

Characterization of Admissible Marking Sets in Petri Nets with Uncontrollable Transitions

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Abstract—This work studies the equivalent transformation from a legal marking set to its admissible marking set. First, the concepts of escaping-marking set and transforming marking set are defined, and two algorithms are provided to compute the transforming marking set and the admissible marking set. Second, the equivalent transformation of a disjunction of linear constraints expressed in terms of generalized mutual exclusion constraints (GMECs) with non-negative weights via uncontrollable transitions is established.

I. INTRODUCTION

DECIDING how to supervise a discrete event system (DES) to reach no forbidden states is an important problem in DES control theory, and is usually called *forbidden state problem* [1]–[20]. Petri nets (PNs), thanks to their graphical representation and powerful algebraic formulation, have been a popular modeling tool to handle it.

Several forbidden state specifications that require a DES to run within a specified set of allowed states can be formalized as a logic expression of linear constraints on the state space of a PN model [12], [13]. Many papers, e.g., [8], [9], [20] were published in the past decades dealing with the case of transitions that are *all* controllable. Obviously, for a Petri net with uncontrollable transitions, the problem is more complex since the uncontrollable transitions cannot be prevented from firing by any external supervisor and thus a forbidden marking may be reached from a legal one by the firing of uncontrollable transitions. Actually, in order to make such a PN satisfy the forbidden state specification, its behavior should be restricted within the admissible marking set instead of the legal marking set [9]. Therefore, the characterization of the admissible

marking set is at the basis of the computation of an optimal (maximally permissive) control law.

This work provides two main contributions to solve this problem. First, we formally introduce the notions of *escaping marking set* and *transforming marking set*. Based on them we show how to characterize the admissible marking set corresponding to a given set of legal markings that is not necessarily convex as required by some other methods [1], [10], [12]–[14], [16], [17]. Second, by focusing on legal sets that consist of the union of a finite number of convex sets, we provide some comments on a previous constraint transformation approach [13]. Then we propose some simple formulas that do not require reachability analysis such that we can compute the equivalent transformation of the disjunction of some linear constraints (given as GMECs with non-negative weights). We show that, based on them, a subset of the admissible marking set can be computed, which exactly coincides with the admissible marking set for some cases. A criterion to judge if a such coincidence occurs is provided.

A. Literature Review

Methods for constraints transformation can be divided in two main groups: those requiring reachability analysis and those not. Methods in the first group [2], [3], [4], [6]–[8] suffer from the state explosion problem and can hardly be applied to large-sized nets. Methods in the second group are [1], [10], [13], [16]. In particular, Moody and Antsaklis [16] deal with convex legal marking sets described by linear constraints called Generalized Mutual Exclusion Constraints (GMECs). They present a method that transforms an inadmissible GMEC into an admissible one. The proposed approach is computationally efficient and the supervisor is also simple in structure but usually it is not maximally permissive (optimal). Basile *et al.* [1] improve the method in [16] by adding two parameters to the matrix containing the uncontrollable column of the plant incidence matrix. Iordache and Antsaklis [10] also improve the method in [16] by using the concepts of firing vector and Parikh vector.

Some interesting results have been provided for some subclasses of Petri nets with uncontrollable transitions [14], [17]. As for Petri nets with general structures, the studies in [13], [16] point out that it is impossible to equivalently transform some linear constraints into admissible ones. Luo *et al.* [13] thereby propose the concept of a weakly admissible constraint. Based on it, they claim that they propose an algorithm that can equivalently transform a linear constraint into a disjunction of weakly admissible ones and an optimal supervisor is designed

Research supported by National Natural Science Foundation of China under Grant 61472361, the Zhejiang Natural Science Foundation under Grant LR14F020001, LY15F030002, and LY15F030000, the State Scholarship Fund of China, Zhejiang Sci. & Tech. Project under Grant 2015C31064, the Opening Project of State Key Laboratory for Manufacturing Systems Engineering under Grant No. sklms2014011, Zhejiang NNST Key Laboratory under Grant 2013E10012, and Zhejiang Gongshang University Innovation Project under Grant CX201411010.

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for the weakly admissible ones. However, counterexamples in [15] and [19] reveal that it is a non-equivalent transformation. Therefore, deciding how to directly transform a given linear constraint equivalently into a logic expression of linear constraints that can describe the entire admissible marking set of the given constraint without the analysis of a reachability set remains an open problem. This paper aims to solve it.

II. PRELIMINARIES

An *ordinary* Petri net (PN) N is a 3-tuple (P, T, F) where P and T are finite, nonempty, and disjoint sets. P is a set of places, and T is a set of transitions. $F \subseteq (P \times T) \cup (T \times P)$ is a set representing all the flow relations. Given a net $N=(P, T, F)$ and a node $x \in P \cup T$, $\bullet x = \{y \in P \cup T | (y, x) \in F\}$ is the preset of x , while $x^\bullet = \{y \in P \cup T | (x, y) \in F\}$ is the postset of x . For any set of places $X \subseteq P \cup T$, it is: $\bullet X = \bigcup_{x \in X} \bullet x$ and $X^\bullet = \bigcup_{x \in X} x^\bullet$. The incidence matrix of N is denoted by $[N]: P \times T \rightarrow \{-1, 0, 1\}$ indexed by P and T such that $[N](p, t) = -1$ if $p \in \bullet t$; $[N](p, t) = 1$ if $p \in t^\bullet$; otherwise $[N](p, t) = 0$, $\forall p \in P$ and $\forall t \in T$.

