The editorial version of the paper can be found at SIAM J. CONTROL OPTIM., Vol. 55, No. 6, pp. 4112–4155

Eprint: https://doi.org/10.1137/15M1034325

OUTPUT FEEDBACK STABILIZATION OF COUPLED REACTION-DIFFUSION PROCESSES WITH CONSTANT PARAMETERS

Y. ORLOV*, A. PISANO†, A. PILLONI ‡, AND E. USAI§

Abstract. The problem of output feedback boundary stabilization is considered for n coupled plants, distributed over the one-dimensional spatial domain $[0, 1]$ where they are governed by linear reaction-diffusion Partial Differential Equations (PDEs). All plants have costant parameters and are equipped with its own scalar boundary control input, acting at one end of the domain. First, a state feedback law is designed to exponentially stabilize the closed-loop system with an arbitrarily fast convergence rate. Then, collocated and anti-collocated observers are designed, using a single boundary measurement for each plant. The exponential convergence of the observed state towards the actual one is demonstrated for both observers, with a convergence rate that can be made as fast as desired. Finally, the state feedback controller and the pre-selected, either collocated or anti-collocated, observer are coupled together to yield an output feedback stabilizing controller. The distinct treatments are proposed separately for the case in which all processes have the same diffusivity and for the more challenging scenario where each process has its own diffusivity. The backstepping method is used for both controller and observer designs. Two main classes of coupled PDEs are studied along the paper. In the first one, all processes possess a Dirichlet-type boundary condition (BC) at the uncontrolled side. With reference to this class, the state feedback and observer-based output feedback designs are successfully solved in both the equi-diffusivity and distinct-diffusivity scenarios, and, particularly, the kernel matrices of the underlying transformations are derived in analytical form by using the method of successive approximations to solve the corresponding kernel PDEs. Thus, the resulting control laws and observers become available in explicit form. The second and more general class of coupled PDEs considered in the paper entails a subset of the processes having Dirichlet-type BCs at the uncontrolled side, whereas all remaining processes possess Neumann-type BCs. With reference to this wider class of systems with heterogenous BCs, it turns out that the state feedback design can only be solved in the equi-diffusivity case, although the resulting kernel matrix is no longer available in explicit form, whereas the same approach yields an overdetermined kernel PDE admitting no solution in the distinct diffusivity case. Anticollocated observer design and output feedback designs are additionally developed in the equi-diffusivity scenario. Interestingly, the observer gains are still available in explicit form also in this case. Capabilities of the proposed synthesis and its effectiveness are supported by a numerical study made for two coupled systems with heterogeneous BCs.

Key words. Reaction-diffusion equation; Boundary Control; Backstepping.

OUTPUT FEEDBACK STABILIZATION OF COUPLED REACTION-DIFFUSION PROCESSES WITH CONSTANT PARAMETERS

Y. ORLOV*, A. PISANO†, A. PILLONI ‡, AND E. USAI§

Abstract. The problem of output feedback boundary stabilization is considered for n coupled plants, distributed over the one-dimensional spatial domain $[0, 1]$ where they are governed by linear reaction-diffusion Partial Differential Equations (PDEs). All plants have costant parameters and are equipped with its own scalar boundary control input, acting at one end of the domain. First, a state feedback law is designed to exponentially stabilize the closed-loop system with an arbitrarily fast convergence rate. Then, collocated and anti-collocated observers are designed, using a single boundary measurement for each plant. The exponential convergence of the observed state towards the actual one is demonstrated for both observers, with a convergence rate that can be made as fast as desired. Finally, the state feedback controller and the pre-selected, either collocated or anti-collocated, observer are coupled together to yield an output feedback stabilizing controller. The distinct treatments are proposed separately for the case in which all processes have the same diffusivity and for the more challenging scenario where each process has its own diffusivity. The backstepping method is used for both controller and observer designs. Two main classes of coupled PDEs are studied along the paper. In the first one, all processes possess a Dirichlet-type boundary condition (BC) at the uncontrolled side. With reference to this class, the state feedback and observer-based output feedback designs are successfully solved in both the equi-diffusivity and distinct-diffusivity scenarios, and, particularly, the kernel matrices of the underlying transformations are derived in analytical form by using the method of successive approximations to solve the corresponding kernel PDEs. Thus, the resulting control laws and observers become available in explicit form. The second and more general class of coupled PDEs considered in the paper entails a subset of the processes having Dirichlet-type BCs at the uncontrolled side, whereas all remaining processes possess Neumann-type BCs. With reference to this wider class of systems with heterogenous BCs, it turns out that the state feedback design can only be solved in the equi-diffusivity case, although the resulting kernel matrix is no longer available in explicit form, whereas the same approach yields an overdetermined kernel PDE admitting no solution in the distinct diffusivity case. Anticollocated observer design and output feedback designs are additionally developed in the equi-diffusivity scenario. Interestingly, the observer gains are still available in explicit form also in this case. Capabilities of the proposed synthesis and its effectiveness are supported by a numerical study made for two coupled systems with heterogeneous BCs.

Key words. Reaction-diffusion equation; Boundary Control; Backstepping.

AMS subject classifications. 93D09, 93B12, 35K05, 35L05.

1. Introduction.

Reaction-diffusion equations are parabolic Partial Differential Equations (PDEs) which often occur in practice, e.g., to model the concentration of one or more substances, distributed in space, under the influence of different phenomena such as local chemical reactions, in which the substances are transformed into each other, and diffusion, which causes the substances to spread out over a surface in space. Certainly, reaction-diffusion PDEs are not confined to chemical applications (see e.g. [25]), but they also describe dynamical processes of non-chemical nature, with examples being found in thermodynamics, biology, geology, physics, ecology, etc. (see e.g. [10, 13]).

In the present work, the problem of output feedback boundary stabilization is considered for two classes of coupled linear reaction-diffusion PDEs having constant diffusion and reaction parameters. In the first class, all processes are governed by Dirichlet boundary conditions at the uncontrolled side provided that only boundary flows are available for measurements.

*CICESE Research Center, Electronics and Telecommunication Department. San Diego, CA 92143-4944, USA. (yorlov@cicese.mx)

†Department of Electrical and Electronic Engineering (DIEE), University of Cagliari, Piazza d'Armi, Cagliari Italy. (pisano@diee.unica.it).

‡Department of Electrical and Electronic Engineering (DIEE), University of Cagliari, Piazza d'Armi, Cagliari Italy. (alessandro.pilloni@diee.unica.it).

§Department of Electrical and Electronic Engineering (DIEE), University of Cagliari, Piazza d'Armi, Cagliari Italy. (eusai@diee.unica.it).

In the second, wider, class of coupled PDEs a subset of them possess Dirichlet BCs, whereas all remaining processes possess Neumann BCs

The adopted treatment does not rely on any discretization or finite-dimensional approximation of the underlying PDEs and it preserves the infinite-dimensional structure of the system during the entire design process. The proposed output feedback synthesis is based on the so-called "backstepping" approach [17]. Basically, the backstepping approach deals with an invertible Volterra integral transformation, mapping the system dynamics onto a predefined exponentially stable target dynamics. Backstepping is a versatile and powerful approach to boundary control and observer design, applicable to a broad spectrum of linear PDEs, and under certain circumstances controllers and observers are derived in explicit forms [17].

1.1. Related literature. The backstepping-based boundary control of scalar reaction-diffusion processes was studied, e.g., in [21], [31] whereas scalar wave processes were studied, e.g., in [18], [33]. Complex-valued PDEs such as the Schrodinger equation were dealt with in [19]. Synergies between the backstepping methodology and the flatness approach were exploited in [22], [23] to control parabolic PDEs with spatially and time-varying coefficients in spatial domains of dimension 2 and higher. In addition, an interesting feature of backstepping is that it admits a synergic integration with robust control paradigms such as the sliding mode control methodology (see, e.g., [11]).

The implementation of backstepping controllers usually requires the full state information. From the practical standpoint, the available measurements of Distributed Parameter Systems (DPSs) are typically located at the boundary of the spatial domain, that motivates the need of the state observer design [37, 12]. For linear infinite-dimensional systems, the Luenberger observer theory was established by replacing matrices with linear operators [7, 20, 12], and the observer design was confined to determining a gain operator that stabilizes the associated observation error dynamics. In contrast to finite-dimensional systems, finding such a gain operator was not trivial even numerically because operators were not generally represented with a finite number of parameters.

Observer design methods that would be capable of yielding the observer gains in the analytical form have only recently been investigated. In this context, the backstepping method appears to be a particularly effective systematic observer design approach [17, 32]. For scalar systems governed by parabolic PDEs defined on a 1-dimensional (1D) spatial domain, a systematic observer design approach, using boundary sensing, is introduced in [32]. Recently, the backstepping-based observer design was presented in [36] for reaction-diffusion processes with spatially-varying reaction coefficients while measuring a certain integral average value of the state of the plant. In [14, 15], backstepping-based observer design was addressed for reaction-diffusion processes evolving in multi-dimensional spatial domains.

More recently, high-dimensional systems of *coupled* PDEs were considered in the backstepping boundary control and observer design settings. The most intensive efforts of current literature were oriented towards coupled hyperbolic processes of the transport-type [2, 8, 9, 42, 43]. In [2], a 2×2 linear hyperbolic system was stabilized by a scalar observer-based output feedback boundary control input, with an additional feature that an unmatched disturbance, generated by an *a-priori* known exosystem, was rejected. In [42], a 2×2 system of coupled linear heterodirectional hyperbolic equations was stabilized by observer-based output feedback. The underlying design was extended in [8] to a particular type of 3×3 linear systems, arising in modeling of multi-phase flow, and to the quasilinear case in [43]. In [9], backstepping observer-based output feedback design was presented for a system of $n + 1$ coupled first-order linear heterodirectional hyperbolic PDEs (n of which featured rightward convecting transport, and one leftward) with a single boundary input.

Some specific results on the backstepping based boundary stabilization of parabolic coupled PDEs have additionally been presented in the literature [1, 35, 41, 39, 40]. In [35], two parabolic reaction-diffusion processes, coupled through the corresponding boundary conditions, were dealt with. The stabilization of the coupled equations was reformulated in terms of the stabilization problem for a unique process, which possessed piecewise-continuous diffusivity and (space-dependent) reaction coefficient and which was viewed as the "cascade" between the two original systems. The problem was then solved by using a scalar boundary control input and by employing a non conventional backstepping approach with a discontinuous kernel function. In [1], the Ginzburg-Landau equation with the imaginary and real parts expanded, thus being specified to a 2×2 parabolic system with equal diffusion coefficients, was dealt with. In [41], the linearized 2×2 model of thermal-fluid convection was treated by using a singular perturbations approach combined with backstepping and Fourier series expansion. In [40], the boundary stabilization of the linearized model of an incompressible magnetohydrodynamic flow in an infinite rectangular 3D channel, also recognized as Hartmann flow, was achieved by reducing the original system to a set of coupled diffusion equations with the same diffusivity parameter and by applying backstepping. In [39], an observer that estimated the velocity, pressure, electric potential and current fields in a Hartmann flow was presented where the observer gains were designed using multi-dimensional backstepping. In [24], a backstepping observer was designed for a system of two diffusion-convection-reaction processes coupled through the corresponding boundary conditions.

The recent authors‘ work [4], which appeared to be more closely related to the present investigation, dealt with the state feedback controller design for coupled reaction-diffusion processes equipped with Neumann (rather than Dirichlet) BCs. The same publication also addressed a state feedback stabilization problem for two coupled reaction-diffusion processes, which were underactuated by a scalar boundary input applied just to one of the processes.

1.2. Results and contributions of the paper. Thus motivated, the primary concern of this work is to extend the backstepping synthesis developed in [32], where explicit stabilizing output feedback boundary controllers were designed for scalar unstable reaction-diffusion processes with constant parameters. Here, a generalization is provided by considering a set of n reaction-diffusion processes, which are coupled through the corresponding reaction terms.

A constructive observer-based output feedback synthesis procedure, with the majority of controllers and observers given in explicit form, presents the main contribution of this work to the existing literature. Under the requirement that the considered multi-dimensional process is fully actuated by a set of n boundary control inputs acting on each subsystem, all these approaches are shown to exponentially stabilize the controlled system with an arbitrarily fast convergence rate. Particularly, in the present paper output feedback stabilizing controllers using both collocated and anti-collocated observers are presented.

The present treatment addresses, side by side, two distinct situations which require quite different solution approaches to be adopted. First, the case where all processes have the same diffusivity parameter ("equi-diffusivity" case) is attacked, and then the more challenging situation where each process possesses its own diffusivity ("distinct-diffusivity" case) is treated.

Particularly, the paper first investigates the class of coupled PDEs where all processes possess a Dirichlet-type boundary condition (BC) at the uncontrolled side. For this class of systems, the state feedback and observer-based output feedback designs (with the observers being developed in the anti-collocated and collocated forms) are successfully solved in both the equi-diffusivity and distinct-diffusivity scenarios, and, particularly, all controllers and observers are available in explicit form.

Successively, the analysis is extended towards the more general class of coupled PDEs where a subset of the processes have Dirichlet-type BCs whereas all remaining processes possess Neumann-type BCs. With reference to this wider class of systems with heterogenous BCs, the state feedback design is only solved in the equi-diffusivity case, although the resulting kernel matrix is no longer available in explicit form, whereas in the distinct diffusivity scenario an overdetermined kernel PDE admitting no solution is obtained. The anti-collocated observer and output feedback designs are additionally developed in the equi-diffusivity scenario. Interestingly, the observer gains are still available in explicit form also for this extended class of coupled PDEs. To the best of authors knowledge, coupled parabolic processes with different types of BCs are studied for the first time in the present paper within the backstepping-based boundary control design framework.

1.3. Organization. The structure of the paper is as follows. After introducing in Subsection 1.4 some notation and a useful Lemma, in Section 2 the problem statement for the class of coupled PDEs with Dirichlet BCs at the uncontrolled side is presented along with the associated assumptions. In Section 3, the resulting state feedback controller synthesis is developed. Sections 4 and 5 present, respectively, the anti-collocated and collocated observer designs. Section 6 develops the output feedback controller design by providing a demonstration of the stable coupling between the designed controllers and observers.

Section 7 deals with the generalized class of coupled PDEs with "heterogenous" BCs. The state feedback design is first addressed, showing that it can only be successfully completed within the equi-diffusivity scenario. Section 7 also contains the underlying anti-collocated observer and output feedback designs, both developed within the same equi-diffusivity scenario. Section 8 discusses some simulation results. Finally, Section 9 collects concluding remarks and features future perspectives of this research.

1.4. Notation and Instrumental Lemma. $L_2(0,1)$ stands for the Hilbert space of square integrable scalar functions $z(\zeta)$ on the domain $(0,1)$ with the corresponding L_2-norm

$$\|z(\cdot)\|_2 = \sqrt{\int_0^1 z^2(\zeta)d\zeta}. \tag{1.1}$$

$\mathrm{H}^\ell(0,1)$, with $\ell = 0,1,2,\ldots$, denotes the Sobolev space of absolutely continuous scalar functions $z(\zeta)$ on the domain $(0,1)$, with square integrable derivatives $z^{(k)}(\varsigma)$ up to order ℓ and the corresponding H^ℓ-norm

$$\|z(\cdot)\|_{\mathrm{H}^\ell} = \sqrt{\sum_{k=0}^{\ell} \|z^{(k)}(\cdot)\|_2^2}. \tag{1.2}$$

Also, the notations

$$L_2^n = \underbrace{L_2(0,1) \times L_2(0,1) \times \ldots \times L_2(0,1)}_{n \ \mathtt{times}}$$

and

$$H^{\ell,n} = \underbrace{H^\ell(0,1) \times H^\ell(0,1) \times \ldots \times H^\ell(0,1)}_{n \ \mathtt{times}}$$

are utilized and

$$\|Z(\cdot)\|_{2,n} = \sqrt{\sum_{i=1}^{n} \|z_i(\cdot)\|_2^2}, \tag{1.3}$$

$$\|W(\cdot)\|_{H^{\ell,n}} = \sqrt{\sum_{i=1}^{n} \|w_i(\cdot)\|^2_{H^\ell}} \tag{1.4}$$

stand, respectively, for the L_2-norm of a vector function $Z(\zeta) = [z_1(\zeta), z_2(\zeta), ..., z_n(\zeta)] \in L_2^n$ and for the Sobolev H^ℓ-norm of a vector function $W(\zeta) = [w_1(\zeta), w_2(\zeta), ..., w_n(\zeta)] \in H^{\ell,n}$.

Throughout, $I_1(\cdot)$ stands for the first order modified Bessel functions of the first kind, and $\mathcal{T}$ denotes the domain

$$\mathcal{T} = \{(x, y) \in \mathbb{R}^2 : 0 \le y \le x \le 1\}. \tag{1.5}$$

Given a generic real-valued square matrix A, the symbol $S[A]$ denotes its symmetric part $S[A] = (A + A^T)/2$. Provided that A is symmetric, the inequality $A > 0$ means that it is positive definite. Just in case, $\sigma_m(A)$ denotes the smallest eigenvalue of A.

Given a real-valued square matrix function $M(x)$ of order n, whose entries $m_{ij}(x)$ are defined on a set X, its $\mathcal{C}^0(X)$-norm is determined by

$$\|M(x)\|_{\mathcal{C}^0(X)} = \max_{i,j=1,2,...,n} \sup_{x\in X} |m_{ij}(x)|. \tag{1.6}$$

Finally, $I_{m\times m}$ stands for the identity matrix of dimension m.

For later use, an instrumental lemma is presented.

LEMMA 1.1. *(cf. [27, Lemma 2]) Let* $b(\zeta) \in \mathrm{L}_2(0, 1)$. *Then, the following inequality*

$$\left[\int_0^1 |b(\zeta)| d\zeta\right]^2 \le \|b(\cdot)\|_2^2 \tag{1.7}$$

holds.

