connection ∇ on the $\mathbb{Z}_2^2$-symmetric space $\mathbb{H}_{2p+1}/H = \mathbb{H}_{2p+1}$ satisfies $T = 0$ and $R = 0$ where T and R are respectively the torsion and the curvature of ∇ if and only if the corresponding linear map $\bigwedge$ satisfies*

$$\begin{cases} \bigwedge(X_{2k+1})(X_{2s}) = C_s^{2k+1} X_{2p+1}, & s = 1, \cdots, p, \quad k = 0, \cdots, p-1, \\ \bigwedge(X_{2k+1})(X_i) = 0, & k = 0, \cdots, p-1, \quad i \notin \{2, \cdots, 2p\}, \end{cases}$$

$$\begin{cases} \bigwedge(X_{2s})(X_{2k+1}) = C_s^{2k+1} X_{2p+1}, & s = 1, \cdots, p, \quad k = 0, \cdots, p-1, k \neq s-1, \\ \bigwedge(X_{2s})(X_{2s-1}) = (C_s^{2k+1} - 1) X_{2p+1}, & s = 1, \cdots, p, \\ \bigwedge(X_{2s})(X_i) = 0, & s = 1, \cdots, p, \quad i \notin \{1, \cdots, 2p-1\}. \end{cases}$$

In fact, we determine in a first step, all the connection adapted to the $\mathbb{Z}_2^2$-symmetric structure and which are torsion-free. In this case, $\bigwedge$ satisfies

$$\bigwedge(X)(Y) - \bigwedge(Y)(X) - [X, Y] = 0, \text{ for any } X, Y \in \mathfrak{h}_{2p+1}.$$

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