A *marking* or *state* of a Petri net $N=(P, T, F)$ is a mapping $m: P \rightarrow \mathbb{N}$ where $\mathbb{N} = \{0, 1, 2, \dots\}$, and $m(p)$ denotes the number of tokens in place p at m . The set of all possible markings of N is defined as $\mathcal{M} = \mathbb{N}^{|P|}$. (N, m_0) is called a net system or marked net given its initial marking m_0 . A transition t is enabled at a marking m , denoted by $m[t]$, if $\forall p \in \bullet t, m(p) > 0$. An enabled transition t at m can fire, resulting in m' , denoted by $m[t]m'$, where $m'(p) = m(p) + [N](p, t)$. A sequence of transitions $\alpha = t_{i1}t_{i2}\dots t_{ik}$, $t_{ij} \in T$, $j \in \overline{N_k} = \{1, 2, \dots, k\}$, is fireable at m if $m[t_{ij}]m_{j+1}$, $j \in \overline{N_k}$, where $m_1 = m$. In such a case, we use $m[\alpha]m_{k+1}$ to denote that m_{k+1} is reachable from m after firing α . $R(N, m_0)$ denotes the set of all reachable markings of N from m_0 .

The transition set T is partitioned in two subsets: T_u is the set of uncontrollable transitions, and T_c is the set of controllable transitions.

The set of reachable markings under the supervision of a policy u in N from m_0 is denoted by $R(N, m_0, u)$. The least permissive supervisory policy, denoted as u_{zero} , disables all controllable transitions. $R(N, m_0, u_{zero})$ is the set of markings uncontrollably reachable from m_0 , where all controllable transitions are disabled. Clearly, it is $R(N, m_0, u_{zero}) \subseteq R(N, m_0, u)$ for any u . Besides, we use $R_t(N, m)$ to denote the set of all reachable markings (including m) of N from m by firing t (once or multiple times).

Given a marking m for N and $P' \subseteq P$, $m|_{P'}$ denotes the restriction of m to P' . Let \mathcal{M} denote the set of all possible markings of N . Given a marking set $Q \subseteq \mathcal{M}$ and $P' \subseteq P$, $Q|_{P'} = \{m|_{P'} | m \in Q\}$ is called the restriction of the marking set Q to P' . Besides, \bar{Q} denotes the complementary set of Q , namely $\mathcal{M} - Q$.

Definition 1: Given a marking set $Q \subseteq \mathcal{M}$ and $p \in P$, p is called an *unrestricted place* if $Q|_{\{p\}} = \mathbb{N}$; otherwise, it is called a *restricted place*. The set of all restricted places with respect to Q is denoted by P_Q . The *all-place-restricted marking set* of Q is defined as $Q^* = Q|_{P_Q}$.

III. ESCAPING AND TRANSFORMING MARKING SET

Definition 2: Let $Q \subseteq \mathcal{M}$ be a marking set for N and t be an uncontrollable transition. The set

$$\Gamma(Q, t) = \{m \in Q | \exists m' \in R_t(N, m), m' \in \bar{Q}\}$$

is called the *escaping-marking set* of Q via t , and

$$Q_t = Q - \Gamma(Q, t)$$

is called the *transforming marking set* of Q via t .

Therefore, $\Gamma(Q, t)$ denotes a subset of Q whose markings lead outside Q by firing uncontrollable transition t (once or multiple times); Q_t denotes a subset of Q in which no marking can “run away from” Q by firing transition t only.

Now, given two sets $Q_1, Q_2 \subseteq \mathcal{M}$ and the set $Q = Q_1 \cup Q_2$ it holds: $Q_t \supseteq (Q_1)_t \cup (Q_2)_t$. In the sequel we provide an algorithm to compute the transforming marking set of a union of two marking sets via an uncontrollable transition. *Note that Q_1^* and Q_2^* are required to be finite marking sets.* To make the following section easy to understand, we provide a case of $Q = Q_1 \cup Q_2$ in Fig. 1, where $Q_1 \cap Q_2 \neq \emptyset$.

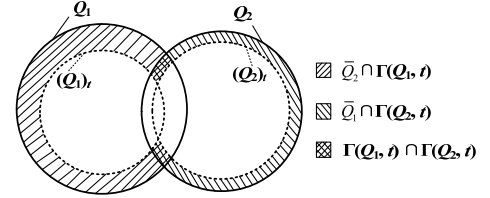


Fig. 1. A case of $Q = Q_1 \cup Q_2$

Algorithm 1: Computation of $(Q_1 \cup Q_2)_t$

Input: Two marking sets Q_1, Q_2 and a transition $t \in T_u$.

Output: A marking set Q_{out}

- 1) $B_1 = (Q_1)_t$;
- 2) $B_2 = (Q_2)_t$;
- 3) $C_1 = \{m \in (Q_1 - B_1) \cap \bar{B}_2 | \exists m' \in R_t(N, m) \cap \bar{Q}_1 \cap B_2 \text{ such that } R_t(N, m) - R_t(N, m') \subseteq Q_1\}$;
- 4) $C_2 = \{m \in (Q_2 - B_2) \cap \bar{B}_1 | \exists m' \in R_t(N, m) \cap \bar{Q}_2 \cap B_1 \text{ such that } R_t(N, m) - R_t(N, m') \subseteq Q_2\}$;
- 5) **while** $C_1 \cup C_2 \neq \emptyset$ **do**
- 6) $B_1 = B_1 \cup C_1$;
- 7) $B_2 = B_2 \cup C_2$;
- 8) $C_1 = \{m \in (Q_1 - B_1) \cap \bar{B}_2 | \exists m' \in R_t(N, m) \cap \bar{Q}_1 \cap B_2 \text{ such that } R_t(N, m) - R_t(N, m') \subseteq Q_1\}$;
- 9) $C_2 = \{m \in (Q_2 - B_2) \cap \bar{B}_1 | \exists m' \in R_t(N, m) \cap \bar{Q}_2 \cap B_1 \text{ such that } R_t(N, m) - R_t(N, m') \subseteq Q_2\}$;
- 10) **end while**
- 11) $Q_{out} = B_1 \cup B_2$;
- 12) **Output:** Q_{out} ;
- 13) **End.**

Theorem 1: Given a marking set $Q = Q_1 \cup Q_2$ for (N, m_0) , where Q_1^* and Q_2^* are finite marking sets, and a transition $t \in T_u$, the output of Algorithm 1 is $Q_{out} = Q_t$.