2. Coupled PDEs with Dirichlet-type BCs. A system of n coupled reaction-diffusion processes, governed by the reaction-diffusion vector PDE

$$Q_t(x, t) = \Theta Q_{xx}(x, t) + \Lambda Q(x, t), \tag{2.1}$$

which is equipped with the BCs

$$Q(0, t) = 0, \tag{2.2}$$
$$Q(1, t) = U(t), \tag{2.3}$$

and subject to the initial condition (IC)

$$Q(x, 0) = Q_0(x) \in H^{4,n}, \tag{2.4}$$

is under investigation. Hereinafter,

$$Q(x, t) = [q_1(x, t),\ q_2(x, t), \ldots, q_n(x, t)]^T \in H^{4,n} \tag{2.5}$$

is the state vector,

$$U(t) = [u_1(t),\ u_2(t), \ldots, u_n(t)]^T \in \mathbb{R}^n \tag{2.6}$$

is the boundary control vector, $\Lambda \in \mathbb{R}^{n\times n}$ is a real-valued square matrix, and $\Theta = diag(\theta_i)$ is the diagonal diffusivity matrix with $\theta_i > 0$ for all $i = 1, 2, ..., n$.

To avoid imposing a restrictive compatibility condition on the initial function (2.4) to satisfy the BCs (2.2)-(2.3), solutions of the Boundary Value Problem (BVP) (2.1)-(2.3) (as well as solutions of any BVP to be used in the sequel) are viewed in the weak sense as those to the variational problem of finding a function $Q(x,t) \in H^{1,n}$ subject to the BCs (2.2)-(2.3) such that

$$\begin{aligned} \int_0^1 W^T(\xi) Q_t(\xi,t) d\xi &= W^T(1) Q_\xi(1,t) - W^T(0) Q_\xi(0,t) - \int_0^1 W_\xi^T(\xi) Q_\xi(\xi,t) d\xi \\ &+ \int_0^1 W^T(\xi) \Lambda Q(\xi,t) d\xi \end{aligned} \tag{2.7}$$

for any $t > 0$ and for any $W(\cdot) \in H^{1,n}$. Such a solution of (2.7), satisfying (2.2)-(2.3), is further referred to as a weak solution of the BVP (2.1)-(2.3) that has become standard in the literature.

If confined to a linear feedback input $U(\cdot)$, the closed-loop system (2.1)-(2.4) is well-known[1] to possess a unique weak solution of class $H^{\ell,n}$ with an arbitrarily large integer ℓ provided that the initial state is of the same class. For technical reasons, the weak solutions of (2.1)-(2.3) are required to evolve in the state space $H^{4,n}$ to guarantee that the corresponding second order spatial derivative evolves in the state space $H^{2,n}$. Due to this, the IC (2.4) has been pre-specified to belong to $H^{4,n}$.

The open-loop system (2.1)-(2.4) (with $U(t) = 0$) possesses arbitrarily many unstable eigenvalues whenever $S[\Lambda]$ has positive and sufficiently large eigenvalues. Since the term $\Lambda Q(x,t)$ is the source of such an instability, the problem then arises to exponentially stabilize the closed-loop system by "reshaping" this term via reversing its effect into a stabilizing one. This problem will be addressed under two distinct scenarios:

i.) *anti-collocated measurement setup*, where the only measurement of the flow $Q_x(0,t)$ is available at the uncontrolled boundary;

ii.) *collocated measurement setup*, where sensing of $Q_x(1,t)$ is available at the controlled boundary only.

To facilitate exposition the treatment is first addressed by deriving a stabilizing control law using the state feedback. Then the corresponding collocated and anti-collocated observers are designed. Finally, feeding the proposed state feedback controller with the state of such an observer, running in parallel, yields an output feedback stabilizing control law.

3. State-feedback controller design. The rationale of the backstepping state feedback boundary control design is to exponentially stabilize system (2.1)-(2.3) by exploiting an invertible transformation

$$Z(x,t) = Q(x,t) - \int_0^x K(x,y) Q(y,t) dy \tag{3.1}$$

with a $n \times n$ kernel matrix function $K(x,y)$. An appropriate choice of the kernel $K(x,y)$ and that of the state feedback input vector U allows one to transform the underlying closed-loop system into the target system

$$Z_t(x,t) = \Theta Z_{xx}(x,t) - C Z(x,t), \tag{3.2}$$

$$Z(0,t) = 0, \tag{3.3}$$

$$Z(1,t) = 0, \tag{3.4}$$

[1] See, e.g., [6] for the Fourier representation of such a solution similar to (3.9) used in the proof of Theorem 3.1.

written in terms of the state vector $Z(x,t) = [z_1(x,t),\, z_2(x,t), \dots, z_n(x,t)]^T$ with $C \in \mathbb{R}^{n\times n}$ being a design matrix parameter, subject to the IC

$$Z(x,0) = Q_0(x) - \int_0^x K(x,y)Q_0(y)dy, \tag{3.5}$$

which follows from (2.4) and (3.1). To ensure that an arbitrary weak solution of the target system BVP (3.2)-(3.5) evolves in the same state space $H^{4,n}$ it suffices to assume that the kernel matrix function $K(x,y)$ is smooth enough in its domain $\mathcal{T}$ defined in (1.5). The validity of this assumption is subsequently verified when the analytical representation of $K(x,y)$ is derived.

With the above consideration in mind, the exponential stability of the target system (3.2)-(3.5) is then ensured with an arbitrarily fast convergence rate by an appropriate choice of the real-valued square matrix $C \in \mathbb{R}^{n\times n}$. The following result is in order.

THEOREM 3.1. *Let matrix C be such that $S[C] > 0$. Then, system* (3.2)-(3.5) *with the IC* (3.5) *in $H^{4,n}$ is exponentially stable in the space $H^{2,n}$ with the decay rate $\sigma_m(S[C])$ so that*

$$\|Z(\cdot,t)\|_{H^{2,n}} \le \|Z(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[C])t}. \tag{3.6}$$

Additionally, the following point-wise estimates

$$\max_{x\in[0,1]} |z_i(x,t)| \le \sqrt{2}\|Z(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[C])t}, \quad i = 1,2,...,n \tag{3.7}$$

$$\max_{x\in[0,1]} |z_{ix}(x,t)| \le \sqrt{2}\|Z(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[C])t}, \quad i = 1,2,...,n \tag{3.8}$$

are in force, where $z_{ix}(x,t)$ denotes the i-th element of $Z_x(x,t)$.

Proof. To begin with, let us note that under the conditions of the theorem a weak solution $Z(x,t)$ of (3.2)-(3.5) admits a Fourier representation

$$Z(x,t) = \sum_{k=1}^{\infty} Z_k(t)\sin(\pi k x), \tag{3.9}$$

where $Z_k(t),\ k = 1,2,\dots$, is a solution of the ODE $\dot{Z}_k = -[(\pi k)^2\Theta + C]Z_k$ (see, e.g., [6, 16] for details). It is then straightforward to verify that the spatial derivatives $Z_x(x,t)$ and $Z_{xx}(x,t)$ constitute weak solutions of the BVPs

$$\begin{aligned} Z_{tx}(x,t) &= \Theta Z_{xxx}(x,t) - CZ_x(x,t), &(3.10)\\ Z_{txx}(x,t) &= \Theta Z_{xxxx}(x,t) - CZ_{xx}(x,t), &(3.11)\\ Z_{xx}(0,t) &= Z_{xx}(1,t) = 0, &(3.12)\end{aligned}$$

inherited from (3.2)-(3.4). Remarkably, the same BCs (3.12) are of Neumann type for the PDE (3.10) in Z_x, and of Dirichlet type for the PDE (3.11) in Z_{xx}.

Taking this into account, let us now consider the Lyapunov functional

$$\begin{aligned} V(t) &= \frac{1}{2}\|Z(\cdot,t)\|^2_{H^{2,n}} = \frac{1}{2}\int_0^1 Z^T(\xi,t)Z(\xi,t)d\xi + \frac{1}{2}\int_0^1 Z_\xi^T(\xi,t)Z_\xi(\xi,t)d\xi \\ &+ \frac{1}{2}\int_0^1 Z_{\xi\xi}^T(\xi,t)Z_{\xi\xi}(\xi,t)d\xi. \end{aligned} \tag{3.13}$$

In light of (3.10)-(3.11), the corresponding time derivative of the Lyapunov functional (3.13) along the solutions of (3.2)-(3.4) and (3.10)-(3.12) is given by

$$\begin{aligned}\dot{V}(t) = \int_0^1 Z^T(\xi,t)\Theta Z_{\xi\xi}(\xi,t)d\xi - \int_0^1 Z^T(\xi,t)CZ(\xi,t)d\xi + \int_0^1 Z_\xi^T(\xi,t)\Theta Z_{\xi\xi\xi}(\xi,t)d\xi \\ - \int_0^1 Z_\xi^T(\xi,t)CZ_\xi(\xi,t)d\xi + \int_0^1 Z_{\xi\xi}^T(\xi,t)\Theta Z_{\xi\xi\xi\xi}(\xi,t)d\xi - \int_0^1 Z_{\xi\xi}^T(\xi,t)CZ_{\xi\xi}(\xi,t)d\xi.\end{aligned} \tag{3.14}$$

The first integral term in the right hand side of equality (3.14), being integrated by parts, is estimated as

$$\begin{aligned}\int_0^1 Z^T(\xi,t)\Theta Z_{\xi\xi}(\xi,t)d\xi &= Z^T(\chi,t)\Theta Z_x(\chi,t)\big|_{\chi=0}^{\chi=1} - \int_0^1 Z_\xi^T(\xi,t)\Theta Z_\xi(\xi,t)d\xi \\ &\le -\theta_m\|Z_x(\cdot,t)\|_{2,n}^2,\end{aligned} \tag{3.15}$$

where relations (3.3), (3.4) and the diagonal form of matrix Θ have been taken into account, and the notation $\theta_m = \min_{1\le i\le n}\theta_i > 0$ has been used. Following the same route, the third and fifth integral terms in the right hand side of (3.14) are estimated as

$$\int_0^1 Z_\xi^T(\xi,t)\Theta Z_{\xi\xi\xi}(\xi,t)d\xi \le -\theta_m\|Z_{xx}(\cdot,t)\|_{2,n}^2, \tag{3.16}$$

$$\int_0^1 Z_{\xi\xi}^T(\xi,t)\Theta Z_{\xi\xi\xi\xi}(\xi,t)d\xi \le -\theta_m\|Z_{xxx}(\cdot,t)\|_{2,n}^2, \tag{3.17}$$

where the BCs (3.12) have been used. To manage the remaining integral terms in the right hand side of (3.14), the well-known property

$$\zeta^T C\zeta \ge \sigma_m(S[C])\zeta^T\zeta \tag{3.18}$$

of the quadratic form $\zeta^T C\zeta$ is exploited with the matrix C, whose symmetric part is positive definite by assumption, and an arbitrary n-dimensional vector ζ. Substituting (3.15)-(3.17) into (3.14), one readily obtains

$$\dot{V}(t) \le -\theta_m\|Z_\xi(\cdot,t)\|_{H^{2,n}}^2 - 2\sigma_m(S[C])V(t) \le -2\sigma_m(S[C])V(t) \tag{3.19}$$

by applying straightforward manipulations, made according to (3.18). By definition of the Lyapunov functional (3.13), relation (3.19) ensures the exponential stability of the target system (3.2)-(3.4) in the space $H^{2,n}$ with the decay rate obeying the estimate (3.6).

It remains to establish the point-wise estimates (3.7) and (3.8). For this purpose, let us note that relation (3.6) remains in force in the component-wise form

$$\|z_i(\cdot,t)\|_{H^2} \le \|Z(\cdot,0)\|_{H^{2,n}}e^{-\sigma_m(S[C])t}, \quad i=1,2,\dots,n \tag{3.20}$$

and due to the trivial inequalities $\|z_i(\cdot,t)\|_2 \le \|z_i(\cdot,t)\|_{H^2}$, $\|z_{ix}(\cdot,t)\|_2 \le \|z_i(\cdot,t)\|_{H^2}$, the next estimates

$$\|z_i(\cdot,t)\|_2 \le \|Z(\cdot,0)\|_{H^{2,n}}e^{-\sigma_m(S[C])t}, \ \|z_{ix}(\cdot,t)\|_2 \le \|Z(\cdot,0)\|_{H^{2,n}}e^{-\sigma_m(S[C])t} \tag{3.21}$$

are in force as well. The point-wise estimate (3.7) is then trivially derived from that obtained by employing Agmon's inequality and utilizing the estimates (3.21):

$$\max_{x\in[0,1]} z_i^2(x,t) \le 2\|z_i(\cdot,t)\|_2\|z_{ix}(\cdot,t)\|_2 \le 2\|Z(\cdot,0)\|_{H^{2,n}}^2 e^{-2\sigma_m(S[C])t}, \ i=1,2,\dots,n. \tag{3.22}$$

To prove (3.8) let us consider an arbitrary *constant* $\bar{x} \in [0,1]$ and write down the trivial relation

$$z_{ix}(\bar{x},t) = z_{ix}(x,t) - \int_{\bar{x}}^{x} z_{i\xi\xi}(\xi,t)d\xi, \quad \bar{x} \in [0,1], \quad i = 1,2,...,n \qquad (3.23)$$

where $z_{ix}(\cdot)$ and $z_{ixx}(\cdot)$ denote the i-th element of vectors $Z_x(\cdot)$ and $Z_{xx}(\cdot)$. Squaring both sides of (3.23) and applying the triangle inequality yield

$$z_{ix}^2(\bar{x},t) \leq 2z_{ix}^2(x,t) + 2\left[\int_{\bar{x}}^{x} z_{i\xi\xi}(\xi,t)d\xi\right]^2, \quad \bar{x} \in [0,1], \quad i = 1,2,...,n. (3.24)$$

By virtue of Lemma 1.1, specified with $b(\cdot) = z_{i\xi\xi}(\cdot)$, the chain of inequalities

$$\begin{aligned} z_{ix}^2(\bar{x},t) &\leq 2z_{ix}^2(x,t) + 2\left[\int_0^1 |z_{i\xi\xi}(\xi,t)|d\xi\right]^2 \\ &\leq 2z_{ix}^2(x,t) + 2\|z_{ixx}(\cdot,t)\|_2^2, \quad \bar{x} \in [0,1], \quad i = 1,2,...,n \qquad (3.25) \end{aligned}$$

is derived from (3.24). Then by integrating both sides of (3.25) with respect to the spatial variable x from 0 to 1 and by exploiting relation (3.20), one gets

$$\begin{aligned} z_{ix}^2(\bar{x},t) \leq 2\|z_{ix}(\cdot,t)\|_2^2 + 2\|z_{ixx}(\cdot,t)\|_2^2 \leq 2\|z_i(\cdot,t)\|_{H^2} \leq 2\|Z(\cdot,0)\|_{H^{2,n}}^2 e^{-2\sigma_m(S[C])t} \\ \bar{x} \in [0,1], \quad i = 1,2,...,n. \qquad (3.26) \end{aligned}$$

By noticing that $\bar{x}$ is an arbitrarily chosen point in the interval [0,1], the point-wise estimate (3.8) is straightforwardly concluded from (3.26). The proof of Theorem 3.1 is thus completed. □

REMARK 1. *It should be pointed out that relations* (3.7) *and* (3.8) *do not truly establish the exponential point-wise decay of $z_i(x,t)$ and $z_{ix}(x,t)$ due to the fact that $\|Z(\cdot,0)\|_{H^{2,n}}$, rather than $|z_i(x,0)|$ and, respectively, $|z_{ix}(x,0)|$, appears in the corresponding right-hand sides of these relations. However, such "quasi-exponential" decays prove to be suitable for establishing the exponential stability of the original system (2.1)-(2.3) in the space $H^{2,n}$ under the output feedback boundary controller to subsequently be designed.*

The BVP governing the kernel matrix function $K(x,y)$ is now derived through the standard procedure adopted in the backstepping design [17]. Next developments closely follow our recent works [3, 4], where the same analysis were conducted for coupled reaction diffusion equations equipped with Neumann rather than Dirichlet BCs.

By applying the Leibnitz differentiation rule to (3.1), spatial derivatives $Z_x(x,t)$ and $Z_{xx}(x,t)$ are readily developed as a straightforward matrix generalization of corresponding well-known scalar counterparts. Furthermore, using (2.1) and applying recursively integration by parts, the time derivative $Z_t(x,t)$ is derived as well. Combining such expressions and performing rather lengthy but straightforward computations (see [3] for more detailed derivations) yield

$$\begin{aligned} &Z_t(x,t) - \Theta Z_{xx}(x,t) + CZ(x,t) \\ &= \left[\Lambda + C + K_y(x,x)\Theta + \Theta K_x(x,x) + \Theta\frac{d}{dx}K(x,x)\right]Q(x,t) \\ &+ \int_0^x \left[\Theta K_{xx}(x,y) - K_{yy}(x,y)\Theta - K(x,y)\Lambda - CK(x,y)\right]Q(y,t)dy \\ &+ \left[\Theta K(x,x) - K(x,x)\Theta\right]Q_x(x,t) + K(x,0)\Theta Q_x(0,t) - K_y(x,0)\Theta Q(0,t). \qquad (3.27) \end{aligned}$$

Clearly, the target system PDE (3.2) requires the right hand side of (3.27) to be identically zero. Considering the homogeneous BC (2.2), this leads to the next relations

$$\begin{aligned}
\Theta K_{xx}(x,y) &- K_{yy}(x,y)\Theta = K(x,y)\Lambda + CK(x,y), && (3.28)\\
\Lambda + C &+ K_y(x,x)\Theta + \Theta K_x(x,x) + \Theta \frac{d}{dx}K(x,x) = 0, && (3.29)\\
\Theta K(x,x) &- K(x,x)\Theta = 0, && (3.30)\\
K(x,0) &= 0. && (3.31)
\end{aligned}$$

As in the Neumann BCs case [3], the main critical feature of (3.28)-(3.31) is in the presence of relation (3.30). While being identically satisfied in the scalar case ($n = 1$) [31], this relation is generally contradictive, and there are two options to fulfill (3.30). One of these options is to impose the constraint that all the coupled processes possess the same diffusivity value θ, i.e.,

$$\Theta = \theta I_{n\times n}. \tag{3.32}$$

An alternative option is to enforce the next constraint

$$K(x,y) = k(x,y)I_{n\times n} \tag{3.33}$$

on the form of the kernel matrix. Assumption (3.33) greatly simplifies the complexity of the underlying backstepping transformation, which is simply determined by a scalar function. This simplification, however, will also bring some constraint on the choice of the matrix C which is no longer an arbitrary design parameter when the relation (3.33) is in force. The above arguments motivate the need of treating separately the equi-diffusivity case, where constraint (3.32) is in force, and the distinct diffusivity case where the kernel matrix is subject to the constraint (3.33).

