

BICONSERVATIVE SURFACES

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ABSTRACT. In this work we obtain some geometric properties of biconservative surfaces in a Riemannian manifold. In particular, we shall study the relationship between biconservative surfaces and the holomorphicity of a generalized Hopf function. Also, we give a complete classification of CMC biconservative surfaces in a 4-dimensional space form.

1. INTRODUCTION

For an immersion $\varphi : M^2 \hookrightarrow N^3(c)$ of an oriented surface in a three-dimensional space form of constant sectional curvature c , the classical Hopf function is defined by

$$(1.1) \quad \Phi(z, \bar{z}) = \langle B(\partial_z, \partial_z), \eta \rangle = \langle A(\partial_z), \partial_z \rangle,$$

where η is the global unit normal vector field, B is the second fundamental form, A is the shape operator and, with respect to isothermal coordinates (x, y) on M^2 ,

$$z = x + iy, \quad \partial_z = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \partial_{\bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

The Hopf function defined in (1.1) is the key ingredient in the proof of the famous Hopf's Theorem: *a CMC immersed sphere in $N^3(c)$ is a round sphere*. Hopf's proof is based on the following facts: (i) Φ is holomorphic if and only if M is CMC; (ii) $\Phi = 0$ at umbilical points; (iii) on a topological sphere any holomorphic quadratic differential vanishes everywhere.

As for immersions $\varphi : M^2 \hookrightarrow N^n(c)$, $n \geq 4$, the function Φ cannot be defined and a fair substitute is the function

$$(1.2) \quad Q(z, \bar{z}) = \langle B(\partial_z, \partial_z), H \rangle = \langle A_H(\partial_z), \partial_z \rangle,$$

where $H = (1/2)$ trace B is the mean curvature vector field. A classical result, see [15, 21], states that if the surface M has parallel mean curvature (PMC) then Q is holomorphic. The function Q in (1.2) was recently used by Loubeau, Oniciuc (see [18]) and, in particular, enabled them to prove several rigidity results for biharmonic surfaces. We point out that the functions Φ , Q above are defined *locally* on the surface M . Indeed, to give them a global meaning, one should consider their associated quadratic differentials (see [16]). However, for the sake of simplicity, we decide to work with functions because most of our calculations and results are of a local nature.

2000 *Mathematics Subject Classification.* 58E20.

Key words and phrases. Biharmonic maps, biconservative immersions, Hopf differential.

Work supported by: PRIN 2010/11 – Varietà reali e complesse: geometria, topologia e analisi armonica N. 2010NNBZ78 003 – Italy; GNSAGA – INdAM, Italy; Romanian National Authority for Scientific Research, CNCS – UEFISCDI, project number PN-II-RU-TE-2011-3-0108.

A natural question is to ask whether the converse of the Hoffman and Yau result holds, that is, if a surface with Q holomorphic is PMC. This question is interesting also for surfaces $M^2 \hookrightarrow N^3(c)$. In this case, it turns out (see Proposition 3.1 below) that Q is holomorphic if and only if the following condition is satisfied :

$$(1.3) \quad A(\text{grad } f) = 0 ,$$

where we have put $H = f \eta$. Since $\det A = K^M - c$, where K^M is the Gaussian curvature of M , it follows that, if $K^M \neq c$, then Q is holomorphic if and only if M is CMC. On the other hand, a solution of (1.3) which is not CMC must have $K^M = c$, and there exist examples. For instance, in \mathbb{R}^3 , we shall show that the cone parametrized by

$$X(u, v) = \left(\frac{1}{k} (1 - \alpha v) \cos(ku), \frac{1}{k} (1 - \alpha v) \sin(ku), \beta v \right) ,$$

where α , k and β are constants satisfying $\alpha^2 + k^2 \beta^2 = k^2$ and $\alpha \neq 0$, has Q holomorphic and it is not CMC (see Section 3 below for details).

The main motivation of this paper is to study the relationship between the holomorphicity of Q and the constancy of the mean curvature in a more general context. More specifically, we shall study immersed surfaces $M^2 \hookrightarrow N^n$, the starting point being the fact that the definition (1.2) of the function Q can also be adapted to a surface in any Riemannian manifold (N^n, h) . Moreover, $Q = 0$ describes the pseudo-umbilical points.

In this more general framework, the family of immersed surfaces $M^2 \hookrightarrow N^n$ which turns out to be suitable for our purposes is that of *biconservative surfaces*. Biconservative immersions are a rapidly developing topic (see Section 2 below for complete definitions and details). In simple words, they are defined as immersions for which the tangential component of the bitension field vanishes. In particular (see Theorem 4.2 below), we shall prove that, if $M^2 \hookrightarrow N^n$ is biconservative, then the holomorphicity of Q is equivalent to the constancy of the mean curvature.

A key ingredient to obtain this result is the symmetric $(0, 2)$ -tensor, associated to any immersion $\varphi : M^2 \hookrightarrow (N^n, h)$, defined by

$$(1.4) \quad S_2 = -2|H|^2 g + 4A_H ,$$

where $g = \varphi^* h = \langle \cdot, \cdot \rangle$. The tensor S_2 is called the *stress-bienergy* tensor and, as we will describe in Section 2, it has a variational meaning. In particular, an immersed surface $\varphi : M^2 \hookrightarrow (N^n, h)$ is biconservative if $\text{div } S_2 = 0$. More specifically, we will show that Theorem 4.2 is a consequence of the fact that, for a biconservative surface, $4|H|^2 = \text{trace } S_2 = \text{constant}$ if and only if $S_2(\partial_z, \partial_z) = 4Q(z, \bar{z})$ is holomorphic.

