Relative-locality effects in Snyder spacetime

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Abstract

Most models of noncommutative geometry and doubly special relativity suggest that the principle of absolute locality should be replaced by the milder notion of relative locality. In particular, they predict the occurrence of a delay in the time of arrival of massless particle of different energies emitted by a distant observer. In this letter, we show that this is not the case with Snyder spacetime, essentially because the Lorentz invariance is not deformed in this case. Distant observers may however measure different times of flight for massive particles.

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1. Introduction

Recently, relative locality [1] has been proposed as a framework for investigating the physical properties of models derived from noncommutative geometry (NCG) [2] and doubly special relativity (DSR) [3], that are assumed to provide an effective description of quantum gravity effects in the limit were $\hbar$ and G become negligible, but their ratio $M_P = \sqrt{\hbar/G}$ stays finite, providing a fundamental energy scale that can deform the kinematics of special relativity.

Although strictly speaking the relativity of locality is only the most remarkable feature of this class of models, the name has been extended to denote a formalism based on the postulate that momentum space is curved and the nontrivial physical effects arising from NCG and DSR (such as deformed dispersion relations, noncommutativity and nonassociativity of the addition of momenta, etc.) are related to nontrivial properties of the geometry of momentum space, as torsion, curvature, and nonmetricity [1], and we conform to this use here. In particular, this framework systematizes several results coming from the study of DSR and fixes some ambiguities present in that formalism. The theory has proven to be very well suited for the description of the physical features of models based on the noncommutative κ-Minkowski spacetime [4].

As mentioned above, the main physical implication of the nontrivial geometry of momentum space consists in the loss of the absolute meaning of the concept of locality, that becomes observer dependent: different observers may not agree on the fact that two events take place at the same spacetime point. One of the most striking consequences of this effect for the phenomenology is the possibility that the speed of massless particles can depend on their energy, in accordance with some predictions of NCG [5] and DSR [6] (although in the latter case this result depends on the details of the dynamics and on the identification of the velocity of particles with the group velocity $\mathbf{v}_i = \partial p_0/\partial p_i$ rather than with the kinematic definition $\mathbf{v}_i = \dot{x}_i/\dot{x}_0$).[1] This effect would imply a time delay in the observation of particles of different energy simultaneously emitted from a distant observer. In particular, in [8] it was shown that this result can be deduced from a purely classical analysis of the wordlines of simultaneously emitted particles of different momenta and from the relativity of that simultaneity for distant observers. The proof does not rely on a specific definition of velocity, and was extended to more general models in [9].

It is interesting to investigate if relative locality implies an energy dependence of the speed of light also in the case of the Snyder model [10]. This is a model of noncommutative geometry whose distinctive feature is the preservation of the Poincaré invariance. Therefore, the dispersion relation for particles is essentially the same as in special relativity, and the speed of light should be independent of the momentum of the particles. More explicitly, the dispersion relation is a function of $p^2 = p_0^2 - p_i^2$. It follows that both definitions of velocity mentioned above give the same result, $\mathbf{v}_i = p_i/p_0$, and in particular massless particles move at the speed of light. Consistency of the relative locality formalism would therefore require that using arguments analogous to those of [8] one can predict for Snyder spacetime a momentum-independent speed of light and hence no delay in the detection of simultaneously emitted particles of different energy. The purpose of this letter is to show that it is indeed so.

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[1] This ambiguity has been clarified using the formalism of relative locality [7].

To obtain this result, we discuss the geometry of the momentum space of the Snyder model,[2] and establish the correct Hamiltonian according to the prescriptions of relative locality, solving the ambiguity related to the fact that, in the context of DSR theories, any function of the Casimir invariant of the Poincaré algebra can be chosen as Hamiltonian for the Snyder model.[3] It follows that the equations of motion for a free particle are equivalent to those of special relativity, giving rise to the same geodesic motion. The equivalence of course disappears in the case of an external coupling.

2. The geometry of the Snyder model

The Snyder model is defined by the Poisson brackets [10]

$$
\begin{aligned}
\{J_{\mu\nu}, J_{\rho\sigma}\} &= \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\rho} J_{\mu\sigma} + \eta_{\nu\sigma} J_{\mu\rho}, \\
\{J_{\mu\nu}, p_\rho\} &= \eta_{\mu\rho} p_\nu - \eta_{\nu\rho} p_\mu, \qquad \{p_\mu, p_\nu\} = 0,
\end{aligned}
\tag{1}
$$

$$
\{J_{\mu\nu}, x_\rho\} = \eta_{\mu\rho} x_\nu - \eta_{\nu\rho} x_\mu, \qquad \{x_\mu, p_\nu\} = \eta_{\mu\nu} - \beta^2 p_\mu p_\nu, \qquad \{x_\mu, x_\nu\} = -\beta^2 J_{\mu\nu}, \tag{2}
$$

where β is a constant of order $1/M_P$, $\mu, \nu = 0, \ldots, 3$, $\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$ is the flat metric and $J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$ are the generators of the Lorentz transformations. Eqs. (1) reproduce the Poincaré algebra, while eqs. (2) describe the action of the Poincaré group on the position coordinates and the noncommutativity of spacetime.

Since the Poincaré algebra is not deformed, the momenta transform in the standard way, in particular are unaffected by translations. For what concerns the position coordinates, the Lorentz transformations act on them in the usual way, while the action of the translations is nontrivial. In fact, from (2) it follows that under a finite translation of parameter a_μ,[4]

$$
x_\mu \to x_\mu + a_\mu - \beta^2 a \cdot p\, p_\mu. \tag{3}
$$

Hence, the effect of a translation depends on the 4-momentum of the particle, as in most examples of DSR and NCG [12,13].

As other models of NCG and DSR related to relative locality, the Snyder model can be realized on a constant curvature momentum space [14,15]. More precisely, the momentum space of the Snyder model can be identified with a hyperboloid embedded in a five-dimensional flat space of coordinates ξ_A ($A = 0, \ldots, 4$), with signature $(1, -1, -1, -1, -1)$, satisfying the constraint $\xi_A^2 = -1/\beta^2$. Since Lorentz invariance is preserved, it is convenient to parametrize the space using isotropic coordinates. This can be done in several different ways, which are however not equivalent.

The simplest parametrization is obtained by identifying the four-dimensional momentum p_μ as $p_\mu = \xi_\mu$. The metric induced on the hyperboloid and its inverse are then

$$
g_{\mu\nu} = \eta_{\mu\nu} - \frac{\beta^2 p_\mu p_\nu}{1 + \beta^2 p^2}, \qquad\qquad g^{\mu\nu} = \eta^{\mu\nu} + \beta^2 p^\mu p^\nu, \tag{4}
$$

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[2] This geometry has previously been investigated from a somewhat different point of view in ref. [11].

[3] Alternatively, one may consider as Hamiltonian the Casimir operator of the full Snyder algebra. However, this does not seem to be a physically relevant choice.

[4] In the following we denote $A \cdot B = \eta^{\mu\nu} A_\mu B_\nu$, $A^2 = A \cdot A$ and $|A| = \sqrt{A^2}$.

where $\eta_{\mu\nu}$ is the four-dimensional Minkowski metric. This parametrization holds for $p^2 > -1/\beta^2$ and is therefore compatible with any value of mass.

Another remarkable possibility are the so-called Beltrami coordinates, defined as $\beta p_\mu = \xi_\mu/\xi_4$, with metrics

$$g_{\mu\nu} = \frac{(1 - \beta^2 p^2)\eta_{\mu\nu} + \beta^2 p_\mu p_\nu}{(1 - \beta^2 p^2)^2}, \qquad g^{\mu\nu} = (1 - \beta^2 p^2)(\eta^{\mu\nu} - \beta^2 p^\mu p^\nu). \qquad (5)$$

In this case, the inverse transformations are $\xi_\mu = p_\mu/\sqrt{1 - \beta^2 p^2}$, $\beta\xi_4 = 1/\sqrt{1 - \beta^2 p^2}$. From the definition follows that the bound $p^2 < 1/\beta^2$ must be satisfied; hence an upper limit exists on the mass of particles. The existence of bounds on the mass or on the energy of particles is a common feature in DSR theories [12].

To complete the definition of the Snyder phase space, one must choose the position coordinates x_μ in such a way that they satisfy the Poisson brackets (2). Calling ζ_A the five-dimensional position coordinates canonically conjugated to the ξ_A, so that $\{\zeta_A, \xi_B\} = \eta_{AB}$, one can show that in the representation (4) the position coordinates are given by $x_\mu = \zeta_\mu - \beta^2 \zeta_A \xi^A \xi_\mu$, while in the Beltrami representation they are given by $x_\mu = \beta(\xi_4 \zeta_\mu - \zeta_4 \xi_\mu)$ and coincide with the generators of translations in the hyperboloid of momenta.

Due to the nontrivial symplectic structure, the classical dynamics of the Snyder model is suitably described in Hamiltonian form. The Hamiltonian can be an arbitrary function of the Casimir operator p^2 of the undeformed Poincaré algebra. To fix this arbitrariness, in relative locality models the Hamiltonian is defined as the square of the geodesic distance in momentum space from the origin to a point parametrized by p_μ [1]. For isotropic coordinates this is very easily computed geometrically, without resorting to the calculation of the geodesics. On a hyperboloid, the distance from the origin along a timelike path is given simply by

$$l = \frac{1}{\beta} \operatorname{arctanh} \frac{|\xi|}{\xi_4} = \frac{1}{\beta} \operatorname{arcsinh} \beta|\xi|. \qquad (6)$$

This can be shown considering the embedding of the Snyder hyperboloid in five-dimensional Minkowski space. Because of the isotropy, one can simply consider a one-dimensional section in two-dimensional embedding space. For timelike geodesics, this is a hyperbola, that can be parametrized by $\beta\xi_0 = \sinh\theta$, $\beta\xi_1 = \cosh\theta$. The arclength is then $l = \int \sqrt{\dot{\xi}_0^2 - \dot{\xi}_1^2}\, d\theta = \beta^{-1} \int \sqrt{\cosh^2\theta - \sinh^2\theta}\, d\theta = \beta^{-1}\theta$, from which (6) readily follows.

Hence the Hamiltonian for a free particle of mass m in the coordinates (4) is

$$H = \frac{\lambda}{2}\left(\frac{\operatorname{arcsinh}^2 \beta|p|}{\beta^2} - m^2\right), \qquad (7)$$

while in Beltrami coordinates it is

$$H = \frac{\lambda}{2}\left(\frac{\operatorname{arctanh}^2 \beta|p|}{\beta^2} - m^2\right), \qquad (8)$$

with λ a Lagrange multiplier enforcing the Hamiltonian constraint. Notice that in the previous literature [16] the Hamiltonian has been chosen to be p^2 or some simple algebraic

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function of it, as $p^2/(1 - \beta^2 p^2)$. Although in the case of free particles the geodesics are not modified, except for a reparametrization of the momentum, in the interacting case important effects may arise.

Of particular interest are the equations of motion that follow from the Hamiltonian (8) and the symplectic structure (2),

$$\dot{x}_\mu = \lambda q_\mu \equiv \lambda \frac{\operatorname{arctanh} \beta|p|}{\beta|p|} p_\mu, \qquad \dot{p}_\mu = \dot{q}_\mu = 0. \tag{9}$$

The geodesic equations for a free particle are identical to those of special relativity when written in terms of the auxiliary variable q_μ if $m \neq 0$ and coincide with them if $m = 0$. It follows that the momentum components are conserved and the geodesics are then given by

$$x_i = \bar{x}_i + \frac{\bar{q}_i}{\bar{q}_0}(x_0 - \bar{x}_0) = \bar{x}_i + \frac{\bar{p}_i}{\bar{p}_0}(x_0 - \bar{x}_0), \tag{10}$$

which are exactly the same as in special relativity.

The Hamiltonian (7) yields slightly more involved equations,

$$\dot{x}_\mu = \lambda r_\mu \equiv \lambda \frac{\sqrt{1 + \beta^2 p^2}}{1 - \beta^2 p^2} \frac{\operatorname{arcsinh} \beta|p|}{\beta|p|} p_\mu, \qquad \dot{p}_\mu = \dot{r}_\mu = 0, \tag{11}$$

but the geodesics are still given by (10). The coincidence of the geodesics of a free particle with those of special relativity can be viewed as a consequence of the Lorentz invariance of the model.

The fact that the Snyder geodesics coincide with those of special relativity is not however sufficient to deduce that the physical predictions are the same in both cases [7]: in fact, due to the curvature of momentum space, one must take into account the presence of relative locality effects. This possibility will be discussed in the next section.

3. Relative locality effects

Following [8], we now use arguments from classical relativistic mechanics and relative locality to evaluate the delay in the time of arrival of massless particles of different momenta emitted simultaneously according to an observer $\mathcal{A}$, as detected by a distant observer $\mathcal{B}$ in Snyder space. It has been shown in [8] that such effect is present in κ-Minkowski spaces, giving rise to the possibility of detecting experimentally signals of the structure of spacetime at Planck scale.

From the Lorentz invariance of the model and its analysis in terms of DSR, we expect that this effect should not arise in the case of Snyder spacetime. However, the nontrivial transformations of spacetime under translations (3) might invalidate this conclusion and it is therefore necessary to analyze in detail this topic.

For simplicity we consider a two-dimensional setting, and denote $x_0 = t$, $x_1 = x$, $p_0 = E$, $p_1 = P$. Without loss of generality we can choose the initial conditions at an event $\mathcal{A}$ with $\bar{x} = 0$, $\bar{t} = 0$.

We suppose that at $\mathcal{A}$ two massless particles are emitted with momenta $P_1 = E_1$ and $P_2 = E_2$, according to the dispersion relation (8). In conformity with (10), their worldlines are given by

$$x_{P_1}^{\mathcal{A}}(t^{\mathcal{A}}) = t^{\mathcal{A}}, \qquad x_{P_2}^{\mathcal{A}}(t^{\mathcal{A}}) = t^{\mathcal{A}}, \tag{13}$$

and of course, according to $\mathcal{A}$, they reach $\mathcal{B}$ at the same instant.

The distant observer $\mathcal{B}$ who detects the particles is related to $\mathcal{A}$ by a translation of parameter $a_\mu = (a, a)$, and therefore the quantities he measures are related to those measured by $\mathcal{A}$ by the transformations (3). As we have seen, in Snyder space the momenta p_μ are invariant under translations, while x and t, according to (3), transform as

$$x^{\mathcal{B}} = x^{\mathcal{A}} + a, \qquad t^{\mathcal{B}} = t^{\mathcal{A}} + a, \tag{14}$$

since $a{\cdot}p = a(E - P)$ vanishes in our setting. Substituting in (13), it follows that

$$x_{P_1}^{\mathcal{B}}(t^{\mathcal{B}}) = x_{P_2}^{\mathcal{B}}(t^{\mathcal{B}}) = t^{\mathcal{B}}, \tag{15}$$

independently of the value of P. Hence, as expected, the two particles will be detected by $\mathcal{B}$ at the same time in contrast with other noncommutative models, where a time delay arises [8,9].

In principle, it can be interesting to consider also the case of massive particles, although an experimental verification does not seem to be realizable in this case. For massive particles in Beltrami parametrization, eq. (8) yields $E = \sqrt{P^2 + \beta^{-2} \tanh^2 \beta m}$. According to $\mathcal{A}$, after a time $t^{\mathcal{A}} = a$, the particles are in a position $x^{\mathcal{A}} = \frac{P}{E} a$. If the observer $\mathcal{B}$ is placed in this position, he is related to $\mathcal{A}$ by a translation with parameter $a_\mu = (a, \frac{P}{E} a)$, and hence due to (3) he measures,

$$t^{\mathcal{B}} = t^{\mathcal{A}} + \frac{a}{\cosh^2 \beta m}, \qquad x^{\mathcal{B}} = x^{\mathcal{A}} + \frac{a}{\cosh^2 \beta m} \frac{P}{E}, \tag{16}$$

since $a{\cdot}p = a(E^2 - P^2)/E = a \tanh^2 \beta m/\beta^2 E$.

Therefore,

$$x^{\mathcal{B}} = \frac{P}{E} t^{\mathcal{B}}, \tag{17}$$

which is the same relation valid in special relativity. Hence, also massive Snyder particles do not present time delay effects except for the trivial ones due to the different velocity. However, a difference arises in the time of flight of the particles as seen from the observer $\mathcal{B}$ with respect to the observer $\mathcal{A}$. In fact, they differ by a factor $1/\cosh^2 \beta m$; this could induce anomalies in the lifetime of unstable massive particles measured by $\mathcal{B}$, analogous to those investigated in [17] in a different setting.

As we have seen, the geodesics do not depend essentially on the coordinates chosen. Therefore the results of this section are valid also in the case of the parametrization (4).

It must be noted that a second kind of effect, called transverse relative locality, has been predicted for some models presenting deformed Lorentz invariance [18]. This effect is detected by an observer boosted in a direction orthogonal to the propagation of massless

particles emitted from a source, who sees a momentum-dependent angle of approach of the particles. This phenomenon essentially depends on the deformation of the Lorentz transformations. Since these transformations are not deformed in the Snyder case, we expect the absence of such effect as well. This can also be confirmed by an explicit calculation.

4. Conclusions

Using the formalism of relative locality, we have shown that in the case of Snyder geometry no effect due to delayed time of arrival of high-energy massless particles is observable, contrary to other models of noncommutative geometry [8,9]. This may have been expected because of the Lorentz invariance of the model and confirms the results one would get from a DSR approach.

However, different times of flight are measured by distant observers for massive particles of different energies, and hence some effects can in principle be detected due to the energy dependence of the lifetime of unstable particles.

It seems therefore that the relative locality effects are greatly mitigated if the Lorentz invariance is not deformed. It is likely that this conclusion can be extended to more general Lorentz-invariant NCG models, like those studied in refs. [19].

For the derivation of the results of this paper, only the metric of the Snyder momentum space was needed. However, in the framework of relative locality one can associate also torsion and nonmetricity to the momentum manifold, depending on the properties of the law of addition of the momenta [1]. We plan to discuss in detail this topic in a future paper.

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