3.1. Equi-diffusivity case. Specializing system (3.28), (3.29), (3.31) in light of the equi-diffusivity constraint (3.32) and exploiting the identity $\frac{d}{dx}K(x,x) = K_x(x,x)+K_y(x,x)$ yield the BVP

$$\begin{aligned}
K_{xx}(x,y) &- K_{yy}(x,y) = \frac{1}{\theta}K(x,y)\Lambda + \frac{1}{\theta}CK(x,y), && (3.34)\\
& \Lambda + C + 2\theta\frac{d}{dx}K(x,x) = 0, && (3.35)\\
& K(x,0) = 0. && (3.36)
\end{aligned}$$

Integrating (3.35) with respect to x gives $K(x,x) = -\frac{1}{2\theta}(\Lambda + C)x + K(0,0)$. It follows from (3.36) that $K(0,0) = 0$, hence relation (3.35) is replaced by

$$K(x,x) = -\frac{1}{2\theta}(\Lambda + C)x. \tag{3.37}$$

The following result is in order.

THEOREM 3.2. *The boundary-value problem* (3.34)*,* (3.36)*,* (3.37) *possesses a solution*

$$K(x,y) = -\sum_{j=0}^{\infty} \frac{(x^2-y^2)^j(2y)}{j!(j+1)!}\left(\frac{1}{4\theta}\right)^{j+1}\left[\sum_{i=0}^{j}\binom{j}{i}C^i\,(\mathit{\Lambda}+C)\,\mathit{\Lambda}^{j-i}\right] \tag{3.38}$$

which is of class $\mathcal{C}^\infty$ in the domain $\mathcal{T}$ defined in (1.5)*.*

Proof. The proof of the present theorem follows the same line of reasoning as that of [3, Th. 1], where the Neumann BCs were in play. Therefore, the detailed proof can straighforwardly be derived from the proof of [3, Th. 1]. □

REMARK 2. *Uniqueness of a solution to some BVPs, similar to* (3.34)*,* (3.36)*,* (3.37)*, has been addressed in the literature (see, e.g., [31, 11]). This valuable issue does not, however, affect the underlying synthesis and it therefore remains beyond the scope of the paper.*

The designed state feedback boundary controller for the equi-diffusivity case takes the form

$$U(t) = \int_0^1 K(1,y)Q(y,t)dy, \tag{3.39}$$

$$K(1,y) = -\sum_{n=0}^{\infty}\left[\frac{2y(1-y^2)^n}{n!(n+1)!}\right]\left(\frac{1}{4\theta}\right)^{n+1}\left[\sum_{i=0}^{n}\binom{n}{i}C^i\left(\Lambda+C\right)\Lambda^{n-i}\right]. \tag{3.40}$$

The following result is in order:

THEOREM 3.3. *Let matrix C be selected in such a manner that $S[C]>0$ and $\sigma_m(S[C])$ is arbitrarily large. Then, the boundary control input* (3.39)-(3.40) *exponentially stabilizes system (2.1)-(2.3) in the space $H^{2,n}$ with the corresponding norm obeying the estimate*

$$\|Q(\cdot,t)\|_{H^{2,n}} \le a\|Q(\cdot,0)\|_{H^{2,n}}e^{-\sigma_m(S[C])t}, \tag{3.41}$$

where a is a positive constant independent of $Q(x,0)$.

Proof. See Subsection 3.1.2 □

REMARK 3. *The plant IC at $x=0$ should meet, of course, the corresponding BC* (2.2) *to avoid the resulting incompatibility of the IC and BC. The additional, and more critical, compatibility issue arises that the plant IC at $x=1$ must match the initial value $U(0)$ of the applied control law in order to avoid initial jumps of the boundary state. The assumption on the IC $Q_0(x)$, needed to solve such an issue, would be certainly restrictive in applications. However, an alternative approach is feasible, as done, e.g., in [43], through adding an appropriate, exponentially vanishing, extra term in the control law* (3.39)-(3.40)*. The next modified control law*

$$U(t) = \int_0^1 K(1,y)Q(y,t)dy + \left[Q_0(1) - \int_0^1 K(1,y)Q_0(y)dy\right]e^{-\gamma t}, \quad \gamma > 0 \tag{3.42}$$

is capable of solving the aforementioned compatibility issue, yielding the required compatibility condition $Q_0(1)=U(0)$. Clearly, the exponential convergence condition (3.41) *is no longer in force due to this modification, and lengthy steps of analysis are needed to assess the resulting closed-loop performance (see [43]). This analysis, however, remains out of the scope of this paper. In the simulation section we will show that adding such an extra term one not only removes the initial jump of the closed-loop system boundary state but one also alleviates the transient peaking as well.*

3.1.1. Inverse transformation and stability issues. Relevant results, concerning the invertibility of the backstepping transformation (3.1) and the smoothness of the inverse kernel matrix, are collected in this subsection to be used in the proof of Theorem 3.3.

Transformation (3.1) is a matrix Volterra integral equation. We look for an inverse transformation in the form

$$Q(x,t) = Z(x,t) + \int_0^x L(x,y)Z(y,t)dy. \tag{3.43}$$

Existence and smoothness properties of $L(x,y)$ are investigated in the following lemma (see, e.g., [9, 38] for the scalar case which is going to be extended to the present vector case).

The following lemma is in order.

LEMMA 3.4. *There exist a kernel matrix $L(x,y)$, of class $\mathcal{C}^\infty(\mathcal{T})$ with the domain $\mathcal{T}$ specified in* (1.5)*, such that the inverse transformation* (3.43) *of* (3.1) *is in force.*

Proof. Substituting (3.1) into (3.43) and performing straightforward manipulations, one derives the integral equation

$$L(x,y) = K(x,y) + \int_y^x L(x,s)K(s,y)ds, \tag{3.44}$$

that implicitly defines the inverse kernel matrix $L(x,y)$ on $\mathcal{T}$. The method of successive approximations is going to be applied to show that a smooth solution to (3.44) exists. Let us start with the initial guess $L^0(x,y) = 0$ and construct the recursive formula

$$L^{j+1}(x,y) = K(x,y) + \int_y^x L^j(x,s)K(s,y)ds, \quad j = 0,1,2,\dots \tag{3.45}$$

Let us denote the difference between two consecutive terms as

$$\Delta L^j(x,y) \quad = \quad L^{j+1}(x,y) - L^j(x,y), \quad j = 0,1,2,\dots \tag{3.46}$$

Then, the next recursion is obtained by (3.45)

$$\Delta L^0(x,y) \quad = \quad L^1(x,y) = K(x,y), \tag{3.47}$$

$$\Delta L^{j+1}(x,y) \quad = \quad \int_y^x \Delta L^j(x,s)K(s,y)ds, \quad j = 0,1,2,\dots \tag{3.48}$$

If the recursion (3.47)-(3.48) converges, a solution $L(x,y)$ to (3.44) takes the form

$$L(x,y) = \sum_{j=0}^{\infty} \Delta L^j(x,y). \tag{3.49}$$

The kernel matrix $K(x,y)$ is continuous (cf. Theorem 3.2), hence its $\mathcal{C}^0$-norm (1.6) admits a uniform upperbound in the compact set $\mathcal{T}$. It means that there exists a positive constant M such that

$$\|\Delta L^0(x,y)\|_{\mathcal{C}^0(\mathcal{T})} = \|K(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M. \tag{3.50}$$

Suppose that

$$\|\Delta L^j(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M^{j+1}\frac{x^j}{j!}. \tag{3.51}$$

Then, by (3.48), (3.50) and (3.51) one derives the next estimate

$$\begin{aligned}\|\Delta L^{j+1}(x,y)\|_{\mathcal{C}^0(\mathcal{T})} &\leq \left|\int_y^x \|\Delta L^{j+1}(x,s)\|_{\mathcal{C}^0(\mathcal{T})}\|K(s,y)\|_{\mathcal{C}^0(\mathcal{T})}ds\right| \\ &\leq \frac{M^{j+2}}{j!}x^j\left|\int_y^x ds\right| = \frac{M^{j+2}}{j!}x^j\,|x-y|\,. \end{aligned} \tag{3.52}$$

Due to the inequalities $0 \leq y \leq x$, which come from the domain definition (1.5), the following estimate

$$x^j\,|x-y| \leq x^{j+1}, \quad (x,y) \in \mathcal{T}, \tag{3.53}$$

is in force. Therefore, combining (3.52) and (3.53), one gets

$$\|\Delta L^{j+1}(x,y)\| \leq M^{j+2}\frac{x^{j+1}}{(j+1)!}. \tag{3.54}$$

By mathematical induction, (3.54) is true for all $j \geq 0$. It then follows from the Weierstrass M-test that the series (3.49) converges absolutely and uniformly in $\mathcal{T}$. Thus, (3.47)-(3.49) is a solution to (3.44) and it thereby implements the inverse transformation (3.43).

Due to the fact that $K(x,y)$ is of class $\mathcal{C}^\infty(\mathcal{T})$, the integral equation (3.44) allows one to conclude that the function $L(x,y)$ is continuous and at least one time continuously differentiable in the domain $\mathcal{T}$. By iterating on the successive differentiation of (3.44), one further derives that $L(x,y)$ is of class $\mathcal{C}^\infty$ in its domain. Lemma 3.4 is proven.
□

By generalizing [34, Th 2.3], it is now proved that the properties $K(x,y) \in \mathcal{C}^\infty(\mathcal{T})$ and $L(x,y) \in \mathcal{C}^\infty(\mathcal{T})$ result in the equivalence of the norms of $Z(x,t)$ and $Q(x,t)$ in $H^{2,n}$.

LEMMA 3.5. *Consider the direct and inverse backstepping transformations* (3.1) *and* (3.43) *with the associated kernel matrices* $K(x,y), L(x,y) \in \mathcal{C}^\infty(\mathcal{T})$ *on the domain* $\mathcal{T}$ *defined in* (1.5). *Then, there are positive constants* b_1 *and* b_2 *such that*

$$\|Q(\cdot,t)\|_{H^{2,n}} \leq b_1\|Z(\cdot,t)\|_{H^{2,n}}, \tag{3.55}$$

$$\|Z(\cdot,t)\|_{H^{2,n}} \leq b_2\|Q(\cdot,t)\|_{H^{2,n}}. \tag{3.56}$$

Proof. To begin with, one notices that properties $K(x,y) \in \mathcal{C}^\infty(\mathcal{T})$ and $L(x,y) \in \mathcal{C}^\infty(\mathcal{T})$ guarantee the existence of positive constants $M_1, M_2, ..., M_8$ such that

$$\|K(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M_1, \qquad \|L(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M_2, \tag{3.57}$$

$$\|K_x(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M_3, \qquad \|L_x(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M_4, \tag{3.58}$$

$$\|K_y(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M_5, \qquad \|L_y(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M_6, \tag{3.59}$$

$$\|K_{xx}(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M_7, \qquad \|L_{xx}(x,y)\|_{\mathcal{C}^0(\mathcal{T})} \leq M_8. \tag{3.60}$$

From relation (3.43) one concludes, after straightforward manipulations (similar to those of [34, Th 2.3]), that

$$\|Q(\cdot,t)\|_{2,n} \leq \|Z(\cdot,t)\|_{2,n} + \|L(x,y)\|_{\mathcal{C}^0(\mathcal{T})}\|Z(\cdot,t)\|_{2,n} \leq (1+M_2)\|Z(\cdot,t)\|_{2,n}. \tag{3.61}$$

Spatial derivatives $Q_x(x,t)$ and $Q_{xx}(x,t)$ are computed by iteratively applying the Leibnitz differentiation rule to (3.43). It yields

$$Q_x(x,t) = Z_x(x,t) + L(x,x)Z(x,t) + \int_0^x L_x(x,y)Z(y,t)dy, \tag{3.62}$$

$$\begin{aligned} Q_{xx}(x,t) &= Z_{xx}(x,t) + \left[\frac{d}{dx}L(x,x)\right] Z(x,t) + L(x,x)Z_x(x,t) + L_x(x,x)Z(x,t) \\ &+ \int_0^x L_{xx}(x,y)Z(y,t)dy. \end{aligned} \tag{3.63}$$

It is concluded from (3.62), (3.57) and (3.58) that

$$\begin{aligned} \|Q_x(\cdot,t)\|_{2,n} &\leq \|Z_x(\cdot,t)\|_{2,n} + \|L(x,x)\|_{\mathcal{C}^0(\mathcal{T})}\|Z(\cdot,t)\|_{2,n} + \|L_x(x,y)\|_{\mathcal{C}^0(\mathcal{T})}\|Z(\cdot,t)\|_{2,n} \\ &\leq \|Z_x(\cdot,t)\|_{2,n} + (M_2+M_4)\|Z(\cdot,t)\|_{2,n}. \end{aligned} \tag{3.64}$$

By applying a similar estimation to (3.63), and noticing that by (3.58) and (3.59) the relation

$$\begin{aligned} \left\|\frac{d}{dx}L(x,x)\right\|_{\mathcal{C}^0(\mathcal{T})} &= \|L_x(x,x)+L_y(x,x)\|_{\mathcal{C}^0(\mathcal{T})} \leq \|L_x(x,x)\|_{\mathcal{C}^0(\mathcal{T})} + \|L_y(x,x)\|_{\mathcal{C}^0(\mathcal{T})} \\ &\leq M_4 + M_6 \end{aligned} \tag{3.65}$$

is in force, one straightforwardly concludes that

$$\|Q_{xx}(\cdot,t)\|_{2,n} \leq \|Z_{xx}(\cdot,t)\|_{2,n} + M_2\|Z_x(\cdot,t)\|_{2,n} + (2M_4+M_6+M_8)\|Z(\cdot,t)\|_{2,n}. \tag{3.66}$$

From (3.61), (3.64) and (3.66) one gets (3.55) with the constant $b_1 = 1 + 2M_2 + 3M_4 + M_6 + M_8\}$. Relation (3.56) is obtained by applying the similar analysis starting from relation (3.1). This concludes the proof of Lemma 3.5.
□

3.1.2. Proof of Theorem 3.3. The backstepping transformation (3.1), (3.38) was derived to map system (2.1)-(2.3) into the target dynamics governed by the PDE (3.2). It remains to prove that the homogenous BCs (3.3)-(3.4) hold as well. Specifying (3.1) with $x = 0$ and $x = 1$, and considering (2.2) and (2.3), yield

$$Z(0,t) = Q(0,t) = 0, \tag{3.67}$$

$$Z(1,t) = Q(1,t) - \int_0^1 K(1,y)Q(y,t)dy = U(t) - \int_0^1 K(1,y)Q(y,t)dy. \tag{3.68}$$

Thus, the boundary control input vector (3.39)-(3.40), where the kernel $K(1,y)$ is readily obtained by specifying (3.38) for $x = 1$, results in the target BVP (3.2)-(3.4) with homogeneous BCs. The exponential stability of (3.2)-(3.4) in the space $H^{2,n}$ was established in Theorem 3.1 provided that $S[C] > 0$. Particularly, relation (3.6) was proven. Coupling (3.6) and (3.55), one derives that

$$\|Q(\cdot,t)\|_{H^{2,n}} \leq b_1\|Z(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[C])t}. \tag{3.69}$$

Specifying (3.56) with $t = 0$ and substituting the resulting relation in (3.69), one obtains (3.41) with the constant $a = b_1 b_2$ which is independent of $Q(x,0)$. This completes the proof of Theorem 3.3.

3.2. Distinct diffusivity case. In the present subsection, boundary stabilization of system (2.1)-(2.3) with distinct diffusivity parameters is addressed by following the previously introduced backstepping design, specified with (3.33). Specializing system (3.28), (3.29), (3.31) in view of the constraint (3.33) on the kernel matrix yields

$$(k_{xx}(x,y) - k_{yy}(x,y))\Theta = k(x,y)(\Lambda + C), \tag{3.70}$$

$$2\frac{d}{dx}k(x,x)\Theta = -(\Lambda + C), \tag{3.71}$$

$$k(x,0) = 0. \tag{3.72}$$

By following [4, Sect. 4], where system (2.1), equipped with homogeneous Neumann BCs, was under investigation, one concludes that to guarantee the solvability of (3.70)-(3.72) the matrix C has to be selected in the constrained form

$$C = -\Lambda + \gamma^* \Theta, \tag{3.73}$$

where γ^* is a scalar parameter.

REMARK 4. *It was proven in [4, Th. 4] that one can always select the parameter γ^* in* (3.73) *large enough such that $S[C] > 0$ and $\sigma_m(S[C])$ is arbitrarily large as well.*

By substituting (3.73) into (3.70) and (3.71), and by performing straightforward manipulations, the kernel function $k(x, y)$ is shown to be a solution to the following BVP

$$\begin{aligned} k_{xx}(x,y) - k_{yy}(x,y) &= \gamma^* k(x,y), && (3.74) \\ k(x,x) = &= -\frac{\gamma^*}{2} x, && (3.75) \\ k(x,0) &= 0, && (3.76) \end{aligned}$$

whose explicit representation

$$k(x,y) = -\gamma^* y \frac{I_1(\sqrt{\gamma^*(x^2 - y^2)})}{\sqrt{\gamma^*(x^2 - y^2)}} \tag{3.77}$$

is extracted from [17].

REMARK 5. *The BVP* (3.74)-(3.76) *is a particular case of* (3.34)*,* (3.36)*,* (3.37)*. It thus follows from Theorem 3.2 that $k(x, y)$ is of class $\mathcal{C}^\infty(\mathcal{T})$ with $\mathcal{T}$ defined in* (1.5)*. Clearly, the inverse transformation of* (3.1)*,* (3.33) *takes the form* (3.43) *specified with $L(x, y) = l(x, y)I_{n\times n}$. By Lemma 3.4 one concludes that $l(x, y)$ is of class $\mathcal{C}^\infty(\mathcal{T})$, too.*

The next theorem specifies the proposed state feedback boundary control design for the distinct diffusivity case.

THEOREM 3.6. *Let matrix C be selected according to* (3.73) *with sufficiently large parameter $\gamma^* > 0$ to ensure that $S[C] > 0$ and $\sigma_m(S[C])$ is arbitrarily large. Then, the boundary control input*

$$U(t) = \int_0^1 k(1,y) Q(y,t) dy, \qquad k(1,y) = -\gamma^* y \frac{I_1(\sqrt{\gamma^*(1 - y^2)})}{\sqrt{\gamma^*(1 - y^2)}} \tag{3.78}$$

exponentially stabilizes system (2.1)-(2.3) in the space $H^{2,n}$ with decay rate

$$\|Q(\cdot,t)\|_{H^{2,n}} \le a \|Q(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[C])t}, \tag{3.79}$$

where a is a positive constant independent of $Q(x, 0)$.

Proof. The form (3.78) of the chosen boundary feedback control is readily justified by specifying (3.67) and (3.68) with (3.33) and noticing that $k(1, y)$ is obtained by specifying (3.77) with $x = 1$. Thus, with the feedback law (3.78) system (2.1)-(2.3) is transferred by (3.1), (3.33) into the target dynamics (3.2)-(3.4) with the matrix C given by (3.73). According to Remark 4, one can always select a large enough parameter γ^* such that $S[C] > 0$ and $\sigma_m(S[C])$ is arbitrarily large. Provided that $S[C] > 0$, the stability of the target dynamics (3.2)-(3.4) in the space $H^{2,n}$ was established in Theorem 3.1. Noticing that Lemma 3.5 is still in force due to Remark 5, the rest of the proof follows that of Theorem 3.3. The stability of the original system (2.1)-(2.3) is then established in the space $H^{2,n}$ by employing (3.79) with the same constant $a = b_1 b_2$. Theorem 3.6 is thus proved. □

4. Observer design for the anti-collocated measurement setup. For system (2.1)-(2.3) of n coupled reaction-diffusion processes with the boundary flow $Q_x(0,t)$ being the only available measurement, the state observer

$$\begin{aligned} \hat{Q}_t(x,t) &= \Theta\hat{Q}_{xx}(x,t) + \Lambda\hat{Q}(x,t) + G(x)\left[Q_x(0,t) - \hat{Q}_x(0,t)\right], & (4.1)\\ \hat{Q}(0,t) &= 0, & (4.2)\\ \hat{Q}(1,t) &= U(t), & (4.3)\\ \hat{Q}(x,0) &= \hat{Q}_0(x) \in H^{4,n} & (4.4) \end{aligned}$$

is proposed, where $\hat{Q}(x,t)$ is the observer state and $G(x)$ is a square matrix of spatially-dependent observer gains to subsequently be designed. The observer is equipped with analogous BCs as those of the original system (2.2)-(2.3). The meaning of the BVP (4.1)-(4.4) is viewed in the weak sense as the system (2.1)-(2.4) is. In order to ensure that the weak solutions of (4.1)-(4.4) evolve in the state space $H^{4,n}$ the IC (4.4) is pre-specified to belong to $H^{4,n}$.

REMARK 6. *As in the state feedback design, initial jumps of the observer state occur if the observer ICs $\hat{Q}_0(x)$ at $x=0$ and $x=1$ do not match the corresponding BCs* (4.2) *and the initial value $U(0)$, resulting from* (4.3)*. The assumption*

$$\hat{Q}_0(1) = U(0) \tag{4.5}$$

on the IC $\hat{Q}_0(x)$ now becomes less restrictive as the observer IC can be set arbitrarily. However, as in Remark 3, an alternative approach can be invoked by adding an appropriate, exponentially vanishing, extra term in the observer dynamics. More precisely, the following modification to eq. (4.3)

$$\hat{Q}(1,t) = U(t) + \left[\hat{Q}_0(1) - U(t)\right]e^{-\beta t}, \quad \beta > 0, \tag{4.6}$$

can solve the mentioned compatibility issue without requiring to put any restriction on the observer IC at $x=1$. Note that a completely analogous modification will also be effective to solve the same compatibility issue for the collocated observer design.

Introduce the estimation error variable

$$\tilde{Q}(x,t) = Q(x,t) - \hat{Q}(x,t), \tag{4.7}$$

and consider the associated BVP

$$\begin{aligned} \tilde{Q}_t(x,t) &= \Theta\tilde{Q}_{xx}(x,t) + \Lambda\tilde{Q}(x,t) - G(x)\tilde{Q}_x(0,t), & (4.8)\\ \tilde{Q}(0,t) &= 0, & (4.9)\\ \tilde{Q}(1,t) &= 0, & (4.10)\\ \tilde{Q}(x,0) &= Q_0(x) - \hat{Q}_0(x), & (4.11) \end{aligned}$$

which is readily derived from (2.1)-(2.4) and (4.1)-(4.4).

To design the observer gain matrix $G(x)$ the backstepping approach is involved for finding out the conditions under which an invertible transformation

$$\tilde{Q}(x,t) = \tilde{Z}(x,t) - \int_0^x P(x,y)\tilde{Z}(y,t)dy, \tag{4.12}$$

with a $n \times n$ matrix kernel function $P(x, y)$, maps the error system (4.8)-(4.10) into the exponentially stable target error BVP

$$\tilde{Z}_t(x,t) = \Theta \tilde{Z}_{xx}(x,t) - \bar{C}\tilde{Z}(x,t), \tag{4.13}$$

$$\tilde{Z}(0,t) = 0, \tag{4.14}$$

$$\tilde{Z}(1,t) = 0. \tag{4.15}$$

To derive the corresponding IC, the inverse transformation of (4.12) comes into play, which takes the form

$$\tilde{Z}(x,t) = \tilde{Q}(x,t) + \int_0^x R(x,y)\tilde{Q}(y,t)dy. \tag{4.16}$$

Specifying (4.16) with $t = 0$ yields

$$\tilde{Z}(x,0) = \tilde{Q}(x,0) + \int_0^x R(x,y)\tilde{Q}(y,0)dy \in H^{4,n}, \tag{4.17}$$

which complements the boundary value problem (4.13)-(4.15).

The meaning of (4.13)-(4.15), (4.17) is also viewed in the weak sense. In the sequel, the BVP, governing the kernel matrix $P(x, y)$, and the tuning rule of selecting the observer gain matrix $G(x)$ are derived.

Spatial differentiation of (4.12) yields

$$\tilde{Q}_x(x,t) = \tilde{Z}_x(x,t) - P(x,x)\tilde{Z}(x,t) - \int_0^x P_x(x,y)\tilde{Z}(y,t)dy. \tag{4.18}$$

By specifying (4.12) and (4.18) with $x = 0$, and substituting (4.14) in the resulting relations, one arrives at

$$\tilde{Q}(0,t) = \tilde{Z}(0,t) = 0, \tag{4.19}$$

$$\tilde{Q}_x(0,t) = \tilde{Z}_x(0,t) - P(0,0)\tilde{Z}(0,t) = \tilde{Z}_x(0,t). \tag{4.20}$$

Specifying (4.12) with $x = 1$, substituting the resulting expression in (4.10), and imposing the BC (4.15), the relation

$$\int_0^1 P(1,y)\tilde{Z}(y,t)dy = 0 \tag{4.21}$$

is obtained to derive the BC

$$P(1,y) = 0. \tag{4.22}$$

By differentiating (4.18) with respect to x, the second-order spatial derivative $\tilde{Q}_{xx}(x,t)$ is readily developed (all spatial differentiations involve the use of the Leibnitz differentiation rule). Differentiating (4.12) in time, substituting (4.13) in the resulting relation, and applying recursively integration by parts, one readily obtains the time derivative $\tilde{Q}_t(x,t)$ as well. Substituting (4.12), (4.19), (4.20) and the obtained expressions of $\tilde{Q}_{xx}(x,t)$ and $\tilde{Q}_t(x,t)$ into (4.8) and performing lengthy but straightforward computations yield

$$\begin{aligned}
&\tilde{Z}_t(x,t) - \Theta\tilde{Z}_{xx}(x,t) + \bar{C}Z(x,t) = \\
&[\Theta P(x,x) - P(x,x)\Theta]\,\tilde{Z}_x(x,t) - [G(x) + P(x,0)\Theta]\,\tilde{Z}_x(0,t) \\
&- \left\{\Theta\left[\frac{d}{dx}P(x,x)\right] + P_y(x,x)\Theta + \Theta P_x(x,x) - \Lambda - \bar{C}\right\}\tilde{Z}(x,t) \\
&- \int_0^x \left[\Theta P_{xx}(x,y) - P_{yy}(x,y)\Theta + P(x,y)\bar{C} + \Lambda P(x,y)\right]\tilde{Z}(y,t)dy
\end{aligned} \tag{4.23}$$

To meet the PDE (4.13) the right-hand side of (4.23) should be identically zero. From this requirement, coupled to the BC (4.22), the BVP

$$\begin{aligned}
\Theta P_{xx}(x,y) - P_{yy}(x,y)\Theta &= -P(x,y)\bar{C} - \Lambda P(x,y), && (4.24)\\
\Theta \frac{d}{dx}P(x,x) + \Theta P_x(x,x) &+ P_y(x,x)\Theta = \Lambda + \bar{C}, && (4.25)\\
P(x,x)\Theta &= \Theta P(x,x), && (4.26)\\
P(1,y) &= 0, && (4.27)
\end{aligned}$$

governing the kernel matrix $P(x,y)$ is derived, and the observer gain tuning condition is obtained in the form

$$G(x) = -P(x,0)\Theta. \tag{4.28}$$

Similarly to system (3.28)-(3.31), that was derived in the state feedback controller design, due to relation (4.26) the boundary value problem (4.24)-(4.27) is overdetermined and it has no solution unless either the equi-diffusivity constraint (3.32) holds true or, alternatively, the relation

$$P(x,y) = p(x,y)I_{n\times n}, \tag{4.29}$$

similar to (3.33), is enforced. Thus, duality between the controller and observer designs is in force, and the observer treatment is then separately studied for the equi-diffusivity and distinct-diffusivity scenarios.

4.1. Equi-diffusivity case. Specializing system (4.24)-(4.27) with the equi-diffusivity constraint (3.32) and exploiting the identity $\frac{d}{dx}P(x,x) = P_x(x,x) + P_y(x,x)$ yield the BVP

$$\begin{aligned}
P_{xx}(x,y) - P_{yy}(x,y) &= -\frac{1}{\theta}\left[P(x,y)\bar{C} + \Lambda P(x,y)\right], && (4.30)\\
2\theta \frac{d}{dx}P(x,x) &= \Lambda + \bar{C}, && (4.31)\\
P(1,y) &= 0, && (4.32)
\end{aligned}$$

whereas the tuning condition (4.28) simplifies to

$$G(x) = -\theta P(x,0). \tag{4.33}$$

Integrating (4.31) with respect to x gives

$$P(x,x) = \frac{1}{2\theta}\left(\Lambda + \bar{C}\right)x + P(0,0). \tag{4.34}$$

Evaluating (4.34) at $x = 1$ yields

$$P(1,1) = \frac{1}{2\theta}\left(\Lambda + \bar{C}\right) + P(0,0). \tag{4.35}$$

On the other hand, by evaluating (4.32) at $y = 1$ it is concluded that $P(1,1) = 0$, thereby obtaining

$$P(0,0) = -\frac{1}{2\theta}\left(\Lambda + \bar{C}\right). \tag{4.36}$$

In light of (4.34) and (4.36) one thus rewrites (4.30)-(4.32) as

$$\begin{aligned} P_{xx}(x,y) - P_{yy}(x,y) &= -\frac{1}{\theta}\left[P(x,y)\bar{C} + \Lambda P(x,y)\right], && (4.37)\\ P(x,x) &= \frac{\Lambda + \bar{C}}{2\theta}(x-1), && (4.38)\\ P(1,y) &= 0. && (4.39)\end{aligned}$$

Conditions (4.37)-(4.39) form a well-posed BVP which admits an analytical solution. The following result is in order.

THEOREM 4.1. *The boundary-value problem* (4.37)-(4.39) *possesses a solution*

$$P(x,y) = -\sum_{j=0}^{\infty} \frac{2(1-x)((1-y)^2-(1-x)^2)^j}{j!(j+1)!}\left(\frac{1}{4\theta}\right)^{j+1}\left[\sum_{i=0}^{j}\binom{j}{i}\mathit{\Lambda}^i(\mathit{\Lambda}+\bar{C})\bar{C}^{j-i}\right] \tag{4.40}$$

which is infinitely times continuously differentiable in the domain $\mathcal{T}$ defined in (1.5).

Proof. By making the invertible change of variables

$$\bar{x} = 1-y, \qquad \bar{y} = 1-x, \tag{4.41}$$

one transforms the boundary-value problem (4.37)-(4.39) into

$$\begin{aligned} \bar{P}_{\bar{x}\bar{x}}(\bar{x},\bar{y}) - \bar{P}_{\bar{y}\bar{y}}(\bar{x},\bar{y}) &= \frac{1}{\theta}\left[\bar{P}(\bar{x},\bar{y})\bar{C} + \Lambda\bar{P}(\bar{x},\bar{y})\right], && (4.42)\\ \bar{P}(\bar{x},\bar{x}) &= -\frac{\Lambda+\bar{C}}{2\theta}\bar{x}, && (4.43)\\ \bar{P}(\bar{x},0) &= 0. && (4.44)\end{aligned}$$

By direct comparison between (4.42)-(4.44) and (3.34), (3.36)-(3.37) one immediately notices that $\bar{P}(\bar{x},\bar{y}) = K(x,y)$ when Λ and C are respectively replaced by $\bar{C}$ and $\mathit{\Lambda}$. Thus, from (3.38), one obtains the solution of (4.42)-(4.44) in the form

$$\bar{P}(\bar{x},\bar{y}) = -\sum_{j=0}^{\infty}\frac{(\bar{x}^2-\bar{y}^2)^j(2\bar{y})}{j!(j+1)!}\left(\frac{1}{4\theta}\right)^{j+1}\left[\sum_{i=0}^{j}\binom{j}{i}\mathit{\Lambda}^i\left(\mathit{\Lambda}+\bar{C}\right)\bar{C}^{j-i}\right]. \tag{4.45}$$

By substituting the change of variables (4.41) into (4.45) one returns back to the original variables x and y, thereby getting the series expansion (4.40) for the Kernel matrix $P(x,y)$ which solves the BVP (4.37)-(4.39). Clearly, due to the smooth change of coordinates (4.41) the solution $P(x,y)$ inherits the smoothness properties of $K(x,y)$ to be of class $\mathcal{C}^\infty(\mathcal{T})$. Theorem 4.1 is thus proven. □

The representation

$$G(x) = \theta\sum_{j=0}^{\infty}\frac{2(1-x)(1-(1-x)^2)^j}{j!(j+1)!}\left(\frac{1}{4\theta}\right)^{j+1}\left[\sum_{i=0}^{j}\binom{j}{i}\mathit{\Lambda}^i(\mathit{\Lambda}+\bar{C})\bar{C}^{j-i}\right] \tag{4.46}$$

of the observer gain matrix is straightforwardly derived by specifying (4.33) with the solution (4.40) to the boundary-value problem (4.37)-(4.39).

The next theorem summarizes the proposed anti-collocated observer design for the equi-diffusivity case.

THEOREM 4.2. *Let matrix $\bar{C}$ be selected such that $S[\bar{C}] > 0$ and $\sigma_m(S[\bar{C}])$ is arbitrarily large. Then, the observer* (4.1)-(4.3)*,* (4.46) *reconstructs the state of system (2.1)-(2.3),* (3.32) *with the associated error decay rate obeying the estimate*

$$\|\tilde{Q}(\cdot,t)\|_{H^{2,n}} \le b\|\tilde{Q}(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[\bar{C}])t}, \tag{4.47}$$

where b is a positive constant independent of $\tilde{Q}(\xi,0)$.

Proof. It was shown in the present section that the backstepping transformation (4.12), (4.40) transfers the error system (4.8)-(4.11) into the exponentially stable target error dynamics (4.13)-(4.15), (4.17), provided that the observer gain $G(x)$ is selected as in (4.46). By straightforwardly specifying relation (3.6) with the state $\tilde{Z}(x,t)$ of the target error dynamics one obtains the estimate

$$\|\tilde{Z}(\cdot,t)\|_{H^{2,n}} \le \|\tilde{Z}(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[\bar{C}])t}. \tag{4.48}$$

Owing on the smoothness properties of $P(x,y)$, established in Theorem 4.1, and applying Lemma 3.4 one concludes that the kernel matrix $R(x,y)$ is of class $\mathcal{C}^\infty(\mathcal{T})$ as well. Thus, Lemma 3.5 is straightforwardly reformulated with reference to the direct and inverse backstepping transformations (4.12) and (4.16) along with the associated smooth kernel matrices $P(x,y)$ and $R(x,y)$. Particularly, relations

$$\|\tilde{Q}(\cdot,t)\|_{H^{2,n}} \le c_1\|\tilde{Z}(\cdot,t)\|_{H^{2,n}}, \tag{4.49}$$

$$\|\tilde{Z}(\cdot,t)\|_{H^{2,n}} \le c_2\|\tilde{Q}(\cdot,t)\|_{H^{2,n}}, \tag{4.50}$$

readily follow from (3.55)-(3.56) for some positive constants c_1 and c_2. Coupling together (4.48) and (4.49), one derives that

$$\|\tilde{Q}(\cdot,t)\|_{H^{2,n}} \le c_1\|\tilde{Z}(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[\bar{C}])t}. \tag{4.51}$$

Finally, specifying (4.50) with $t = 0$, and substituting the resulting relation in (4.51), one obtains (4.47) with the constant $b = c_1c_2$ which is independent on $\tilde{Q}(x,0)$. Theorem 4.2 is proven. □

4.2. Distinct diffusivity case. In the present subsection, the anti-collocated observer design is addressed by dispensing with the equi-diffusivity requirement (3.32) (i.e., all processes possess their own distinct diffusivity parameters) and by introducing an extra constraint (4.29) on the kernel matrix $P(x,y)$ of the backstepping transformation (4.12).