If $\varphi : M^2 \hookrightarrow N^3(c)$ is an immersed surface in a three-dimensional space form, then

$$(1.5) \quad \frac{1}{4} \text{div } S_2 = A(\text{grad } f) + f \text{grad } f ,$$

which implies that CMC surfaces are automatically biconservative. In [4], followed by [13], the authors provided the complete classification of biconservative surfaces in a 3-dimensional space form which are not CMC.

In codimension greater than or equal to 2, PMC surfaces in space forms are automatically biconservative, while CMC surfaces are not. Therefore, it makes sense to study CMC biconservative surfaces. In this paper, we shall consider the simplest case of codimension 2 and prove the following theorem:

Theorem 5.1. *Let $M^2 \hookrightarrow N^4(c)$ be a CMC biconservative surface in a space form of constant sectional curvature $c \neq 0$. Then M^2 is PMC.*

Thus, there could exist a CMC biconservative surface in $N^4(c)$, which is not PMC, only when $c = 0$, and in this case we shall show that, locally, the surface is given by

$$X(u, v) = (\gamma(u), v + a) = (\gamma^1(u), \gamma^2(u), \gamma^3(u), v + a), \quad a \in \mathbb{R},$$

where $\gamma : I \rightarrow \mathbb{R}^3$ is a curve in \mathbb{R}^3 with constant curvature.

2. BIHARMONIC MAPS, STRESS-ENERGY TENSORS AND BICONSERVATIVE IMMERSIONS

As described by Hilbert in [14], the *stress-energy* tensor associated to a variational problem is a symmetric 2-covariant tensor S which is conservative at critical points, i.e. with $\operatorname{div} S = 0$. In the context of harmonic maps $\varphi : (M, g) \rightarrow (N, h)$ between two Riemannian manifolds, that is critical points of the *energy* functional

$$(2.1) \quad E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 dv_g,$$

the stress-energy tensor was studied in detail by Baird and Eells in [1] (see also [20] and [2]). Indeed, the Euler-Lagrange equation associated to the energy functional (2.1) is equivalent to the vanishing of the tension field $\tau(\varphi) = \operatorname{trace} \nabla d\varphi$ (see [10]), and the tensor

$$S = \frac{1}{2} |d\varphi|^2 g - \varphi^* h$$

satisfies $\operatorname{div} S = -\langle \tau(\varphi), d\varphi \rangle$. Therefore, $\operatorname{div} S = 0$ when the map is harmonic.

Remark 2.1. We point out that, in the case of isometric immersions, the condition $\operatorname{div} S = 0$ is always satisfied, since $\tau(\varphi)$ is normal.

A natural generalization of harmonic maps are the so-called *biharmonic maps*: these maps are the critical points of the bienergy functional (as suggested by Eells–Lemaire [9])

$$(2.2) \quad E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 dv_g.$$

In [12] G. Jiang showed that the Euler-Lagrange equation associated to $E_2(\varphi)$ is given by the vanishing of the bitension field

$$(2.3) \quad \tau_2(\varphi) = -\Delta \tau(\varphi) - \operatorname{trace} R^N(d\varphi, \tau(\varphi))d\varphi,$$

where Δ is the rough Laplacian on sections of $\varphi^{-1}(TN)$ that, for a local orthonormal frame $\{e_i\}_{i=1}^m$ on M , is defined by

$$\Delta = - \sum_{i=1}^m \{ \nabla_{e_i}^\varphi \nabla_{e_i}^\varphi - \nabla_{\nabla_{e_i}^M e_i}^\varphi \}.$$

The curvature operator on (N, h) , which also appears in (2.3), can be computed by means of

$$R^N(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]} .$$

The study of the stress-energy tensor for the bienergy was initiated in [11] and afterwards developed in [17]. Its expression is

$$(2.4) \quad S_2(X, Y) = \frac{1}{2} |\tau(\varphi)|^2 \langle X, Y \rangle + \langle d\varphi, \nabla \tau(\varphi) \rangle \langle X, Y \rangle - \langle d\varphi(X), \nabla_Y \tau(\varphi) \rangle - \langle d\varphi(Y), \nabla_X \tau(\varphi) \rangle ,$$

and it satisfies the condition

$$(2.5) \quad \operatorname{div} S_2 = -\langle \tau_2(\varphi), d\varphi \rangle ,$$

thus conforming to the principle of a stress-energy tensor for the bienergy.

If $\varphi : (M, g) \hookrightarrow (N, h)$ is an isometric immersion, then (2.5) becomes

$$(\operatorname{div} S_2)^\# = -\tau_2(\varphi)^\top ,$$

where $\#$ denotes the musical isomorphism sharp. This means that isometric immersions with $\operatorname{div} S_2 = 0$ correspond to immersions with vanishing tangential part of the corresponding bitension field.

In particular, an isometric immersion $\varphi : (M, g) \hookrightarrow (N, h)$ is called *biconservative* if $\operatorname{div} S_2 = 0$. The study of biconservative immersions is quite rich from the analytical point of view, because one has to understand whether certain fourth order differential equations admit solutions which do not correspond to minimal immersions. Moreover, in suitable equivariant context (see [19]) such a study is the starting point to obtain non-existence of proper biharmonic immersions.

In this paper, we restrict our attention to isometric immersions $\varphi : M^2 \hookrightarrow (N^n, h)$ from a surface in an n -dimensional Riemannian manifold. In this case $\tau(\varphi) = 2H$, H being the mean curvature vector field, and, if we denote by A_H the shape operator in the direction of H , we can easily check that the stress energy tensor S_2 given in (2.4) becomes

$$S_2 = -2|H|^2 g + 4A_H ,$$

which justifies (1.4). Then, in the case of surfaces $M^2 \hookrightarrow N^n(c)$ in a space of constant sectional curvature c , a straightforward calculation gives

$$(2.6) \quad \frac{1}{2} \operatorname{div} S_2 = 2 \operatorname{trace} A_{\nabla_{(\cdot)}^\perp H}(\cdot) + \operatorname{grad} |H|^2 ,$$

which becomes, if $n = 3$, condition (1.5) in the introduction.

3. SURFACES IN $N^3(c)$ WITH Q HOLOMORPHIC

Let $M^2 \hookrightarrow N^3(c)$ be an oriented surfaces in a three-dimensional space form of constant sectional curvature c and denote by $g = \langle, \rangle$ the induced metric on M^2 . By assumption, M^2 is orientable and then it is a one-dimensional complex manifold. If we consider local

isothermal coordinates $(U; x, y)$, then $g = \lambda^2(dx^2 + dy^2)$ for some positive function λ on U and $\{\partial_x, \partial_y\}$ is positively oriented. Let us denote, as usual,

$$z = x + iy, \quad \partial_z = \frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Also, let η be the global unit normal vector field, B the second fundamental form and A the shape operator. Then the mean curvature vector field can be written as $H = f \eta$, where f is the mean curvature function. As we mentioned in the introduction, for surfaces $M^2 \hookrightarrow N^3(c)$ the function

$$\Phi(z, \bar{z}) = \langle B(\partial_z, \partial_z), \eta \rangle = \langle A(\partial_z), \partial_z \rangle$$

is holomorphic if and only if the surface is CMC. In fact, a straightforward computation, taking into account Codazzi's equation, gives

$$(3.1) \quad 8 \partial_{\bar{z}} \langle A(\partial_z), \partial_z \rangle = 2 \lambda^2 (f_x - i f_y) = 4 \lambda^2 \partial_z(f).$$

Differently, if we consider the function

$$Q(z, \bar{z}) = \langle B(\partial_z, \partial_z), H \rangle = \langle A_H(\partial_z), \partial_z \rangle,$$

taking into account (3.1), we obtain

$$\begin{aligned} 8 \partial_{\bar{z}} \langle B(\partial_z, \partial_z), H \rangle &= 8 (\partial_{\bar{z}} f) \Phi + 8 f (\partial_{\bar{z}} \Phi) \\ &= (f_x + i f_y) [\langle A(\partial_x), \partial_x \rangle - \langle A(\partial_y), \partial_y \rangle - 2i \langle A(\partial_x), \partial_y \rangle] + 2 \lambda^2 f (f_x - i f_y) \\ &= 2 f_x \langle A(\partial_x), \partial_x \rangle + 2 f_y \langle A(\partial_y), \partial_x \rangle - 2i (f_x \langle A(\partial_x), \partial_y \rangle + f_y \langle A(\partial_y), \partial_y \rangle) \\ (3.2) \quad &= 2 \lambda^2 \langle A(\text{grad } f), \partial_x \rangle - 2 \lambda^2 i \langle A(\text{grad } f), \partial_y \rangle, \end{aligned}$$

where in the third equality we have used that $2 \lambda^2 f = \langle A(\partial_x), \partial_x \rangle + \langle A(\partial_y), \partial_y \rangle$. From (3.2) we have the following proposition.

Proposition 3.1. *Let $M^2 \hookrightarrow N^3(c)$ be an oriented surface in a space form of constant sectional curvature c . If f is constant, then $Q(z, \bar{z})$ is holomorphic. Conversely, if $Q(z, \bar{z})$ is holomorphic, then $A(\text{grad } f) = 0$.*

As a consequence of (3.2), if we denote by K^M the Gaussian curvature of the surface M , we obtain

Proposition 3.2. *Let $M^2 \hookrightarrow N^3(c)$ be an oriented surface in a space form of constant sectional curvature c .*

- (a) *If $\det(A) = K^M - c \neq 0$, then f is constant if and only if $Q(z, \bar{z})$ is holomorphic;*
- (b) *If f is not constant and $Q(z, \bar{z})$ is holomorphic, then $K^M = c$.*

A natural question is whether there exists a surface satisfying condition (b) of Proposition 3.2. In the next proposition we will show that such surfaces do exist in \mathbb{R}^3 and we characterize them completely.

Proposition 3.3. *Let $M^2 \hookrightarrow \mathbb{R}^3$ be an oriented flat surface. Assume that $|\text{grad } f| > 0$ and $f > 0$ on M . If $Q(z, \bar{z})$ is holomorphic, then (locally) the surface can be parametrized by*

$$X(u, v) = \left(\frac{1}{k} (1 - \alpha v) \cos(ku), \frac{1}{k} (1 - \alpha v) \sin(ku), \beta v \right),$$

where α , k and β are constants satisfying $\alpha^2 + k^2 \beta^2 = k^2$ and $\alpha \neq 0$.

Proof. Since $Q(z, \bar{z})$ is holomorphic, we deduce that $A(\text{grad } f) = 0$. Let $\{e_1, e_2\}$ be a local orthonormal frame of principal directions such that

$$Ae_1 = 0, \quad Ae_2 = 2fe_2.$$

From $A(\text{grad } f) = 2f(e_2f)e_2 = 0$ we have that $e_2f = 0$. Next, from the Codazzi equation

$$\nabla_{e_1}A(e_2) - \nabla_{e_2}A(e_1) = A[e_1, e_2],$$

we find

$$(e_1f)e_2 + f\omega_2^1(e_1)e_1 - f\omega_2^1(e_2)e_2 = 0,$$

where ω_i^j are the connection 1-forms defined by $\nabla e_i = \omega_i^j e_j$. Thus

$$\begin{cases} \omega_2^1(e_1) = 0 \\ \omega_2^1(e_2) = (e_1f)/f = e_1(\ln f) \neq 0. \end{cases}$$

From here, since

$$B(e_1, e_1) = 0, \quad B(e_1, e_2) = 0, \quad B(e_2, e_2) = 2f\eta,$$

we obtain

$$(3.3) \quad \begin{aligned} \bar{\nabla}_{e_1}e_1 &= 0, & \bar{\nabla}_{e_1}e_2 &= 0, \\ \bar{\nabla}_{e_2}e_1 &= -e_1(\ln f)e_2, & \bar{\nabla}_{e_2}e_2 &= e_1(\ln f)e_1 + 2f\eta, \\ \bar{\nabla}_{e_1}\eta &= -Ae_1 = 0, & \bar{\nabla}_{e_2}\eta &= -Ae_2 = -2fe_2, \end{aligned}$$

where $\bar{\nabla}$ denotes the connection in \mathbb{R}^3 . Let now γ be an integral curve of e_1 parametrized by arc-length, that is $\gamma(v) = \bar{a} + v\bar{b}$, with \bar{a} and \bar{b} constant vectors of \mathbb{R}^3 with $|\bar{b}| = 1$. Let $p_0 \in M$ and $\sigma(u)$ an integral curve of e_2 with $\sigma(0) = p_0$. Then the surface can be locally parametrized by

$$(3.4) \quad X(u, v) = \sigma(u) + v\bar{b}(u),$$

where $\bar{b}(u) = e_1(\sigma(u))$. Now, from (3.3),

$$\bar{b}'(u) = \frac{d\bar{b}}{du} = \bar{\nabla}_{e_2}e_1 = -e_1(\ln f)e_2$$

and, as $[e_1, e_2] = -e_1(\ln f)e_2$,

$$e_2(e_1(\ln f)) = e_1(e_2(\ln f)) - [e_1, e_2] \ln f = 0.$$

We can then put $e_1(\ln f)|_{\sigma(u)} = \alpha = \text{constant} \neq 0$ and

$$(3.5) \quad \bar{b}'(u) = -\alpha e_2.$$

Next, by construction, $\sigma'(u) = e_2(\sigma(u))$ and this implies that

$$\bar{\nabla}_{\sigma'}e_2 = \bar{\nabla}_{e_2}e_2 = \alpha e_1 + 2f\eta = kN,$$

where N is a unit normal vector field along σ and $k = \sqrt{\alpha^2 + 4f^2} = \text{constant} \neq 0$ along $\sigma(u)$. Moreover,

$$\bar{\nabla}_{\sigma'}N = \frac{1}{k}\bar{\nabla}_{e_2}(\alpha e_1 + 2f\eta) = \frac{1}{k}(-\alpha^2 e_2 - 4f^2 e_2) = -k e_2,$$

which implies that σ is the arc-length parametrization of a circle in \mathbb{R}^3 of radius $1/k$, that is, up to a global isometry of \mathbb{R}^3 ,

$$\sigma(u) = \left(\frac{1}{k} \cos(ku), \frac{1}{k} \sin(ku), 0 \right).$$

From (3.5) we then get

$$\bar{b}(u) = \left(-\frac{\alpha}{k} \cos(ku) + \beta_1, -\frac{\alpha}{k} \sin(ku) + \beta_2, \beta \right)$$

where, since $|\bar{b}(u)| = 1$, $\beta_1 = \beta_2 = 0$ and $\alpha^2 + k^2 \beta^2 = k^2$. Replacing the expression of $\sigma(u)$ and $\bar{b}(u)$ in (3.4) we find the desired expression. \square

Remark 3.4. Proposition 3.3 can also be stated for surfaces in $N^3(c)$, with $c \neq 0$. For example, it can be proved that a non constant mean curvature surface $M^2 \hookrightarrow \mathbb{S}^3 \subset \mathbb{R}^4$, with $Q(z, \bar{z})$ holomorphic, is (locally) parametrized by

$$\begin{aligned} X(u, v) = & \frac{1}{k} \left(\cos(ku), \sin(ku), \sqrt{k^2 - 1}, 0 \right) \cos v \\ & + \frac{1}{k} \left(-\alpha \cos(ku), -\alpha \sin(ku), \frac{\alpha}{\sqrt{k^2 - 1}}, k\beta \right) \sin v \end{aligned}$$

where α , k and β are constants satisfying $\beta^2(k^2 - 1) = k^2 - \alpha^2 - 1$ and $k > \sqrt{1 + \alpha^2}$, $\alpha \neq 0$.

We end this section with the following general result about immersed surfaces in a space form.

Proposition 3.5. *Let $M^2 \hookrightarrow N^3(c)$ be an immersed orientable surface. Assume that M^2 is a topological sphere. Then M^2 is CMC if and only if $Q(z, \bar{z})$ is holomorphic.*

Proof. If M^2 is CMC from Proposition 3.1 $Q(z, \bar{z})$ is holomorphic. Conversely, if $Q(z, \bar{z})$ is holomorphic and M^2 is a topological sphere, then $Q(z, \bar{z})$ must vanish. This implies that M^2 is pseudo-umbilical, that is, for all tangent vector fields X, Y to M the second fundamental form B satisfies $\langle B(X, Y), H \rangle = |H|^2 \langle X, Y \rangle$ (see [7]). Let $H = f\eta$, where η is the unit normal vector field. If $f = 0$, then M is minimal, thus CMC. We can then assume that there exists $p_0 \in M$ such that $|f(p_0)| > 0$ and put $A = \{p \in M : |f(p)| = |f(p_0)|\}$. The subset A is non-empty and closed in M^2 . We shall prove that A is also open. Indeed, let $p_1 \in A$. As $f(p_1) \neq 0$ there exists an open neighborhood U of p_1 such that $f(p) \neq 0$ for all $p \in U$. Since M^2 is pseudo-umbilical and $f(p) \neq 0$ for any $p \in U$ we conclude that U is umbilical in $N^3(c)$ and this implies that $f|_U = \text{constant}$, that is $U \subset A$. \square

Remark 3.6. It is not difficult to prove that, in general, if $M^2 \hookrightarrow (N^n, h)$ is pseudo-umbilical and biconservative, then it is CMC (see [3], Proposition 2.5).

4. DIVERGENCE FREE SYMMETRIC TENSORS

We begin with two rather general properties that could be interesting by themselves.

Proposition 4.1. *Let T be a symmetric $(0, 2)$ -tensor field on a Riemannian surface (M^2, g) and set $t = \text{trace } T$. Assume that M^2 is orientable and $\text{div } T = 0$. Then*

- (a) $T(\partial_z, \partial_z)$ is holomorphic if and only if $t = \text{constant}$;
- (b) T is a Codazzi tensor if and only if $t = \text{constant}$.

Proof. (a) - We consider local isothermal coordinates and use the same notations as in Section 3. Then

$$\begin{aligned} T(\partial_z, \partial_z) &= \frac{1}{4} T(\partial_x - i\partial_y, \partial_x - i\partial_y) \\ &= \frac{1}{4} \{T(\partial_x, \partial_x) - T(\partial_y, \partial_y) - 2i T(\partial_x, \partial_y)\}. \end{aligned}$$

Thus

$$8 \partial_{\bar{z}} T(\partial_z, \partial_z) = A + iB ,$$

where we have set

$$A = \partial_x T(\partial_x, \partial_x) - \partial_x T(\partial_y, \partial_y) + 2\partial_y T(\partial_x, \partial_y)$$

and

$$B = \partial_y T(\partial_x, \partial_x) - \partial_y T(\partial_y, \partial_y) - 2\partial_x T(\partial_x, \partial_y) .$$

We now show that $A = B = 0$ if and only if $t = \text{constant}$.

First, using the standard formula for the covariant derivative of a $(0, 2)$ -tensor, the term A can be rewritten as

$$\begin{aligned} A &= (\nabla_{\partial_x} T)(\partial_x, \partial_x) + 2T(\nabla_{\partial_x} \partial_x, \partial_x) - (\nabla_{\partial_x} T)(\partial_y, \partial_y) - 2T(\nabla_{\partial_x} \partial_y, \partial_y) \\ &\quad + 2(\nabla_{\partial_y} T)(\partial_x, \partial_y) + 2T(\nabla_{\partial_y} \partial_x, \partial_y) + 2T(\partial_x, \nabla_{\partial_y} \partial_y) \\ &= (\nabla_{\partial_x} T)(\partial_x, \partial_x) - (\nabla_{\partial_x} T)(\partial_y, \partial_y) + 2(\nabla_{\partial_y} T)(\partial_y, \partial_x) + 2T(\nabla_{\partial_x} \partial_x, \partial_x) + 2T(\partial_x, \nabla_{\partial_y} \partial_y) . \end{aligned}$$

Next, from $\text{div } T = 0$, we obtain

$$0 = \text{div } T(\partial_x) = (\nabla_{\partial_x} T)(\partial_x, \partial_x) + (\nabla_{\partial_y} T)(\partial_y, \partial_x) ,$$

which implies that

$$(4.1) \quad (\nabla_{\partial_y} T)(\partial_y, \partial_x) = -(\nabla_{\partial_x} T)(\partial_x, \partial_x) .$$

Using (4.1), the expression of A becomes

$$\begin{aligned} A &= -(\nabla_{\partial_x} T)(\partial_x, \partial_x) - (\nabla_{\partial_x} T)(\partial_y, \partial_y) + 2T(\nabla_{\partial_x} \partial_x, \partial_x) + 2T(\partial_x, \nabla_{\partial_y} \partial_y) \\ &= -\partial_x T(\partial_x, \partial_x) + 2T(\partial_x, \nabla_{\partial_x} \partial_x) - \partial_x T(\partial_y, \partial_y) + 2T(\partial_y, \nabla_{\partial_x} \partial_y) \\ &\quad + 2T(\nabla_{\partial_x} \partial_x, \partial_x) + 2T(\partial_x, \nabla_{\partial_y} \partial_y) \\ &= -\partial_x(\lambda^2 t) + 4T(\nabla_{\partial_x} \partial_x, \partial_x) + 2T(\nabla_{\partial_x} \partial_y, \partial_y) + 2T(\nabla_{\partial_y} \partial_y, \partial_x) . \end{aligned}$$

Since (x, y) are isothermal coordinates, computing the Christoffel symbols we have

$$\begin{aligned} \nabla_{\partial_x} \partial_y &= \nabla_{\partial_y} \partial_x = \frac{1}{\lambda} (\lambda_y \partial_x + \lambda_x \partial_y) , \\ \nabla_{\partial_x} \partial_x &= \frac{1}{\lambda} (\lambda_x \partial_x - \lambda_y \partial_y) , \quad \nabla_{\partial_y} \partial_y = \frac{1}{\lambda} (-\lambda_x \partial_x + \lambda_y \partial_y) , \end{aligned}$$

from which

$$A = -\lambda^2 \partial_x t .$$

In a similar way, we obtain

$$B = \lambda^2 \partial_y t .$$

(b) - Let $\{X_1, X_2\}$ be a local orthonormal frame on M^2 . The tensor T is a Codazzi tensor if

$$(\nabla_{X_1} T)(X_2, X_i) = (\nabla_{X_2} T)(X_1, X_i) , \quad i = 1, 2 .$$

By definition of covariant derivative of a $(0, 2)$ -tensor, we have

$$\begin{aligned}
(\nabla_{X_1} T)(X_2, X_1) - (\nabla_{X_2} T)(X_1, X_1) &= X_1 T(X_1, X_2) - T(\nabla_{X_1} X_2, X_1) - T(\nabla_{X_1} X_1, X_2) \\
&\quad - X_2 T(X_1, X_1) + 2T(\nabla_{X_2} X_1, X_1) \\
&= X_1 T(X_1, X_2) + X_2 T(X_2, X_2) - T(\nabla_{X_1} X_1, X_2) \\
&\quad - T(\nabla_{X_1} X_2, X_1) + 2T(\nabla_{X_2} X_1, X_1) - X_2 t.
\end{aligned}
\tag{4.2}$$

Next, from $\operatorname{div} T = 0$, we obtain

$$X_1 T(X_1, X_2) + X_2 T(X_2, X_2) = T(\nabla_{X_1} X_1, X_2) + T(\nabla_{X_1} X_2, X_1) + 2T(\nabla_{X_2} X_2, X_2),$$

that substituted in (4.2) gives

$$\begin{aligned}
(\nabla_{X_1} T)(X_2, X_1) - (\nabla_{X_2} T)(X_1, X_1) &= 2T(\nabla_{X_2} X_2, X_2) + 2T(\nabla_{X_2} X_1, X_1) - X_2 t \\
&= -X_2 t,
\end{aligned}
\tag{4.3}$$

where, for the last equality, we have used the fact that

$$T(\nabla_{X_2} X_2, X_2) + T(\nabla_{X_2} X_1, X_1) = \omega_2^1(X_2) T(X_1, X_2) - \omega_2^1(X_2) T(X_2, X_1) = 0.$$

In a similar way, one can prove that

$$(\nabla_{X_1} T)(X_2, X_2) - (\nabla_{X_2} T)(X_1, X_2) = X_1 t.$$

□

Let now $M^2 \hookrightarrow N^n$ be a surface in an n -dimensional Riemannian manifold and denote by g the induced metric. Then the stress energy tensor S_2 is a symmetric $(0, 2)$ -tensor on M^2 which is defined, according to (1.4), by

$$S_2 = -2|H|^2 g + 4A_H.$$

Taking the trace of S_2 we have

$$4|H|^2 = \operatorname{trace} S_2.$$

Moreover, since $g(\partial_z, \partial_z) = 0$, we have that

$$S_2(\partial_z, \partial_z) = 4 \langle A_H(\partial_z), \partial_z \rangle = 4Q(z, \bar{z}).$$

Thus, as a direct consequence of Proposition 4.1, we obtain the following theorem.

Theorem 4.2. *Let $M^2 \hookrightarrow (N^n, h)$ be a biconservative surface in an n -dimensional Riemannian manifold. Then*

- (a) $Q(z, \bar{z})$ is holomorphic if and only if $|H| = \text{constant}$;
- (b) S_2 is a Codazzi tensor if and only if $|H| = \text{constant}$.

Corollary 4.3. *Let $M^2 \hookrightarrow (N^n, h)$ be a CMC biconservative surface. If M^2 is a topological sphere, then M^2 is pseudo-umbilical.*

Remark 4.4. The previous corollary is a converse of Remark 3.6.

5. CMC BICONSERVATIVE SURFACES IN $N^4(c)$

In this section we shall study the case of surfaces $M^2 \hookrightarrow N^4(c)$ in a 4-dimensional space form of constant sectional curvature c . In this setting, using (2.6), a surface is biconservative if

$$(5.1) \quad 2 \operatorname{trace} A_{\nabla_{(\cdot)}^\perp H}(\cdot) + \operatorname{grad} |H|^2 = 0.$$

Clearly, a surface with parallel mean curvature vector field (PMC), that is such that $\nabla^\perp H = 0$, is biconservative and such surfaces are classified. In fact, D. Hoffman, in his doctoral thesis [Stanford University, 1971], classified PMC surfaces in \mathbb{R}^4 . Later, his result was extended to any space form of any dimension in [6, 21], yielding the following classification of PMC surfaces in $N^4(c)$ (see also [8]):

- (i) minimal surfaces in $N^4(c)$;
- (ii) CMC surfaces (including minimal) in a hypersphere of $N^4(c)$;

A natural question is when a CMC biconservative surface $M^2 \hookrightarrow N^4(c)$ is PMC. As any pseudo-umbilical surface in a four dimensional space form is CMC if and only if it is PMC (see [5]), from Corollary 4.3 it follows that a CMC biconservative surface in $N^4(c)$ which is topologically a sphere is PMC. When M is not necessary a sphere, we provide the following answer

Theorem 5.1. *Let $M^2 \hookrightarrow N^4(c)$ be a CMC biconservative surface in a space form of constant sectional curvature $c \neq 0$. Then M^2 is PMC.*

Proof. If $M^2 \hookrightarrow N^4(c)$ is pseudo-umbilical, then, as we have already said, M^2 is PMC (see [5]). Thus we can assume that the surface is not pseudo-umbilical and that the pseudo-umbilical points are isolated. The last assertion is a consequence of the fact that, from Theorem 4.2, $Q(z, \bar{z})$ is holomorphic and $Q(z, \bar{z}) = 0$ precisely at pseudo-umbilical points. Now, assume that $\nabla^\perp H \neq 0$. Therefore, there exists an open subset $U \subset M$ such that $\nabla^\perp H(p) \neq 0$ for all $p \in U$, and $k_1(p) \neq k_2(p)$, where we have denoted by k_1 and k_2 the eigenvalues of the endomorphism A_H . Let $\{E_1, E_2\}$ be a local orthonormal frame field (defined on U) such that

$$A_{\frac{H}{|H|}}(E_1) = \lambda_1 E_1, \quad A_{\frac{H}{|H|}}(E_2) = \lambda_2 E_2,$$

where $\lambda_i = k_i/|H|$, $i = 1, 2$. Moreover, put $E_3 = H/|H|$ and define E_4 such that it is a unit normal vector field orthogonal to E_3 . Then the frame

$$(5.2) \quad \{E_1, E_2, E_3, E_4\}$$

can be extended to a local frame of $N^4(c)$ defined on an open subset V of $N^4(c)$. Let us denote, for convenience, $A_i = A_{E_i}$, $i = 3, 4$, and note that

$$\operatorname{trace} A_4 = \langle A_4(E_1), E_1 \rangle + \langle A_4(E_2), E_2 \rangle = \langle B(E_1, E_1), E_4 \rangle + \langle B(E_2, E_2), E_4 \rangle = 2\langle H, E_4 \rangle = 0.$$

Also, let ω_A^B be the connection 1-forms on V corresponding to $\{E_1, E_2, E_3, E_4\}$, that is

$$\nabla^N E_A = \omega_A^B E_B.$$

Then

$$\nabla_{E_1}^\perp E_3 = \omega_3^4(E_1) E_4, \quad \nabla_{E_2}^\perp E_3 = \omega_3^4(E_2) E_4.$$

As M is CMC, then the biconservative condition (5.1) becomes

$$(5.3) \quad \begin{aligned} \text{trace } A_{\nabla_{(\cdot)}^\perp E_3}(\cdot) &= A_{\nabla_{E_1}^\perp E_3}(E_1) + A_{\nabla_{E_2}^\perp E_3}(E_2) \\ &= \omega_3^4(E_1) A_4 E_1 + \omega_3^4(E_2) A_4 E_2 = 0, \end{aligned}$$

which can be written as the system

$$(5.4) \quad \begin{cases} \omega_3^4(E_1) \langle A_4 E_1, E_1 \rangle + \omega_3^4(E_2) \langle A_4 E_2, E_1 \rangle = 0 \\ \omega_3^4(E_1) \langle A_4 E_1, E_2 \rangle + \omega_3^4(E_2) \langle A_4 E_2, E_2 \rangle = 0. \end{cases}$$

If we regard system (5.4) as a linear system in the variables $\omega_3^4(E_1)$ and $\omega_3^4(E_2)$, then, since $(\omega_3^4(E_1))^2 + (\omega_3^4(E_2))^2 > 0$ on U (it represents the square of the norm of $\nabla^\perp E_3$), we must have

$$0 = \langle A_4 E_1, E_1 \rangle \langle A_4 E_2, E_2 \rangle - (\langle A_4 E_1, E_2 \rangle)^2 = -(\langle A_4 E_1, E_1 \rangle)^2 - (\langle A_4 E_1, E_2 \rangle)^2 = -|A_4 E_1|^2,$$

where in the second equality we have used that $\text{trace } A_4 = 0$. Moreover,

$$|A_4 E_2|^2 = |A_4 E_1|^2,$$

thus $|A_4|^2 = 2|A_4 E_1|^2 = 0$, that is $A_4 = 0$. The second fundamental form B of the immersed surfaces becomes

$$B(E_1, E_1) = \lambda_1 E_3, \quad B(E_2, E_2) = \lambda_2 E_3, \quad B(E_1, E_2) = 0.$$

Using the connection form ω_1^2 we obtain

$$\begin{aligned} \nabla_{E_1} E_1 &= \omega_1^2(E_1) E_2, & \nabla_{E_1} E_2 &= -\omega_1^2(E_1) E_1, \\ \nabla_{E_2} E_1 &= \omega_1^2(E_2) E_2, & \nabla_{E_2} E_2 &= -\omega_1^2(E_2) E_1, \end{aligned}$$

where ∇ denotes the Levi-Civita connection on M^2 . The Codazzi equation, when X, Y, Z are tangent vector fields and η is normal, takes the form

$$(5.5) \quad \begin{aligned} X \langle B(Y, Z), \eta \rangle &- \langle B(\nabla_X Y, Z), \eta \rangle - \langle B(Y, \nabla_X Z), \eta \rangle - \langle B(Y, Z), \nabla_X^\perp \eta \rangle \\ &= Y \langle B(X, Z), \eta \rangle - \langle B(\nabla_Y X, Z), \eta \rangle - \langle B(X, \nabla_Y Z), \eta \rangle - \langle B(X, Z), \nabla_Y^\perp \eta \rangle. \end{aligned}$$

We now use (5.5) four times:

- replacing $X = E_1, Y = E_2, Z = E_1$ and $\eta = E_3$ gives

$$(5.6) \quad E_2 \lambda_1 = (\lambda_1 - \lambda_2) \omega_1^2(E_1);$$

- replacing $X = E_1, Y = E_2, Z = E_2$ and $\eta = E_3$ we obtain

$$(5.7) \quad E_1 \lambda_2 = (\lambda_1 - \lambda_2) \omega_1^2(E_2);$$

- replacing $X = E_1, Y = E_2, Z = E_1$ and $\eta = E_4$ yields

$$(5.8) \quad \lambda_1 \omega_3^4(E_2) = 0;$$

- replacing $X = E_1, Y = E_2, Z = E_2$ and $\eta = E_4$ reduces to

$$(5.9) \quad \lambda_2 \omega_3^4(E_1) = 0.$$

Moreover, since $2H = (\lambda_1 + \lambda_2)E_3$, then

$$|\lambda_1 + \lambda_2| = 2|H| = \text{constant} \neq 0.$$

Assume that $\omega_3^4(E_1) \neq 0$, then, from (5.9), $\lambda_2 = 0$ on a open subset and, taking into account (5.7), $\omega_1^2(E_2) = 0$. Next, since $\lambda_1 = \text{constant}$, (5.6) and (5.8) give $\omega_1^2(E_1) = 0$ and $\omega_3^4(E_2) = 0$ respectively. Thus the surface M is flat.

Similarly, if $\omega_3^4(E_2) \neq 0$, we obtain that $\lambda_1 = 0$, the surface is flat and also $\omega_3^4(E_1) = 0$.

Finally, replacing in the Gauss equation

$$\langle R^N(X, Y)Z, W \rangle = \langle R(X, Y)Z, W \rangle + \langle B(X, Z), B(Y, W) \rangle - \langle B(X, W), B(Y, Z) \rangle$$

$X = W = E_1$ and $Y = Z = E_2$, we get

$$(5.10) \quad c = \lambda_1 \lambda_2 = 0,$$

which is a contradiction. \square

Because of Theorem 5.1, we are left to analyze the case of CMC biconservative surfaces into \mathbb{R}^4 . In this case, we have the following explicit description.

Proposition 5.2. *Let $M^2 \hookrightarrow \mathbb{R}^4$ be a proper (non PMC) biconservative surface with constant mean curvature different from zero. Then, locally, the surface is given by*

$$(5.11) \quad X(u, v) = (\gamma(u), v + a) = (\gamma^1(u), \gamma^2(u), \gamma^3(u), v + a), \quad a \in \mathbb{R},$$

where $\gamma : I \rightarrow \mathbb{R}^3$ is a curve in \mathbb{R}^3 parametrized by arc-length, with constant non-zero curvature, and non-zero torsion. Conversely, a surface in \mathbb{R}^4 , parametrized by (5.11), is a proper biconservative surface with constant mean curvature different from zero.

Proof. We use the local frame defined in (5.2). Then $2|H| = |\lambda_1 + \lambda_2|$ and, using (5.10), we can assume that $\lambda_1 = \text{constant} \neq 0$ and $\lambda_2 = 0$. Then

$$\nabla_{E_1}^{\mathbb{R}^4} E_2 = \nabla_{E_2}^{\mathbb{R}^4} E_2 = 0.$$

Thus E_2 is the restriction to M^2 of a constant vector field of \mathbb{R}^4 which, up to an isometry of \mathbb{R}^4 , we can choose as

$$E_2 = (0, 0, 0, 1).$$

Let now σ be an integral curve of E_2 parametrized by arc-length, that is

$$\sigma(v) = (b_1, b_2, b_3, v + a), \quad b_1, b_2, b_3, a \in \mathbb{R}.$$

Let $p_0 = (b_1, b_2, b_3, a) \in M$ and

$$\gamma(u) = (\gamma^1(u), \gamma^2(u), \gamma^3(u), \gamma^4(u)),$$

be an integral curve of E_1 with $\gamma(0) = p_0$. As $\langle \gamma'(u), E_2 \rangle = 0$, we get $\gamma^4(u) = a$. Thus γ lies in a hyperplane of \mathbb{R}^4 orthogonal to E_2 . Then the surface can be locally parametrized by

$$(5.12) \quad X(u, v) = (\gamma^1(u), \gamma^2(u), \gamma^3(u), v + a).$$

Using the calculation in the proof of Theorem 5.1, we easily see that

$$\begin{cases} \nabla_{E_1}^{\mathbb{R}^4} E_1 = \lambda_1 E_3 \\ \nabla_{E_1}^{\mathbb{R}^4} E_3 = -\lambda_1 E_1 + \omega_3^4(E_1) E_4 \\ \nabla_{E_1}^{\mathbb{R}^4} E_4 = -\omega_3^4(E_1) E_3, \end{cases}$$

where, using Ricci's equation

$$R^\perp(E_1, E_2)E_3 = 0,$$

we obtain $E_2(\omega_3^4(E_1)) = 0$, that is $\omega_3^4(E_1)$ depends only on u . We can assume that $\lambda_1 > 0$. Thus $\{E_1, E_3, E_4\}$ is a Frenet's frame along $\gamma(u)$ and so γ is a curve with constant curvature $k = \lambda_1$ and torsion given by $\tau = \tau(u) = \omega_3^4(E_1)$. Moreover, since $\nabla_{E_1}^\perp H = |H|\nabla_{E_1}^\perp E_3 = (k\tau)/2 E_4$ and the surface is not PMC, we conclude that $\tau \neq 0$ at any point of M .

We now prove the converse. For this, suppose that the surface is parametrized by (5.11) and let $\{T = (\gamma'(u), 0), N(u), B(u)\}$ be a Frenet frame along γ . Note that $\{N(u), B(u)\}$ is an orthonormal frame of the normal bundle of the surface. Then a straightforward computation gives:

$$\nabla_{\partial_u}^{\mathbb{R}^4} \partial_u = X_{uu} = k N(u), \quad \nabla_{\partial_u}^{\mathbb{R}^4} \partial_v = X_{uv} = 0, \quad \nabla_{\partial_v}^{\mathbb{R}^4} \partial_v = X_{vv} = 0,$$

from which

$$B(\partial_u, \partial_u) = k N(u), \quad B(\partial_u, \partial_v) = 0, \quad B(\partial_v, \partial_v) = 0.$$

Furthermore,

$$\nabla_{\partial_u}^\perp N(u) = \tau(u) B(u), \quad \nabla_{\partial_v}^\perp N(u) = 0.$$

Thus $H = (k/2) N(u)$ and $\nabla_{\partial_u}^\perp H = (k\tau/2) B(u)$, showing that the surface is CMC but not PMC. Finally, since the surface is CMC from (5.1), to prove that it is biconservative we only need to check that

$$A_{\nabla_{\partial_u}^\perp N}(\partial_u) + A_{\nabla_{\partial_v}^\perp N}(\partial_v) = \tau(u) A_B(\partial_u) = 0.$$

Indeed,

$$A_B(\partial_u) = -\left(\nabla_{\partial_u}^{\mathbb{R}^4} B(u)\right)^\top = (\tau(u) N(u))^\top = 0.$$

□

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