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Boundary control of coupled reaction-diffusion processes with constant parameters

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Abstract

The problem of boundary stabilization is considered for some classes of coupled parabolic linear PDEs of the reaction-diffusion type. With reference to n coupled equations, each one equipped with a scalar boundary control input, a state feedback law is designed with actuation at only one end of the domain, and exponential stability of the closed-loop system is proven. The treatment is addressed separately for the case in which all processes have the same diffusivity and for the more challenging scenario where each process has its own diffusivity and a different solution approach has to be taken. The backstepping method is used for controller design, and, particularly, the kernel matrix of the transformation is derived in explicit form of series of Bessel-like matrix functions by using the method of successive approximations to solve the corresponding PDE. Thus, the proposed control laws become available in explicit form. Additionally, the stabilization of an underactuated system of two coupled reaction-diffusion processes is tackled under the restriction that only a scalar boundary input is available. Capabilities of the proposed synthesis and its effectiveness are supported by numerical studies made for three coupled systems with distinct diffusivity parameters and for underactuated linearized dimensionless temperature-concentration dynamics of a tubular chemical reactor, controlled through a boundary at low fluid superficial velocities when convection terms become negligible.

Keywords: Coupled reaction-diffusion processes; Boundary control; Backstepping.

1 Introduction

The problem of boundary stabilization is considered for some classes of coupled linear parabolic Partial Differential Equations (PDEs) in a finite spatial domain $x \in [0, 1]$. Particularly, by exploiting the so-called "backstepping" approach [8], [22], this work is devoted to "approximation-free" control synthesis not relying on any discretization or finite-dimensional approximation.

The backstepping-based boundary control problem for scalar heat processes was studied, e.g., in [11], [22]. Several classes of scalar wave processes were studied, e.g., in [9], [21], whereas complex-valued, PDEs such as the Schrodinger equation were also dealt with by means of such an approach [10]. Synergies between the backstepping methodology and the flatness-based approach were studied in [12], [13] with reference to the case of spatially- and time- varying coefficients and covering spatial domains of dimension 2 and higher. In particular, in the latter situation conditions on the target system arise that somewhat resemble those considered in the remainder of the present paper. The backstepping methodology was also applied to observer design for linear parabolic PDEs with non constant coefficients in one- and multi-dimensional spatial domains [20] and [7].

More recently, high-dimensional systems of coupled PDEs are being considered in the backstepping-based boundary control setting. The most intensive efforts of current literature are however oriented towards coupled hyperbolic processes of the transport-type [1,4,5,24,25]. The state feedback design in [24], which admits stabilization of 2×2 linear heterodirectional hyperbolic systems, was extended in [4] to a particular type of 3×3 linear systems, arising in modeling of multiphase flow, and to the quasilinear case in [25]. In [1], a 2×2 linear hyperbolic system was stabilized by a single boundary

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control input, with the additional feature that an unmatched disturbance, generated by an *a-priori* known exosystem, is rejected. In [5], a system of $n+1$ coupled first-order hyperbolic linear PDEs with a single boundary input was studied.

In a recent publication [23], two parabolic reaction-diffusion processes coupled through the corresponding boundary conditions were dealt with. The stabilization of the coupled equations is reformulated in terms of the stabilization problem for a unique process, with piecewise continuous diffusivity and (space-dependent) reaction coefficient, which can be viewed as the "cascade" between the two original systems. The problem is solved by using a unique control input acting only at a boundary. A non conventional backstepping approach with a discontinuous kernel function was employed under a certain inequality constraint involving the diffusivity parameters of the two systems and the corresponding lengths of their spatial domains.

Some specific results concerning the backstepping based boundary stabilization of parabolic coupled PDEs have additionally been presented in the literature [2,26–28]. In [2], the Ginzburg-Landau equations, which represent a 2×2 system with equal diffusion coefficients when the imaginary and real parts are expanded, was dealt with. In [26], the linearized 2×2 model of thermal-fluid convection, which entails very dissimilar diffusivity parameters, has been treated by using a singular perturbations approach combined with backstepping and Fourier series expansion. In [27], an observer that estimates the velocity, pressure, electric potential and current fields in a Hartmann flow was presented where the observer gains were designed using multi-dimensional backstepping. In [28], the boundary stabilization of the linearized model of an incompressible magnetohydrodynamic flow in an infinite rectangular 3D channel, also recognized as Hartmann flow, was achieved by reducing the original system to a set of coupled diffusion equations with the same diffusivity parameter and by applying backstepping.

It is of interest to note that the multidimensional transformation considered in the present work generalizes the bi-dimensional backstepping transformation used in [2]. Apart from this, the set of linear coupled kernel PDEs that was derived in [27,28] for the magnetohydrodynamic channel flow is another inspiration for the present investigation. An additional interesting feature of backstepping, which further motivates our work, is that it admits an easy synergic integration with robust control paradigms such as the sliding mode control methodology (see, e.g., [6]).

Thus motivated, the primary concern of this work is to extend the backstepping synthesis developed in [22], where stabilizing boundary controllers were designed for scalar unstable reaction-diffusion processes. Here, a generalization is provided by considering a set of n reaction-diffusion processes, which are coupled through the corresponding reaction terms. The motivation behind the present investigation comes from chemical processes [14] where coupled temperature-concentration parabolic PDEs occur to describe the process dynamics.

A constructive synthesis procedure, with all boundary controllers given in explicit form, presents the main contribution of the paper to the existing literature. As shown in the paper, this generalization is far from being trivial because the underlying backstepping-based treatment gives rise to more complex development of finding out an explicit solution in the form of Bessel-like matrix series.

The present treatment addresses, side by side, two distinct situations which require quite different solution approaches to be adopted. First, the case where all processes have the same diffusivity ("equi-diffusivity" case, recently announced in [3]) is attacked, and then the more challenging scenario where each process possesses its own diffusivity ("distinct-diffusivity" case) is treated. Under the requirement that the considered multi-dimensional process is fully actuated by a set of n boundary control inputs acting on each subsystem, all these approaches are shown to exponentially stabilize the controlled system with an arbitrarily fast convergence rate.

Apart from this, the stabilization problem of an underactuated system of 2 coupled reaction-diffusion processes, which is relevant to regulation of tubular chemical reactors [14], is addressed under the restriction that only a unique scalar boundary input is available whereas the overall system features a certain minimum-phase property and it meets an additional restriction in the form of a suitable inequality involving both the plant and controller parameters. Exponential stability of the closed loop system is achieved in this case as well, but unlike the previously developed approaches the associated convergence rate cannot be made arbitrarily fast anymore.

The structure of the paper is as follows. In Section 2, the problem statement is presented and the underlying backstepping transformation is introduced. In Section 3, the "equi-diffusivity" scenario is investigated. Explicit solution of the kernel PDE is given for both the direct and inverse transformations, and the resulting boundary control design is presented. In Section 4, the "distinct-diffusivity" case is dealt with, which involves a simplified backstepping transformation defined by a scalar kernel function rather than a matrix one. Section 5 investigates the stabilization problem of an underactuated system of 2 coupled reaction-diffusion processes where only a unique scalar manipulable boundary input is available. Section 6 presents some simulation results. Finally, Section 7 collects concluding remarks and features future perspectives of this research.

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1.1 Notation

The notation used throughout is fairly standard. $L_2(0,1)$ stands for the Hilbert space of square integrable scalar functions $z(\zeta)$ on $(0,1)$ with the corresponding norm

$$\|z(\cdot)\|_2 = \sqrt{\int_0^1 z^2(\zeta)d\zeta}. \quad (1)$$

Also, the notation

$$[L_2(0,1)]^n = \underbrace{L_2(0,1) \times L_2(0,1) \times \ldots \times L_2(0,1)}_{n \ \texttt{times}} \text{ and}$$

$$\|Z(\cdot)\|_{2,n} = \sqrt{\sum_{i=1}^{n} \|z_i(\cdot)\|_2^2} \quad (2)$$

is adopted for the corresponding norm of a generic vector function $Z(\zeta) = [z_1(\zeta), z_2(\zeta), ..., z_n(\zeta)] \in [L_2(0,1)]^n$.

$J_1(\cdot)$ and $J_2(\cdot)$ ($I_1(\cdot)$ and $I_2(\cdot)$) stand for the first and second order (modified) Bessel functions of the first kind.

With reference to a generic real-valued square matrix A of dimension n, $S[A]$ denotes its symmetric part $S[A] = (A + A^T)/2$, and $\sigma_i(A)$ $(i = 1, 2, ..., n)$ the corresponding eigenvalues. Provided that A is also symmetric and positive definite, $\sigma_m(A)$ and $\sigma_M(A)$ denote respectively the smallest and largest eigenvalues of A, i.e., $\sigma_m(A) = \min_{1\le i\le n} \sigma_i(A)$, $\sigma_M(A) = \max_{1\le i\le n} \sigma_i(A)$. Finally, $I_{n\times n}$ stands for the identity matrix of dimension n.

2 Problem formulation and backstepping transformation

A n-dimensional system of coupled reaction-diffusion processes is under investigation. Throughout, it is governed by the parabolic PDE

$$Q_t(x,t) = \Theta Q_{xx}(x,t) + \Lambda Q(x,t) \quad (3)$$

and equipped with Neumann-type boundary conditions

$$Q_x(0,t) = 0, \quad (4)$$
$$Q_x(1,t) = U(t), \quad (5)$$

where $Q(x,t) = [q_1(x,t),\ q_2(x,t), \ldots, q_n(x,t)]^T \in [L_2(0,1)]^n$ is the vector collecting the state of all systems, $U(t) = [u_1(t),\ u_2(t), \ldots, u_n(t)]^T \in \Re^n$ is the vector collecting all the manipulable boundary control signals, $\Theta \in \Re^{n\times n}$ is the diagonal diffusivity matrix of the form $\Theta = diag(\theta_i)$, with $\theta_i > 0\ \forall i = 1, 2, ..., n$, $\Lambda \in \Re^{n\times n}$ is a real-valued square matrix whose elements are denoted as λ_{ij}, with $i, j = 1, 2, \ldots, n$.

The open-loop system (3)-(5) (with $U(t) = 0$) possesses arbitrarily many unstable eigenvalues when the matrix $S[\Lambda]$ has sufficiently large positive eigenvalues. Since the term $\Lambda Q(x,t)$ is the source of instability, the natural objective for a boundary feedback is to "reshape" (or cancel) this term by reversing its effect into a stabilizing one. Thus motivated, our objective is to exponentially stabilize system (3)-(5) by using an invertible backstepping transformation

$$Z(x,t) = Q(x,t) - \int_0^x K(x,y)Q(y,t)dy \quad (6)$$

with a $n \times n$ kernel matrix function $K(x,y)$. The entries $k_{ij}(x,y)$ $(i,j = 1,2,\ldots,n)$ of $K(x,y)$ are selected in such a manner that the underlying closed-loop system is transformed into the target one

$$Z_t(x,t) = \Theta Z_{xx}(x,t) - CZ(x,t), \quad (7)$$
$$Z_x(0,t) = 0, \quad (8)$$
$$Z_x(1,t) = 0, \quad (9)$$

written in terms of the state vector $Z(x,t) = [z_1(x,t), z_2(x,t), \ldots, z_n(x,t)]^T \in [L_2(0,1)]^n$. The exponential stability of the target system (7)-(9) is then ensured with an arbitrarily fast convergence rate by an appropriate choice of the real-valued square matrix $C \in \Re^{n\times n}$ with entries c_{ij}, $i,j = 1,2,\ldots,n$.

The PDE governing the kernel matrix function $K(x,y)$ is now derived through the standard procedure adopted in the backstepping design [8]. By applying the Leibnitz differentiation rule to (6), spatial derivatives $Z_x(x,t)$ and $Z_{xx}(x,t)$ are readily developed as a straightforward matrix generalization of corresponding well-known scalar counterparts. Furthermore, using (3) and applying recursively integration by parts, the time derivative $Z_t(x,t)$ is derived as well. Combining such expressions, and performing rather lengthy but straightforward computations (see [3] for more detailed derivations), yield

$$\begin{aligned}
&Z_t(x,t) - \Theta Z_{xx}(x,t) + CZ(x,t)\\
&= \left[\Lambda + C + K_y(x,x)\Theta + \Theta K_x(x,x) + \Theta \tfrac{d}{dx} K(x,x)\right]\\
&\times Q(x,t) + \textstyle\int_0^x \left[\Theta K_{xx}(x,y) - K_{yy}(x,y)\Theta - K(x,y)\Lambda\right.\\
&\left. -CK(x,y)\right] Q(y,t)dy + \left[\Theta K(x,x) - K(x,x)\Theta\right] Q_x(x,t)\\
&+K(x,0)\Theta Q_x(0,t) - K_y(x,0)\Theta Q(0,t).
\end{aligned} \quad (10)$$

Clearly, the target system's PDE (7) requires that the right hand side of (10) has to be identically zero. Employing the homogeneous BC (4), this leads to the fol-

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lowing relations

$$\Theta K_{xx}(x,y) - K_{yy}(x,y)\Theta - K(x,y)\Lambda - CK(x,y) = 0, \tag{11}$$

$$\Lambda + C + K_y(x,x)\Theta + \Theta K_x(x,x) + \Theta\frac{d}{dx}K(x,x) = 0, \tag{12}$$

$$\Theta K(x,x) - K(x,x)\Theta = 0, \tag{13}$$

$$K_y(x,0)\Theta = 0. \tag{14}$$

The main critical feature of (11)-(14) is in the presence of relation (13). While being identically satisfied in the scalar case when $n = 1$ [22], this relation is in general contradictive, and there are two options to fulfill (13). One of these options is to impose the constraint that all the coupled processes possess the same diffusivity value $\theta_i = \theta$, $i = 1, 2, ..., n$, so that

$$\Theta = \theta I_{n\times n}. \tag{15}$$

An alternative option is to enforce the next constraint on the form of the kernel matrix

$$K(x,y) = k(x,y)I_{n\times n}. \tag{16}$$

Assumption (16) greatly simplifies the complexity of the underlying backstepping transformation, which is determined by a scalar function. This simplification, however, will also bring some constraint on the choice of the matrix C when the relation (16) is in force. Solution of the kernel PDE (11), (12), (14) under the additional constraints (15) or (16) will be addressed in Sections 3 and 4.

2.1 Stability of the target system dynamics

The following result is in force.

Theorem 1 *Consider the target system (7)-(9). If the matrix $S[C]$ is positive definite then system (7)-(9) is exponentially stable in the space $[L_2(0,1)]^n$ with the convergence rate specified by*

$$\|Z(\cdot,t)\|_{2,n} \le \|Z(\cdot,0)\|_{2,n}e^{-\sigma_m(S[C])t}. \tag{17}$$

Proof The detailed proof can be found in [3]. ■

3 Stabilization in the "equi-diffusivity" case

Boundary stabilization of system (3)-(5) under the constraint (15) is addressed by following the previously introduced backstepping design [3] with the corresponding treatment being included in the present work for the sake of completeness.

3.1 Explicit solution of the relevant kernel boundary-value problem

Specializing system (11), (12), (14) in light of the actual form (15) of the diffusivity matrix Θ yields

$$K_{xx}(x,y) - K_{yy}(x,y) = \frac{1}{\theta}K(x,y)\Lambda + \frac{1}{\theta}CK(x,y), \tag{18}$$

$$\Lambda + C + 2\theta\frac{d}{dx}K(x,x) = 0, \tag{19}$$

$$K_y(x,0) = 0. \tag{20}$$

Integrating (19) with respect to x gives $K(x,x) = -\frac{1}{2\theta}(\Lambda + C)x + K(0,0)$. Substituting the boundary conditions (4) and (8) into the relation $Z_x(0,t) = Q_x(0,t) - K(0,0)Q(0,t)$, which is obtained by spatial differentiation of (6) at $x = 0$, one derives that

$$K(0,0) = 0. \tag{21}$$

Hence, relation (19) is replaced by

$$K(x,x) = -\frac{1}{2\theta}(\Lambda + C)x. \tag{22}$$

The following result is in order.

Theorem 2 *The problem (18), (20), (22) possesses a solution*

$$K(x,y) = -\sum_{j=0}^{\infty}\frac{(x^2-y^2)^j(2x)}{j!(j+1)!}\left(\frac{1}{4\theta}\right)^{j+1} \times\left[\sum_{i=0}^{j}\binom{j}{i}C^i(\varLambda + C)\varLambda^{j-i}\right] \tag{23}$$

which is of class C^∞ in the domain $0 \le y \le x \le 1$.

Proof The detailed proof is presented in [3]. ■

Remark 1 *If the condition $\varLambda C = C\varLambda$ holds, then (23) simplifies to*

$$K(x,y) = -\sum_{j=0}^{\infty}\frac{(x^2-y^2)^j(2x)}{j!(j+1)!}\left[\frac{\varLambda + C}{4\theta}\right]^{j+1}. \tag{24}$$

In the scalar case $n = 1$, relation (24) specifies to that obtained in [22].

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Remark 2 *Uniqueness of the solution* (23) *to the kernel PDE* (18), (20), (22) *can be proven following the same steps as, e.g., in [6, Lemma 2.1]. The complete treatment is , however, beyond the scope of the present paper as it does not impact the underlying closed-loop stability result, and it is skipped for brevity.*

Finally, let us show that the transformation (6) is invertible, and its inverse is representable in the form

$$Q(x,t) = Z(x,t) + \int_0^x L(x,y)Z(y,t)dy. \tag{25}$$

By performing analogous developments as those made for the derivation of the gain kernel PDE (18), (20), (22), the next PDE is obtained

$$L_{xx}(x,y) - L_{yy}(x,y) = -\frac{1}{\theta}L(x,y)C - \frac{1}{\theta}\Lambda L(x,y), \tag{26}$$

$$L(x,x) = -\frac{1}{2\theta}\left(\Lambda + C\right)x, \tag{27}$$

$$L_y(x,0) = 0, \tag{28}$$

governing $L(x,y)$. By comparison between (18), (20), (22) and (26)-(28) one immediately notice that in this case $L(x,y) = -K(x,y)$ when Λ and C are replaced by $-\Lambda$ and $-C$. To reproduce the latter conclusion it suffices to explicitly denote the dependence of the solutions $L(x,y) = L(x,y;\Lambda,C)$ and $K(x,y) = K(x,y;\Lambda,C)$ on Λ and C and verify that the substitution $L(x,y;\Lambda,C) = -K(x,y;-\Lambda,-C)$ transfers (26)-(28) into (18), (20), (22).

3.2 Boundary controller design

The next result specifies the proposed boundary control design and summarizes the first stability result of this paper.

Theorem 3 *Let matrix C be selected in such a manner that $S[C]$ is positive definite whereas $\sigma_m(S[C])$ is arbitrarily large. Then, the boundary control input*

$$U(t) = -\frac{1}{2\theta}\left(\Lambda + C\right)Q(1,t) + \int_0^1 K_x(1,y)Q(y,t)dy, \tag{29}$$

$$K_x(1,y) = -\sum_{j=0}^{\infty}\left[\frac{2(1-y^2)^j + 4j(1-y^2)^{j-1}}{j!(j+1)!}\right] \times \left(\frac{1}{4\theta}\right)^{j+1}\left[\sum_{i=0}^{j}\binom{j}{i}C^i\left(\Lambda + C\right)\Lambda^{j-i}\right], \tag{30}$$

exponentially stabilizes system (3)-(5) in the space $[L_2(0,1)]^n$ with an arbitrarily fast convergence rate in accordance with

$$\|Q(\cdot,t)\|_{2,n} \le A\|Q(\cdot,0)\|_{2,n}e^{-\sigma_m(S[C])t}, \tag{31}$$

where A is a positive constant independent of $Q(x,0)$.

Proof The backstepping transformation (6), (23) was derived to map system (3)-(5) into the target dynamics governed by (7). It remains to prove that the homogenous BCs (8)-(9) hold as well. Spatial differentiation of (6) at $x = 0$ and $x = 1$ yields

$$Z_x(0,t) = Q_x(0,t) - K(0,0)Q(0,t)$$
$$Z_x(1,t) = Q_x(1,t) - K(1,1)Q(1,t) - \int_0^1 K_x(1,y)Q(y,t)dy.$$

The boundary conditions (4) and (5) and relation (22), coupled together, ensure that $K(0,0) = 0$ and $K(1,1) = -\frac{1}{2\theta}\left(\Lambda + C\right)$, thereby yielding

$$Z_x(0,t) = 0$$
$$Z_x(1,t) = U(t) + \frac{1}{2\theta}\left(\Lambda + C\right)Q(1,t) - \int_0^1 K_x(1,y)Q(y,t)dy.$$

Thus, the boundary control input vector (29)-(30), where the kernel spatial derivative $K_x(1,y)$ is obtained by differentiating (23) with respect to x at $x = 1$, results in the target dynamics (7)-(9) with homogeneous BCs.

Recall that the exponential stability of (7)-(9) was guaranteed by Theorem 1 provided that $S[C]$ is positive definite. With this in mind, it is followed [22] to derive analogous convergence properties for the original system (3)-(5) as well. The estimates $\|K(x,y)\| \le Me^{2Mx}$ and $\|L(x,y)\| \le Me^{2Mx}$ are established for some positive constant M by generalizing [22] where the scalar counterparts of such estimates were obtained. A straightforward generalization of [22, Th 4] yields that the above two upper estimates, coupled together, establish the equivalence of norms of $Z(x,t)$ and $Q(x,t)$ in $[L_2(0,1)]^n$ thereby ensuring that there exists a positive constant A independent of $Q(\xi,0)$ such that (31) straightforwardly follows from (17). This completes the proof of Theorem 3. ■

4 Stabilization in the distinct diffusivity case

In the present section, boundary stabilization of system (3)-(5) is addressed by following the previously introduced backstepping design specified with (16). Relation

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(15) is no longer in force, and now all processes possess their own distinct diffusivity parameter. As noted in Section 2, constraint (16) has to be brought into play in order to ensure that the stabilization problem is solvable through the backstepping route.

Let us now specialize system (11), (12), (14) by considering the constraint (16) on the kernel matrix:

$$(k_{xx}(x,y)-k_{yy}(x,y))\Theta = k(x,y)(\Lambda + C) \tag{32}$$
$$\Lambda + C + 2\frac{d}{dx}k(x,x)\Theta = 0 \tag{33}$$
$$k_y(x,0) = 0. \tag{34}$$

Being represented in the component-wise form, relation (32) gives rise to n independent scalar PDEs of the form

$$k_{xx}(x,y)-k_{yy}(x,y) = k(x,y)\left(\frac{\lambda_{ii}+c_{ii}}{\theta_i}\right),\ i=1,2,..,n \tag{35}$$

and to the constraints

$$\lambda_{ij}+c_{ij}=0, \quad i,j=1,2,...,n, \quad i\neq j. \tag{36}$$

In turns, relation (33), represented in the component-wise form, results in the same constraints (36) and additionally imposes the next scalar relations

$$\frac{d}{dx}k(x,x) = \frac{1}{2}\left(\frac{\lambda_{ii}+c_{ii}}{\theta_i}\right), \quad i=1,2,...,n. \tag{37}$$

It is clear that a solution may only exist if the constants $\frac{\lambda_{ii}+c_{ii}}{\theta_i}$ in the right hand sides of (35) and (37) possess the same value for all $i=1,2,...,n$. Therefore, the next constraints

$$c_{ii} = \gamma^*\theta_i - \lambda_{ii}, \quad i=1,2,...,n, \tag{38}$$
$$c_{ij} = -\lambda_{ij}, \quad i,j=1,2,...,n, \quad i\neq j, \tag{39}$$

on the elements of the matrix C must be imposed with an arbitrary constant γ^*, thereby yielding the kernel PDE

$$k_{xx}(x,y)-k_{yy}(x,y) = \gamma^* k(x,y), \tag{40}$$
$$k_y(x,0) = 0 \tag{41}$$
$$\frac{d}{dx}k(x,x) = -\frac{\gamma^*}{2}. \tag{42}$$

Integrating (42) with respect to x gives the relation $k(x,x) = -\frac{\gamma^*}{2}x + k(0,0)$ whereas the additional relation $k(0,0)=0$ is deduced by specifying the derivation of formula (21) to the current case.

System (40)-(42) can thus be specified to the boundary-value problem

$$k_{xx}(x,y)-k_{yy}(x,y) = \gamma^* k(x,y), \tag{43}$$
$$k_y(x,0) = 0 \tag{44}$$
$$k(x,x) = -\frac{\gamma^*}{2}x, \tag{45}$$

whose explicit solution

$$k(x,y) = -\gamma^* x\frac{I_1(\sqrt{\gamma^*(x^2-y^2)})}{\sqrt{\gamma^*(x^2-y^2)}} \tag{46}$$

is extracted from [22]. By making lengthy but straightforward computations, the kernel PDE of the inverse transformation can be derived as follows:

$$l_{xx}(x,y)-l_{yy}(x,y) = -\gamma^* l(x,y), \tag{47}$$
$$l_y(x,0) = 0, \tag{48}$$
$$l(x,x) = -\frac{\gamma^*}{2}x, \tag{49}$$

whose explicit solution is also drawn from [22] in the form

$$l(x,y) = -\gamma^* x\frac{J_1(\sqrt{\gamma^*(x^2-y^2)})}{\sqrt{\gamma^*(x^2-y^2)}}. \tag{50}$$

4.1 Controller design

Clearly, relations (38)-(39) require the γ^*-dependent matrix C to be selected in the form

$$C = -\Lambda + \gamma^*\Theta. \tag{51}$$

The next condition ensures that matrix $S[C]$ is positive definite.

Condition 1 *The scalar parameter κ and the design parameter γ^* are respectively chosen according to*

$$\kappa > \max_{1\le i\le n} |\sigma_i(-S[\Lambda])|, \tag{52}$$
$$\gamma^* > \frac{\sigma_M(-S[\Lambda]+\kappa I_{n\times n})+\kappa}{\sigma_m(\Theta)}, \quad \sigma_m(\Theta) = \min_{1\le i\le n}\theta_i. \tag{53}$$

The proposed boundary control design is specified for the distinct diffusivity case as follows

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Theorem 4 *Let matrix C be selected according to (51) and let Condition 1 be satisfied. Then, the boundary control input*

$$U(t) = -\frac{\gamma^*}{2}Q(1,t) + \int_0^1 k_x(1,y)Q(y,t)dy, \tag{54}$$

$$k_x(1,y) = -\gamma^* \frac{I_1(\sqrt{\gamma^*(1-y^2)})}{\sqrt{\gamma^*(1-y^2)}} - \gamma^* \frac{I_2(\sqrt{\gamma^*(1-y^2)})}{1-y^2}, \tag{55}$$

exponentially stabilizes system (3)-(5) in the space $[L_2(0,1)]^n$ with an arbitrarily fast convergence rate

$$\|Q(\cdot,t)\|_{2,n} \le A\|Q(\cdot,0)\|_{2,n} e^{-\sigma_m(S[C])t}, \tag{56}$$

where A is a positive constant independent of $Q(\xi,0)$.

Proof Noticing that $k(1,1) = -\frac{\gamma^*}{2}$ by virtue of (45), the form of the chosen boundary feedback control is justified by following the same line of reasoning used in the beginning of the proof of Theorem 3. The stability properties of the target dynamics (7)-(9) are established in Theorem 1, that requires $S[C]$ to be positive definite. Now let us show that selecting the matrix C as in (51), with the scalar parameter γ^* chosen according to (52)-(53), ensures that $S[C]$ is positive definite and $\sigma_m(S[C])$ is arbitrarily large.

Since Θ is a diagonal matrix, and γ^* is a scalar, it follows from (51) that $S[C] = -S[\Lambda] + \gamma^*\Theta$. Matrix $S[C]$ is positive definite iff the quadratic form $p^T S[C]p$ takes positive value for every nontrivial real-valued column vector p of dimension n. The quadratic form $p^T S[C]p$ can be expanded as follows by adding and subtracting to $S[C]$ the dummy quantity $\kappa I_{n\times n}$

$$\begin{aligned} p^T S[C]p &= p^T\left(-S[\Lambda] + \gamma^*\Theta + \kappa I_{n\times n} - \kappa I_{n\times n}\right)p \\ &= p^T\left(-S[\Lambda] + \kappa I_{n\times n}\right)p + \gamma^* p^T\Theta p - \kappa p^T p. \end{aligned} \tag{57}$$

It is well-known that adding $\kappa I_{n\times n}$ to any matrix shifts the corresponding eigenvalues by κ, which results in the eigenvalues of matrix $-S[\Lambda] + \kappa I_{n\times n}$ to be located at $k + \sigma_i(-S[\Lambda])$, $i = 1,2,...,n$. Therefore, condition (52) guarantees that the symmetric matrix $-S[\Lambda] + \kappa I_{n\times n}$ is positive definite. In light of this, the estimate

$$p^T S[C]p \ge \left[-\sigma_M(-S[\Lambda] + \kappa I_{n\times n}) + \gamma^*\sigma_m(\Theta) - \kappa\right] p^T p \tag{58}$$

can be derived from (57) by exploiting well-known properties of quadratic norms.

By taking into account that $\sigma_i(\Theta) = \theta_i$, it follows from (53) that the right hand side of (58) is strictly positive, thus ensuring that matrix $S[C]$ is positive definite. Since (58) holds for an arbitrary nontrivial $p \in \Re^n$, and its right hand side grows unbounded with increasing γ^*, one concludes that the smallest eigenvalue $\sigma_m(S[C])$ of $S[C]$ can be made arbitrarily large. Thus, the exponential stability of the target system's dynamics (7)-(9) is established with an arbitrarily fast convergence rate in accordance with Theorem 1.

The rest of the proof follows [22] to derive analogous convergence properties for the original system (3)-(5) as well. As shown in [22, Th.2, Th.3], both the kernel functions (46) and (50) are bounded according to the estimates $|k(x,y)| \le Me^{2Mx}$ and $|l(x,y)| \le Me^{2Mx}$ where M is a positive constant. [22, Th.4] states that those two upperbounds, coupled together, establish the equivalence between norms of $Z(x,t)$ and $Q(x,t)$ in $[L_2(0,1)]^n$ which means that there exists a positive constant A independent of $Q(\xi,0)$ such that the estimate (56) is in force as a direct consequence of (17). Theorem 4 is thus proved. ■

5 Underactuated boundary stabilization of two coupled distinct diffusion processes

Let us now consider a 2-dimensional system of coupled reaction-diffusion processes

$$q_{1t}(x,t) = \theta_1 q_{1xx}(x,t) + \lambda_{11}q_1(x,t) + \lambda_{12}q_2(x,t), \tag{59}$$
$$q_{2t}(x,t) = \theta_2 q_{2xx}(x,t) + \lambda_{21}q_1(x,t) + \lambda_{22}q_2(x,t), \tag{60}$$

equipped with Neumann-type boundary conditions

$$q_{1x}(0,t) = q_{2x}(0,t) = 0, \tag{61}$$
$$q_{1x}(1,t) = u_1(t), \quad q_{2x}(1,t) = 0, \tag{62}$$

where $q_i(x,t) \in L_2(0,1)$, $i = 1,2$, are the state variables and $u_1(t)$ is the manipulable boundary input acting on the q_1-subsystem only. To add practical value to the present investigation it is worth noticing that such a system represents linearized dimensionless dynamics of a tubular chemical reactor controlled through a boundary at low fluid superficial velocities when convection terms become negligible (cf. that of [14]). Thus interpreted, the meaning of the two state variables becomes normalized temperature and reactant concentration, respectively.

In contrast to the investigation of Section 4, where independent boundary actuation of each subsystem was available, the present system of two coupled diffusion processes is underactuated by a unique boundary control input applied to subsystem (59). It it easy to check that system (59)-(62) can be rewritten in the form (3)-(5) where $Q(x,t) = [q_1(x,t),\ q_2(x,t)]^T$, $U(t) = [u_1(t), 0]^T$, and

$$\Theta = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}. \tag{63}$$

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The next "minimum phase" assumption is imposed on the system to ensure that the q_2 subsystem (60) of (59)-(62) is asymptotically stable when $q_1(x,t)=0$.

Assumption 1 *The parameter λ_{22} is negative.*

Our objective is to exponentially stabilize system (59)-(62) by applying the invertible backstepping transformation (6) specified with

$$K(x,y) = \begin{bmatrix} k(x,y) & 0 \\ 0 & 0 \end{bmatrix}. \tag{64}$$

It follows from (6) and (64) that

$$z_1(x,t) = q_1(x,t) - \int_0^x k(x,y) q_1(y,t) dy, \tag{65}$$
$$z_2(x,t) = q_2(x,t), \tag{66}$$

i.e., the second state variable of the target dynamics is the same as that of the original system (59)-(62).

The main difference from the developments of the previous sections comes from the fact that relation (11) will be now in general impossible to fulfill. As a consequence, the target system dynamics will contain an additional integral term in contrast to (7), and it will take the form of a Partial Integro-Differential Equation (PIDE). It is worth to remark that the presence of extra integral terms in the target system is not unusual in backstepping designs when dealing with terms that cannot be compensated otherwise (see e.g. [5], [25], [29]).

The next lemma presents the derivation of the target system dynamics in the present underactuated scenario.

Lemma 1 *The backstepping transformation* (6)*,* (64)*, where $k(x,y)$ is the solution* (46) *to the boundary-value problem* (43)*-*(45)*, transfers system* (59)*-*(62) *into the target system dynamics*

$$Z_t(x,t) = \Theta Z_{xx}(x,t) - CZ(x,t) + \int_0^x \begin{bmatrix} -\lambda_{12} k(x,y) z_2(y,t) \\ \lambda_{21} l(x,y) z_1(y,t) \end{bmatrix} dy, \tag{67}$$

where $Z(x,t) = [z_1(x,t),\ z_2(x,t)]^T \in [L_2(0,1)]^2$ is the corresponding state vector, $C = C(\gamma^*) = \{c_{ij}\} \in \Re^{2\times 2}$ is the γ^*-dependent real-valued matrix given by

$$C(\gamma^*) = -\Lambda + \gamma^* \begin{bmatrix} \theta_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\lambda_{11} + \gamma^*\theta_1 & -\lambda_{12} \\ -\lambda_{21} & -\lambda_{22} \end{bmatrix}, \tag{68}$$

$\gamma^* \in \Re$ is an adjustable design parameter and $l(x,y)$ is the solution (50) *to the boundary value problem* (47)*-* (49)

Proof To support the derivation of (67), the previous multidimensional matrix-based treatment is kept to take advantage of the computations previously made. Particularly, relation (10) is still valid and the kernel conditions (12)-(14) are going to be considered and specialized to the current scenario. As for relation (11), it will be now in general impossible to fulfill and a new form of it, with the right-hand side not being identically zero anymore, will be derived and employed within the present proof.

Relations (12) and (64) yield

$$\frac{d}{dx} k(x,x) = \left(\frac{\lambda_{11} + c_{11}}{2\theta_1} \right), \tag{69}$$
$$\lambda_{12} + c_{12} = 0, \quad \lambda_{21} + c_{21} = 0, \quad \lambda_{22} + c_{22} = 0. \tag{70}$$

The following relation

$$c_{11} = \gamma^* \theta_1 - \lambda_{11}, \tag{71}$$

which involves an arbitrary constant γ^*, must then be enforced. By inspection, relations (70) -(71) result in the constrained form (68) of the γ^*-dependent matrix $C(\gamma^*)$ with a unique free parameter $\gamma^* \in \Re$ which is available for design.

The "critical" relation (13) is automatically satisfied due to (64), and relation (14) yields (44).

By taking into account the constraint (64) on the kernel matrix one derives that

$$\Theta K_{xx}(x,y) - K_{yy}(x,y)\Theta - K(x,y)\Lambda - CK(x,y) = \begin{bmatrix} \theta_1 k_{xx}(\cdot) - \theta_1 k_{yy}(\cdot) - (\lambda_{11} + c_{11}) k(\cdot) & -\lambda_{12} k(\cdot) \\ -c_{21} k(\cdot) & 0 \end{bmatrix}. \tag{72}$$

Zeroing the first diagonal element in the right hand side of (72) yields the scalar PDE (43) After employing simple manipulations, analogous to those made in Section 4, and considering as well (69) and (71), the kernel boundary-value PDE problem (43)-(45) is thus verified for the kernel function $k(x,y)$ so that while being a solution of (43)-(45), it is given by (46).

Zeroing the off-diagonal elements in the right hand side of (72) requires that both the coefficients λ_{12} and c_{21} should be identically zero (and, by (70), the same for λ_{21}). This would clearly trivialize the underlying stabilization problem (see Remark 3). Therefore, as apparent from (10), there will be an additional entry in the target dynamics in contrast to (7)-(9) since the right hand side of (11) cannot be made identically zero anymore.

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By considering (10) along with relations (72), (43) and (70), it follows that

$$Z_t(x,t) = \Theta Z_{xx}(x,t) - CZ(x,t) + \int_0^x \begin{bmatrix} 0 & -\lambda_{12}k(x,y) \\ \lambda_{21}k(x,y) & 0 \end{bmatrix} Q(y,t)dy = \Theta Z_{xx}(x,t) - CZ(x,t) + \int_0^x \begin{bmatrix} -\lambda_{12}k(x,y)q_2(y,t) \\ \lambda_{21}k(x,y)q_1(y,t) \end{bmatrix} dy. \quad (73)$$

To rewrite (73) entirely in terms of Z-coordinates, the identity

$$\int_0^x k(x,y)q_1(y,t)dy = \int_0^x l(x,y)z_1(y,t)dy \quad (74)$$

is employed. Relation (74) is derived by summing (65) and the associated inverse transformation

$$q_1(x,t) = z_1(x,t) + \int_0^x l(x,y)z_1(y,t)dy \quad (75)$$

and canceling the identical terms in the resulting equality. Substituting (74) into the last term of (73) the target dynamics PIDE (67) is obtained. Lemma 1 is proved. □

Remark 3 *It has been demonstrated within the proof of Lemma 1 that in order to obtain a target system dynamics equivalent to* (7)-(9) *both the coupling coefficients* λ_{12} *and* λ_{21} *must be zero, i.e., the original system* (59)-(62) *should already be decoupled. Clearly, this would have trivialized the underlying result, which is why such restriction has not been made and the more involved target dynamics PIDE* (67) *has been brought into play.*

The subsequent synthesis involves the next condition, which ensures the asymptotic stability of the target system dynamics. This condition relies on the feasibility problem of seeking a solution to a nonlinear inequality subject to a positive definiteness constraint on a certain parameter-dependent matrix.

Condition 2 *The nonlinear inequality*

$$\sigma_m(S[C(\gamma^*)]) > \bar{\lambda}_M \gamma^* e^{2\gamma^*} \quad (76)$$

with $\bar{\lambda}_M = \max\{|\lambda_{12}|, |\lambda_{21}|\}$ *possesses a solution* γ^* *such that the symmetric* γ^**-dependent matrix*

$$S[C(\gamma^*)] = \begin{bmatrix} -\lambda_{11} + \gamma^*\theta_1 & -\frac{\lambda_{12}+\lambda_{21}}{2} \\ -\frac{\lambda_{12}+\lambda_{21}}{2} & -\lambda_{22} \end{bmatrix} \quad (77)$$

is positive definite.

It is worth noticing that the smallest (real, and positive) eigenvalue $\sigma_m(S[C(\gamma^*)])$ of matrix $S[C(\gamma^*)]$ in Condition 2 admits the explicit representation

$$\sigma_m(S[C(\gamma^*)]) = \frac{T}{2} - \sqrt{\frac{T^2}{4} - D}, \quad (78)$$

where

$$T = -\lambda_{11} + \gamma^*\theta_1 - \lambda_{22}, \quad (79)$$

$$D = -\lambda_{22}\left(-\lambda_{11} + \gamma^*\theta_1\right) - \frac{(\lambda_{12}+\lambda_{21})^2}{4}, \quad (80)$$

are, respectively, the trace and determinant of $S[C(\gamma^*)]$.

5.1 Controller design

The next result specifies the proposed boundary control design for the distinct diffusivity case with $n = 2$ and a scalar input only.

Theorem 5 *Consider system* (59)-(62) *with Assumption 1 and let Condition 2 hold. Then, the boundary control input*

$$u_1(t) = -\frac{\gamma^*}{2}q_1(1,t) + \int_0^1 k_x(1,y)q_1(y,t)dy, \quad (81)$$

$$k_x(1,y) = -\gamma^* \frac{I_1(\sqrt{\gamma^*(1-y^2)})}{\sqrt{\gamma^*(1-y^2)}} - \gamma^* \frac{I_2(\sqrt{\gamma^*(1-y^2)})}{1-y^2}, \quad (82)$$

exponentially stabilizes system (59)-(62) *in the space* $[L_2(0,1)]^2$ *with the convergence rate given by*

$$\|Q(\cdot,t)\|_{2,2} \le A\|Q(\cdot,0)\|_{2,2}e^{-M(\gamma^*)t}, \quad (83)$$

where A *is a positive constant independent of* $Q(x,0)$ *and*

$$M(\gamma^*) = \sigma_m(S[C(\gamma^*)]) - \bar{\lambda}_M\gamma^* e^{2\gamma^*} \quad (84)$$

Proof The form of the proposed boundary feedback control is justified by following the same line of reasoning as that made in the beginning of the proof of Theorem 3. It guarantees that the target dynamics PIDE (67) is actually equipped with the homogeneous BCs

$$Z_x(0,t) = Z_x(1,t) = 0, \quad (85)$$

The asymptotic stability of the target system dynamics PIDE (67), specified with the BCs (85), is investigated by means of the candidate Lyapunov function

$$V(t) = \frac{1}{2}\int_0^1 Z^T(x,t)Z(x,t)dx = \frac{1}{2}||Z(\cdot,t)||^2_{2,2}, \quad (86)$$

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whose time derivative along the solutions of (67), (85) takes the form

$$\dot{V}(t) = \int_0^1 Z^T(x,t) Z_t(x,t) dx$$
$$= \int_0^1 Z^T(x,t)\Theta Z_{xx}(x,t)dx - \int_0^1 Z^T(x,t)CZ(x,t)dx$$
$$+ \int_0^1 Z^T(x,t)\left(\int_0^x \begin{bmatrix} -\lambda_{12}k(x,y)z_2(y,t) \\ \lambda_{21}l(x,y)z_1(y,t)\end{bmatrix} dy\right) dx. \tag{87}$$

The first two terms in the right hand side of (87) can be estimated as follows (cf. [3, Th. 2]):

$$\int_0^1 Z^T(x,t)\Theta Z_{xx}(x,t)dx \leq -\sigma_m(\Theta)||Z_\xi(\cdot,t)||^2_{2,2}, \tag{88}$$

$$\int_0^1 Z^T(x,t)CZ(x,t)dx \leq -\sigma_m(S[C(\gamma^*)])||Z(\cdot,t)||^2_{2,2}, \tag{89}$$

where $\sigma_m(\Theta) = min\{\theta_1,\theta_2\}$. By construction, both $\sigma_m(\Theta)$ and $\sigma_m(S[C(\gamma^*)])$ are strictly positive quantities. To estimate the third term in the right hand side of (87), which is sign-indefinite, the relations

$$|k(x,y)| \leq He^{2Hx}, \quad |l(x,y)| \leq He^{2Hx}, \tag{90}$$
$$H = \gamma^* = \frac{\lambda_{11}+c_{11}}{\theta_1}, \tag{91}$$

established in [22, Th.2, Th.3], are subsequently exploited. Within the (x,y) domain of interest (for which $0 \leq x \leq 1$) the worst case value $x = 1$ can be considered in (90), i.e.:

$$|k(x,y)| \leq \gamma^* e^{2\gamma^*}, \quad |l(x,y)| \leq \gamma^* e^{2\gamma^*}. \tag{92}$$

The third term of (87) is expanded as follows:

$$\int_0^1 Z^T(x,t)\left(\int_0^x \begin{bmatrix} -\lambda_{12}k(x,y)z_2(y,t) \\ \lambda_{21}l(x,y)z_1(y,t)\end{bmatrix} dy\right) dx$$
$$= \lambda_{21}\int_0^1 z_2(x,t)\left(\int_0^x l(x,y)z_1(y,t)dy\right)dx$$
$$- \lambda_{12}\int_0^1 z_1(x,t)\left(\int_0^x k(x,y)z_2(y,t)dy\right)dx. \tag{93}$$

By virtue of (92), the magnitude of the first term in the right hand side of (93) can be estimated by means of the next chain of inequalities

$$\left|\lambda_{21}\int_0^1 z_2(x,t)\left(\int_0^x l(x,y)z_1(y,t)dy\right)dx\right|$$
$$\leq |\lambda_{21}|\left|\int_0^1 |z_2(x,t)|\left(\int_0^x |l(x,y)||z_1(y,t)|dy\right)dx\right|$$
$$\leq |\lambda_{21}|\gamma^* e^{2\gamma^*}\left|\int_0^1 |z_2(x,t)|\left(\int_0^x |z_1(y,t)|dy\right)dx\right|$$
$$\leq |\lambda_{21}|\gamma^* e^{2\gamma^*}\left|\int_0^1 |z_2(x,t)|\left(\int_0^1 |z_1(y,t)|dy\right)dx\right|. \tag{94}$$

Using the triangle and Holder inequalities, the integrand in the last row of (94) is manipulated to

$$|z_2(x,t)|\left(\int_0^1 |z_1(y,t)|dy\right) \leq \frac{1}{2}\left[z_2^2(x,t) + \left(\int_0^1 |z_1(y,t)|dy\right)^2\right] \leq \frac{1}{2}\left[z_2^2(x,t) + \|z_1(\cdot,t)\|_2^2\right]. \tag{95}$$

Substituting (95) into (94) one concludes that

$$\left|\lambda_{21}\int_0^1 z_2(x,t)\left(\int_0^x l(x,y)z_1(y,t)dy\right)dx\right|$$
$$\leq \frac{1}{2}|\lambda_{21}|\gamma^* e^{2\gamma^*}\int_0^1 \left[z_2^2(x,t) + \|z_1(\cdot,t)\|_2^2\right]dx$$
$$= \frac{1}{2}|\lambda_{21}|\gamma^* e^{2\gamma^*}\left(\|z_1(\cdot,t)\|_2^2 + \|z_2(\cdot,t)\|_2^2\right). \tag{96}$$

By performing analogous manipulations, the last term in the right hand side of (93) is straightforwardly shown to obey the estimate

$$\left|\lambda_{12}\int_0^1 z_1(x,t)\left(\int_0^x k(x,y)z_2(y,t)dy\right)dx\right|$$
$$\leq \frac{1}{2}|\lambda_{12}|\gamma^* e^{2\gamma^*}\left(\|z_1(\cdot,t)\|_2^2 + \|z_2(\cdot,t)\|_2^2\right). \tag{97}$$

Combining (96) and (97) yields

$$\left|\int_0^1 Z^T(x,t)\left(\int_0^x \begin{bmatrix} -\lambda_{12}k(x,y)z_2(y,t) \\ \lambda_{21}l(x,y)z_1(y,t)\end{bmatrix} dy\right) dx\right|$$
$$\leq \bar{\lambda}_M\gamma^* e^{2\gamma^*}\left(\|z_1(\cdot,t)\|_2^2 + \|z_2(\cdot,t)\|_2^2\right)$$
$$= \bar{\lambda}_M\gamma^* e^{2\gamma^*}\|Z(\cdot)\|^2_{2,2}, \tag{98}$$

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where $\bar{\lambda}_M = \max\{|\lambda_{12}|, |\lambda_{21}|\}$. Therefore, combining (98), (88) and (89), one further elaborates (87) by getting the next final estimate of $\dot{V}(t)$:

$$\dot{V}(t) \leq -\sigma_m(\Theta)||Z_\xi(\cdot,t)||^2_{2,2} - (\sigma_m(S[C(\gamma^*)]) \\ -\bar{\lambda}_M \gamma^* e^{2\gamma^*}\Big) \|Z(\cdot)\|^2_{2,2} \leq -2M(\gamma^*)V(t), \quad (99)$$

where $M(\gamma^*)$ is given in (84). Thus, under condition (76) (which implies that $M(\gamma^*) > 0$), the exponential stability of the target dynamics (67), (85) is concluded. Following [22], analogous exponential convergence properties, as specified in (83), are ensured for the original system (59)-(62) as well, according to the supporting arguments given in the concluding part of the proof of Theorem 4. Theorem 5 is proven. ■

Remark 4 *The stabilization result just demonstrated relies on the nonlinear inequality* (76) *to possess a feasible solution. The feasibility of such a solution, which critically affects the subsequent stability analysis, intrinsically depends on the plant parameters and there exist some actual plants for which no constant* γ^**, satisfying* (76)*, can be found. However, the numerical evidences of Subsection 6.2 show that the proposed synthesis can be applied to successfully stabilize a physically relevant class of underactuated coupled reaction-diffusion processes. It is also worth to stress that Condition* (76) *is only sufficient for an underactuated boundary stabilizing synthesis to exist due to heavily conservative estimations made within the Lyapunov based convergence proof. Finally, it should be pointed out that the developments of the Section 5 don't really hinge on having constant coefficients and may be likely extended to more general scenarios where the coefficients of* (59)-(60) *are spatially and/or time varying.*

6 Simulation results

To support the theory developed, capabilities of the the proposed boundary synthesis are tested in simulation runs. First, the boundary stabilization of three coupled PDEs with distinct diffusivity parameters is treated, and then the underactuated boundary stabilization of two coupled processes is dealt with. To solve the closed-loop PDEs a standard finite-difference approximation method is used in all simulations by discretizing the spatial solution domain $x \in [0,1]$ into a finite number of N uniformly spaced solution nodes $x_i = ih$, $h = 1/(N+1)$, $i = 1, 2, ..., N$. The value $N = 40$ is set and the resulting discretized system of ODEs is then solved in the Matlab-Simulink environment by using the fixed-step Runge-Kutta method with the fixed step $T_s = 10^{-4}$.

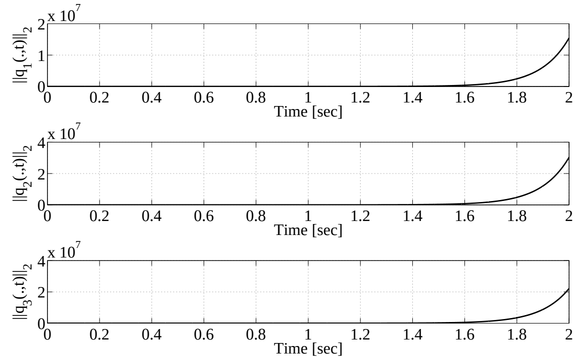

Fig. 1. TEST 1. Temporal evolution of the norms $\|q_i(\cdot,t)\|_2$, $i = 1, 2, 3$, in the open loop test.

6.1 *TEST 1: fully actuated case*

System (3)-(5) of three ($n = 3$) coupled reaction-diffusion processes, specified with the parameters

$$\Theta = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 5 & 1 \end{bmatrix}, \quad (100)$$

is first considered for simulation purposes. The initial conditions are set as $q_1(x,0) = q_3(x,0) = 2 + 2cos(\pi x)$, $q_2(x,0) = 5cos(\pi x)$. Matrix Λ possesses a real positive eigenvalue and the system in the open-loop (i.e., with $u_1(t) = u_2(t) = u_3(t) = 0$) is unstable, as displayed in the Figure 1 which shows the diverging temporal evolutions of the state norms $\|q_1(\cdot,t)\|_2$, $\|q_2(\cdot,t)\|_2$ and $\|q_3(\cdot,t)\|_2$. The boundary controller (54)-(55) is implemented by selecting the parameter $\gamma^* = 5$ as prescribed in Condition 1 to fulfill the requirement $S[C] > 0$, where C is given in (51). The converging spatiotemporal evolutions of the states in the closed-loop is shown in Figure 2 as well as the associated norm $\|Q(\cdot,t)\|_{2,3}$ is. As expected, this associated norm monotonically tends to zero. Figure 3 displays the time histories of the three control inputs $u_i(t)$ $(i = 1, 2, 3)$ showing the initial peaking, and subsequent convergence to zero, which are typical for the backstepping design.

6.2 *TEST 2: underactuated case*

Next, the underactuated system (59)-(60), specified with the parameters $\theta_1 = 9$, $\theta_2 = 1$, $\lambda_{11} = 3$, $\lambda_{12} = \lambda_{21} = 1$ and $\lambda_{22} = -5$, is under numerical study. The initial conditions are set as $q_1(x,0) = 2 + 2cos(\pi x)$, $q_2(x,0) = 5cos(\pi x)$. The considered system in the open-loop (i.e., with $u_1(t) = 0$) is unstable since the Λ matrix possesses a positive eigenvalue. The unstable behaviour of the open-loop plant is displayed in the Figure 4, which shows the diverging spatiotemporal evolutions of the states $q_1(x,t)$ and $q_2(x,t)$.

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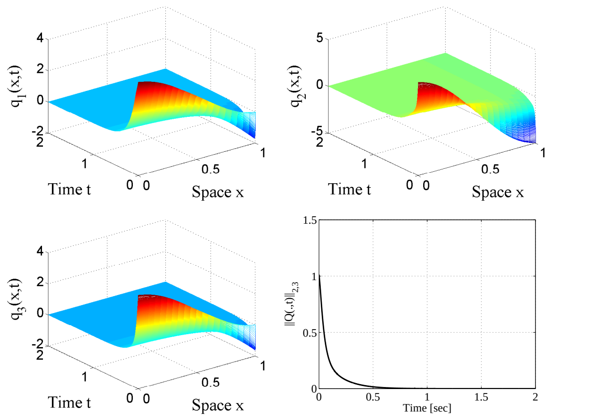

Fig. 2. TEST 1. Spatiotemporal evolution of the states $q_i(x,t)$, $i=1,2,3$, in the closed-loop test and (bottom-right) time profile of the corresponding norm $\|Q(\cdot,t)\|_{2,3}$

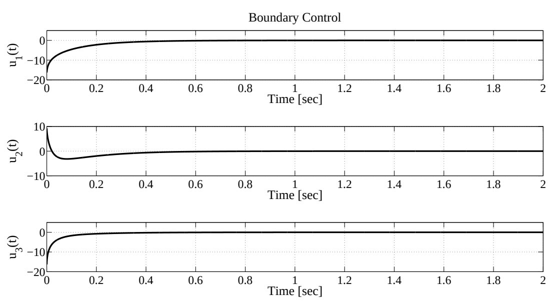

Fig. 3. TEST 1. Temporal evolution of the boundary controls $u_i(t)$, $i=1,2,3$.

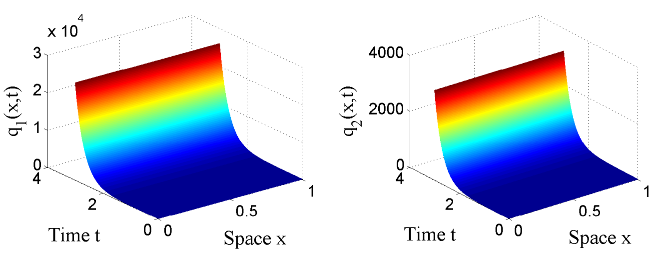

Fig. 4. TEST 2. Spatiotemporal evolution of $q_1(x,t)$ and $q_2(x,t)$ in the open loop.

Clearly, Assumption 1 holds true, and the boundary controller (81)-(82) is implemented by selecting the parameter $\gamma^* = 0.7$. With the adopted choice of γ^* it turn out that $\sigma_m(S[C(\gamma^*)]) = 2.58$, whereas the right hand side of (76) takes the value 2.52, hence Condition 2 is satisfied thereby ensuring that the closed-loop system meets desired exponential stability properties according to Theorem 5.

Figure 5 shows the resulting stable spatiotemporal evolutions of the state variables $q_1(x,t)$ and $q_2(t)$ in the closed-loop, which both vanish in L_2 norm as shown in the Figure 6. The time evolution of the boundary control input $u_1(t)$ is displayed in the Figure 7.

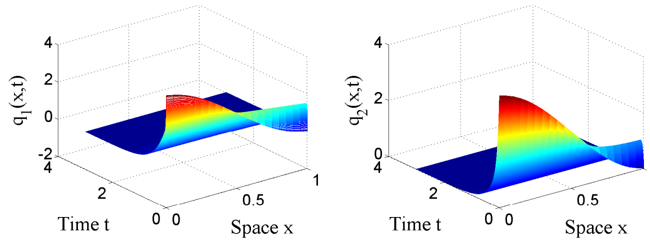

Fig. 5. TEST 2. Spatiotemporal evolution of $q_1(x,t)$ and $q_2(x,t)$in the closed-loop test.

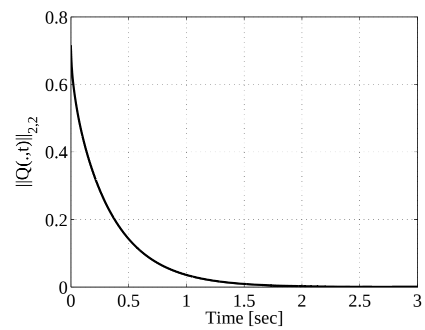

Fig. 6. TEST 2. Vector norm $\|Q(\cdot)\|_{2,2}$ in the closed-loop test.

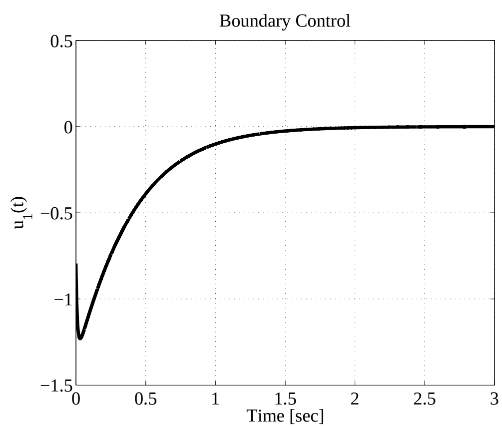

Fig. 7. TEST 2. Time evolution of the boundary control input $u_1(t)$.

7 Conclusions

The backstepping-based boundary stabilization of certain classes of unstable coupled parabolic linear PDEs was tackled, and explicit state feedback boundary controllers were derived to attain the exponential decay of the closed-loop system in the state space $[L_2(0,1)]^n$. These results provide a non trivial multidimensional counterpart to the "scalar" ($n=1$) treatment previously developed in [22]. Addressing the observer-based output feedback design, dealing with spatially-dependent

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parameters, and including the convection terms in the coupled PDEs, are among the most interesting lines of future related investigations. It is also of interest to deepen the present investigation on the underactuated case where only one scalar manipulable input variable is available, by generalizing the 2-dimensional problem statement, studied in the present work, towards higher dimensional scenarios. Additionally, integration with other design methodologies such as the sliding mode approaches, will be pursued as well to enhance the underlying robustness features. Particularly, recent investigations of [16]-[19] are hoped to complement the presented approaches by integrating them with suitably designed second-order sliding mode based boundary controllers in order to deal with the control of perturbed coupled PDEs.

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