By specializing the BVP (4.24)-(4.27) with the constraint (4.29), and applying the identity $\frac{d}{dx}p(x,x) = p_x(x,x) + p_y(x,x)$, one obtains

$$(p_{xx}(x,y) - p_{yy}(x,y))\,\Theta \;=\; -p(x,y)(\Lambda + \bar{C}), \tag{4.52}$$

$$2\frac{d}{dx}p(x,x)\Theta \;=\; \Lambda + \bar{C}, \tag{4.53}$$

$$p(1,y) \;=\; 0, \tag{4.54}$$

whereas the observer gain (4.28) specializes to

$$G(x) \;=\; -\Theta p(x,0). \tag{4.55}$$

The BVP (4.52)-(4.54) shares the same structure of (3.70)-(3.72). Thus, its solvability is addressed by following [4, Sect. 4] thereby arriving in analogy with (3.73) to the constrained form

$$\bar{C} = -\Lambda + \bar{\gamma}^*\Theta \tag{4.56}$$

of the matrix $\bar{C}$ in the target error dynamics (4.13)-(4.15), where $\bar{\gamma}^* \in \mathbb{R}$ is a design parameter. Substituting (4.56) into (4.52) and (4.53) it yields the scalar BVP

$$\begin{aligned} p_{xx}(x,y) - p_{yy}(x,y) &= -\bar{\gamma}^* p(x,y), && (4.57) \\ \frac{d}{dx} p(x,x) &= \frac{\bar{\gamma}^*}{2}, && (4.58) \\ p(1,y) &= 0. && (4.59) \end{aligned}$$

Integrating (4.58) with respect to x gives the relation $p(x,x) = \frac{\bar{\gamma}^*}{2} x + p(0,0)$ whereas another relation $p(0,0) = -\frac{\bar{\gamma}^*}{2}$ is deduced from (4.59) by noticing that $p(1,1) = 0$.

System (4.57)-(4.59) can thus be specified to the BVP

$$\begin{aligned} p_{xx}(x,y) - p_{yy}(x,y) &= -\bar{\gamma}^* p(x,y), && (4.60) \\ p(x,x) &= \frac{\bar{\gamma}^*}{2}(x-1), && (4.61) \\ p(1,y) &= 0, && (4.62) \end{aligned}$$

whose solution

$$p(x,y) = -\bar{\gamma}^*(1-x) \frac{I_1(\sqrt{\bar{\gamma}^*(2x - x^2 + y^2 - 2y)})}{\sqrt{\bar{\gamma}^*(2x - x^2 + y^2 - 2y)}} \tag{4.63}$$

is well-known from [17]. The representation

$$G(x) \quad = \quad \Theta \bar{\gamma}^*(1-x) \frac{I_1(\sqrt{\bar{\gamma}^* x(2-x)})}{\sqrt{\bar{\gamma}^* x(2-x)}} \tag{4.64}$$

of the observer gain is straightforwardly derived by specifying (4.55) with the solution (4.63) to the BVP (4.60)-(4.62) evaluated in $y = 0$.

The next theorem specifies the proposed anti-collocated observer design for the distinct diffusivity case.

THEOREM 4.3. *Let the constant $\bar{\gamma}^*$ be chosen large enough such that $S[\bar{C}] > 0$ and $\sigma_m(S[\bar{C}])$ is arbitrarily large where $\bar{C}$ is given in* (4.56)*. Then, the observer* (4.1)-(4.3)*,* (4.64) *reconstructs the state of system (2.1)-(2.3) with the observation error decay obeying the estimate*

$$\|\tilde{Q}(\cdot,t)\|_{H^{2,n}} \leq b \|\tilde{Q}(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[\bar{C}])t}, \tag{4.65}$$

with a positive constant b, independent of $\tilde{Q}(x,0)$.

Proof. The backstepping transformation (4.12), (4.29), specified with (4.63), transfers the error system (4.8)-(4.11) into the target error dynamics (4.13)-(4.15), (4.17), where $\bar{C}$ is given by (4.56), and the observer gain $G(x)$ is selected as in (4.64). According to Remark 4, one can always select the parameter $\bar{\gamma}^*$ large enough such that $S[\bar{C}] > 0$ and $\sigma_m(S[\bar{C}])$ is arbitrarily large. Thus, exponential stability of the target error dynamics is in force with an arbitrarily fast convergence rate according to estimate (4.48). Since the BVP (4.60)-(4.62) is a particular instance of (4.37)-(4.39), its solution (4.29) is guaranteed by Theorem 4.1 to be of class $\mathcal{C}^\infty(\mathcal{T})$. Clearly, the inverse backstepping transformation takes the form (4.16), specified with $R(x,y) = r(x,y) I_{n\times n}$, and by a straightforward extension of Lemma 3.4, one concludes that $r(x,y)$ is of class $\mathcal{C}^\infty(\mathcal{T})$, too. The rest of the proof follows the same steps used in the proof of Theorem 4.2. Particularly, relations (4.49)-(4.50) are shown to be in force for some positive constants c_1 and c_2. These relations, along with (4.48), result in estimate (4.65). This concludes the proof of Theorem 4.3. □

5. Observer design for the collocated measurement setup. In the present section the state observer design for system (2.1)-(2.4) is addressed and solved under the assumption that only the boundary flow $Q_x(1,t)$ at the controlled side of the spatial domain is available for measurements. The design closely follows that of Section 4, with a slightly different form of the backstepping transformations used. All similar developments to those of the anti-collocated scenario will be skipped. The proposed collocated observer takes the form

$$\begin{aligned} \hat{Q}_t(x,t) &= \Theta\hat{Q}_{xx}(x,t) + \Lambda\hat{Q}(x,t) + G(x)\left[Q_x(1,t) - \hat{Q}_x(1,t)\right], && (5.1)\\ \hat{Q}(0,t) &= 0, && (5.2)\\ \hat{Q}(1,t) &= U(t), && (5.3)\\ \hat{Q}(x,0) &= \hat{Q}_0(x) \in H^{4,n}, && (5.4) \end{aligned}$$

where $G(x)$ is a square matrix of observer gain functions to subsequently be designed.

The observation error variable (4.7) is governed by the BVP

$$\begin{aligned} \tilde{Q}_t(x,t) &= \Theta\tilde{Q}_{xx}(x,t) + \Lambda\tilde{Q}(x,t) - G(x)\tilde{Q}_x(1,t), && (5.5)\\ \tilde{Q}(0,t) &= 0, && (5.6)\\ \tilde{Q}(1,t) &= 0, && (5.7)\\ \tilde{Q}(x,0) &= Q_0(x) - \hat{Q}_0(x) \in H^{4,n}. && (5.8) \end{aligned}$$

To design the observer gain $G(x)$ extra conditions are to be involved under which an invertible transformation

$$\tilde{Q}(x,t) = \tilde{Z}(x,t) - \int_x^1 P(x,y)\tilde{Z}(y,t)dy \qquad (5.9)$$

maps the error BVP (5.5)-(5.8) into the exponentially stable target error dynamics (4.13)-(4.15). The IC (4.15) is rewritten as

$$\tilde{Z}(x,0) = \tilde{Q}(x,0) + \int_x^1 R(x,y)\tilde{Q}(y,0)dy, \qquad (5.10)$$

where $R(x,y)$ is the kernel matrix of the inverse transformation

$$\tilde{Z}(x,t) = \tilde{Q}(x,t) + \int_x^1 R(x,y)\tilde{Q}(y,t)dy. \qquad (5.11)$$

Note that the integration interval adopted in (5.9) and (5.11) is different from that of (4.12) and (4.16), which constitutes the main observer design difference between the anti-collocated and collocated case. Due to this difference, the domain of the kernel matrices $P(x,y)$ and $R(x,y)$ is actually given by the set

$$\mathcal{T}_1 = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le y \le 1\} \qquad (5.12)$$

which is symmetrical to the domain (1.5), considered in the previous sections. Apart from these minor differences, the subsequent treatment follows the same line of reasoning used before.

As assumed throughout, the meaning of the BVPs (5.1)-(5.4), (5.5)-(5.8), and that of (4.13)-(4.15), (5.10) are viewed in the weak sense and the weak solutions $\hat{Q}_(x,t)$, $\tilde{Q}(x,t)$, $\tilde{Z}(x,t)$ are required to evolve in the state space $H^{4,n}$. Due to this, the corresponding ICs (5.4), (5.8) and (4.17) are pre-specified to belong to $H^{4,n}$.

Similar developments to those of Section 4, which are skipped for brevity, yield the following BVP

$$\begin{aligned} \Theta P_{xx}(x,y) - P_{yy}(x,y)\Theta &= -P(x,y)\bar{C} - \Lambda P(x,y), && (5.13)\\ \Theta \frac{d}{dx}P(x,x) + \Theta P_x(x,x) &+ P_y(x,x)\Theta = -\Lambda - \bar{C}, && (5.14)\\ P(x,x)\Theta &= \Theta P(x,x), && (5.15)\\ P(0,y) &= 0, && (5.16) \end{aligned}$$

governing the kernel matrix $P(x,y)$, and the observer gain tuning condition

$$G(x) = P(x,1)\Theta \qquad (5.17)$$

is involved. Due to relation (5.15), the BVP (5.13)-(5.16) admits a solution iff either the equi-diffusivity constraint (3.32) holds or relation (4.29) is enforced. This is in analogy to the BVP (3.28)-(3.31), that was involved in the state feedback controller design, and in analogy to the BVP (4.24)-(4.27), that was employed in the anti-collocated observer design. These two separate scenarios are investigated independently.

5.1. Equi-diffusivity case. Specializing system (5.13)-(5.16) with the equi-diffusivity constraint (3.32) and exploiting the identity $\frac{d}{dx}P(x,x) = P_x(x,x) + P_y(x,x)$ yield after straightforward manipulations

$$\begin{aligned} P_{xx}(x,y) - P_{yy}(x,y) &= -\frac{1}{\theta}\left[P(x,y)\bar{C} + \Lambda P(x,y)\right], && (5.18)\\ P(x,x) &= -\frac{\Lambda + \bar{C}}{2\theta}x, && (5.19)\\ P(0,y) &= 0. && (5.20) \end{aligned}$$

Conditions (5.18)-(5.20) form a well-posed BVP which admits an analytical solution as shown in the following theorem.

THEOREM 5.1. *The BVP* (5.18)-(5.20) *possesses a solution*

$$P(x,y) = -\sum_{j=0}^{\infty} \frac{2x(y^2-x^2)^j}{j!(j+1)!}\left(\frac{1}{4\theta}\right)^{j+1}\left[\sum_{i=0}^{j}\binom{j}{i}\mathit{\Lambda}^i(\mathit{\Lambda}+\bar{C})\bar{C}^{j-i}\right] \qquad (5.21)$$

which is infinitely times continuously differentiable in the domain (5.12).

Proof. By making the invertible change of variables

$$\bar{x} = y, \qquad \bar{y} = x, \qquad (5.22)$$

one transforms (5.18)-(5.20) into

$$\begin{aligned} \bar{P}_{\bar{x}\bar{x}}(\bar{x},\bar{y}) - \bar{P}_{\bar{y}\bar{y}}(\bar{x},\bar{y}) &= \frac{1}{\theta}\left[\bar{P}(\bar{x},\bar{y})\bar{C} + \Lambda \bar{P}(\bar{x},\bar{y})\right], && (5.23)\\ \bar{P}(\bar{y},\bar{y}) &= -\frac{\Lambda + \bar{C}}{2\theta}\bar{y}, && (5.24)\\ \bar{P}(\bar{x},0) &= 0. && (5.25) \end{aligned}$$

The BC (5.24) can be rewritten in the equivalent form

$$\bar{P}(\bar{x},\bar{x}) = -\frac{\Lambda + \bar{C}}{2\theta}\bar{x}. \qquad (5.26)$$

Substituting $\bar{C}$ and Λ into the BVP (3.34), (3.36), (3.37) for Λ and C, respectively, one arrives at the BVP (5.23), (5.25), (5.26), thereby establishing the relation $\bar{P}(\bar{x},\bar{y}) = K(x,y)$ between the solutions of these BVPs. With this in mind, the solution representation (3.38) allows one to reproduce the solution of (5.23), (5.25), (5.26) in the form

$$\bar{P}(\bar{x},\bar{y}) = -\sum_{j=0}^{\infty} \frac{(\bar{x}^2-\bar{y}^2)^j(2\bar{y})}{j!(j+1)!}\left(\frac{1}{4\theta}\right)^{j+1}\left[\sum_{i=0}^{j}\binom{j}{i}\Lambda^i\left(\Lambda+\bar{C}\right)\bar{C}^{j-i}\right] \qquad (5.27)$$

By substituting the change of variables (5.22) into (5.27) one returns back to the original variables x and y to obtain the series expansion (5.21) of the kernel matrix $P(x,y)$ which solves the BVP (5.18)-(5.20). Due to the smooth change of coordinates (5.22) the solution $P(x,y)$ inherits the smoothness properties of $K(x,y)$, and therefore it proves to be of class $\mathcal{C}^{\infty}(\mathcal{T}_1)$. Theorem 5.1 is thus proven □

The observer gain representation

$$G(x) = P(x,1)\theta = -\theta\sum_{n=0}^{\infty}\frac{2x(1-x^2)^n}{n!(n+1)!}\left(\frac{1}{4\theta}\right)^{n+1}\left[\sum_{i=0}^{n}\binom{n}{i}\Lambda^i(\Lambda+\bar{C})\bar{C}^{n-i}\right] \quad (5.28)$$

is straightforwardly derived by specifying (5.17) according to the solution representation (5.21) for the BVP (5.18)-(5.20).

The next theorem summarizes the proposed anti-collocated observer design for the equi-diffusivity case.

THEOREM 5.2. *Let matrix $\bar{C}$ be selected such that $S[\bar{C}] > 0$ and $\sigma_m(S[\bar{C}])$ is arbitrarily large. Then, the observer* (5.1) -(5.3)*,* (5.28) *reconstructs the state of system (2.1)-(2.3),* (3.32) *with decay rate specified by* (4.47)*, where b is a positive constant independent of $\tilde{Q}(\xi,0)$.*

Proof. The proof is nearly the same as that of Theorem 4.2 and it is therefore omitted. □

5.2. Distinct diffusivity case. In the present subsection, the collocated observer design is addressed in the distinct diffusivity scenario. The content of this section, being similar to that of Section 4.2, is not accompanied with design details as they can straightforwardly be derived from the corresponding anti-collocated design.

The observer gain takes the form

$$G(x) \quad = \quad -\bar{\gamma}^* x\frac{I_1(\sqrt{\bar{\gamma}^*(1-x^2)})}{\sqrt{\bar{\gamma}^*(1-x^2)}}\Theta, \qquad (5.29)$$

and the next result is in force.

THEOREM 5.3. *Let the constant $\bar{\gamma}^*$ be chosen large enough to ensure that $S[\bar{C}] > 0$ with $\bar{C}$ given in* (4.56) *and $\sigma_m(S[\bar{C}])$ is arbitrarily large. Then, the observer* (5.1)-(5.3)*,* (5.29) *reconstructs the state of system (2.1)-(2.3) with the observation error decay obeying the estimate* (4.65)*, where b is a positive constant independent of $\tilde{Q}(x,0)$.*

Proof. The proof is identical to that of Theorem 4.3. □

6. Output-feedback stabilization. In this section, the anti-collocated and collocated backstepping observers of Sections 4 and 5 are combined with their natural dual backstepping controllers of Section 3 to present the output feedback exponential stabilization of system (2.1)-(2.4).

6.1. Anti-collocated measurement setup. The following result is in order

THEOREM 6.1. *Consider system (2.1)-(2.4) driven by the controller*

$$U(t) = \int_0^1 K(1,y)\hat{Q}(y,t)dy \tag{6.1}$$

and fed by observer (4.1)-(4.4), (4.46). *Let the matrices C and $\bar{C}$ be selected such that $S[C] > 0$ and $S[\bar{C}] > 0$, and let $K(1,y)$ be given by* (3.40). *Then, the closed-loop system (2.1)-(2.4),*(4.1)-(4.3), (4.46), (6.1) *is exponentially stable in the space $H^{2,n} \times H^{2,n}$.*

Proof.

Lengthy but straightforward manipulations show that the backstepping transformation

$$\hat{Z}(x,t) = \hat{Q}(x,t) - \int_0^x K(x,y)\hat{Q}(y,t)dy \tag{6.2}$$

maps the observer dynamics (4.1)-(4.3) into the system

$$\hat{Z}_t(x,t) = \Theta \hat{Z}_{xx}(x,t) - \bar{C}\hat{Z}(x,t) + F_1(x)\tilde{Z}_x(0,t), \tag{6.3}$$
$$\hat{Z}(0,t) = 0, \tag{6.4}$$
$$\hat{Z}(1,t) = 0, \tag{6.5}$$

with

$$F_1(x) = \left[G(x) - \int_0^x K(x,y)G(y)dy\right]. \tag{6.6}$$

The $\tilde{Z}(x,t)$-system, governed by (4.13)-(4.15), is exponentially stable in the space $H^{2,n}$ as well as the homogeneous part of the $\hat{Z}(x,t)$-system (6.3)-(6.5) is if considered separately with the external term $\tilde{Z}_x(0,t)$ deliberately set to zero. Following [32, Sect. 5.1], one notices that the interconnection of the two systems in the ($\hat{Z}$, $\tilde{Z}$) coordinates is in cascade form, and it was shown in Theorem 3.1 that all entries of the forcing term $\tilde{Z}_x(0,t)$ escape “quasi-exponentially” to zero according to (3.8) (see Remark 1). Owing on the boundedness and smoothness of $G(x)$ and $K(x,y)$ in the corresponding domains, the function $F_1(x)$ is bounded and smooth in its domain $0 \le x \le 1$, too. Thus, the combined ($\hat{Z}$, $\tilde{Z}$)-system straightforwardly proves to be exponentially stable in the space $H^{2,n} \times H^{2,n}$. As a result, the ($\hat{Q}$, $\tilde{Q}$)-system is exponentially stable in the same space since it is related to ($\hat{Z}$, $\tilde{Z}$) by the invertible coordinate transformations (4.12) and (6.2) whose kernel matrix gains $P(x,y)$ and $K(x,y)$, along with the corresponding inverse transformation matrices $R(x,y)$ and $L(x,y)$, belong to $\mathcal{C}^\infty(\mathcal{T})$, where $\mathcal{T}$ is defined in (1.5). Indeed, a straightforward generalization of Lemma 3.5 shows that these smoothness properties guarantee the equivalence between norms of ($\hat{Z}$, $\tilde{Z}$) and ($\hat{Q}$, $\tilde{Q}$) in the space $H^{2,n} \times H^{2,n}$, directly ensuring the exponential stability of the closed-loop system of interest in the space $H^{2,n} \times H^{2,n}$. Theorem 6.1 is proven. □

The proof of the stable coupling of the controller to the anti-collocated observer, designed in the distinct-diffusivity case, follows the same line of reasoning and it is skipped for brevity.

6.2. Collocated measurement setup. The following theorem is in force

THEOREM 6.2. *Consider system (2.1)-(2.4) driven by the controller*

$$U(t) = \int_0^1 K(1,y)\hat{Q}(y,t)dy \tag{6.7}$$

and fed by observer (5.1)-(5.3)*,* (5.28)*. Let the matrices C and $\bar{C}$ be selected such that $S[C] > 0$ and $S[\bar{C}] > 0$, and let $K(1, y)$ be given by* (3.40)*. Then, the closed-loop system (2.1)-(2.4),*(5.1)-(5.3)*,* (5.28)*,* (6.7) *is exponentially stable in the space $H^{2,n} \times H^{2,n}$.*

Proof. One shows that the backstepping transformation (6.2) maps the observer dynamics (5.1)-(5.3) into the BVP

$$\begin{aligned}
\hat{Z}_t(x,t) &= \Theta \hat{Z}_{xx}(x,t) - \bar{C}\hat{Z}(x,t) + F_1(x)\tilde{Z}_x(1,t), && (6.8)\\
\hat{Z}(0,t) &= 0, && (6.9)\\
\hat{Z}(1,t) &= 0, && (6.10)
\end{aligned}$$

which only differs from (6.3)-(6.5) in that $\tilde{Z}_x(1,t)$ rather than $\tilde{Z}_x(0,t)$ enters the corresponding PDE as an external input premultiplied by the smooth matrix gain $F_1(x)$. The rest of the proof follows the same steps and reasonings used in the proof of Theorem 6.1. The proof of Theorem 6.2 is thus concluded. □

REMARK 7. *The previously addressed compatibility issue (see Remark 3) also arises in the output feedback case. To solve it, both the output feedback controllers using the anti-collocated and collocated observer can be modified in the same manner. Particularly, the next modified control*

$$U(t) = \int_0^1 K(1,y)\hat{Q}(y,t)dy + \left[Q_0(1) - \int_0^1 K(1,y)\hat{Q}_0(y)dy\right] e^{-\delta t}, \quad \delta > 0 \qquad (6.11)$$

which is intended to replace either (6.1) *or* (6.7) *can be implemented to solve the aforementioned issue.*

7. Coupled PDEs with heterogenous BCs. In this section we investigate the more general class of processes governed by n coupled reaction-diffusion PDEs which possess different BCs at the uncontrolled side. Particularly, we address the scenario in which, among the coupled reaction-diffusion subsystems, $n_1 > 0$ processes are subject to Dirichlet BCs whereas $n_2 = n - n_1 > 0$ processes are subject to Neumann BCs. We consider the following BVP

$$\begin{aligned}
Q_{1t}(x,t) &= \Theta_1 Q_{1xx}(x,t) + \Lambda_{11}Q_1(x,t) + \Lambda_{12}Q_1(x,t), && (7.1)\\
Q_{2t}(x,t) &= \Theta_2 Q_{2xx}(x,t) + \Lambda_{21}Q_1(x,t) + \Lambda_{22}Q_1(x,t), && (7.2)
\end{aligned}$$

which is equipped with the BCs and ICs

$$\begin{aligned}
Q_1(0,t) &= 0, && (7.3)\\
Q_{2x}(0,t) &= 0, && (7.4)\\
Q(1,t) &= U(t), && (7.5)\\
Q(x,0) &= Q_0(x) \in H^{4,n}, && (7.6)
\end{aligned}$$

where

$$\begin{aligned}
Q_1(x,t) &= \left[q_{11}(x,t),\ q_{12}(x,t), \ldots, q_{1n_1}(x,t)\right]^T, && (7.7)\\
Q_2(x,t) &= \left[q_{21}(x,t),\ q_{22}(x,t), \ldots, q_{2n_2}(x,t)\right], && (7.8)\\
Q(x,t) &= \left[Q_1^T(x,t),\ Q_2^T(x,t)\right]^T, && (7.9)
\end{aligned}$$

Θ_1 and Θ_2 are diagonal definite-positive matrices of dimension n_1 and n_2, respectively, whereas Λ_{ij} (i, j=1,2) are arbitrary matrices of appropriate dimension. Define

$$\Theta = \begin{bmatrix} \Theta_1 & 0 \\ 0 & \Theta_2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}, \qquad (7.10)$$

7.1. State feedback stabilization. The target system is chosen in the form of the BVP composed of the PDE (3.2) and the BCs

$$
\begin{aligned}
Z_1(0,t) &= 0, && (7.11)\\
Z_{2x}(0,t) &= 0, && (7.12)\\
Z(1,t) &= 0 && (7.13)
\end{aligned}
$$

The resulting BVP is conveniently expressed in the expanded form

$$
\begin{aligned}
Z_{1t}(x,t) &= \Theta_1 Z_{1xx}(x,t) - C_{11}Z_1(x,t) - C_{12}Z_1(x,t), && (7.14)\\
Z_{2t}(x,t) &= \Theta_2 Z_{2xx}(x,t) - C_{21}Z_1(x,t) - C_{22}Z_1(x,t), && (7.15)\\
Z_1(0,t) &= 0, && (7.16)\\
Z_{2x}(0,t) &= 0, && (7.17)\\
Z(1,t) &= 0, && (7.18)
\end{aligned}
$$

where $Z(x,t) = [Z_1^T(x,t)Z_2^T(x,t)]$, C_{11}, C_{12}, C_{21}, C_{22} and

$$
C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \tag{7.19}
$$

is the design matrix of an appropriate dimension.

The exponential stability of the target system (7.14)-(7.18) is ensured with an arbitrarily fast convergence rate by an appropriate choice of the real-valued matrices $C \in \mathbb{R}^{n\times n}$. The following result is in order.

THEOREM 7.1. *Let matrix C be such that $S[C] > 0$. Then, system* (7.14)-(7.18) *is exponentially stable in the space $H^{2,n}$ with the decay rate $\sigma_m(S[C])$ according to* (3.6)*, and the estimates* (3.7)-(3.8) *are in force where $z_{ix}(x,t)$ denotes the i-th element of $Z_x(x,t)$.*

Proof. Following the same line of reasoning, used in the proof of Theorem 3.1, one shows that the spatial derivatives $Z_x(x,t)$ and $Z_{xx}(x,t)$ constitute weak solutions of the BVPs composed of the PDEs (3.10), (3.11) and the BCs

$$
Z_{1xx}(0,t) = Z_{2xxx}(0,t) = Z_{xx}(1,t) = 0, \tag{7.20}
$$

inherited from (3.2), (7.11)-(7.13). The Lyapunov functional (3.13) is then involved, whose time derivative along the weak solutions in question is given by (3.14). The first integral term in the right hand side of equality (3.14), being integrated by parts, is estimated as follows

$$
\begin{aligned}
\int_0^1 Z^T(\xi,t)\Theta Z_{\xi\xi}(\xi,t)d\xi &= Z^T(\chi,t)\Theta Z_x(\chi,t)\Big|_{\chi=0}^{\chi=1} - \int_0^1 Z_\xi^T(\xi,t)\Theta Z_\xi(\xi,t)d\xi\\
&= Z^T(1,t)\Theta Z_x(1,t) - Z_1^T(0,t)\Theta_1 Z_{1x}(0,t)\\
&- Z_2^T(0,t)\Theta_2 Z_{2x}(0,t) - \int_0^1 Z_\xi^T(\xi,t)\Theta Z_\xi(\xi,t)d\xi\\
&\leq -\theta_m\|Z_x(\cdot,t)\|_{2,n}^2, && (7.21)
\end{aligned}
$$

where the BCs (7.11), (7.13) and the diagonal form of matrix Θ have been taken into account, and the same notation $\theta_m = \min_{1\le i\le n}\theta_i > 0$ has been used. The third and fifth integral terms in the right hand side of (3.14) are estimated as follows

$$\begin{aligned}
\int_0^1 Z_\xi^T(\xi,t)\Theta Z_{\xi\xi\xi}(\xi,t)d\xi &= Z_x^T(\chi,t)\Theta Z_{xx}(\chi,t)\big|_{\chi=0}^{\chi=1} - \int_0^1 Z_{\xi\xi}^T(\xi,t)\Theta Z_{\xi\xi}(\xi,t)d\xi \\
&= Z_x^T(1,t)\Theta Z_{xx}(1,t) - Z_{1x}^T(0,t)\Theta_1 Z_{1xx}(0,t) \\
&- Z_{2x}^T(0,t)\Theta_2 Z_{2xx}(0,t) - \int_0^1 Z_{\xi\xi}^T(\xi,t)\Theta Z_{\xi\xi}(\xi,t)d\xi \\
&\leq -\theta_m \|Z_{xx}(\cdot,t)\|_{2,n}^2, \qquad (7.22) \\
\int_0^1 Z_{\xi\xi}^T(\xi,t)\Theta Z_{\xi\xi\xi\xi}(\xi,t)d\xi &= Z_{xx}^T(\chi,t)\Theta Z_{xxx}(\chi,t)\big|_{\chi=0}^{\chi=1} - \int_0^1 Z_{\xi\xi\xi}^T(\xi,t)\Theta Z_{\xi\xi\xi}(\xi,t)d\xi \\
&= Z_{xx}^T(1,t)\Theta Z_{xxx}(1,t) - Z_{1xx}^T(0,t)\Theta_1 Z_{1xxx}(0,t) \\
&- Z_{2xx}^T(1,t)\Theta_2 Z_{2xxx}(0,t) - \int_0^1 Z_{\xi\xi\xi}^T(\xi,t)\Theta Z_{\xi\xi\xi}(\xi,t)d\xi \\
&\leq -\theta_m \|Z_{xxx}(\cdot,t)\|_{2,n}^2, \qquad (7.23)
\end{aligned}$$

where the BCs (7.20) have been used. From this point on, the proof follows the same steps as those in the last part of the proof of Theorem 3.1. This concludes the proof. □

To transfer the BVP (7.1)-(7.9) into the target system BVP, the same invertible backstepping transformation (3.1) is applied with the $n \times n$ kernel matrix function $K(x,y)$, whose elements are denoted as $k_{ij}(t),\ i,j = 1,2,...,n$. This matrix is decomposed as follows

$$K(x,y) = \begin{bmatrix} K_1^1(x,y) & K_2^1(x,y) \\ K_1^2(x,y) & K_2^2(x,y) \end{bmatrix} = \begin{bmatrix} K_1(x,y) & K_2(x,y) \end{bmatrix} \qquad (7.24)$$

where the involved matrices

$$K_1^1(x,y) \in \mathbf{R}^{n_1\times n_1},\ K_2^1(x,y) \in \mathbf{R}^{n_1\times n_2},\ K_1^2(x,y) \in \mathbf{R}^{n_2\times n_1},\ K_2^2(x,y) \in \mathbf{R}^{n_2\times n_2}, \qquad (7.25)$$

play an important role in the sequel and $K_1(x,y) \in \mathbf{R}^{n\times n_1}$, $K_2(x,y) \in \mathbf{R}^{n\times n_2}$ denote the first n_1 columns and the last n_2 columns of matrix $K(x,y)$, respectively. Note that the generic name $K(x,y)$ is used in the present Section to denote the kernel matrix, as it was previously done in Sections 3 and 6. Clearly, within the present section the kernel matrix will be governed by a different BVP, and thus its expression will be different from that obtained in Sections 3 and 6.

The backstepping transformation (3.1) is thus rewritten in the expanded form

$$\begin{aligned}
Z_1(x,t) &= Q_1(x,t) - \int_0^x K_1^1(x,y)Q_1(y,t)dy - \int_0^x K_2^1(x,y)Q_2(y,t)dy \quad (7.26) \\
Z_2(x,t) &= Q_2(x,t) - \int_0^x K_1^2(x,y)Q_1(y,t)dy - \int_0^x K_2^2(x,y)Q_2(y,t)dy \quad (7.27)
\end{aligned}$$

By applying similar manipulations as those made in Section 3, the resulting kernel BVP takes the form

$$\Theta K_{xx}(x,y) - K_{yy}(x,y)\Theta = K(x,y)\Lambda + CK(x,y), \tag{7.28}$$

$$\Lambda + C + K_y(x,x)\Theta + \Theta K_x(x,x) + \Theta\frac{d}{dx}K(x,x) = 0, \tag{7.29}$$

$$\Theta K(x,x) - K(x,x)\Theta = 0, \tag{7.30}$$

$$K_1(x,0) = 0. \tag{7.31}$$

$$K_{2y}(x,0) = 0. \tag{7.32}$$

The only difference between the obtained BVP (7.28)-(7.32) and the BVP (3.28)-(3.31) previously derived in the Dirichlet BC case is in the relations (7.31) and (7.32), defining the BCs of the underlying BVP along the line $y = 0$.

Solvability of the BVP (7.28)-(7.32) is subsequently studied in the two cases of the equi-diffusivity, where the constraint (3.32) is imposed on the diffusivity parameters, and of the distinct diffusivity, where relation (3.32) is no longer in force.

In contrast to the Dirichlet BC case studied previously, in the present "heterogeneous BCs" setting the distinct-diffusivity case yields an overdetermined kernel PDE that possesses no solution.

7.1.1. Equi-diffusivity case. Specifying the BVP (7.28)-(7.32) in light of the equi-diffusivity constraint (3.32), and performing straightforward manipulations, yield

$$K_{xx}(x,y) - K_{yy}(x,y) = \frac{1}{\theta}K(x,y)\Lambda + \frac{1}{\theta}CK(x,y), \tag{7.33}$$

$$K(x,x) = -\frac{1}{2\theta}(\Lambda + C)\,x + K(0,0) \tag{7.34}$$

$$K_1(x,0) = 0. \tag{7.35}$$

$$K_{2y}(x,0) = 0. \tag{7.36}$$

Let us presently motivate the crucial constraint $K(0,0) = 0$ that was earlier required in Subsection 3.1 where only Dirichlet BCs were dealt with. It follows from (7.35) that

$$K_1(0,0) = 0, \tag{7.37}$$

i.e. the first n_1 columns of matrix $K(0,0)$ are identically zero. To derive the additional constraints on $K(0,0)$, the spatial derivative of the backstepping transformation (7.26)-(7.27) is first computed by applying the Leibnitz differentiation rule and then specified at $x = 0$:

$$Z_{1x}(0,t) = Q_{1x}(0,t) - K_1^1(0,0)Q_1(0,t) - K_2^1(0,0)Q_2(0,t) \tag{7.38}$$

$$Z_{2x}(0,t) = Q_{2x}(0,t) - K_1^2(0,0)Q_1(0,t) - K_2^2(0,0)Q_2(0,t). \tag{7.39}$$

Substituting the BCs (7.3)-(7.4) and (7.16)-(7.17) into (7.38)-(7.39) yields

$$Z_{1x}(0,t) = Q_{1x}(0,t) - K_2^1(0,0)Q_2(0,t) \tag{7.40}$$

$$0 = K_2^2(0,0)Q_2(0,t) \tag{7.41}$$

It follows from (7.41) that $K_2^2(0,0)$ must be zero, whereas relation (7.40) does not impose any constraint on $K_2^1(0,0)$. We thus make the deliberate choice $K_2^1(0,0) = 0$, yielding the final relation $K(0,0) = 0$. It is worth noticing that any alternative value of $K_2^1(0,0)$ is compatible with (7.40), and it could be a valid choice as well provided that the resulting kernel BVP is feasible.

Thus, relation (7.34) of the kernel BVP is replaced by

$$K(x,x) \quad = \quad -\frac{1}{2\theta}\,(\Lambda + C)\,x \tag{7.42}$$

LEMMA 7.2. *The BVP* (7.33)*,* (7.35)-(7.36)*,* (7.42) *admits a solution which is of class* $\mathcal{C}^\infty$ *in the domain* (1.5)*.*

Proof. Inspired from [31], the substitution

$$\xi = x + y, \qquad \eta = x - y, \tag{7.43}$$

of the independent variables is adopted to represent the BVP (7.33), (7.35)-(7.36), (7.42) in terms of

$$G(\xi,\eta) = K(x,y) = K\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) \tag{7.44}$$

as follows

$$G_{\xi\eta}(\xi,\eta) \quad = \quad \frac{1}{4\theta}G(\xi,\eta)\Lambda + \frac{1}{4\theta}CG(\xi,\eta), \tag{7.45}$$

$$G(\xi,0) \quad = \quad -\frac{1}{4\theta}\,(\Lambda + C)\,\xi, \tag{7.46}$$

$$G_1(\xi,\xi) \quad = \quad 0, \tag{7.47}$$

$$G_{2\xi}(\xi,\xi) \quad = \quad G_{2\eta}(\xi,\xi), \tag{7.48}$$

where $G_1(x,y)$ and $G_2(x,y)$ are such that

$$G(x,y) = \left[\begin{array}{cc} G_1(x,y) & G_2(x,y) \end{array}\right], \quad G_1(x,y) \in \mathbf{R}^{n\times n_1}, \;\; G_2(x,y) \in \mathbf{R}^{n\times n_2} \tag{7.49}$$

Integrating (7.45) with respect to η from 0 to η, and considering the relation $G_\xi(\xi,0) = -\frac{1}{4\theta}(\mathit{\Lambda} + C)$, which is straightforwardly obtained from (7.46), we get

$$G_\xi(\xi,\eta) = -\frac{1}{4\theta}(\mathit{\Lambda} + C) + \frac{1}{4\theta}\int_0^\eta \left[G(\xi,s)\mathit{\Lambda} + CG(\xi,s)\right] ds \tag{7.50}$$

Integrating (7.50) with respect to the first argument from η to ξ and applying straightforward manipulations yield

$$G(\xi,\eta) \quad - \quad G(\eta,\eta) = -\frac{1}{4\theta}(\mathit{\Lambda} + C)(\xi-\eta) + \frac{1}{4\theta}\int_\eta^\xi \left\{\int_0^\eta \left[G(\tau,s)\mathit{\Lambda} + CG(\tau,s)\right] ds\right\} d\tau \tag{7.51}$$

We are now in a position to derive an explicit form of $G(\eta,\eta)$. The first n_1 columns of $G(\eta,\eta)$, i.e. the matrix $G_1(\eta,\eta)$, are identically zero due to (7.47). To derive $G_2(\eta,\eta)$, we use (7.48) to write

$$\frac{d}{d\xi}G_2(\xi,\xi) = G_{2\xi}(\xi,\xi) + G_{2\eta}(\xi,\xi) = 2G_{2\xi}(\xi,\xi) \tag{7.52}$$

Let us note that $G_2(x,y) = G(x,y)T_{n_1n_2}$, where

$$T_{n_1n_2} = \left[\begin{array}{c} 0_{n_1,n_2} \\ I_{n2} \end{array}\right] \tag{7.53}$$

is a projection operator which allows one to extract the corresponding last n_2 columns. Taking this into account and setting $\eta = \xi$ one derives from (7.52) that

$$G_{2\eta}(\eta,\eta) = -\frac{1}{4\theta}(\Lambda + C)T_{n_1n_2} + \frac{1}{4\theta}\int_0^{\eta}[G(\eta,s)\Lambda + CG(\eta,s)]\,dsT_{n_1n_2}. \tag{7.54}$$

Using (7.54), relation (7.50) can be represented in the form of the differential equation

$$\frac{d}{d\eta}G_2(\eta,\eta) = -\frac{1}{2\theta}(\Lambda + C)T_{n_1n_2} + \frac{1}{2\theta}\int_0^{\eta}[G(\eta,s)\Lambda + CG(\eta,s)]\,dsT_{n_1n_2} \tag{7.55}$$

Integrating both sides of (7.55) with respect to η yields

$$G_1(\eta,\eta) = 0 \tag{7.56}$$

$$G_2(\eta,\eta) = -\frac{1}{2\theta}(\Lambda + C)\eta T_{n_1n_2} + \frac{1}{2\theta}\int_0^{\eta}\left\{\int_0^{\tau}[G(\tau,s)\Lambda + CG(\tau,s)]\,ds\right\}d\tau T_{n_1n_2} \tag{7.57}$$

Now involving the projection operator

$$E_{n_1n_2} = [0_{n_1+n_2,n_1}T_{n_1n_2}] \tag{7.58}$$

let us rewrite (7.56)-(7.57) in the compact form

$$G(\eta,\eta) = -\frac{1}{2\theta}(\Lambda + C)\eta E_{n_1n_2} + \frac{1}{2\theta}\int_0^{\eta}\left\{\int_0^{\tau}[G(\tau,s)\Lambda + CG(\tau,s)]\,ds\right\}d\tau E_{n_1n_2}. \tag{7.59}$$

Substituting (7.59) into (7.51) for $G(\eta,\eta)$ yields

$$\begin{aligned} G(\xi,\eta) &= -\frac{1}{2\theta}(\Lambda + C)\eta E_{n_1n_2} + \frac{1}{2\theta}\int_0^{\eta}\left\{\int_0^{\tau}[G(\tau,s)\Lambda + CG(\tau,s)]\,ds\right\}d\tau E_{n_1n_2} \\ &+ \frac{1}{4\theta}(\Lambda + C)(\xi-\eta) + \frac{1}{4\theta}\int_{\eta}^{\xi}\left\{\int_0^{\eta}[G(\tau,s)\Lambda + CG(\tau,s)]\,ds\right\}d\tau \end{aligned} \tag{7.60}$$

Next the method of successive approximations is used to show that equation (**??**) has a smooth solution. By letting

$$G^0(\xi,\eta) = 0 \tag{7.61}$$

the recursive formula is set up for (7.60) as follows

$$\begin{aligned} G^{n+1}(\xi,\eta) &= -\frac{1}{2\theta}(\Lambda + C)\eta E_{n_1n_2} + \frac{1}{2\theta}\int_0^{\eta}\left\{\int_0^{\tau}[G^n(\tau,s)\Lambda + CG^n(\tau,s)]\,ds\right\}d\tau E_{n_1n_2} \\ &+ \frac{1}{4\theta}(\Lambda + C)(\xi-\eta) + \frac{1}{4\theta}\int_{\eta}^{\xi}\left\{\int_0^{\eta}[G^n(\tau,s)\Lambda + CG^n(\tau,s)]\,ds\right\}d\tau \end{aligned} \tag{7.62}$$

Clearly if this recursion converges, the limiting relation

$$G(\xi,\eta) = \lim_{n\to\infty} G^n(\xi,\eta) \tag{7.63}$$

determines a solution of (7.60).

Denote the difference between two consecutive terms as

$$\Delta G^n(\xi,\eta) \quad = \quad G^{n+1}(\xi,\eta) - G^n(\xi,\eta) \tag{7.64}$$

Then, the next recursion

$$\Delta G^0(\xi,\eta) \quad = \quad G^1(\xi,\eta) = -\frac{1}{2\theta}(\Lambda + C)\eta E_{n_1 n_2} + \frac{1}{4\theta}(\Lambda + C)(\xi - \eta) \tag{7.65}$$

$$\begin{aligned}\Delta G^{n+1}(\xi,\eta) \quad = \quad & \frac{1}{2\theta}\int_0^\eta \left\{\int_0^\tau \left[\Delta G^n(\tau,s)\Lambda + C\Delta G^n(\tau,s)\right] ds\right\} d\tau E_{n_1 n_2} \\ & + \frac{1}{4\theta}\int_\eta^\xi \left\{\int_0^\eta \left[\Delta G^n(\tau,s)\Lambda + C\Delta G^n(\tau,s)\right] ds\right\} d\tau\end{aligned} \tag{7.66}$$

is concluded from (7.61)-(7.62), and (7.63) can be rewritten in the form

$$G(\xi,\eta) = \sum_{n=0}^{\infty} \Delta G^n(\xi,\eta) \tag{7.67}$$

Since $0 \le y \le x \le 1$, the ξ- and η-variables are located within $|\eta| \le 1$, $|\xi| \le 2$. Furthermore,

$$\|E_{n_1 n_2}\| \le 1 \tag{7.68}$$

due to (7.58), and using (7.65) and (7.68), one can readily show that

$$\|\Delta G^0(\xi,\eta)\| \le \frac{1}{2\theta}(\|\Lambda\| + \|C\|)\|E_{n_1 n_2}\| + \frac{1}{2\theta}(\|\Lambda\| + \|C\|) \le \frac{1}{\theta}(\|\Lambda\| + \|C\|) = M \tag{7.69}$$

Suppose that

$$\|\Delta G^n(\xi,\eta)\| \le M^{n+1}\frac{(\xi+\eta)^n}{n!} \tag{7.70}$$

Then applying computations, similar to those used in [3] for deriving equations (55)-(56), yield

$$\|\Delta G^{n+1}(\xi,\eta)\| \le M^{n+2}\frac{(\xi+\eta)^{n+1}}{(n+1)!} \tag{7.71}$$

Thus by mathematical induction, (7.71) is true for all $n > 0$. It then follows from the Weierstrass M-test that the series (7.67) converges absolutely and uniformly in $0 \le \eta \le \xi \le 2$.

To complete the proof it remains to note that being given by the integral equality (7.60), the continuous function $G(\xi,\eta)$ is at least twice continuously differentiable in the domain $0 \le \eta \le \xi \le 2$. Moreover, by iterating on the successive differentiation of (7.60), one concludes that $G(\xi,\eta)$ is of class $\mathcal{C}^\infty$ in its domain. By virtue of (7.44), the corresponding solution is thus shown to be of class $\mathcal{C}^\infty(\mathcal{T})$. This concludes the proof. □

Under certain conditions, the state feedback boundary controller

$$U(t) \quad = \quad \int_0^1 K(1,y)Q(y,t)dy, \tag{7.72}$$

is shown in the following Theorem to be exponentially stabilizing in the equi-diffusivity case.

THEOREM 7.3. *Let matrix C be selected in such a manner that $S[C] > 0$ whereas $\sigma_m(S[C])$ is arbitrarily large. Then, the boundary control input* (7.72)*, where $K(x, y)$ is a solution to the BVP* (7.33), (7.35)-(7.36), (7.42)*, exponentially stabilizes system* (7.1)-(7.9) *in the space $H^{2,n}$ with the corresponding norm obeying the estimate*

$$\|Q(\cdot, t)\|_{H^{2,n}} \leq a\|Q(\cdot, 0)\|_{H^{2,n}} e^{-\sigma_m(S[C])t}, \tag{7.73}$$

where a is a positive constant independent of $Q(x, 0)$.

Proof. The proof is based on the demonstration of Lemma 7.2, and it is formally identical to that of Theorem (3.3). □

7.1.2. Distinct diffusivity case. We now show that in the distinct diffusivity case, where the constraint (3.33) is imposed on the form of the kernel matrix to verify the relation (7.30), the BVP (7.28)-(7.32) is overdetermined and it possesses no solution. For the sake of simplicity this analysis is illustrated for two ($n = 2$) coupled processes with $n_1 = n_2 = 1$ and

$$\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}. \tag{7.74}$$

The general treatment is essentially the same.

The matrix PDE (7.28), specified with $n = 2$, yields the four scalar relations

$$k_{xx}(x, y) - k_{yy}(x, y) = \frac{\lambda_{11} + c_{11}}{\theta_1} k(x, y), \tag{7.75}$$

$$k_{xx}(x, y) - k_{yy}(x, y) = \frac{\lambda_{22} + c_{22}}{\theta_2} k(x, y), \tag{7.76}$$

$$\lambda_{12} + c_{12} = 0, \tag{7.77}$$

$$\lambda_{21} + c_{21} = 0, \tag{7.78}$$

that imply the constraints

$$\frac{\lambda_{11} + c_{11}}{\theta_1} = \frac{\lambda_{22} + c_{22}}{\theta_2} = \gamma, \tag{7.79}$$

$$c_{12} = -\lambda_{12}, \tag{7.80}$$

$$c_{21} = -\lambda_{21}. \tag{7.81}$$

on the coefficients of the C matrix. The BCs (7.29) yields the four relations

$$\frac{d}{dx} k(x, x) = \frac{\lambda_{11} + c_{11}}{2\theta_1}, \tag{7.82}$$

$$\frac{d}{dx} k(x, x) = \frac{\lambda_{22} + c_{22}}{2\theta_2}, \tag{7.83}$$

$$\lambda_{12} + c_{12} = 0, \tag{7.84}$$

$$\lambda_{21} + c_{21} = 0, \tag{7.85}$$

which yield the same constraints (7.79)- (7.81). Finally, the BCs (7.31)-(7.32) lead to the corresponding BCs

$$k(x, 0) = 0. \tag{7.86}$$

$$k_y(x, 0) = 0. \tag{7.87}$$

It yields the overall BVP

$$\begin{aligned} k_{xx}(x,y) - k_{yy}(x,y) &= \gamma k(x,y), & (7.88)\\ \frac{d}{dx}k(x,x) &= \frac{1}{2}\gamma, & (7.89)\\ k(x,0) &= 0, & (7.90)\\ k_y(x,0) &= 0, & (7.91) \end{aligned}$$

which is clearly overdetermined since the separate BVPs (7.88)-(7.90) and (7.88)-(7.89), (7.91) admit distinct unique solutions (see [17]). Thus in the distinct diffusivity case, the state feedback design becomes unfeasible within the present approach.

7.2. Anticollocated observer design and output feedback stabilization. For system (7.1)-(7.5) of n coupled reaction-diffusion processes with the boundary flow $Q_{1x}(0,t)$ and the boundary state $Q_2(0,t)$ being the only available measurements, the state observer

$$\begin{aligned} \hat{Q}_{1t}(x,t) &= \Theta_1\hat{Q}_{1xx}(x,t) + \Lambda_{11}\hat{Q}_1(x,t) + \Lambda_{12}\hat{Q}_1(x,t) + G_{11}(x)[Q_{1x}(0,t) - \hat{Q}_{1x}(0,t)]\\ &+ G_{12}(x)[Q_2(0,t) - \hat{Q}_2(0,t)], & (7.92)\\ \hat{Q}_{2t}(x,t) &= \Theta_2\hat{Q}_{2xx}(x,t) + \Lambda_{21}\hat{Q}_1(x,t) + \Lambda_{22}\hat{Q}_1(x,t) + G_{21}(x)[Q_{1x}(0,t) - \hat{Q}_{1x}(0,t)]\\ &+ G_{22}(x)[Q_2(0,t) - \hat{Q}_2(0,t)], & (7.93) \end{aligned}$$

is equipped with the BCs and arbitrary IC:

$$\begin{aligned} \hat{Q}_1(0,t) &= 0, & (7.94)\\ \hat{Q}_{2x}(0,t) &= M(Q_2(0,t) - \hat{Q}_2(0,t)), & (7.95)\\ Q(1,t) &= U(t), & (7.96)\\ \hat{Q}(x,0) &= \hat{Q}_0(x) \in H^{4,n}. & (7.97) \end{aligned}$$

Hereinafter, $\hat{Q}(x,t) = [\hat{Q}_1^T(x,t)\hat{Q}_2^T(x,t)]$ is the state estimate, M is a constant square matrix of dimension n_2 to subsequently be designed and

$$G(x) = \begin{bmatrix} G_1^1(x) & G_2^1(x) \\ G_1^2(x) & G_2^2(x) \end{bmatrix} = \begin{bmatrix} G_1(x) & G_2(x) \end{bmatrix} \qquad (7.98)$$

is a square matrix of spatially-dependent observer gains to subsequently be designed as well, where the dimensions of the involved sub-matrices are

$$G_1^1(x) \in \mathbf{R}^{n_1\times n_1},\ G_2^1(x) \in \mathbf{R}^{n_1\times n_2},\ G_1^2(x) \in \mathbf{R}^{n_2\times n_1},\ G_2^2(x) \in \mathbf{R}^{n_2\times n_2}, \qquad (7.99)$$

and $G_1(x) \in \mathbf{R}^{n\times n_1}$, $G_2(x) \in \mathbf{R}^{n\times n_2}$ denote the first n_1 columns and the last n_2 columns of matrix $G(x)$, respectively.

The meaning of the BVP (7.92)-(7.97) is viewed in the weak sense similar to that of system (7.1)-(7.5) is. In order to ensure that the weak solutions of (7.92)-(7.96) evolve in the state space $H^{4,n}$ the IC (7.97) is pre-specified to belong to $H^{4,n}$.

The observation error variables

$$\begin{aligned} \tilde{Q}_1(x,t) &= Q_1(x,t) - \hat{Q}_1(x,t), & (7.100)\\ \tilde{Q}_2(x,t) &= Q_2(x,t) - \hat{Q}_2(x,t), & (7.101)\\ \tilde{Q}(x,t) = [\tilde{Q}_1^T(x,t)\tilde{Q}_2^T(x,t)] &= Q(x,t) - \hat{Q}(x,t), & (7.102) \end{aligned}$$

are defined, on the basis of which the associated BVP

$$\begin{aligned}
\tilde{Q}_{1t}(x,t) &= \Theta_1\tilde{Q}_{1xx}(x,t)+\Lambda_{11}\tilde{Q}_1(x,t)+\Lambda_{12}\tilde{Q}_1(x,t)+G_{11}(x)\tilde{Q}_{1x}(0,t) \\
&+ G_{12}(x)\tilde{Q}_2(0,t), \qquad (7.103)\\
\tilde{Q}_{2t}(x,t) &= \Theta_2\tilde{Q}_{2xx}(x,t)+\Lambda_{21}\tilde{Q}_1(x,t)+\Lambda_{22}\tilde{Q}_1(x,t)+G_{21}(x)\tilde{Q}_{1x}(0,t) \\
&+ G_{22}(x)\tilde{Q}_2(0,t), \qquad (7.104)\\
\tilde{Q}_1(0,t) &= 0, \qquad (7.105)\\
\tilde{Q}_{2x}(0,t) &= -M\tilde{Q}_2(0,t), \qquad (7.106)\\
\tilde{Q}(1,t) &= 0, \qquad (7.107)\\
\tilde{Q}(x,0) &= Q_0(x)-\hat{Q}_0(x), \qquad (7.108)
\end{aligned}$$

is readily derived from (7.1)-(7.5) and (7.92)-(7.97).

The procedure for designing the gain matrices M and $G(x)$ is essentially the same as that adopted in Sect. 4 to deal with the Dirichlet BC case.

The backstepping transformation (4.12), with a $n\times n$ matrix kernel function $P(x,y)$, is employed to map the error system (7.103)-(7.107) into the target error BVP

$$\begin{aligned}
\tilde{Z}_t(x,t) &= \Theta\tilde{Z}_{xx}(x,t)-\bar{C}\tilde{Z}(x,t), \qquad (7.109)\\
\tilde{Z}_1(0,t) &= 0, \qquad (7.110)\\
\tilde{Z}_{2x}(0,t) &= 0, \qquad (7.111)\\
\tilde{Z}(1,t) &= 0. \qquad (7.112)
\end{aligned}$$

where $\bar{C}$ is to be designed in accordance with Theorem 7.1 for enforcing to enforce the exponential stability of the target error BVP. The IC (4.17) and the BC (4.22) on the kernel matrix $P(x,y)$ are derived by similar computations to those made in Section 4.1.

Performing lengthy but straightforward computations,which are similar to those of Section 4.1, one derives the next relation

$$\begin{aligned}
&\tilde{Z}_t(x,t)-\Theta\tilde{Z}_{xx}(x,t)+\bar{C}Z(x,t)= \\
&\left[\Theta P(x,x)-P(x,x)\Theta\right]\tilde{Z}_x(x,t)-\left[G(x)+P(x,0)\Theta\right]\tilde{Z}_x(0,t)-\left[G(x)-P_y(x,0)\Theta\right]\tilde{Z}(0,t)\\
&-\left\{\Theta\left[\frac{d}{dx}P(x,x)\right]+P_y(x,x)\Theta+\Theta P_x(x,x)-\Lambda-\bar{C}\right\}\tilde{Z}(x,t)\\
&-\int_0^x\left[\Theta P_{xx}(x,y)-P_{yy}(x,y)\Theta+P(x,y)\bar{C}+\Lambda P(x,y)\right]\tilde{Z}(y,t)dy \qquad (7.113)
\end{aligned}$$

which, in contrast to its counterpart (4.23) that was previously obtained in the Dirichlet BC case, contains the additional term $\left[G(x)-P_y(x,0)\Theta\right]\tilde{Z}(0,t)$. This term is no longer identically zero due to the different nature of the present plant and target BCs. It is clear that in order to simplify (7.113) to the PDE (4.13) the right-hand side of (7.113) should be identically zero.

Let us further exploit the decomposition

$$P(x,y)=\begin{bmatrix} P_1^1(x,y) & P_2^1(x,y)\\ P_1^2(x,y) & P_2^2(x,y)\end{bmatrix}=\begin{bmatrix} P_1(x,y) & P_2(x,y)\end{bmatrix} \qquad (7.114)$$

of the n-th order kernel square matrix $P(x,y)$ where the dimensions of the involved matrices are as follows

$$P_1^1(x,y)\in\mathbf{R}^{n_1\times n_1},\ P_2^1(x,y)\in\mathbf{R}^{n_1\times n_2},\ P_1^2(x,y)\in\mathbf{R}^{n_2\times n_1},\ P_2^2(x,y)\in\mathbf{R}^{n_2\times n_2}, \qquad (7.115)$$

and $P_1(x,y) \in \mathbf{R}^{n\times n_1}$, $P_2(x,y) \in \mathbf{R}^{n\times n_2}$ denote the first n_1 columns and the last n_2 columns of matrix $P(x,y)$, respectively.

Due to the BC (7.111), only the first n_1 entries $\tilde{Z}_{1x}(0,t)$ of vector $\tilde{Z}_x(0,t)$ are nonzero, due to which the term $[G(x)+P(x,0)\Theta]\,\tilde{Z}_x(0,t)$ in the right-hand side of (7.113) will be identically zero provided that the first n_1 columns of matrix $[G(x)+P(x,0)\Theta]$ are forced to be identically zero as well. This yields the constraint relation

$$G_1(x) \quad = \quad -P_1(x,0)\Theta_1. \tag{7.116}$$

Similarly, due to the BC (7.110), only the last n_2 entries $\tilde{Z}_2(0,t)$ of vector $\tilde{Z}(0,t)$ are nonzero, due to which the term $[G(x)-P_y(x,0)\Theta]\,\tilde{Z}(0,t)$ will be identically zero provided that the last n_2 columns of matrix $[G(x)-P_y(x,0)\Theta]$ are identically zero as well, that yields the constraint relation

$$G_2(x) \quad = \quad P_{2y}(x,0)\Theta_2. \tag{7.117}$$

Additionally, one can derive from (4.18), evaluated at $x=0$, the relation

$$\tilde{Q}_x(0,t) \quad = \quad \tilde{Z}_x(0,t) - P(0,0)\tilde{Z}(0,t). \tag{7.118}$$

Substituting the BCs (7.105)-(7.106) and (7.110)-(7.111) into (7.118), and noticing that the equality $\tilde{Z}(0,t)=\tilde{Q}(0,t)$ is straihforwardly obtained by specifying (4.12) with $x=0$, one derives the condition

$$[M - P_2^2(0,0)]\tilde{Z}_2(0,t) = 0, \tag{7.119}$$

resulting in the additional observer gain tuning rule

$$M = P_2^2(0,0). \tag{7.120}$$

Thus, in order to nullify the right-hand side of (7.113), the BVP

$$\begin{aligned}
\Theta P_{xx}(x,y) - P_{yy}(x,y)\Theta &= -P(x,y)\bar{C} - \Lambda P(x,y), && (7.121)\\
\Theta\frac{d}{dx}P(x,x) + \Theta P_x(x,x) &+ P_y(x,x)\Theta = \Lambda + \bar{C}, && (7.122)\\
P(x,x)\Theta &= \Theta P(x,x), && (7.123)\\
P(1,y) &= 0, && (7.124)
\end{aligned}$$

governing the kernel matrix $P(x,y)$, is straightforwardly derived, and the observer gain tuning rules are obtained in the form

$$\begin{aligned}
G_1(x) &= -P_1(x,0)\Theta_1, && (7.125)\\
G_2(x) &= P_{2y}(x,0)\Theta_2, && (7.126)\\
M &= P_2^2(0,0). && (7.127)
\end{aligned}$$

Since the equi-diffusivity case is the only one for which a viable solution for the state-fedback design has been found, the observer design is now going to be finalized, along with the associated output feedback controller design, for the equi-diffusivity scenario only.

7.2.1. Anti-collocated observer and output feedback designs in the equi-diffusivity case. Specializing system (7.121)-(7.124) with the equi-diffusivity constraint (3.32) and exploiting the identity $\frac{d}{dx}P(x,x) = P_x(x,x) + P_y(x,x)$ yield the same BVP (4.37)-(4.39) that was obtained in the Dirichlet BC case[2] whereas the tuning conditions (7.125)-(7.127) are specified to

$$\begin{aligned} G_1(x) &= -\theta P_1(x,0), && (7.128)\\ G_2(x) &= \theta P_{2y}(x,0), && (7.129)\\ M &= P_2^2(0,0) = -\frac{1}{2\theta}\left(\Lambda_{22} + \bar{C}_{22}\right). && (7.130)\end{aligned}$$

where the BC (4.38) has been used to derive the right-hand side of relation (7.130),

By similar computations as those made in Section 4.1, relation (4.31) is manipulated to (4.38), thus yielding the same BVP (4.37)-(4.39) that was shown in Theorem 4.1 to admit the explicit solution (4.40). Thus, the observer gains become available in explicit form in the present "heterogenous BCs" case as they were in the previously studied Dirichlet BC case.

The following result summarizes the proposed anti-collocated observer design.

THEOREM 7.4. *Let matrix $\bar{C}$ be selected such that $S[\bar{C}] > 0$ and $\sigma_m(S[\bar{C}])$ is arbitrarily large. Then, the observer (7.92)-(7.97),* (7.125)-(7.127)*, where $P(x,y)$ is the explicit solution* (4.40) *to the BVP* (4.37)-(4.39) *reconstructs the state of system* (7.1)-(7.5) *with the associated error decay rate $\sigma_m(S[\bar{C}]$, obeying the estimate*

$$\|\tilde{Q}(\cdot,t)\|_{H^{2,n}} \le b\|\tilde{Q}(\cdot,0)\|_{H^{2,n}} e^{-\sigma_m(S[\bar{C}])t}, \tag{7.131}$$

and a positive constant a b, which is independent of $\tilde{Q}(\xi,0)$.

Proof. The proof is based on the developments made in the present subsection and is formally identical to that of Theorem (4.2). □

The output feedback controller is designed, combining the state feedback controller (7.72), analyzed in Theorem 7.3, and the anti-collocated observer (7.92)-(7.97), (7.125)-(7.127), analyzed in Theorem 7.4. The over-all analysis is presented next.

THEOREM 7.5. *Consider system* (7.1)-(7.9) *driven by the controller*

$$U(t) = \int_0^1 K(1,y)\hat{Q}(y,t)dy \tag{7.132}$$

and fed by observer (7.92)-(7.97), (7.125)-(7.127)*. Let the matrices C and $\bar{C}$ be selected such that $S[C] > 0$ and $S[\bar{C}] > 0$, let $K(1,y)$ be obtained from the solution K(x,y) to the BVP* (7.33), (7.35)-(7.36), (7.42)*, and let $P(x,y)$ be the explicit solution* (4.40) *to the BVP* (4.37)-(4.39)*. Then, the closed-loop system* (7.1)-(7.9)*, (7.92)-(7.97),* (7.125)-(7.127), (7.132) *is exponentially stable in the space $H^{2,n} \times H^{2,n}$.*

Proof. Lengthy but straightforward manipulations show that the backstepping transformation

$$\hat{Z}(x,t) = \hat{Q}(x,t) - \int_0^x K(x,y)\hat{Q}(y,t)dy \tag{7.133}$$

[2]The same BVP was also obtained in [5], where the anti-collocated observer design for the Neumann BC case was addressed in the equi-diffusivity scenario.

maps the observer dynamics (7.92)-(7.96) into the system

$$\hat{Z}_t(x,t) = \Theta\hat{Z}_{xx}(x,t) - \bar{C}\hat{Z}(x,t) + F_1(x)\left[\tilde{Z}_{1x}^T(0,t)\tilde{Z}_2^T(0,t)\right]^T, \quad (7.134)$$

$$\hat{Z}_1(0,t) = 0, \quad (7.135)$$

$$\hat{Z}_{2x}(0,t) = 0, \quad (7.136)$$

$$\hat{Z}(1,t) = 0, \quad (7.137)$$

with

$$F_1(x) = \left[G(x) - \int_0^x K(x,y)G(y)dy\right]. \quad (7.138)$$

The $\tilde{Z}(x,t)$-system, governed by (7.109)-(7.112), is exponentially stable in the space $H^{2,n}$ as well as the homogeneous part of the $\hat{Z}(x,t)$-system (7.134)-(7.137) is if considered separately with the external term $\left[\tilde{Z}_{1x}^T(0,t)\tilde{Z}_2^T(0,t)\right]^T$ deliberately set to zero. Following [32, Sect. 5.1], one notices that the interconnection of the two systems in the ($\hat{Z}$, $\tilde{Z}$) coordinates is in cascade form, and it was shown in Theorem 7.1 that all entries of the forcing term $\left[\tilde{Z}_{1x}^T(0,t)\tilde{Z}_2^T(0,t)\right]^T$ escape "quasi-exponentially" to zero according to (3.8) (see Remark 1). The rest of the proof follows the same arguments used in the proof of Theorem 6.1. Theorem 7.5 is thus proved. □

8. Simulation results. To validate the performance of the proposed state and output-feedback designs for coupled reaction-diffusion processes, hereinafter $n = 2$ coupled processes with heterogenous BCs at the uncontrolled side are considered for simulation purposes. The considered plant is governed by the BVP (7.1)-(7.9) specialized with $n_1 = n_2 = 1$ and with parameters

$$\Theta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} 20 & 5 \\ 5 & 20 \end{bmatrix} \quad (8.1)$$

The ICs are set to $q_1(x,0) = \sin(\pi \cdot x)$, $q_2(x,0) = \cos(\pi/2 \cdot x)$ so as they meet the underlying BCs at $x = 0$. For solving the underlying BVP, a standard finite-difference approximation method is used where the spatial domain $x \in [0,1]$ is discretized into $N = 20$ uniformly spaced solution nodes $x_i = ih$, $h = 1/(N+1)$, $i = 1, 2, ..., N$. The resulting 20-th order discretized system is subsequently solved in the MatLab/Simulink environment by fixed-step Runge-Kutta ODE4 method with sampling period $T_s = 10^{-6}$.

The plant is intentionally chosen to be open-loop unstable, and Figure 8.1 shows the diverging spatiotemporal evolution of the states $q_1(x,t)$, and $q_2(x,t)$ in the open-loop test.

We now implement the state-fedback boundary control input (7.72), with the design matrix $C = I_{2\times 2}$ that meets the conditions of Theorem 7.3. It is worth to remark that the Kernel BVP (7.33), (7.35)-(7.36), (7.42) does not posses an explicit solution. Thus, the gain matrix $K(1,y)$ is evaluated numerically. The profiles of the corresponding kernel gain functions are displayed in the Figure 8.2.

Figure 8.3 shows the closed-loop spatiotemporal evolution of the systems states $q_1(x,t)$ (left plots) and $q_2(x,t)$ (right plots) under the state-feedback boundary controller (7.72). Particularly, the top plots show the exponentially stable long-term evolution, whereas the lower plots focus on the initial transient, showing the typical initial peaking of the state, especially felt near the controlled boundary $x = 1$.

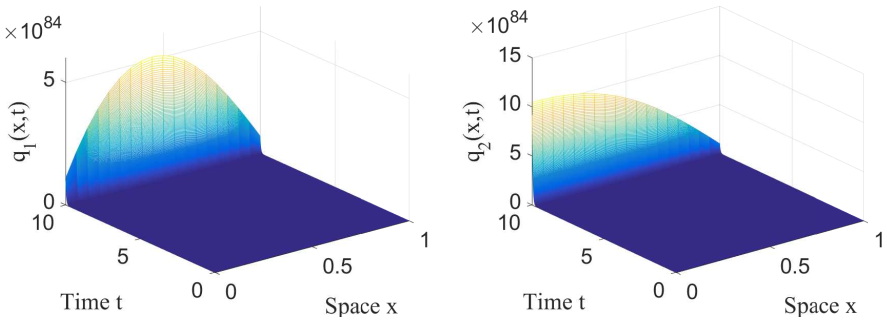

FIG. 8.1. *Spatiotemporal evolution of $q_1(x,t)$ (left plot), and $q_2(x,t)$ (right plot) in the open-loop test.*

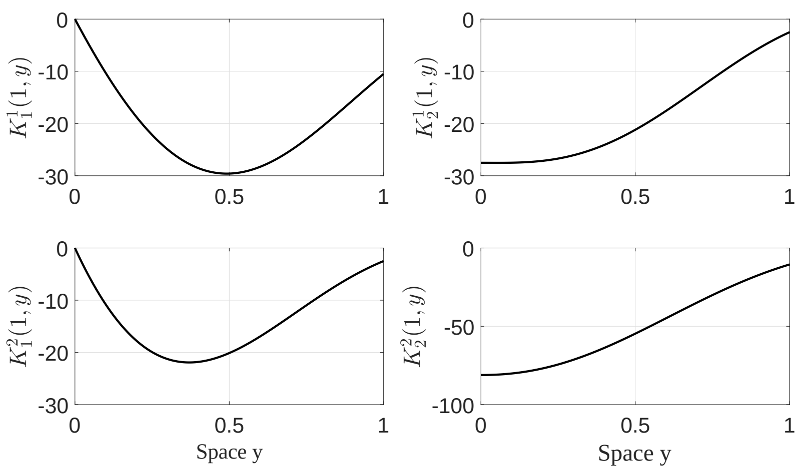

FIG. 8.2. *Spatial distributions of the entries of the Kernel matrix $K(1,y)$.*

To highlight the effect of the vanishing extra term, mentioned in the Remark 3, added to the control law to solve the compatibility issue, a test has been made by implementing the state feedback law (3.42) with $\gamma = 50$. In particular, in Figure 8.4 the spatiotemporal profiles of $q_1(x,t)$ and $q_2(x,t)$ with the controller (3.42) are displayed. These plots, and their comparison with the bottom plots of Figure 8.3, clearly show that the vanishing extra term in the control law successfully solves the compatibility issue and provides peaking attenuation as well.

Figure 8.5 shows the exponentially-decaying temporal profile of the Sobolev norm $\|Q(\cdot,t)\|_{H^{2,2}}$, which confirms the theoretical findings. Finally, in Figure 8.6 the time evolutions of the boundary control laws $u_1(t)$ and $u_2(t)$ without, and with, the vanishing extra term are displayed. As it can be seen, thanks to the vanishing extra term, the initial peaking of the control law, typical of backstepping-based designs, is attenuated significantly.

The response of the closed-loop system with the output-feedback stabilizer (7.132), combined with the anti-collocated observer (7.92)-(7.97) with the design parameter $\bar{C} = 100 \cdot I_{2\times 2}$, is now discussed. Simulations have been performed with the observer's states initialized as follows: $\hat{q}_1(x,0) = \hat{q}_2(x,0) = 0$, that match the underlying BCs at $x = 0$. Figure 8.7 displays the long-term spatiotemporal evolution of the state variables $q_1(x,t)$, and

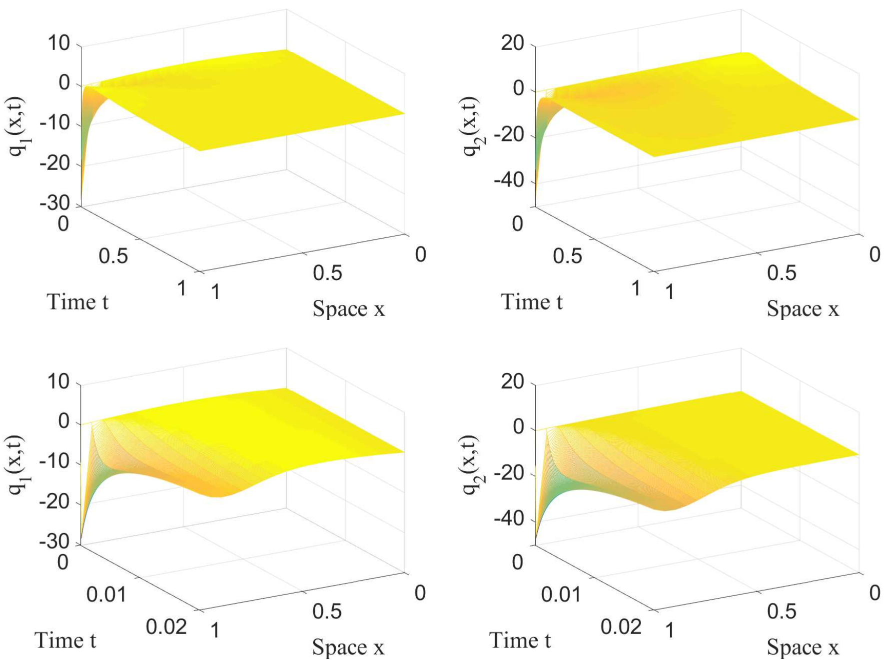

FIG. 8.3. *Spatiotemporal evolutions of* $q_1(x,t)$ *(left plots) and* $q_2(x,t)$ *(right plots) in the closed-loop test with the state-feedback controller* (7.72). *Top plots: long-term evolution. