



# Quantum field theory in generalised Snyder spaces



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## ABSTRACT

We discuss the generalisation of the Snyder model that includes all possible deformations of the Heisenberg algebra compatible with Lorentz invariance and investigate its properties. We calculate perturbatively the law of addition of momenta and the star product in the general case. We also undertake the construction of a scalar field theory on these noncommutative spaces showing that the free theory is equivalent to the commutative one, like in other models of noncommutative QFT.

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## 1. Introduction

Snyder spacetime [1] was introduced in 1947 as an attempt to avoid UV divergences in QFT. By assuming a noncommutative structure of spacetime, and hence a deformation of the Heisenberg algebra, it was possible to define a discrete spacetime without breaking the Lorentz invariance, opening the possibility to smoothen the short-distance behavior of quantum field theory.

The proposal was then forgotten for many years, until more recent times, when noncommutative geometry has become an important field of research [2]. New models have been introduced, in particular the Moyal plane [3] and  $\kappa$ -Minkowski geometry [4], and the formalism of Hopf algebras has been applied to their study [5]. However, contrary to Snyder's, the new models either break or deform the Lorentz group action on phase space.

In spite of the renewed interest in spacetime noncommutativity and of its preservation of spacetime symmetries, relatively few investigations have been dedicated to the original proposal of Snyder from the point of view of noncommutative geometry, except for a series of papers [6–8], where the model was extended to include more general Lorentz-invariant models, as the one proposed by Maggiore [9], and the star product, coproduct and antipodes of its Hopf algebra were calculated. The model was also investigated in [10], where it was considered from a geometrical point of view as a coset in momentum space, with results equivalent to those of refs. [6,7]. Also the construction of QFT on Snyder spacetime was undertaken in these papers.

However, some basic properties of the Hopf algebra formalism for Snyder spaces have not yet been investigated: for example the twist has not been explicitly calculated. Also the investigation of QFT has only been sketched. In [7] a scalar field theory was defined in terms of the star product, but no explicit calculation was carried on. Moreover, a non-Hermitian representation was used, that complicates the formalism. Other approaches to Snyder QFT are based on a five-dimensional formalism [11,10].

Several efforts have also been devoted to the study of the classical and quantum aspects of the model from a phenomenological point of view, without resorting to the formalism of noncommutative geometry, especially in the nonrelativistic 3D limit [12–14]. The most interesting results in this context are the clarification of its lattice-like properties, leading to deformed uncertainty relations, and the study of the corrections induced on the energy spectrum of some simple physical systems.

In this letter, we extend previous investigations on the noncommutative geometry of the Snyder model in two directions: first, we further generalise the model to include in the defining commutation relations all the terms compatible with undeformed Lorentz invariance. Among these generalisations, some have peculiar properties, for example it is possible to construct models that describe a commutative spacetime, but nevertheless display nontrivial commutation relations between positions and momenta, leading to deformed addition rules for momenta and nonlocal behavior in field theory.

Moreover, we improve the results of ref. [7] on QFT, adopting a Hermitian representation of the noncommutative coordinates and showing that with this definition the free field theory of a scalar

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particle is equivalent to the one in noncommutative spacetime, similarly to other well-known models [15,16].

## 2. Snyder space and its generalisation

We define generalised Snyder space as a deformation of ordinary phase space, generated by noncommutative coordinates  $\bar{x}_\mu$  and momenta  $p_\mu$  that span a deformed Heisenberg algebra  $\mathcal{H}(\bar{x}, p)$ ,

$$\begin{aligned} [\bar{x}_\mu, \bar{x}_\nu] &= i\beta M_{\mu\nu} \psi(\beta p^2), & [p_\mu, p_\nu] &= 0, \\ [p_\mu, \bar{x}_\nu] &= -i\varphi_{\mu\nu}(\beta p^2), \end{aligned} \quad (1)$$

together with Lorentz generators  $M_{\mu\nu}$  that satisfy the standard relations