Proof: See Appendix. ■

IV. FROM LEGAL MARKING SET TO ADMISSIBLE MARKING SET

The sets of forbidden states and allowed states are called the forbidden marking set and the legal marking set, and are denoted as \mathcal{M}_F and \mathcal{L} , respectively. It is: $\mathcal{M}_F \subseteq \mathcal{M}$, $\mathcal{L} \subseteq \mathcal{M}$, and $\mathcal{L} = \mathcal{M} - \mathcal{M}_F$.

Definition 3: Let $Q \subseteq \mathcal{M}$ be a marking set. $A(Q) = \{m \in Q \mid R(N, m, u_{zero}) \subseteq Q\}$ is called the *admissible marking set* with respect to Q and $M_{WF}(Q) = \{m \in Q \mid \exists m' \in R(N, m, u_{zero}), m' \in \bar{Q}\}$ is called the *weakly forbidden marking set* with respect to Q .

Given a legal marking set \mathcal{L} for a net, we use \mathcal{A} to denote $A(\mathcal{L})$ and \mathcal{M}_{WF} to denote $M_{WF}(\mathcal{L})$.

According to Definition 3, a legal marking set \mathcal{L} can be partitioned into two subsets, namely the admissible marking set \mathcal{A} and the weakly forbidden marking set \mathcal{M}_{WF} . The markings in \mathcal{A} can never reach a forbidden marking by firing uncontrollable transitions only; while those in \mathcal{M}_{WF} can.

Definition 4 [9]: Let $\mathcal{L} \subseteq \mathcal{M}$ be a legal marking set, \mathcal{A} be the admissible marking set with respect to \mathcal{L} , and u be a control policy. The policy u is called *optimal (maximally permissive)* if $R(N, m_0, u) = R(N, m_0) \cap \mathcal{A}$.

Therefore, a control policy u is optimal, or maximally permissive, if it guarantees that the controller only forbids those markings that are not in the admissible marking set of \mathcal{L} .

In the rest of the paper we make the following assumption:
A1. for any legal marking set \mathcal{L} , its all-place-restricted marking set \mathcal{L}^ is a finite marking set.*

The following algorithm provides a way to compute the admissible marking set of a Petri net system, given a set of legal markings \mathcal{L} . It is based on the idea of iteratively computing the transforming marking set via each uncontrollable transition.

Algorithm 2: Computation of an admissible marking set

Input: An ordinary PN system (N, m_0) and a legal marking set \mathcal{L} .

Output: The admissible marking set \mathcal{A} of \mathcal{L} .

- 1) $Q = \mathcal{L}$;
- 2) **while** $\exists t \in T_u, \Gamma(Q, t) \neq \emptyset$ **do**
- 3) $Q = Q_i$;
- 4) **end while**
- 5) **Let** $\mathcal{A} = Q$;
- 6) **End.**

Theorem 2: Given an ordinary PN system (N, m_0) and a legal marking set \mathcal{L} as the inputs, under Assumption A1 the output of Algorithm 2 is equal to the admissible marking set of \mathcal{L} .

Proof: See [18, Section I] ■

V. LINEAR CONSTRAINT TRANSFORMATION

A linear constraint (ω, k) , which is expressed in terms of a GMEC with non-negative weights, requires the markings m of a PN to satisfy $\omega \bullet m \leq k$ where $k \in \mathbb{N}$ and ω is a weight vector

from P to \mathbb{N} . Let $W = \{(\omega_1, k_1), (\omega_2, k_2), \dots, (\omega_n, k_n)\}$, $n \in \mathbb{N}^+ = \{1, 2, 3, \dots\}$ denote a set of linear constraints. The disjunction of the constraints in W is denoted as $\vee(W)$, that is, $\vee_{(\omega, k) \in W} \omega \bullet m \leq k$.

We use $Q_{(\omega, k)} = \{m \in \mathcal{M} \mid \omega \bullet m \leq k\}$ to denote a marking set whose elements meet a linear constraint (ω, k) , and $Q_{\vee(W)} = \bigcup_{(\omega, k) \in W} Q_{(\omega, k)}$ to denote a set whose elements meet the disjunction of linear constraints in W . Obviously, for any (ω, k) in this paper, $Q_{(\omega, k)}^*$ is a finite marking set.

Definition 5 [14]: Given a PN (N, m_0) with a linear constraint (ω, k) , the weight of transitions is defined as a row vector, that is, $\varpi = \omega \bullet [N]$ where $[N]$ denotes the incidence matrix of N .

For any t , $\varpi(t)$ is a measure of the effect of t on the constraint. In more detail, if t is enabled at m and its firing leads to m' , it holds that $\varpi(t) = \omega \bullet m' - \omega \bullet m$.

The following properties are straightforward from Def. 5.

Property 1: Let $Q_{(\omega, k)}$ be a marking set for (N, m_0) and $t \in T_u$ such that $\varpi(t) \geq 0$. Then $\forall m \in \bar{Q}_{(\omega, k)}, R_t(N, m) \subseteq \bar{Q}_{(\omega, k)}$.

Property 2: Let $Q_{(\omega, k)}$ be a marking set for (N, m_0) and $t \in T_u$ such that $\varpi(t) \leq 0$. $\forall m \in Q_{(\omega, k)}, R_t(N, m) \subseteq Q_{(\omega, k)}$.

A. Some Comments on [13]

Luo *et al.* [13] gave an algorithm to equivalently transform a given linear constraint into a disjunction of weakly admissible ones [13]. However, the obtained disjunction of weakly admissible constraints describes a marking set that may be just a subset of the admissible marking set of the given linear constraint. In other words, the transformation method in [13] is not an equivalent one, contrary to what they claim in [13]. Counterexamples to [13] have been independently proposed by Wang *et al.* [19] and Ma *et al.* [15].

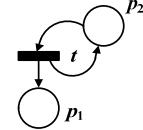


Fig. 2. A PN with $(\omega, 1): m(p_1) \leq 1$

The reason why the transformation [13] may fail is that the approach in [13] is based on the assumption that:

$$A(Q_{\vee(W)}) = \bigcup_{(\omega, k) \in W} A(Q_{(\omega, k)}). \quad (1)$$

Equation (1) does not hold in general. Indeed, assume that we have one uncontrollable transition t only and the disjunction of two linear constraints. Let Q_1 and Q_2 be the legal marking sets corresponding to such constraints. In this case equation (1) would imply that $(Q_1 \cup Q_2)_t = (Q_1)_t \cup (Q_2)_t$, that is not true as discussed in Section III.

B. Equivalent Transformation of a Linear Constraint

Definition 6: Consider a PN with set of places P and set of uncontrollable transitions T_u . Let Ω be the set of all possible linear constraints over P . The *uncontrollable transition gain transformation* (UTGT) is a function $\rho: \Omega \times T_u \times P \rightarrow \Omega$ defined as follows:

$$\forall (\omega, k) \in \Omega, \forall t \in T_u, \forall p \in P, (\omega', k') = \rho((\omega, k), t, p) \text{ where}$$

$$\begin{cases} k' = k \\ \forall p' \in P, \quad \omega'(p') = \begin{cases} \omega(p') & p' \neq p \vee p' \notin \bullet t \\ \omega(p') + \varpi(t) & p' = p \wedge p' \in \bullet t \setminus \bullet \\ k+1 & p' = p \wedge p' \in \bullet t \cap t \end{cases} \end{cases}$$

Definition 7: Given an uncontrollable transition t and a linear constraint (ω, k) , $\varrho((\omega, k), t)$ is defined as

$$\varrho((\omega, k), t) = \begin{cases} \{(\omega, k)\} & \varpi(t) \leq 0 \\ \bigcup_{p \in \bullet t} \{(\omega, k), t, p\} & \varpi(t) > 0 \end{cases}$$

where ρ is defined in Def. 6.

Note that the definitions of $\rho((\omega, k), t, p)$ and $\varrho((\omega, k), t)$ in this work are different from those in [13]. In particular, $\rho((\omega, k), t, p)$ differs from [13] in the case that p is both an input and an output place of t , while $\varrho((\omega, k), t)$ now also deals with the case of $\varpi(t) \leq 0$.

As an example, $(\omega, 1): m(p_1) \leq 1$ is a linear constraint for the net in Fig. 2, where t is an uncontrollable transition. According to our definition $\rho((\omega, 1), t, p_2) = (\omega', 1): m(p_1) + 2m(p_2) \leq 1$, while $\rho((\omega, 1), t, p_2) = (\omega'', 1): m(p_1) + m(p_2) \leq 1$ according to [13].

The following theorem proves that the disjunction of constraints provided by Def. 7 is the equivalent transformation of a linear constraint via an uncontrollable transition.

Theorem 3: For any $t \in T_u$ and any $(\omega, k) \in \Omega$, it holds:

$$(Q_{(\omega, k)})_t = Q_{\vee(W)}, \text{ where } W = \varrho((\omega, k), t).$$

Proof: See [18, Section II]. ■

C. Equivalent Transformation of Disjunction of Linear Constraints

For a disjunction of linear constraints, the equivalent transformation via an uncontrollable transition is now presented. In particular, given a set of markings Q defined as the disjunction of n linear constraints, we provide a characterization of Q_t , where t is the uncontrollable transition. As discussed in detail in the following, such a characterization depends on the effect of t on the individual constraints, namely on the sign of $\varpi_i(t)$, $\forall i \in \{1, 2, \dots, n\}$.

Theorem 4: Let $Q = Q_{(\omega_1, k_1)} \cup Q_{(\omega_2, k_2)}$ be a marking set for (N, m_0) and $t \in T_u$. $Q_t = (Q_{(\omega_1, k_1)})_t \cup (Q_{(\omega_2, k_2)})_t$ if $\varpi_1(t) \geq 0$ and $\varpi_2(t) \geq 0$.

Proof: Let $Q = Q_{(\omega_1, k_1)} \cup Q_{(\omega_2, k_2)}$ and t be the inputs of Algorithm 1. First, we have $B_1 = (Q_{(\omega_1, k_1)})_t$ and $B_2 = (Q_{(\omega_2, k_2)})_t$. Next, it is easy to see that $C_1 \subseteq \Gamma(Q_{(\omega_1, k_1)}, t) \cap \overline{(Q_{(\omega_2, k_2)})_t}$. Here, we use D to denote $\Gamma(Q_{(\omega_1, k_1)}, t) \cap \overline{(Q_{(\omega_2, k_2)})_t}$. Clearly, we have

$D = D_1 \cup D_2$, where $D_1 = \Gamma(Q_{(\omega_1, k_1)}, t) \cap \overline{Q_{(\omega_2, k_2)}}$ and $D_2 = \Gamma(Q_{(\omega_1, k_1)}, t) \cap \Gamma(Q_{(\omega_2, k_2)}, t)$. 1) Consider $m \in D_1$. By Property 1, we have $\forall m \in D_1, R_t(N, m) \subseteq \overline{Q_{(\omega_2, k_2)}}$ since $\varpi_2(t) \geq 0$. Hence, $\forall m \in D_1, R_t(N, m) \subseteq \overline{(Q_{(\omega_2, k_2)})_t}$; 2) Consider $m \in D_2$. We can see that $\forall m \in D_2$, once the net evolves from m to $m' \in \overline{D_2}$, $m' \in D_1 \cup D_3$ holds, where $D_3 = \Gamma(Q_{(\omega_2, k_2)}, t) \cap \overline{Q_{(\omega_1, k_1)}}$. Hence, it is easy to conclude that $\forall m \in D_2, R_t(N, m) \subseteq \overline{(Q_{(\omega_2, k_2)})_t}$. As a result, we

have $\forall m \in D, R_t(N, m) \subseteq \overline{(Q_{(\omega_2, k_2)})_t}$. This implies that $C_1 = \emptyset$. Similarly, we have $C_2 = \emptyset$. Finally, we have $Q_{out} = (Q_{(\omega_1, k_1)})_t \cup (Q_{(\omega_2, k_2)})_t$. According to Theorem 1, we have $Q_t = Q_{out} = (Q_{(\omega_1, k_1)})_t \cup (Q_{(\omega_2, k_2)})_t$. ■

Corollary 1: Let $W = \{(\omega_1, k_1), (\omega_2, k_2), \dots, (\omega_n, k_n)\}$, $n \in \mathbb{N}^+$ be a set of linear constraints for (N, m_0) and $t \in T_u$. $(Q_{\vee(W)})_t = \bigcup_{(\omega, k) \in W} (Q_{(\omega, k)})_t$ if $\varpi_i(t) \geq 0, \forall i \in \{1, 2, \dots, n\}$.

Theorem 5: Let $Q = Q_{(\omega_1, k_1)} \cup Q_{(\omega_2, k_2)}$ be a marking set for (N, m_0) and $t \in T_u$. $Q_t = Q_{(\omega_1, k_1)} \cup Q_{(\omega_2, k_2)}$ if $\varpi_1(t) \leq 0$ and $\varpi_2(t) \leq 0$.

Proof: See [18, Section III]. ■

Corollary 2: Let $W = \{(\omega_1, k_1), (\omega_2, k_2), \dots, (\omega_n, k_n)\}$, $n \in \mathbb{N}^+$ be a set of linear constraints for (N, m_0) and $t \in T_u$. $(Q_{\vee(W)})_t = Q_{\vee(W)}$ if $\varpi_i(t) \leq 0, \forall i \in \{1, 2, \dots, n\}$.

Definition 8: Given two linear constraints (ω_i, k_i) and (ω_j, k_j) for a PN (N, m_0) and $t \in T_u$, $C_{i \rightarrow j} = \{m \in \Gamma(Q_{(\omega_i, k_i)}, t) \cap \overline{Q_{(\omega_j, k_j)}} \mid \exists m' \in R_t(N, m) \cap \overline{Q_{(\omega_i, k_i)}} \cap Q_{(\omega_j, k_j)} \text{ such that } R_t(N, m) - R_t(N, m') \subseteq Q_{(\omega_i, k_i)}\}$ is called the *complementary-marking set* from (ω_i, k_i) to (ω_j, k_j) .

Property 3: $C_{i \rightarrow j} = \emptyset$ if $\varpi_i(t) \leq 0$ or $\varpi_j(t) \geq 0$.

Under the condition that $\varpi_i(t) > 0$ and $\varpi_j(t) < 0$, the complementary-marking set $C_{i \rightarrow j}$ describes the markings in $\Gamma(Q_{(\omega_i, k_i)}, t) \cap \overline{Q_{(\omega_j, k_j)}}$, which satisfy that once they reach a marking outside of $Q_{(\omega_i, k_i)}$ by firing t , the reachable marking belongs to $Q_{(\omega_j, k_j)}$, or in other words, once they “run away from” $Q_{(\omega_i, k_i)}$ by firing t , they “enter” $Q_{(\omega_j, k_j)}$.

Property 4: Given two linear constraints (ω_i, k_i) and (ω_j, k_j) for a PN (N, m_0) and $t \in T_u$ with $\varpi_i(t) > 0$ and $\varpi_j(t) < 0$, $C_{i \rightarrow j}$ can be described by the following logic expression of linear constraints:

$$\bigvee_{\lambda=1}^n \Delta_\lambda, \quad (2)$$

where Δ_λ represents

$$\begin{cases} \forall p \in \bullet t, m(p) \geq \lambda \\ \omega_j \bullet m > k_j \\ \omega_i \bullet m + (\lambda - 1)\varpi_i(t) \leq k_i \\ \omega_i \bullet m + \lambda\varpi_i(t) > k_i \\ \omega_j \bullet m + \lambda\varpi_j(t) \leq k_j \end{cases}$$

and $n \in \mathbb{N}^+, n \leq k_i/\varpi_i(t) + 1$.

Proof: Δ_1 deals with the markings in $\overline{Q_{(\omega_j, k_j)}} \cap Q_{(\omega_i, k_i)}$ satisfying that a) they can reach a marking outside of $Q_{(\omega_i, k_i)}$ after firing t once and b) the reachable marking belongs to $Q_{(\omega_j, k_j)}$. Δ_2 deals with the markings in $\overline{Q_{(\omega_j, k_j)}} \cap Q_{(\omega_i, k_i)}$ satisfying that a) they can reach a marking outside of $Q_{(\omega_i, k_i)}$ after firing t twice and b) the reachable marking belongs to $Q_{(\omega_j, k_j)}$, and so forth. Hence, it is clear that (2) describes the markings in $\Gamma(Q_{(\omega_i, k_i)}, t) \cap \overline{Q_{(\omega_j, k_j)}}$, which satisfy that once they reach a marking outside of $Q_{(\omega_i, k_i)}$ by firing t , the reachable marking belongs to $Q_{(\omega_j, k_j)}$. ■

In what follows, $C_{i \rightarrow j}$ can also be used to denote (2) for simplification.

Theorem 6: Let $Q = Q_{(\omega_1, k_1)} \cup Q_{(\omega_2, k_2)}$ be a marking set for (N, m_0) and $t \in T_u$. $Q_t = (Q_{(\omega_1, k_1)})_t \cup Q_{(\omega_2, k_2)} \cup C_{1 \rightarrow 2}$ if $\varpi_1(t) > 0$ and $\varpi_2(t) < 0$.

Proof: Let $Q = Q_{(\omega_1, k_1)} \cup Q_{(\omega_2, k_2)}$ and t be the inputs of Algorithm 1. While $B_1 = (Q_{(\omega_1, k_1)})_t$ and $B_2 = Q_{(\omega_2, k_2)}$ after execution of Steps 1 and 2, we have $C_1 = \{m \in \Gamma(Q_{(\omega_1, k_1)}, t) \cap \bar{Q}_{(\omega_2, k_2)} \mid \exists m' \in R_t(N, m) \cap \bar{Q}_{(\omega_1, k_1)} \cap Q_{(\omega_2, k_2)} \text{ such that } R_t(N, m) - R_t(N, m') \subseteq Q_{(\omega_1, k_1)}\}$ and $C_2 = \emptyset$. C_1 is exactly the complementary-marking set $C_{1 \rightarrow 2}$. While $C_1 \neq \emptyset$, Steps 5 to 10 can run again. Then, we have $B_1 = (Q_{(\omega_1, k_1)})_t \cup C_{1 \rightarrow 2}$ and B_2 is still equal to $Q_{(\omega_2, k_2)}$. Here, we use C_1' and C_2' to denote C_1 and C_2 in the second execution. Since B_2 is not expanded, it can be inferred that $C_1' = \emptyset$. Clearly, $C_2' = \emptyset$. Therefore, we have $Q_{out} = (Q_{(\omega_1, k_1)})_t \cup Q_{(\omega_2, k_2)} \cup C_{1 \rightarrow 2}$. It is clear that $Q_t = (Q_{(\omega_1, k_1)})_t \cup Q_{(\omega_2, k_2)} \cup C_{1 \rightarrow 2}$ due to Theorem 1. ■

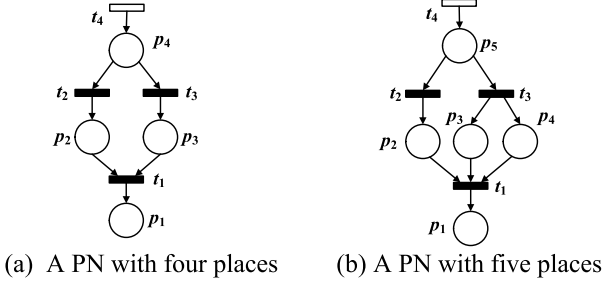


Fig. 3. Two PN with t_1 – t_3 being uncontrollable

As an example, consider the net in Fig. 3(a) with t_1 – t_3 being uncontrollable. Let $Q = Q_{(\omega_1, 1)} \cup Q_{(\omega_2, 1)}$ be a marking set for the net, where $(\omega_1, 1): m(p_1) + m(p_3) \leq 1$ and $(\omega_2, 1): m(p_1) + m(p_2) + m(p_4) \leq 1$. It is: $\varpi_1(t_3) = 1 > 0$ and $\varpi_2(t_3) = -1 < 0$. According to Theorem 6, we have $Q_{t3} = (Q_{(\omega_1, 1)})_{t3} \cup Q_{(\omega_2, 1)} \cup C_{1 \rightarrow 2}$ where

$$C_{1 \rightarrow 2} = \left\{ \begin{array}{l} m(p_4) \geq 1 \\ m(p_1) + m(p_2) + m(p_4) > 1 \\ m(p_1) + m(p_3) \leq 1 \\ m(p_1) + m(p_3) + 1 > 1 \\ m(p_1) + m(p_2) + m(p_4) - 1 \leq 1 \end{array} \right\} \vee \left\{ \begin{array}{l} m(p_4) \geq 2 \\ m(p_1) + m(p_2) + m(p_4) > 1 \\ m(p_1) + m(p_3) + 1 \leq 1 \\ m(p_1) + m(p_3) + 2 > 1 \\ m(p_1) + m(p_2) + m(p_4) - 2 \leq 1 \end{array} \right\}$$

The first conjunction of linear constraints of $C_{1 \rightarrow 2}$ describes the markings in $\Gamma(Q_{(\omega_1, 1)}, t_3) \cap \bar{Q}_{(\omega_2, 1)}$ that “run away from” $Q_{(\omega_1, 1)}$ but “enter” $Q_{(\omega_2, 1)}$ after firing t_3 once, and the second one describes those that “run away from” $Q_{(\omega_1, 1)}$ after firing t_3 twice and “enter” $Q_{(\omega_2, 1)}$.

Here, $C_{1 \rightarrow 2}$ can be reduced into:

$$\left\{ \begin{array}{l} m(p_4) \geq 1 \\ m(p_1) + m(p_3) = 1 \\ m(p_1) + m(p_2) + m(p_4) = 2 \end{array} \right\} \vee \left\{ \begin{array}{l} m(p_4) \geq 2 \\ m(p_1) + m(p_3) = 0 \\ 1 < m(p_1) + m(p_2) + m(p_4) \leq 3 \end{array} \right\}$$

Corollary 3: Let $Q = Q_{(\omega_1, k_1)} \cup Q_{(\omega_2, k_2)}$ be a marking set for (N, m_0) and $t \in T_u$. $Q_t = (Q_{(\omega_1, k_1)})_t \cup (Q_{(\omega_2, k_2)})_t \cup C_{1 \rightarrow 2} \cup C_{2 \rightarrow 1}$.

Proof: Straightforward from Theorems 4 to 6. ■

Corollary 4: Let $W = \{(\omega_1, k_1), (\omega_2, k_2), \dots, (\omega_n, k_n)\}$, $n \in \mathbb{N}^+$, be a set of linear constraints for (N, m_0) and $t \in T_u$. $(Q_{\vee(W)})_t = \bigcup_{k \in W} (Q_{(\omega, k)})_t \cup \bigcup_{(i, j) \in E} C_{i \rightarrow j}$, where $E = \{(i, j) \mid i, j \in \{1, 2, \dots, n\}, \varpi_i(t) > 0 \text{ and } \varpi_j(t) < 0\}$.

Proof: According to Properties 1 and 2 as well as Corollary 3, the conclusion obviously holds. ■

According to Corollary 4, in the case of $n > 2$, any couple of linear constraints, which are different in the signs of the weights of the considered transition, may generate a non-empty complementary-marking set. In other words, all of the complementary-marking sets generated by such couples of linear constraints should be considered in the final transformed result.

Consider the net in Fig. 3(b) with t_1 – t_3 being uncontrollable. Let $Q = Q_{(\omega_1, 3)} \cup Q_{(\omega_2, 3)} \cup Q_{(\omega_3, 3)}$ be a marking set for the net, where $(\omega_1, 3): m(p_1) + m(p_2) + m(p_5) \leq 3$, $(\omega_2, 3): m(p_1) + m(p_3) \leq 3$ and $(\omega_3, 3): m(p_1) + m(p_4) \leq 3$. Since $\varpi_1(t_3) < 0$, $\varpi_2(t_3) > 0$ and $\varpi_3(t_3) > 0$, we have $Q_{t3} = Q_{(\omega_1, 3)} \cup (Q_{(\omega_2, 3)})_{t3} \cup (Q_{(\omega_3, 3)})_{t3} \cup C_{2 \rightarrow 1} \cup C_{3 \rightarrow 1}$ according to Corollary 4, where

$$(Q_{(\omega_2, 3)})_{t3} = Q_{(\omega_2', 3)}, (\omega_2', 3): m(p_1) + m(p_3) + m(p_5) \leq 3 \text{ and } (Q_{(\omega_3, 3)})_{t3} = Q_{(\omega_3', 3)}, (\omega_3', 3): m(p_1) + m(p_4) + m(p_5) \leq 3.$$

For convenience, let

$$a = m(p_1) + m(p_2) + m(p_5); b = m(p_1) + m(p_3); \text{ and } c = m(p_1) + m(p_4).$$

Then, we have

$$C_{2 \rightarrow 1}:$$

$$\left\{ \begin{array}{l} m(p_5) \geq 1 \\ b = 3 \\ a = 4 \end{array} \right\} \vee \left\{ \begin{array}{l} m(p_5) \geq 2 \\ b = 2 \\ 3 < a \leq 5 \end{array} \right\} \vee \left\{ \begin{array}{l} m(p_5) \geq 3 \\ b = 1 \\ 3 < a \leq 6 \end{array} \right\} \vee \left\{ \begin{array}{l} m(p_5) \geq 4 \\ b = 0 \\ 3 < a \leq 7 \end{array} \right\}$$

$$C_{3 \rightarrow 1}:$$

$$\left\{ \begin{array}{l} m(p_5) \geq 1 \\ c = 3 \\ a = 4 \end{array} \right\} \vee \left\{ \begin{array}{l} m(p_5) \geq 2 \\ c = 2 \\ 3 < a \leq 5 \end{array} \right\} \vee \left\{ \begin{array}{l} m(p_5) \geq 3 \\ c = 1 \\ 3 < a \leq 6 \end{array} \right\} \vee \left\{ \begin{array}{l} m(p_5) \geq 4 \\ c = 0 \\ 3 < a \leq 7 \end{array} \right\}$$

D. Implementation of Algorithm 2

As discussed in Section IV, Algorithm 2 has to be used to characterize the admissible marking set. This implies that the transformations provided in the previous subsection needs to be implemented iteratively using Corollary 4 to determine the new set Q_t at each step. Obviously, this may not be feasible since at a certain point GMECs with negative weights may appear. In particular, this happens when some complementary marking set is not empty. We suggest two solutions to overcome such an issue. The first one consists in neglecting these sets and going ahead with the procedure. However, this solution results in a subset of the admissible marking set. The second solution consists in finding some priority among transitions when implementing Algorithm 2. However, in both cases no guarantee of success can be given. Note that a flag variable may be easily added as an output of Algorithm 2 that keeps track of the fact that some complementary marking set has been neglected at intermediate step.

An important remark should be done concerning the comparison between our approach and the approach by Luo *et al.* in [13]. When the Luo’s approach is used, the complementary marking sets are always neglected, while our approach allows taking them into account in several cases. Therefore, the use of our approach never leads to an

approximation of the admissible marking set that is worse than the result of the Luo's approach. Moreover, based on our results, we can always establish if the proposed transformations provide a characterization of the admissible marking set or only an approximation of it.

We conclude this subsection with a brief analysis of the computational complexity of the proposed approach. Algorithm 2 requires the implementation of some simple formulas at Step 2. The complexity of implementing such formulas can be considered linear with respect to the number of places and the number of uncontrollable transitions. However, the number of iterations of Step 2 may in the worst case grow exponentially with the number of uncontrollable transitions.

VI. CONCLUSIONS AND FUTURE WORK

The main contribution of this paper consists in an approach to characterize the admissible marking set in the case of legal marking sets given as the union of a finite number of GMECs with non-negative weights. The advantage of such an approach is that it does not require reachability analysis. Its main limitation is that it provides an exact characterization of the admissible marking set only when a certain number of sets (called complementary marking sets) are empty, or are eventually not empty only at the last iteration of the procedure. Relaxing the assumption that the weights of the GMECs are non-negative will be the main goal of our future research since this will allow us to overcome the limitation mentioned above.

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APPENDIX

Proof of Theorem 1: The proof includes two parts, namely, $Q_i \supseteq Q_{out}$, and $Q_i \subseteq Q_{out}$. First, we prove that $Q_i \supseteq Q_{out}$. Being $Q_i \subseteq Q$, it is obvious that $B_1 \subseteq Q_i$ and $B_2 \subseteq Q_i$ in Steps 1 and 2, i.e., the markings in both B_1 and B_2 can never "run away from" $Q_i \cup Q_2$ by firing t . C_1 denotes a subset of $(Q_i - B_1) \cap B_2$, where once the markings "run away from" Q_i by firing t , they enter B_2 . Since the markings in B_2 can never "run away from" $Q_i \cup Q_2$ by firing t , $C_1 \subseteq Q_i$. Similarly, $C_2 \subseteq Q_i$. Then, since the markings in C_1 are added to B_1 and those in C_2 are added to B_2 , as stated in Steps 6 and 7, we still have $B_1 \subseteq Q_i$ and $B_2 \subseteq Q_i$. With the repeated execution of Steps 5 to 10, B_1 and B_2 are expanded and $B_1 \subseteq Q_i$ and $B_2 \subseteq Q_i$ always hold. When B_1 and B_2 can never be expanded, $Q_{out} = B_1 \cup B_2$. Hence, $Q_i \supseteq Q_{out}$ holds.

Now we prove that $Q_i \subseteq Q_{out}$. By contradiction, suppose that $Q_i \not\subseteq Q_{out}$. Then there exists a marking $m \in Q_i \cap \overline{Q_{out}}$. Since $m \in Q_i$, we have $R_t(N, m) \subseteq Q_i \cup Q_2$. Moreover, since $m \in \overline{Q_{out}}$, we accordingly have the following three cases: 1) $m \in \overline{Q_{out}} \cap Q_1 \cap Q_2$; 2) $m \in \overline{Q_{out}} \cap Q_2 \cap Q_1$; and 3) $m \in \overline{Q_{out}} \cap Q_1 \cap Q_2$.

Case 1: It is easy to know that $Q_{out} \cap Q_1 \subseteq \Gamma(Q_1, t)$. Thus, $m \in Q_2 \cap \Gamma(Q_1, t)$. Clearly, m can reach a marking outside of Q_i by firing t only. Since $R_t(N, m) \subseteq Q_i \cup Q_2$, we can conclude that once the net evolves from m to $m' \in Q_i$, $m' \in Q_2$ holds. Moreover, we have $m' \in Q_1 \cap \Gamma(Q_2, t)$ since otherwise $m \in Q_{out}$. For the same reason, the net can evolve from m' to a marking outside of Q_2 by firing t only, and once a marking $m'' \in Q_2$ is reached from m' , we have $m'' \in Q_2 \cap \Gamma(Q_1, t)$. As a result, we can conclude that t can continuously fire infinite times from m .

Case 2: Similar to Case 1. We can conclude that t can continuously fire infinite times from m .

Case 3: It is easy to know that $m \in \Gamma(Q_1, t) \cap \Gamma(Q_2, t)$. Since $m \in \Gamma(Q_2, t)$, we know that m can reach a marking outside of Q_2 by firing t only. Since $R_t(N, m) \subseteq Q_i \cup Q_2$, we can conclude that once a marking $m' \in Q_2$ is reached from m , $m' \in Q_1$ holds. Moreover, we have $m' \in Q_2 \cap \Gamma(Q_1, t)$ since otherwise $m \in Q_{out}$. As discussed in Case 1, since $m' \in Q_2 \cap \Gamma(Q_1, t)$, we can conclude that t can continuously fire infinite times from m .

Cases 1 to 3 imply that $\Gamma(Q_1, t) \neq \emptyset$ and $\Gamma(Q_2, t) \neq \emptyset$, and since t can continuously fire infinite times, we can know that $\exists p \in t^*$, p is a restricted place both under Q_1 and Q_2 and the number of tokens in p can increase infinitely with the firing of t . This means both Q_1^* and Q_2^* are infinite sets, which contradicts the fact that they are finite. Hence, $Q_i \subseteq Q_{out}$. Therefore, $Q_i = Q_{out}$ holds since $Q_i \supseteq Q_{out}$ and $Q_i \subseteq Q_{out}$.