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On the sliding-mode control of fractional-order nonlinear uncertain dynamics

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SUMMARY

This paper deals with applications of sliding-mode based fractional control techniques to address tracking and stabilization control tasks for some classes of nonlinear uncertain fractional-order systems. Both single-input and multi-input systems are considered. A second-order sliding-mode approach is taken, in suitable combination with PI-based design, in the single-input case, while the unit-vector approach is the main tool of reference in the multi-input case. Sliding manifolds containing fractional derivatives of the state variables are used in the present work. Constructive tuning conditions for the control parameters are derived by Lyapunov analysis, and the convergence properties of the proposed schemes are supported by simulation results. Copyright © John Wiley & Sons, Ltd.

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1. INTRODUCTION

Fractional-order systems (FOSs), i.e. dynamical systems described using fractional (or, more precisely, non-integer) order derivative and integral operators, are studied with growing interest in recent years. It has been pointed out that a large number of physical phenomena can be modeled effectively by means of fractional-order models [19]. Known examples are found in the areas of bioengineering [13], transport phenomena [1, 32], economy [22], mechanics [2], and others (see [19, 33]).

The long-range temporal or spatial hereditary phenomena inherent to the FOSs present unique and intriguing peculiarities, not supported by their integer-order counterpart, which raise numerous challenges and opportunities related to the development of control and estimation methodologies involving fractional order dynamics [29, 11, 18, 9, 10].

Although fractional calculus has been previously combined with the sliding-mode control methodology in the controller design for conventional integer-order systems [7, 4], sliding mode control (SMC) has been applied to fractional-order systems only recently, see [21, 8, 7]. In [21] perfectly known linear multivariable dynamics were studied, and a first-order sliding mode stabilizing controller was suggested. Sliding manifolds containing fractional-order derivatives were used in [21] in combination with conventional relay control techniques. The same type of sliding manifolds has been later used, along with second-order sliding mode control methodologies, to address control, observation and fault detection tasks for certain classes of uncertain linear FOS [15, 16]. Among the recent works on first-order SMC for fractional-order dynamics we mention [23], where a class of nonlinear multi-input FOS with uncertain control matrix was dealt with under the requirement that a “sufficiently accurate” estimation of the uncertain control matrix is known in advance. In [8] perfectly known nonlinear single-input fractional-order dynamics expressed in a form that can be considered as a fractional-order version of the chain-of-integrators “Brunowsky” normal form were studied.

In this paper, the tracking control problem for a class of fractional-order uncertain single-input processes in canonical Brunovsky form is studied first. Sliding-mode based tracking control of fractional-order systems expressed in such canonical form, which generalizes to the fractional-order systems setting the widely studied corresponding integer-order counterpart, was already studied in earlier works [8, 23] by means of first-order sliding-mode control techniques suitably tailored to the fractional systems setting. In [8], with reference to a more general non-commensurate form of the considered class of systems, a discontinuous control law was suggested under the strong requirement that neither uncertainties nor perturbations were admitted to affect the plant to be controlled. In [23], such results were improved in different directions. First of all, uncertainties and perturbations were admitted, satisfying smoothness restrictions similar to those considered in the present work. As for the control law, the authors presented a technique inspired to the first-order (i.e., relay-based) sliding mode control approach. Interestingly, the control input was continuous and belonging to the class $C^{1-\alpha}$, where $\alpha \in (0, 1)$ is the commensurate order of differentiation. Thus, when α approaches the unit value an almost-discontinuous control input is obtained. The authors of [23] recognized this fact suggesting the use of smooth approximations of the discontinuous sign function to alleviate the chattering phenomenon originated by the hard nonlinearity in the definition of the control law. In the present paper we follow a different approach based on the main novelty of using the second-order sliding mode approach, rather than the first-order sliding mode one, along with a special ad-hoc definition of the sliding manifold, different from that used in [8, 23]. Second-order sliding mode algorithms (see e.g. [17]) actually constitute one of the most popular and widely used sliding-mode based approaches, as they solve the chattering issue (due to higher smoothness in the corresponding control laws, as compared to the conventional first-order sliding mode control algorithms) and simultaneously provide higher control accuracy. Thanks to the combined use of the second order sliding mode approach and the specially designed sliding surface, in this paper we achieve the goal of robust tracking of desired state reference trajectories by means of a control law which is of class C^1 whatever the commensurate order of differentiation is, thereby improving the smoothness of the control as compared to [23].

Additionally, a class of uncertain multi-input FOSs, whose dynamics is affected by a state- and time-dependent uncertain nonlinearity and whose high-frequency gain control matrix is also uncertain, is dealt with. A generalization of the “unit vector” control strategy [3] is suggested to stabilize the states of the system. The main improvement against the related result presented in [23] is that we have relaxed the admitted class of uncertain high-frequency gain control matrices. More precisely, while in [23] it was required to know a “sufficiently accurate” invertible approximation of the HFG matrix, here we do consider it as completely uncertain and we only assume that its symmetric part is positive definite with a known lower bound to its positive real eigenvalues. This lowers significantly the amount of knowledge required on the controlled plant.

Preliminary results were given in our earlier works [30] and [31] for the single input and multi input cases, respectively. As compared to [30] we study here a state tracking problem, whereas a simpler state stabilization problem was considered previously. In addition, we allow uncertain nonlinearities to enter the system dynamics. As for the MIMO case, in comparison to [31] we broaden in this work the controlled class of plants by including state and time dependent nonlinearities, whereas a drift term depending only on the time variable, but not on the system’s state, was considered previously.

The paper is structured as follows. In the next Section 2 the main definitions and properties of fractional order derivatives and integrals are recalled, with emphasis on their compositions which play an important role in our successive developments. In the Sections 3 and 4 the previously outlined single-input and multi-input cases are considered. Lyapunov based analysis supports the claimed convergence properties in both cases. Section 5 presents some computer simulations, including comparative performance analyses with respect to existing controllers. Concluding remarks and perspectives for next research are given in the final Section 6.

2. FRACTIONAL OPERATORS AND THEIR PROPERTIES

In the present paper, all fractional integrals and derivatives are defined with lower terminal (limit) equal to zero. In order to make the notation less cumbersome and more elegant, this will not be emphasized further in the text.

Definition 1. (Left) **Riemann-Liouville fractional integral** of order $\alpha > 0$ of a given signal $f(t)$ at time instant $t \geq 0$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t - \tau)^{\alpha-1} d\tau, \quad (1)$$

where Γ denotes the Euler gamma function [9].

For integer values of α , (1) reduces to the well-known Cauchy repeated integration formula (see [18]). It can also be shown that when α approaches zero the fractional integral (1) reduces to the identity operator (in the weak sense, see [20]). In the current paper, fractional integral of order zero is taken by definition to be the identity operator, i.e.

$$I^0 f(t) = f(t). \quad (2)$$

Definition 2. (Left) **Riemann-Liouville fractional derivative** of order $\alpha > 0$ of a given signal $f(t)$ at time instant $t \geq 0$ is defined as the n^{th} derivative of the left Riemann-Liouville fractional integral of order $n - \alpha$, where n is the smallest integer greater than, or equal to, α

$${}^{RL}D^\alpha f(t) = \left(\frac{d}{dt}\right)^n I^{n-\alpha} f(t). \quad (3)$$

Definition 3. (Left) **Caputo fractional derivative** of order $\alpha > 0$ of a given signal $f(t)$ at time instant $t \geq 0$ is defined as the left Riemann-Liouville fractional integral of order $n - \alpha$ of the n^{th} derivative of $f(t)$, where n is the smallest integer greater than, or equal to, α

$${}^C D^\alpha f(t) = I^{n-\alpha} \left(\frac{d}{dt}\right)^n f(t). \quad (4)$$

It is of interest to note that for $\alpha = n$ (n being an integer) both Riemann-Liouville and Caputo derivative coincide with the “classical” derivative of order n . This is a direct consequence of (2). Also, for $\alpha \in (0, 1)$ the two previously defined fractional derivatives are related by the following expression

$${}^{RL}D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{f(0)}{t^\alpha} + {}^C D^\alpha f(t). \quad (5)$$

A similar relation also hold in a more general case of arbitrary positive α [9]. Relation (5) claims that the two fractional derivative definitions differ by a decaying term depending on the initial conditions. When all initial conditions are zero, Riemann-Liouville and Caputo operator coincide.

The following useful properties of the fractional integral and differential operators will be used in the sequel. The proofs can be found in a number of well-known textbooks (see e.g. Kilbas, *et al* [9] and Podlubny [18]).

Lemma 1. *The Riemann-Liouville fractional integral satisfies the **semigroup property**. Let $\alpha > 0$ and $\beta > 0$, then*

$$I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) = I^{\alpha+\beta} f(t) \quad (6)$$

Lemma 2. *The Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$ is the left inverse of the Riemann-Liouville fractional integral of the same order,*

$${}^{RL}D^\alpha I^\alpha f(t) = f(t), \quad (7)$$

for almost all $t \geq 0$. The opposite is, however, not true, since

$$I^\alpha {}^{RL}D^\alpha f(t) = f(t) - \frac{f_{1-\alpha}(0)}{\Gamma(\alpha)} t^{\alpha-1}, \quad (8)$$

where $f_{1-\alpha}(0) = \lim_{t \rightarrow 0} I^{1-\alpha} f(t)$.

Lemma 3. *The following is true when applying fractional integral operation to the Caputo fractional derivative of the same order*

$$I^\alpha {}^C D^\alpha f(t) = f(t) - f(0). \quad (9)$$

It is important to notice that, unlike the classical derivative, the fractional derivatives do not commute. For any positive α and β

$${}^{RL}D^\alpha {}^{RL}D^\beta f(t) = {}^{RL}D^{\alpha+\beta} f(t) - \sum_{j=1}^n \frac{{}^{RL}D^{\alpha-j} f(0)}{\Gamma(1-j-\alpha)} t^{-j-\alpha}, \quad (10)$$

with n being the smallest integer less than or equal to β [9]. The similar expression can be derived for Caputo derivatives also, using (10) and (5). Thus, in general

$${}^{RL}D^\alpha {}^{RL}D^\beta f(t) \neq {}^{RL}D^\beta {}^{RL}D^\alpha f(t) \neq {}^{RL}D^{\alpha+\beta} f(t), \quad (11)$$

$${}^C D^\alpha {}^C D^\beta f(t) \neq {}^C D^\beta {}^C D^\alpha f(t) \neq {}^C D^{\alpha+\beta} f(t). \quad (12)$$

However, by definition the following equalities hold for all $n \in \mathbb{N}$, $\alpha \in \mathbb{R}^+$, and for any signal $f(t)$

$$\frac{d^n}{dt^n} {}^{RL}D^\alpha f(t) = {}^{RL}D^{n+\alpha} f(t), \quad (13)$$

$${}^C D^\alpha \frac{d^n}{dt^n} f(t) = {}^C D^{n+\alpha} f(t). \quad (14)$$

In some applications of fractional calculus, fractional derivatives of some are sequentially applied multiple times to the same signal. The combined action of these multiple derivative operators forms a separate “higher order” derivative operator called *sequential derivative*, see [18]. Such sequential derivatives, formed by multiple application of the Caputo derivative, will be utilized in the present paper in accordance with the next definition.

Definition 4. Sequential Caputo fractional derivative of order $\alpha \in (0, 1)$ and multiplicity $n \in \mathbb{N}$ of a given signal $f(t)$ at time instant $t \geq 0$ is defined as n -times repeated Caputo derivative of order α , i.e.

$${}^C \mathcal{D}^{n,\alpha} f(t) = \underbrace{{}^C D^\alpha {}^C D^\alpha \dots {}^C D^\alpha}_{n \text{ times}} f(t) \quad (15)$$

Note that the sequential Caputo derivative of α and multiplicity n is different from the Caputo derivative of the order $n\alpha$. However, assuming all initial condition of signal f are zero, the two definitions coincide. Under the same restriction on the initial conditions, all previously introduced definitions of fractional derivatives are equivalent. In fact, in the case of zero initial conditions, all fractional operators commute and meet the semi-group property, and any fractional derivative can

be seen as both left and right inverse to the Riemann-Liouville fractional integral. The following relations then hold

$${}^{RL}D^\alpha f(t) = {}^C D^\alpha f(t) = {}^C \mathcal{D}^{1,\alpha} f(t), \quad (16)$$

$$D^{n\alpha} f(t) = {}^C \mathcal{D}^{n,\alpha} f(t), \quad (17)$$

$$D^\alpha D^\beta f(t) = D^{\alpha+\beta} f(t), \quad (18)$$

$$D^\alpha I^\alpha f(t) = I^\alpha D^\alpha f(t) = f(t), \quad (19)$$

where D^α denotes a fractional derivative of any type (Riemann-Liouville, Caputo or sequential Caputo).

The next Lemma, that will be instrumental in the present treatment, was proven in [15].

Lemma 4. *Consider an arbitrary signal $z(t) \in \mathbb{R}$. Let $\beta \in (0, 1)$. If there exists $T < \infty$ such that*

$$I^\beta z(t) = 0 \quad \forall t \geq T \quad (20)$$

then

$$\lim_{t \rightarrow \infty} z(t) = 0. \quad (21)$$

3. FRACTIONAL SLIDING-MODE CONTROL FOR NONLINEAR SINGLE-INPUT FOS

We consider nonlinear uncertain commensurate-order fractional systems governed by the “chain of (fractional) integrators” dynamic model

$$\begin{aligned} {}^C D^\alpha x_i &= x_{i+1}, \quad i = 1, 2, \dots, n-1, \\ {}^C D^\alpha x_n &= f(\mathbf{x}, t) + u(t) + \psi(t). \end{aligned} \quad (22)$$

where $\alpha \in (0, 1)$ is the commensurate order of differentiation, vector $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)] \in \mathbb{R}^n$ collects the process internal variables (pseudo-states), $u(t) \in \mathbb{R}$ is the control input, $\psi(t) \in \mathbb{R}$ is an exogenous disturbance, and $f(\mathbf{x}, t) : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ is a nonlinear function referred to as the “drift term”.

Regarding the process model (22), several notes and clarifications are in order. First, the variables x_i are denoted as the “internal” variables, or pseudo states, because the notion of state variables

is often inappropriate and generally not used in the context of FOS. Fractional order systems are infinitely dimensional, and any actual set of process states would have to be of infinite cardinality. The notion of pseudo states was considered originally in [24], and used later in numerous publications, including [25, 23, 34, 35]. For a more recent and detailed discussion regarding the nature of initial conditions in fractional order systems, the reader is referred to [26, 27, 28].

Caputo definition of fractional derivatives is utilized in (22) for convenience, as it allows to take into account a finite and physically meaningful initial condition $\mathbf{x}(0)$ for the pseudo-states [18, 9]. Although it is true that the Caputo derivative must be used with care in modeling and identification of physical systems, see [26] and references therein, the Caputo definition can be used freely when analyzing robust control strategies. In particular, any influence of the past process history which has not been taken into account can effectively be merged into the “disturbance term” ψ , which is supposed to fulfill the following assumption

Assumption 1. *The exists an a priori known constant M and a time instant $t_\psi \geq 0$ such that*

$$\left| \frac{d}{dt} \psi(t) \right| \leq M, \quad t \geq t_\psi. \quad (23)$$

Assume that the uncertain drift term $f(\mathbf{x}, t)$ is imprecisely known by means of a certain estimate $\hat{f}(\mathbf{x}, t)$. Denote

$$\epsilon(\mathbf{x}, t) = f(\mathbf{x}, t) - \hat{f}(\mathbf{x}, t) \quad (24)$$

and assume the following

Assumption 2. *There exist an a priori known constant W and a time instant $t_\epsilon \geq 0$ such that*

$$\left| \frac{d}{dt} \epsilon(\mathbf{x}, t) \right| \leq W, \quad t \geq t_\epsilon \quad (25)$$

Let a sufficiently smooth reference trajectory $x_{1r}(t)$ be given. Denote

$$\begin{aligned} \mathbf{x}_r(t) &= [x_{1r}(t), x_{2r}(t), \dots, x_{nr}(t)]^T \\ &= [x_{1r}(t), {}^C D^\alpha x_{1r}, \dots, {}^C D^\alpha x_{(n-1)r}]^T \\ &= [x_{1r}(t), {}^C \mathcal{D}^{1,\alpha} x_{1r}, \dots, {}^C \mathcal{D}^{n-1,\alpha} x_{1r}]^T \end{aligned} \quad (26)$$

The reference trajectory $x_{1r}(t)$ is supposed to fulfill the next smoothness restriction

Assumption 3. *The exist an a-priori known constant X_r and a time instant t_γ such that*

$$|{}^C\mathcal{D}^{n,\alpha}x_{1r}(t)| \leq X_r, \quad t \geq t_\gamma. \quad (27)$$

Define the tracking error vector of the pseudo-state

$$\mathbf{e}(t) = [e_1(t), e_2(t), \dots, e_n(t)] = \mathbf{x}(t) - \mathbf{x}_r(t) \quad (28)$$

The aim is that of finding a control law capable of steering the tracking error vector $\mathbf{e}(t)$ of the closed loop process to the origin regardless of the assumed uncertainties and perturbations. By straightforward computations, one obtains the error dynamics

$$\begin{aligned} {}^C D^\alpha e_i &= e_{i+1}, \quad i = 1, 2, \dots, n-1 \\ {}^C D^\alpha e_n &= f(\mathbf{x}, t) + u(t) + \psi(t) - {}^C \mathcal{D}^{n,\alpha}x_{1r}(t). \end{aligned} \quad (29)$$

Consider the fractional order sliding variable

$$\sigma(t) = I^{(1-\alpha)} \left[e_n(t) + \sum_{i=1}^{n-1} c_i e_i(t) \right], \quad (30)$$

where the constants c_1, c_2, \dots, c_{n-1} are selected in such a way that all the roots p_i of the polynomial

$$P(s) = s^{(n-1)} + \sum_{i=0}^{n-2} c_{i+1} s^i = \prod_{i=1}^{n-1} (s - p_i) \quad (31)$$

satisfy the next relation

$$\alpha \frac{\pi}{2} < \arg(p_i) \leq \pi. \quad (32)$$

The stability of system (29) once constrained to evolve along the sliding manifold $\sigma(t) = 0$ is analyzed in the next Lemma 5. A controller capable of steering the considered dynamics onto the sliding manifold in finite time will be illustrated later on.

Lemma 5. *Consider system (22) and let the zeroing of the sliding variable (30) be fulfilled starting from the finite moment t_1 , i.e. let*

$$\sigma(t) = 0, \quad t \geq t_1, \quad t_1 < \infty, \quad (33)$$

with the c_i parameters in (30) satisfying (31)-(32). Then, the next conditions hold

$$\lim_{t \rightarrow \infty} e_i(t) = 0, \quad i = 1, 2, \dots, n \quad (34)$$

Proof of Lemma 5 Define the quantity

$$\xi(t) = e_n(t) + \sum_{i=1}^{n-1} c_i e_i(t). \quad (35)$$

By taking into account Lemma 4 specialized with $\beta = 1 - \alpha$ and $z(t) = \xi(t)$, it yields that the finite time zeroing of $\sigma(t)$ guarantees that signal $\xi(t)$ decays asymptotically to zero. We then simply derive from (35) that

$$e_n(t) = - \sum_{i=1}^{n-1} c_i e_i(t) + \xi(t) \quad (36)$$

where

$$\lim_{t \rightarrow \infty} \xi(t) = 0. \quad (37)$$

Now, in light of (36) we rewrite the first $n - 1$ equations of (29) as

$$\begin{aligned} {}^C D^\alpha e_i &= e_{i+1}, \quad i = 1, 2, \dots, n - 2 \\ {}^C D^\alpha e_{n-1} &= - \sum_{i=1}^{n-1} c_i e_i(t) + \xi(t) \end{aligned} \quad (38)$$

and notice that (38) form a reduced-order (as compared to (29)) fractional order system with an asymptotically decaying input term $\xi(t)$. It readily follows from (31)-(32) that system (38) is Mittag-Leffler stable when $\xi(t) = 0$ (see [18]), thereby the input decay property (37) implies the same for the error variables $e_i(t)$ with $i = 1, 2, \dots, n - 1$. We now conclude from (36) that $e_n(t)$ asymptotically decays, too. Lemma 5 is proved. \square

It is worth to remark that the enforcement of conditions (35), (37) actually “cancels” the last equation of (29) by making the system to behave as the reduced order one (38). We seek for a control law expressed in the form

$$u(t) = u^p(t) + u^i(t) + u^{eq}(t) \quad (39)$$

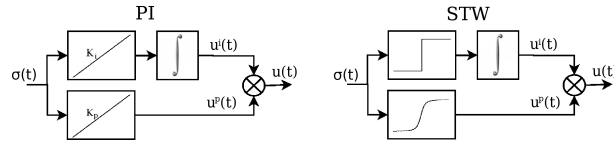


Figure 1. Architecture Comparison between linear (left) and nonlinear PI.

where $u^p(t)$ and $u^i(t)$ are, respectively, combined linear/nonlinear proportional and integral control actions taking the form

$$u^p(t) = -k_1\sigma - k_2|\sigma|^{1/2}\text{sign}(\sigma) \quad (40)$$

$$\dot{u}^i(t) = -k_3\sigma - k_4\text{sign}(\sigma), \quad u^i(0) = 0 \quad (41)$$

and $u^{eq}(t)$ is a control component that will be specified later on. By setting constants k_2 and k_4 to zero then the two control components (40)-(41) reduces to the standard PI controller. On the other hand, by setting k_1 and k_3 to zero one obtains the well-known “super-twisting” second-order sliding mode controller (see [12]). The similarities between a classical PI controller and the super-twisting (STW) one are evident (see Figure 1) in that they both possess a static component (a pure gain, for the PI, and a nonlinear gain with infinite slope at 0 for the STW) and an integral component (a pure integration, for the PI, and the integration of the sign of the error variable, for the STW). A further novelty here is the use of such a combined PI/sliding mode algorithm with a fractional order sliding variable σ .

We are now in position to state the next result.

Theorem 1. Consider system (22) along with the sliding variable (30)-(32), and let Assumption 1 be in force. Then, the control law (39)-(41), specified with

$$u^{eq}(t) = -\hat{f}(\mathbf{x}, t) - \sum_{i=1}^{n-1} c_i e_{i+1}(t) + {}^C\mathcal{D}^{n,\alpha} x_{1r}, \quad (42)$$

and with the tuning parameters chosen according to

$$k_1 > 0, \quad k_2 > 2\sqrt{\rho}, \quad k_4 > \rho, \quad (43)$$

$$k_3 > k_1^2 \frac{k_2^2 + \frac{5}{2}[(\frac{1}{4}k_2^2 - \rho) + k_2k_4]}{(\frac{1}{4}k_2^2 - \rho) + k_2k_4}. \quad (44)$$

where

$$\rho > M + W, \tag{45}$$

provides the asymptotic decay of the pseudo-state $\mathbf{x}(t)$.

Proof of Theorem 1 By virtue of Definition 2, specified with $n = 1$ and $f(t) = e_n(t) + \sum_{i=1}^{n-1} c_i e_i(t)$, and exploiting as well the linearity of the fractional derivative operator, one can easily derive that

$$\frac{d}{dt}\sigma(t) = {}^{RL}D^\alpha \left[e_n(t) + \sum_{i=1}^{n-1} c_i e_i(t) \right] = {}^{RL}D^\alpha e_n(t) + \sum_{i=1}^{n-1} c_i {}^{RL}D^\alpha e_i(t) \tag{46}$$

In light of relation (5), eq. (46) can be rewritten in terms of Caputo derivatives as follows

$$\frac{d}{dt}\sigma(t) = {}^C D^\alpha e_n(t) + \sum_{i=1}^{n-1} c_i {}^C D^\alpha e_i(t) + \varphi(t) \tag{47}$$

where

$$\varphi(t) = \frac{1}{\Gamma(1-\alpha)} \frac{e_n(0) + \sum_{i=1}^{n-1} c_i e_i(0)}{t^\alpha} = \frac{K_0}{t^\alpha} \tag{48}$$

with implicitly defined constant $K_0 = \frac{e_n(0) + \sum_{i=1}^{n-1} c_i e_i(0)}{\Gamma(1-\alpha)}$.

The system equations (22) can be now substituted into (47), yielding the simplified expression

$$\dot{\sigma}(t) = f(\mathbf{x}, t) + u(t) + \psi(t) + \sum_{i=1}^{n-1} c_i e_{i+1}(t) + \varphi(t) - {}^C D^{n,\alpha} x_{1r}(t) \tag{49}$$

Although the disturbance (48) and all its time derivatives are unbounded at $t = 0$, one has that the first-order time derivative

$$\frac{d}{dt}\varphi(t) = -\frac{\alpha K_0}{t^{\alpha+1}} \tag{50}$$

is bounded, in magnitude, along any time interval $t \in [t_1, \infty)$, $t_1 > 0$, according to

$$\left| \frac{d}{dt}\varphi(t) \right| \leq \frac{\alpha K_1}{t_1^{\alpha+1}} \equiv \Psi_1, \quad K_1 = \left| e_n(0) + \sum_{i=1}^{n-1} c_i e_i(0) \right|. \tag{51}$$

We now substitute the control (39)-(42) into (49), yielding

$$\frac{d}{dt}\sigma = -k_1\sigma - k_2|\sigma|^{1/2} \text{sign}(\sigma) + u^i(t) + \psi(t) + \varphi(t) + \varepsilon(\mathbf{x}, t) \tag{52}$$

$$\frac{d}{dt}u^i = -k_3\sigma - k_4 \text{sign}(\sigma) \tag{53}$$

Define

$$z(\mathbf{x}, t) = u^i(t) + \psi(t) + \varphi(t) + \varepsilon(\mathbf{x}, t) \quad (54)$$

and rewrite (52)-(53) as

$$\frac{d}{dt}\sigma = -k_1\sigma - k_2|\sigma|^{1/2}\text{sign}(\sigma) + z(\mathbf{x}, t) \quad (55)$$

$$\frac{d}{dt}z = -k_3\sigma - k_4\text{sign}(\sigma) + \frac{d}{dt}\psi(t) + \frac{d}{dt}\varphi(t) + \frac{d}{dt}\varepsilon(\mathbf{x}, t) \quad (56)$$

Notice that, by Assumptions 1 and 2 and by relation (51), the perturbation terms in (53) fulfill the next estimation

$$\left| \frac{d}{dt}\psi(t) + \frac{d}{dt}\varphi(t) + \frac{d}{dt}\varepsilon(\mathbf{x}, t) \right| \leq M + \Psi_1 + W, \quad t \geq \max\{t_\psi, t_\varepsilon\} > 0 \quad (57)$$

Since $\varphi(t)$ is asymptotically vanishing along with its time derivative $\frac{d}{dt}\varphi(t)$, it readily follows that there exist a finite moment $t_2 > t_1 > 0$ such that $|\frac{d}{dt}\psi(t) + \frac{d}{dt}\varphi(t) + \frac{d}{dt}\varepsilon(\mathbf{x}, t)| \leq \rho$ at every $t \geq t_2$ thus it can be set ρ as in (45) by neglecting the bound on $\frac{d}{dt}\varphi(t)$.

Stability of the dynamics (55)-(57) was already investigated in the literature (cfr. [14], Th. 5), where, particularly, the global finite time stability of the uncertain system trajectories was demonstrated by means of a positive definite and radially-unbounded non-smooth Lyapunov function which specifies as follows in the present context

$$V = \xi^T \Pi \xi, \quad \xi = \begin{bmatrix} |\sigma|^{1/2}\text{sign}(\sigma) \\ \sigma \\ z \end{bmatrix}, \quad \Pi = \frac{1}{2} \begin{bmatrix} (4k_4 + k_2^2) & k_1k_2 & -k_2 \\ k_1k_2 & 2k_3 + k_1^2 & -k_1 \\ -k_2 & -k_1 & 2 \end{bmatrix}. \quad (58)$$

It turns out after the appropriate computations (cfr. [14], Proof of Th. 5) that the tuning conditions (43)-(45) imply the existence of a positive constant γ_1 such that

$$\frac{d}{dt}V \leq -\gamma_1\sqrt{V}, \quad t \geq t_2. \quad (59)$$

Inequality (59) guarantees the global **finite time** convergence of V to zero, and, hence, the same property for the $\sigma(t)$ and $z(t)$ variables. By (52), the finite time convergence to zero of $\frac{d}{dt}\sigma(t)$ can be easily concluded, too. The asymptotic decay of $\mathbf{x}(t)$, thus, readily follows from Lemma 4.

Theorem 1 is proven. \square .

4. FRACTIONAL UNIT-VECTOR CONTROL OF A CLASS OF NONLINEAR UNCERTAIN
MULTI-INPUT FOS

A class of multi-input dynamics is under investigation. More precisely, we consider a commensurate fractional-order linear multivariable square system affected by an unknown perturbation

$${}^C D^\alpha \mathbf{x}(t) = \mathbf{B}\mathbf{u}(t) + \boldsymbol{\psi}(\mathbf{x}, t), \tag{60}$$

where $\alpha \in (0, 1)$ is the non-integer order of the system, $\mathbf{x} = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$ is the pseudo-state vector, $\mathbf{u} = [u_1, u_2, \dots, u_n]^T \in \mathbb{R}^n$ is the input vector, $\boldsymbol{\psi} = [\psi_1, \psi_2, \dots, \psi_n]^T \in \mathbb{R}^n$ is an uncertain disturbance vector, and \mathbf{B} is an uncertain, nonsingular, control matrix.

We cast the next assumptions:

Assumption 4. A lower bound Λ_m to the eigenvalues of the uncertain symmetric matrix

$$\mathbf{G} = \frac{\mathbf{B} + \mathbf{B}^T}{2} \tag{61}$$

is known a priori such that

$$\Lambda_m \leq \min_i \lambda_{\mathbf{G}}^i, \quad i = 1, 2, \dots, n, \tag{62}$$

where $\lambda_{\mathbf{G}}^i$ denotes the i -th eigenvalue of the matrix \mathbf{G} .

Assumption 5. There exist a-priori known functions $\Psi_i(\mathbf{x}, t)$ and finite time instant t_ψ such that

$$|{}^{RL}D^{1-\alpha} \psi_i(\mathbf{x}, t)| \leq \Psi_i(\mathbf{x}, t), \quad t \geq t_\psi \quad i = 1, 2, \dots, n, \tag{63}$$

and define

$$\Psi_M(\mathbf{x}, t) = \sqrt{\sum_{i=1}^n \Psi_i^2(\mathbf{x}, t)}, \tag{64}$$

in such a way that

$$\|{}^{RL}D^{1-\alpha} \boldsymbol{\psi}(\mathbf{x}, t)\|_2 \leq \Psi_M(\mathbf{x}, t) \quad t \geq t_\psi \tag{65}$$

The next controller is suggested:

$$\mathbf{u}(t) = -\frac{1}{\Lambda_m} I^{1-\alpha} \left\{ (\Psi_M(\mathbf{x}, t) + \eta_1) \frac{\mathbf{x}}{\|\mathbf{x}\|_2} + \eta_2 \mathbf{x} \right\}, \tag{66}$$

where η_1 and η_2 are positive tuning constants.

Theorem 2. Consider system (60), satisfying the Assumptions 4 and 5. Then, the controller (66) with $\eta_1 > 0$ and $\eta_2 \geq 0$ provides for the global finite-time convergence of the pseudo-state vector $\mathbf{x}(t)$ to the origin.

Proof

For $\alpha \in (0, 1)$ one can easily prove that combination of the Riemann-Liouville differential operator ${}^{RL}D^{1-\alpha}$ with the Caputo operator of the complement order ${}^C D^\alpha$ yields the “standard” first order differential. By Definitions 2 and 3, taking into considerations the semi-group property of Lemma 1 and the fact that the first derivative is the left inverse of the first-order integration operator (stated in a more general form by Lemma 2), it follows that for any x

$${}^{RL}D^{1-\alpha} {}^C D^\alpha \mathbf{x} = \frac{d}{dt} I^\alpha I^{1-\alpha} \frac{d}{dt} \mathbf{x} = \frac{d}{dt} I^1 \frac{d}{dt} \mathbf{x} = \frac{d}{dt} \mathbf{x} \quad (67)$$

Thus, by applying the operator ${}^{RL}D^{1-\alpha}$ to both sides of (60) it yields

$$\dot{\mathbf{x}}(t) = {}^{RL}D^{1-\alpha} \mathbf{B}\mathbf{u}(t) + {}^{RL}D^{1-\alpha} \psi(\mathbf{x}, t). \quad (68)$$

By substituting the controller equation (66) into the first term in the right hand side of (68) one obtains

$${}^{RL}D^{1-\alpha} \mathbf{B}\mathbf{u}(t) = -\frac{\Psi_M(\mathbf{x}, t) + \eta_1}{\Lambda_m} \mathbf{B} \frac{\mathbf{x}}{\|\mathbf{x}\|_2} - \eta_2 \frac{\mathbf{B}}{\Lambda_m} \mathbf{x}, \quad (69)$$

Consider the Lyapunov function $V = \frac{1}{2} \mathbf{x}^T \mathbf{x} = \frac{1}{2} \|\mathbf{x}\|_2^2$, whose time derivative along the solutions of (68)-(69) is

$$\dot{V} = \mathbf{x}^T \left[\psi_d(\mathbf{x}, t) - \frac{\Psi_M(\mathbf{x}, t) + \eta_1}{\Lambda_m} \mathbf{B} \frac{\mathbf{x}}{\|\mathbf{x}\|_2} - \eta_2 \frac{\mathbf{B}}{\Lambda_m} \mathbf{x} \right] \quad (70)$$

where

$$\psi_d(\mathbf{x}, t) = {}^{RL}D^{1-\alpha} \psi(\mathbf{x}, t) \quad (71)$$

Rewrite (70) as

$$\dot{V} = -\frac{\Psi_M(\mathbf{x}, t) + \eta_1}{\Lambda_m} \frac{1}{\|\mathbf{x}\|_2} \mathbf{x}^T \mathbf{B} \mathbf{x} - \frac{\eta_2}{\Lambda_m} \mathbf{x}^T \mathbf{B} \mathbf{x} + \mathbf{x}^T \psi_d(\mathbf{x}, t) \quad (72)$$

By exploiting the following trivial chain of relations

$$\begin{aligned} \mathbf{x}^T \mathbf{B} \mathbf{x} &= \mathbf{x}^T \left(\frac{\mathbf{B} + \mathbf{B}^T}{2} \right) \mathbf{x} + \mathbf{x}^T \left(\frac{\mathbf{B} - \mathbf{B}^T}{2} \right) \mathbf{x} \\ &= \mathbf{x}^T \mathbf{G} \mathbf{x} \geq \min_i \lambda_G^i \|\mathbf{x}\|_2^2 \geq \Lambda_m \|\mathbf{x}\|_2^2 \end{aligned} \quad (73)$$

that follows from basic properties of quadratic forms and skew-symmetric matrices, one can manipulate (72) as

$$\dot{V} \leq -[\Psi_M(\mathbf{x}, t) + \eta_1] \|\mathbf{x}\|_2 - \eta_2 \|\mathbf{x}\|_2^2 + \mathbf{x}^T \boldsymbol{\psi}_d(\mathbf{x}, t) \quad (74)$$

By applying the Cauchy-Schwartz inequality to the last term in (74), and taking into account (71) and (65), it yields

$$|\mathbf{x}^T \boldsymbol{\psi}_d(\mathbf{x}, t)| \leq \|\mathbf{x}\|_2 \|\boldsymbol{\psi}_d(\mathbf{x}, t)\|_2 \leq \Psi_M(\mathbf{x}, t) \|\mathbf{x}\|_2. \quad (75)$$

By combining (74) and (75) it yields that

$$\dot{V} \leq -\eta_1 \|\mathbf{x}\|_2 - \eta_2 \|\mathbf{x}\|_2^2 = -\eta_1 \sqrt{2V} - 2\eta_2 V \quad (76)$$

which guarantees, by the comparison Lemma, the finite time convergence to zero of $V(t)$ and thus the same behavior for the entries of the pseudo-state vector $\mathbf{x}(t)$. The Theorem is proven. \square

5. SIMULATIONS

5.1. Single-input case

Consider system (22) of dimension $n = 3$, fractional order $\alpha = 0.5$ and with $f(\mathbf{x}, t) = x_1^2 \sqrt{|x_2|} + x_3 |x_3|$. The input disturbance is set as $\psi(t) = 0.2 \sin(5\pi t)$, which is infinitely times continuously differentiable, and let the reference signal be $x_{1r} = \sin(0.1\pi t)$.

Let us design the combined second-order sliding mode/PI controller (39)-(42) with $\hat{f}(\mathbf{x}, t) = 0$, i.e. assuming that the nonlinear function $f(\mathbf{x}, t)$ is totally uncertain. The polynomial $P(\cdot)$ (31) is selected with two coinciding zeros at $p = -\lambda = -3$. Consequently, $c_1 = \lambda^2 = 9$ and $c_2 = 2\lambda = 6$. The upper bound on the disturbance time derivative (see Assumption 1) is taken as $M = 4$. The upper bound W on the time-derivative of the error $\epsilon(\cdot)$ (25) is not straightforward to evaluate by

means of analytic computations, and the value $W = 5$ was found appropriate after few trial and error tests. Thus, one gets the value $\rho = 9$ for the constant entering into the controller tuning rules. It gives rise, according to (43) and (44), to the parameter setting $k_2 = 7$, $k_4 = 10$, $k_1 = 1$, $k_3 = 4$. The simulation results with the initial conditions $x_1(0) = x_2(0) = x_3(0) = -0.1$ are presented in Figures 2 and 3. The good tracking performance of the closed loop system are illustrated in Figure 2, where the time evolutions of the pseudo-states and their reference profiles are shown. Time history of the control signal is depicted in Figure 3-left. Note the slow harmonic oscillations, due to the chosen reference profiles, and the fast ones due to the instantaneous compensation of the disturbance $\psi(t)$. The sliding variable σ is shown in Figure 3-right, from which the finite-time convergence to the chosen sliding manifold is apparent.

Initial peaking of the control signal (not fully depicted in Figure 3-Left) is caused by the equivalent control component $u^{eq}(t)$, as defined in (42), and is particularly affected by the chosen value of λ . By changing λ , the peaking amplitude may be affected. As depicted in Figure 4-left, reducing the value of λ causes the initial peak of the control signal to correspondingly decrease. At the same time, the chosen value of λ affects the convergence speed of the pseudo states tracking errors, the larger λ is, the faster the convergence. This trade-off is investigated in Figure 4-right, therefore a proper design compromise has to be found.

From now on, performance comparisons are made with respect to the sliding-mode based scheme proposed by Valério and Sá da Costa in [23]. The sensitivity of both schemes to measurement noise will be investigated as well. The approach proposed in Section 3 of [23] is now specialized to the problem under consideration. The sliding variable takes a more general form

$$\bar{\sigma}(t) = \left({}^C D_t^\beta + \lambda \right)^{(n-1)\frac{\alpha}{\beta}} e(t), \quad (77)$$

where β is an arbitrary coefficient such that $\alpha/\beta \in \mathbb{N}$. In our approach, the sliding variable is a function of the pseudo-state tracking errors, and the same happens in (77) if we choose $\beta = \alpha$. Therefore, this is the choice which will be utilized in the sequel, for the sake of comparison. The

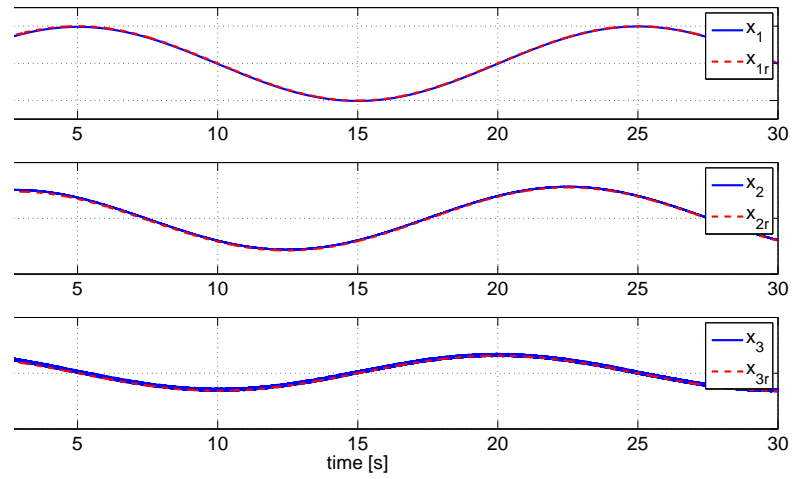


Figure 2. Single input case with noise-free measurements. The pseudo state components and their reference profiles.

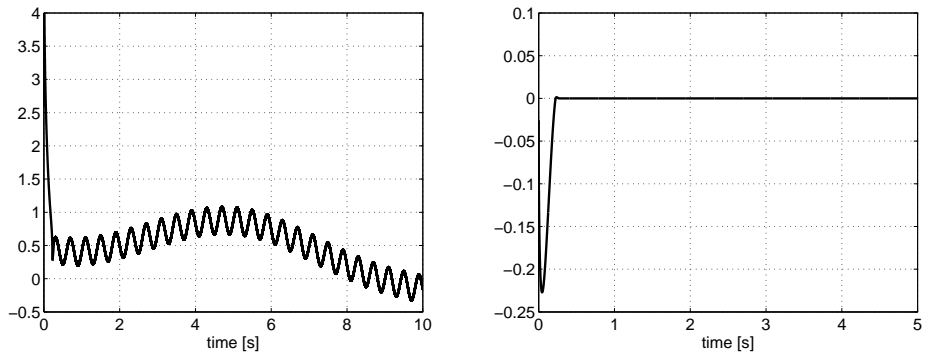


Figure 3. Single input case with noise-free measurements. (Left) The control signal. (Right) The sliding variable σ .

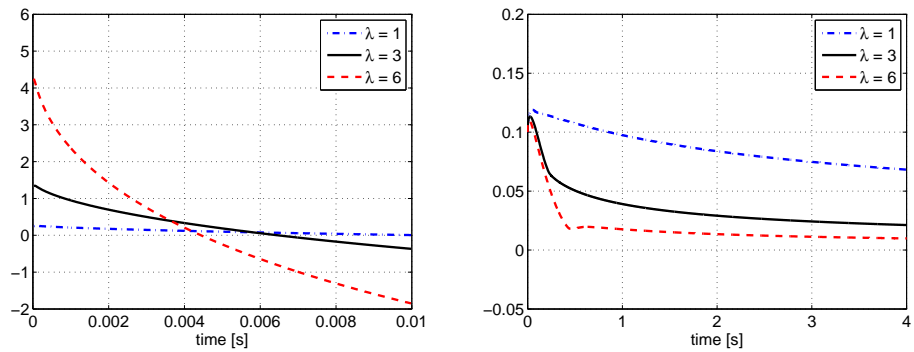


Figure 4. Single input case with noise-free measurements for different values of λ . (Left) Initial time evolution of the control signal $u(t)$. (Right) Tracking error e_1 of the first pseudo-state component.

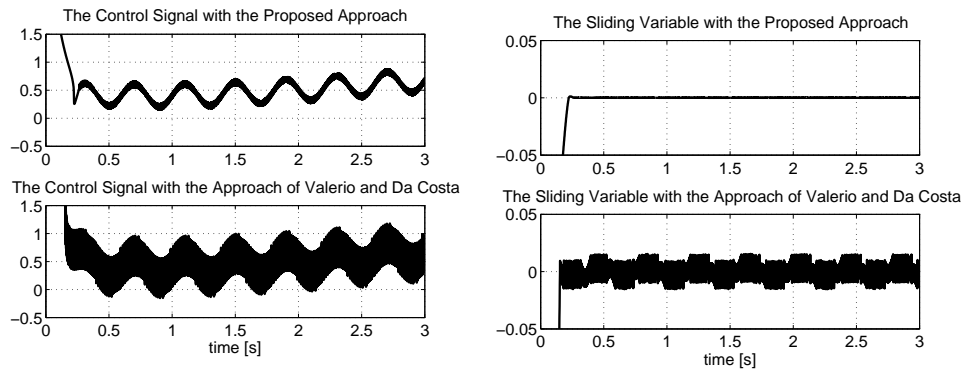


Figure 5. Single input case with noise-free measurements. Comparison of the control signal and the sliding variable.

control law suggested in [23] takes the form

$$u(t) = u^{sm}(t) + u^{eq}(t). \quad (78)$$

Due to the choice $\beta = \alpha$, the equivalent control component, $u^{eq}(t)$, is the same as in (42). The sliding-mode control component is, however, different. It takes the form,

$$u^{sm}(t) = -k_5 {}_0I_t^{1-\alpha} \text{sign}(\bar{\sigma}), \quad (79)$$

with $k_5 > 0$ has to be taken large enough according to the actual uncertainty bounds.

Figure 5 compares the control signal and the sliding variable time evolution using the proposed approach and that of [23]. For the sake of comparison, the discontinuous control gains are set to the same values, $k_3 = k_5 = 10$, and also λ was set to 3 in both cases. Both simulations were performed with sampling time $T = 0.001$ seconds. It is apparent that the approach proposed in this paper provides higher smoothness of the control signal, as well as higher accuracy in maintaining the system on the sliding manifold.

In order to improve the smoothness of the control signal, the utilization of soft-sign function

$$\text{soft_sign}_\theta(s) = \begin{cases} \frac{s}{\theta} & |s| < \theta \\ \text{sign}(s) & |s| \geq \theta \end{cases} \quad (80)$$

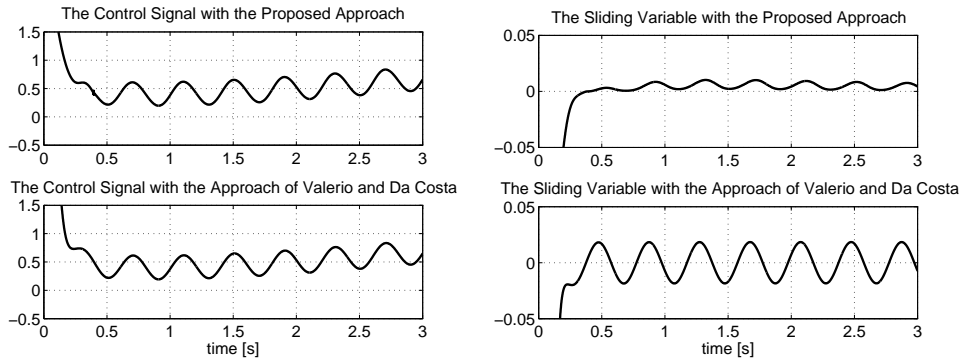


Figure 6. Single input case with noise-free measurements and soft-sign function with $\theta = 0.25$. (Left) Control signals. (Right) Sliding variables.

was recommended in [23]. By adopting this modification smoother control profiles are obtained, as shown in Figure 6-left. Although the smoothness of control signals is now comparable, the scheme proposed in the present paper results in a more accurate sliding motion, see 6-right.

Let us now consider the case of noisy measurements. A uniformly distributed random noise with maximal amplitude 0.01 was added to the pseudo-state variables. The resulting control signal and sliding variable time evolutions are displayed in Figure 7. The two control signals exhibit a comparable amount of chattering. However, the sliding motion is more accurate using the controller proposed in the current work.

In short, the main pros of the approach here presented are the higher degree of smoothness of the control law and the higher accuracy of the resulting sliding motion. Both these aspects contribute to achieve improved chattering alleviation features as compared to the existing schemes. The propagation of the noise towards the plant input seems comparable. The main drawback of the scheme here proposed, as compared with the scheme in [23], is that it requires more restrictive assumptions on the uncertainties, notably we require *constant* upper bounds to the time derivatives of the uncertainties (see (23), (27) and (25)) whereas the approach in [23] may allow time-varying and state dependent uncertainty upper bounds as well (even if, likely for the simplicity sake, this option is not exploited in [23]).

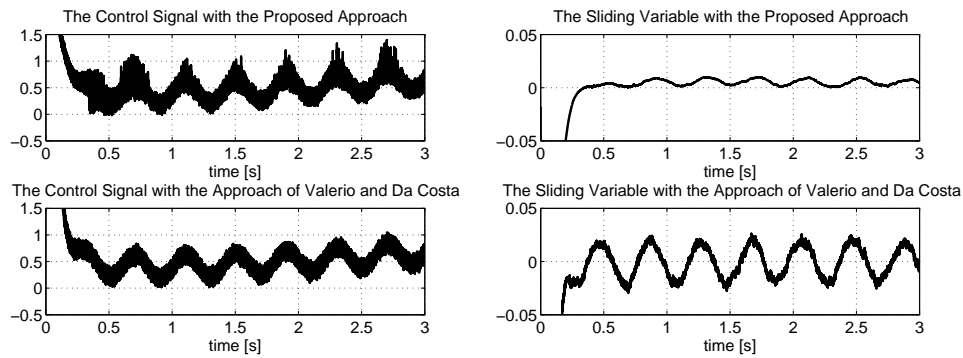


Figure 7. Single input case with uniform measurement noise and soft-sign function with $\theta = 0.25$. (Left) Control signals. (Right) Sliding variables.

5.2. Multi-input case

Consider system (60), with commensurate order $\alpha = 0.5$, dimension $n = 3$, and with the control matrix and disturbance vectors taken as

$$\mathbf{B} = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \quad \psi(\mathbf{x}, t) = \mathbf{x} + \frac{1}{2} \begin{bmatrix} \sin(0.4\pi t) \\ \sin(\pi t) \\ 1 \end{bmatrix}. \quad (81)$$

The bound Ψ_M in (64) is set to 4, and Λ_m in (62) is set to 0.2 (the minimal eigenvalue of $\mathbf{G} = \frac{1}{2}(\mathbf{B} + \mathbf{B}^T)$ is in fact near the value 0.46). Controller (66) has been applied with gains $\eta_1 = 6$ and $\eta_2 = 1$. The initial conditions are $x_1(0) = x_2(0) = 1$. Figure 8 shows the trajectories of the states. The attainment of the finite time convergence property is apparent from the given plots. The control signals are shown in Figure 9.

6. CONCLUSIONS

Fractional sliding-mode controllers are proposed for some classes of commensurate single-input and multi-input fractional order dynamics subject to uncertainties and disturbances. A second-order sliding-mode approach is suitably combined with PI-based design in the single-input case, while the unit-vector approach is the main tool of reference in the multi-input case. Among the most interesting directions for next researches, managing wider classes of fractional dynamics (e.g.

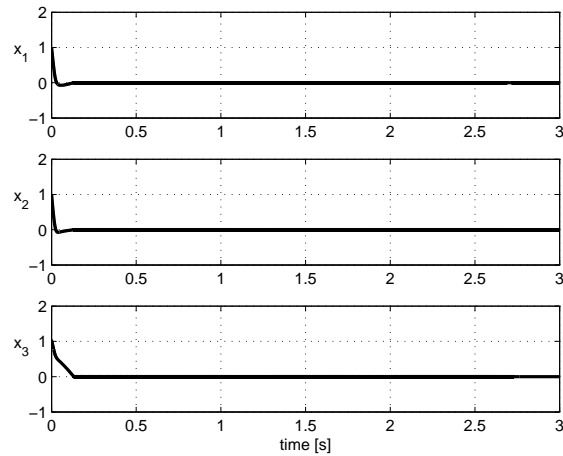


Figure 8. Multi input case. Time evolution of the process pseudo-state variables.

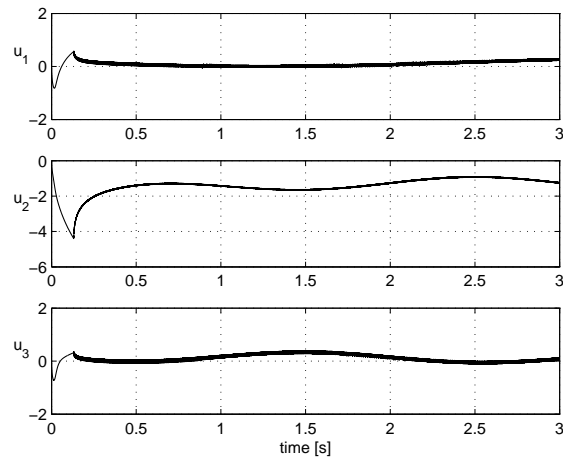


Figure 9. Multi input case. Time evolution of the control signals.

non commensurate ones) appears of great interest. Furthermore, the development of theoretical and practical tools for implementing the suggested controllers in a sampled data environment also appears an important task deserving research efforts.

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