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}), \\ [M_{\mu\nu}, p_\lambda] &= i(\eta_{\mu\lambda} p_\nu - \eta_{\nu\lambda} p_\mu), \\ [M_{\mu\nu}, \bar{x}_\lambda] &= i(\eta_{\mu\lambda} \bar{x}_\nu - \eta_{\nu\lambda} \bar{x}_\mu), \end{aligned} \quad (2)$$

where the functions  $\psi(\beta p^2)$  and  $\varphi_{\mu\nu}(\beta p^2)$  are constrained so that the Jacobi identities hold,  $\beta$  is a constant of the order of  $1/M_{Pl}^2$ , and  $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ . The commutation relations (1)–(2) generalise those of the Snyder spaces originally investigated in [6,7], that are recovered for  $\psi = \text{const}$ . Special cases are the Snyder realisation [1], and the Maggiore realisation [9].

We recall that in its undeformed version, the Heisenberg algebra  $\mathcal{H}(x, p)$  is generated by commutative coordinates  $x_\mu$  and momenta  $p_\mu$ , satisfying

$$[x_\mu, x_\nu] = [p_\mu, p_\nu] = 0, \quad [p_\mu, x_\nu] = -i\eta_{\mu\nu}. \quad (3)$$

The action of  $x_\mu$  and  $p_\mu$  on functions  $f(x)$  belonging to the enveloping algebra  $\mathcal{A}$  generated by the  $x_\mu$  is defined as

$$x_\mu \triangleright f(x) = x_\mu f(x), \quad p^\mu \triangleright f(x) = -i \frac{\partial f(x)}{\partial x_\mu}. \quad (4)$$

The noncommutative coordinates  $\bar{x}_\mu$  and the Lorentz generators  $M_{\mu\nu}$  in (1)–(2) can be expressed in terms of commutative coordinates  $x_\mu$  and momenta  $p_\mu$  as [6,7]

$$\bar{x}_\mu = x_\mu \varphi_1(\beta p^2) + \beta x \cdot p p_\mu \varphi_2(\beta p^2) + \beta p_\mu \chi(\beta p^2), \quad (5)$$

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu. \quad (6)$$

Notice that the function  $\chi$  does not appear in the defining relations (1)–(2), but takes into account ambiguities arising from operator ordering of  $x_\mu$  and  $p_\mu$  in equation (5).

In terms of the realisation (5), the functions  $\varphi_{\mu\nu}$  in (1) read

$$\varphi_{\mu\nu} = \eta_{\mu\nu} \varphi_1 + \beta p_\mu p_\nu \varphi_2, \quad (7)$$

while the Jacobi identities are satisfied if

$$\psi = -2\varphi_1 \varphi_1' + \varphi_1 \varphi_2 - 2\beta p^2 \varphi_1' \varphi_2, \quad (8)$$

where the prime denotes a derivative with respect to  $\beta p^2$ . In particular, the function  $\psi$  does not depend on the function  $\chi$ .

Inserting (8) into (1), it is easy to check that the coordinates  $\bar{x}_\mu$  are commutative for  $\varphi_2 = \frac{2\varphi_1' \varphi_1}{\varphi_1 - 2\beta p^2 \varphi_1'}$ , and correspond to Snyder spaces for  $\varphi_2 = \frac{1+2\varphi_1' \varphi_1}{\varphi_1 - 2\beta p^2 \varphi_1'}$ . In particular, the Snyder realisation [1] is recovered for  $\varphi_1 = \varphi_2 = 1$ , and the Maggiore realisation [9] for  $\varphi_1 = \sqrt{1 - \beta p^2}$ ,  $\varphi_2 = 0$ . Another interesting exact realisation of generalised Snyder spaces is obtained for  $\psi = s = \text{const}$ , and reads

$$\bar{x}_\mu = x_\mu + \frac{\beta s}{4} K_\mu, \quad (9)$$

where  $K_\mu = x_\mu p^2 - 2x \cdot p p_\mu$  are the generators of special conformal transformations in momentum space, satisfying  $[K_\mu, K_\nu] = 0$ .

The algebra (1)–(2) includes as special cases both commutative spaces,  $\psi = 0$ , and Snyder spaces,  $\psi = 1$ . Since the Lorentz transformations are not deformed, its Casimir operator is the same as for the Poincaré algebra,  $C = p^2$ .

## 3. Coproduct and star product

The generalised Snyder spaces defined above can be investigated using the Hopf-algebra formalism developed in refs. [17–19] and shortly reviewed in [20], to which we refer for more details.<sup>1</sup> In this way, one can deduce the properties of the algebra associated to Snyder space starting from its realisation (5).

It can be shown that for a general Hopf algebra  $\mathcal{A}$  [17–19],

$$e^{ik \cdot \bar{x}} \triangleright e^{iq \cdot x} = e^{i\mathcal{P}(k,q) \cdot x + i\mathcal{Q}(k,q)}, \quad (10)$$

and

$$e^{ik \cdot \bar{x}} \triangleright 1 = e^{i\mathcal{K}(k) \cdot x + i\mathcal{L}(k)}, \quad (11)$$

where eqs. (10) and (11) can be seen as the defining relations for the functions  $\mathcal{P}$ ,  $\mathcal{Q}$ ,  $\mathcal{K}$  and  $\mathcal{L}$ .

It is easily seen that  $\mathcal{P}_\mu(\lambda k, q)$  satisfies the differential equation [17–19]

$$\frac{d\mathcal{P}_\mu(\lambda k, q)}{d\lambda} = k_\alpha \varphi_\mu^\alpha \left( \mathcal{P}(\lambda k, q) \right), \quad (12)$$

where  $\lambda$  is a real parameter and

$$\mathcal{P}_\mu(k, 0) = \mathcal{K}_\mu(k), \quad \mathcal{P}_\mu(0, q) = q_\mu. \quad (13)$$

Analogously, it can be shown that  $\mathcal{Q}_\mu(\lambda k, q)$  satisfies the differential equation

$$\frac{d\mathcal{Q}_\mu(\lambda k, q)}{d\lambda} = k_\alpha \chi^\alpha \left( \mathcal{P}(\lambda k, q) \right), \quad (14)$$

with  $\chi^\alpha \equiv \beta p^\alpha \chi(\beta p^2)$  and

$$\mathcal{Q}(k, 0) = \mathcal{L}(k), \quad \mathcal{Q}(0, q) = 0. \quad (15)$$

The generalised addition of momenta  $k_\mu$  and  $q_\mu$  is then defined as

$$k_\mu \oplus q_\mu = \mathcal{D}_\mu(k, q), \quad \text{with } \mathcal{D}_\mu(k, 0) = k_\mu, \quad \mathcal{D}_\mu(0, q) = q_\mu, \quad (16)$$

where the function  $\mathcal{D}_\mu(k, q)$  can be obtained from  $\mathcal{P}_\mu(k, q)$  as [7, 8,22]

$$\mathcal{D}_\mu(k, q) = \mathcal{P}_\mu(\mathcal{K}^{-1}(k), q), \quad (17)$$

and the function  $\mathcal{K}_\mu^{-1}(k)$  is the inverse map of  $\mathcal{K}(k)$ , i.e.  $\mathcal{K}_\mu^{-1}(\mathcal{K}(k)) = k_\mu$ . Remarkably, from (12) and (7) it follows that  $\mathcal{P}_\mu(k, q)$  and hence  $\mathcal{D}_\mu(k, q)$  do not depend on the function  $\chi$  in (5).

From (11), it is also possible to calculate the star product of two plane waves, that turns out to be

$$e^{ik \cdot x} \star e^{iq \cdot x} = e^{i\mathcal{D}(k,q) \cdot x + i\mathcal{G}(k,q)}, \quad (18)$$

where

$$\mathcal{G}(k, q) = \mathcal{Q}(\mathcal{K}^{-1}(k), q) - \mathcal{Q}(\mathcal{K}^{-1}(k), 0). \quad (19)$$

Note that  $\mathcal{G}$  vanishes if  $\chi = 0$ .

<sup>1</sup> A more rigorous treatment, including the full phase space, is given by the Hopf algebroid formalism [21]. However, we shall not need it in the following.

Finally, the coproduct for the momenta  $\Delta p_\mu$  can be written as usual in terms of  $\mathcal{D}_\mu(k, q)$  as

$$\Delta p_\mu = \mathcal{D}_\mu(p \otimes 1, 1 \otimes p). \quad (20)$$

The previous definitions imply that the addition of momenta and the coproduct do not depend on  $\chi$ . Following the steps sketched above, the coproducts of momenta were found for special cases in ref. [7]: for example, for the Snyder realisation [1],

$$\Delta p_\mu = \frac{1}{1 - \beta p_\alpha \otimes p^\alpha} \left( p_\mu \otimes 1 - \frac{\beta}{1 + \sqrt{1 + \beta p^2}} p_\mu p_\alpha \otimes p^\alpha + \sqrt{1 + \beta p^2} \otimes p_\mu \right). \quad (21)$$

Finally, we recall that the antipodes for Snyder space are trivial [7],

$$S(p_\mu) = -p_\mu, \quad S(M_{\mu\nu}) = -M_{\mu\nu}. \quad (22)$$

#### 4. First order expansion

The study of the Hopf algebra for the generalised Snyder model is difficult, but can be tackled using a perturbative approach, by expanding the realisation (5) of the noncommutative coordinates in powers of  $\beta$ . The expansion gives

$$\bar{x}_\mu = x_\mu + \beta (s_1 x_\mu p^2 + s_2 x \cdot p p_\mu + c p_\mu) + \mathcal{O}(\beta^2), \quad (23)$$

with independent parameters  $s_1, s_2, c$ . The commutation relations do not depend on the parameter  $c$  and to first order are given by

$$[\bar{x}_\mu, \bar{x}_\nu] = i\beta s M_{\mu\nu} + \mathcal{O}(\beta^2), \quad [p_\mu, \bar{x}_\nu] = -i[\eta_{\mu\nu}(1 + \beta s_1 p^2) + \beta s_2 p_\mu p_\nu] + \mathcal{O}(\beta^2), \quad (24)$$

where  $s = s_2 - 2s_1$ .

The models of refs. [6,7] are recovered for  $s_2 = 1 + 2s_1$ . Moreover, for  $s_1 = 0, s_2 = 1$ , eqs. (23)–(24) reproduce the exact Snyder realisation, while for  $s_1 = -\frac{1}{2}, s_2 = 0$  they give the first-order expansion of the Maggiore realisation. For  $s_2 = 2s_1$ , spacetime is commutative to first order in  $\beta$ , while for  $s_1 = -s/4, s_2 = s/2, c = 0$  one gets the exact realisation (9).

From (12) one can calculate the first order expression for the function  $\mathcal{P}_\mu(k, q)$  in the general case, which reads

$$\begin{aligned} \mathcal{P}_\mu(k, q) &= k_\mu + q_\mu \\ &+ \beta \left[ \left( s_1 q^2 + \left( s_1 + \frac{s_2}{2} \right) k \cdot q + \frac{s_1 + s_2}{3} k^2 \right) k_\mu \right. \\ &\left. + s_2 \left( k \cdot q + \frac{k^2}{2} \right) q_\mu \right] + \mathcal{O}(\beta^2), \end{aligned} \quad (25)$$

from where it follows that

$$\mathcal{K}_\mu^{-1}(k) = k_\mu - \frac{\beta}{3} (s_1 + s_2) k^2 k_\mu + \mathcal{O}(\beta^2). \quad (26)$$

These results allow one to write down the generalised addition law of the momenta  $k_\mu$  and  $q_\mu$  to first order

$$\begin{aligned} (k \oplus q)_\mu &= \mathcal{D}_\mu(k, q) \\ &= k_\mu + q_\mu + \beta \left[ s_2 k \cdot q q_\mu + s_1 q^2 k_\mu + \left( s_1 + \frac{s_2}{2} \right) k \cdot q k_\mu \right. \\ &\left. + \frac{s_2}{2} k^2 q_\mu \right] + \mathcal{O}(\beta^2). \end{aligned} \quad (27)$$

It is interesting to remark that for  $s_2 = 2s_1 \neq 0$ , although spacetime is commutative up to the first order in  $\beta$ , the addition of momenta is deformed,

$$(k \oplus q)_\mu \neq k_\mu + q_\mu. \quad (28)$$

The Lorentz transformations of momenta are instead not deformed, and denoting them by  $\Lambda(\xi, p)$ , with  $\xi$  the rapidity parameter, the law of addition of momenta implies that

$$\Lambda(\xi, k \oplus q) = \Lambda(\xi_1, k) \oplus \Lambda(\xi_2, q) \quad (29)$$

is satisfied for  $\xi_1 = \xi_2 = \xi$ . Hence there are no backreaction factors in the sense of refs. [23,24]. This means that in composite systems the boosted momenta of the single particles are independent of the momenta of the other particles in the system. This confirms the results obtained from general arguments in [25] and [20].

The coproduct to first order can be read from (27) and is given by

$$\begin{aligned} \Delta p_\mu &= \Delta_0 p_\mu + \beta \left[ s_1 p_\mu \otimes p^2 + s_2 p_\alpha \otimes p^\alpha p_\mu \right. \\ &\left. + \left( s_1 + \frac{s_2}{2} \right) p_\mu p_\alpha \otimes p^\alpha + \frac{s_2}{2} p^2 \otimes p_\mu \right] + \mathcal{O}(\beta^2). \end{aligned} \quad (30)$$

#### 5. Field theory for the Snyder realisation

The scalar field theory on Snyder spacetime was investigated in [7] using the Snyder realisation [1],

$$\bar{x}_\mu = x_\mu + \beta x \cdot p p_\mu. \quad (31)$$

However, this realisation is not Hermitian, and hence also the resulting action functional is not Hermitian. This causes some problems, in particular in the definition of a measure for the integral, while, as we shall show, with a Hermitian realisation the free field action reduces to the usual commutative form. Similar considerations have been made in [16] for the  $\kappa$ -Minkowski case.

The realisation (31) can be made Hermitian by adding its adjoint, yielding

$$\bar{x}^\mu = x^\mu + \frac{\beta}{2} (x \cdot p p^\mu + p^\mu p \cdot x) = x^\mu + \beta x \cdot p p^\mu - \frac{5i}{2} \beta p^\mu, \quad (32)$$

where we have used the canonical commutation relations to rearrange the expression.

For the Snyder realisation, one obtains [7]

$$\mathcal{P}_\mu(k, q) = \frac{q_\mu + \left[ \frac{\sin \sqrt{\beta k^2}}{\sqrt{\beta k^2}} + \frac{k \cdot q}{k^2} (\cos \sqrt{\beta k^2} - 1) \right] k_\mu}{\cos \sqrt{\beta k^2} - \frac{k \cdot q}{k^2} \sqrt{\beta k^2} \sin \sqrt{\beta k^2}}, \quad (33)$$

and

$$\mathcal{K}_\mu(k) = \frac{\tan \sqrt{\beta k^2}}{\sqrt{\beta k^2}} k_\mu, \quad (34)$$

from which it follows that

$$\mathcal{D}_\mu(k, q) = \frac{1}{1 - \beta k \cdot q} \left[ \left( 1 - \frac{\beta k \cdot q}{1 + \sqrt{1 + \beta k^2}} \right) k_\mu + \sqrt{1 + \beta k^2} q_\mu \right], \quad (35)$$

Using eqs. (14) and (19), one can compute the functions  $\mathcal{Q}$  and  $\mathcal{G}$  for the Hermitian realisation (32), obtaining

$$\mathcal{Q}(k, q) = \frac{5i}{2} \ln \left[ \cos \sqrt{\beta k^2} - \frac{k \cdot q}{k^2} \sqrt{\beta k^2} \sin \sqrt{\beta k^2} \right], \quad (36)$$

and

$$\mathcal{G}(k, q) = \frac{5i}{2} \ln[1 - \beta k \cdot q]. \quad (37)$$

According to (18), the star product for plane waves in the Hermitian realisation (32) is therefore

$$e^{ik \cdot x} \star e^{iq \cdot x} = \frac{e^{i\mathcal{D}(k, q) \cdot x}}{(1 - \beta k \cdot q)^{5/2}}, \quad (38)$$

with  $\mathcal{D}_\mu(k, q)$  given by (35).

The action of a noncommutative scalar field  $\bar{\phi}(\bar{x})$  on the identity (ground state) of  $\mathcal{A}$  is defined as

$$\bar{\phi}(\bar{x}) \triangleright 1 = \phi(x), \quad \bar{\phi}(\bar{x})^2 \triangleright 1 = (\phi \star \phi)(x). \quad (39)$$

The action functional for a noninteracting massive real scalar field can then be defined as

$$\begin{aligned} S[\phi] &= \frac{1}{2} \int d^4 \bar{x} (\partial_\mu \bar{\phi} \partial^\mu \bar{\phi} + m^2 \bar{\phi}^2) \triangleright 1 \\ &= \frac{1}{2} \int d^4 x (\partial_\mu \phi \star \partial^\mu \phi + m^2 \phi \star \phi) \end{aligned} \quad (40)$$

To write the action in simpler form, we compute the star product of two real scalar fields  $\phi(x)$  and  $\psi(x)$  by expanding in Fourier series,

$$\phi(x) = \int d^4 k \tilde{\phi}(k) e^{ikx}. \quad (41)$$

Then

$$\begin{aligned} \int d^4 x \psi(x) \star \phi(x) &= \int d^4 x \int d^4 k d^4 q \tilde{\psi}(k) \tilde{\phi}(q) e^{ik \cdot x} \star e^{iq \cdot x} \\ &= \int d^4 k d^4 q \tilde{\psi}(k) \tilde{\phi}(q) \int d^4 x \frac{e^{i\mathcal{D}(k, q) \cdot x}}{(1 - \beta k \cdot q)^{5/2}} \\ &= \int d^4 k d^4 q \tilde{\psi}(k) \tilde{\phi}(q) \frac{\delta^{(4)}(\mathcal{D}(k, q))}{(1 - \beta k \cdot q)^{5/2}}. \end{aligned} \quad (42)$$

Now, since  $\mathcal{D}_\mu(k, q)$  vanishes only for  $q_\mu = -k_\mu$ ,

$$\delta^{(4)}(\mathcal{D}(k, q)) = \frac{\delta^{(4)}(q + k)}{\left| \det \left( \frac{\partial \mathcal{D}_\mu(k, q)}{\partial q_\nu} \right) \right|_{q=-k}}. \quad (43)$$

On the other hand,

$$\left. \frac{\partial \mathcal{D}_\mu(k, q)}{\partial q_\nu} \right|_{q=-k} = \frac{1}{1 + \beta k^2} \left( \sqrt{1 + \beta k^2} \delta_\mu^\nu - \frac{\beta k_\mu k^\nu}{1 + \sqrt{1 + \beta k^2}} \right), \quad (44)$$

and then

$$\left| \det \left( \frac{\partial \mathcal{D}_\mu(k, q)}{\partial q_\nu} \right) \right|_{q=-k} = \frac{1}{(1 + \beta k^2)^{5/2}}. \quad (45)$$

The term in (45) cancels with the one coming from the star product in (42), and finally one obtains

$$\int d^4 x \psi(x) \star \phi(x) = \int d^4 x \psi(x) \phi(x). \quad (46)$$

Hence, the integral of a star product of two fields in the Hermitian realisation can be reduced to the integral of an ordinary product of commutative functions. The same happens in the Moyal [15] and  $\kappa$ -Minkowski case [16]. We conjecture that this is a universal property of noncommutative models in a Hermitian realisation, although we are not able to prove it.

In particular, for a free scalar field, the action  $S[\phi]$  can then be written as

$$\begin{aligned} S[\phi] &= \frac{1}{2} \int d^4 x (\partial_\mu \phi \star \partial^\mu \phi + m^2 \phi \star \phi) \\ &= \frac{1}{2} \int d^4 x (\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2). \end{aligned} \quad (47)$$

## 6. Field theory for the linearised theory

The previous calculations can be extended to the generalised Snyder models by a perturbative expansion in  $\beta$  up to first order. This is a good approximation for small momenta, although cannot be trusted in the ultraviolet region.

For simplicity, we shall limit our consideration to the Snyder models of ref. [7], neglecting the generalisation of section 2. In this case  $s_2 = 1 + 2s_1$ , and the realisation (23) reduces to

$$\bar{x}^\mu = x^\mu (1 + \beta s_1 p^2) + \beta (1 + 2s_1) x \cdot p p^\mu + \mathcal{O}(\beta^2). \quad (48)$$

The Snyder realisation discussed in the previous section is obtained for  $s_1 = 0$ . In general (48) is not Hermitian, but a Hermitian realisation can be obtained as before by redefining

$$\begin{aligned} \bar{x}^\mu &= x^\mu (1 + \beta s_1 p^2) + \beta (1 + 2s_1) x \cdot p p^\mu - i\beta \left( \frac{5}{2} + 6s_1 \right) p^\mu \\ &\quad + \mathcal{O}(\beta^2). \end{aligned} \quad (49)$$

For this realisation, eq. (25) reduces to

$$\begin{aligned} \mathcal{P}_\mu(k, q) &= k_\mu + q_\mu + \beta \left[ s_1 q^2 + \left( \frac{1}{2} + 2s_1 \right) k \cdot q \right. \\ &\quad \left. + \left( \frac{1}{3} + s_1 \right) k^2 \right] k_\mu + \beta (1 + 2s_1) \left[ \frac{k^2}{2} + k \cdot q \right] q_\mu \\ &\quad + \mathcal{O}(\beta^2), \end{aligned} \quad (50)$$

and

$$\mathcal{K}_\mu(k) = \left[ 1 + \beta \left( \frac{1}{3} + s_1 \right) k^2 \right] k_\mu + \mathcal{O}(\beta^2), \quad (51)$$

from which it follows that

$$\begin{aligned} \mathcal{D}_\mu(k, q) &= k_\mu + q_\mu + \beta \left[ s_1 q^2 + \left( \frac{1}{2} + 2s_1 \right) k \cdot q \right] k_\mu \\ &\quad + \beta (1 + 2s_1) \left[ \frac{k^2}{2} + k \cdot q \right] q_\mu + \mathcal{O}(\beta^2). \end{aligned} \quad (52)$$

One can now compute the linearised  $\mathcal{Q}$  and  $\mathcal{G}$  from (14) and (19), obtaining

$$\mathcal{Q}(k, q) = -i\beta \left( \frac{5}{2} + 6s_1 \right) \left( \frac{k^2}{2} + k \cdot q \right) + \mathcal{O}(\beta^2), \quad (53)$$

and

$$\mathcal{G}(k, q) = -i\beta \left( \frac{5}{2} + 6s_1 \right) k \cdot q + \mathcal{O}(\beta^2). \quad (54)$$

The star product for the Hermitian realisation (49) is therefore

$$e^{ik \cdot x} \star e^{iq \cdot x} = \left[ 1 + \beta \left( \frac{5}{2} + 6s_1 \right) k \cdot q \right] e^{i\mathcal{D}(k, q) \cdot x} + \mathcal{O}(\beta^2), \quad (55)$$

where  $\mathcal{D}$  is given in eq. (52).

The product of fields can now be computed as before expanding in noncommutative plane waves, and gives

$$\int d^4x \psi(x) \star \phi(x) = \int d^4k d^4q \tilde{\psi}(k) \tilde{\phi}(q) \left[ 1 + \beta \left( \frac{5}{2} + 6s_1 \right) k \cdot q + \mathcal{O}(\beta^2) \right] \delta^{(4)}(\mathcal{D}(k, q)). \quad (56)$$

Now, it is easy to see that  $\mathcal{D}(k, q)$  vanishes only for  $q_\mu = -k_\mu$ , and then

$$\delta^{(4)}(\mathcal{D}(k, q)) = \frac{\delta^{(4)}(q + k)}{\left| \det \left( \frac{\partial \mathcal{D}_\mu(k, q)}{\partial q_\nu} \right) \right|_{q=-k}}. \quad (57)$$

On the other hand,

$$\frac{\partial \mathcal{D}_\mu(k, q)}{\partial q_\nu} \Big|_{q=-k} = \delta_\mu^\nu - \beta \left[ \left( \frac{1}{2} + s_1 \right) k^2 \delta_\mu^\nu + \left( \frac{1}{2} + 2s_1 \right) k_\mu k^\nu \right] + \mathcal{O}(\beta^2), \quad (58)$$

and hence

$$\left| \det \left( \frac{\partial \mathcal{D}_\mu(k, q)}{\partial q_\nu} \right) \right|_{q=-k} = 1 - \beta \left( \frac{5}{2} + 6s_1 \right) k^2 + \mathcal{O}(\beta^2). \quad (59)$$

Again the corrections coming from (55) and (59) cancel and one recovers (46), showing that also for general realisations at the linear level the integral of the star product in a Hermitian realisation is equivalent to the ordinary product. In particular, eq. (47) still holds and the free field propagator coincides with the commutative one.

Interaction terms can be added to the action through the star product; for example, a cubic interaction can be described by

$$S^{(3)} = \int d^4x \phi \star (\phi \star \phi). \quad (60)$$

Notice that because of the nonassociativity of the star product the ordering of the products is important; one may also define the interaction using symmetrised forms of (60).

The computation of the star products in (60) involves the evaluation of the vertex operator  $\mathcal{D}_\mu^{(3)}(k_1, k_2, k_3) = \mathcal{D}_\mu(k_3, \mathcal{D}(k_2, \mathcal{D}(k_1)))$ . This can be easily done using the results of the previous sections. We leave the calculation of these terms and of loop corrections to future investigations.

## 7. Conclusions

In this paper we have extended the study of the Snyder model to its most general realisations compatible with undeformed Lorentz invariance. Some of the new models have peculiar properties, for example they can display commutative spacetime geometry, but nontrivial Heisenberg algebra, obeying nonstandard laws for the addition of momenta.

We have also constructed a scalar QFT on Snyder spaces along the lines of [7]. The main improvement of our approach has been

the use of a Hermitian representation for the noncommutative spacetime coordinates. This has allowed us to show that the non-interacting terms in the action can be reduced to the ordinary noncommutative form, as in other noncommutative models [15,16]. This might be a universal property of noncommutative models with Hermitian action, and it would be interesting to further investigate the origin of this property. The interacting theory can also be studied in our formalism, and we plan to pursue this topic in future work.

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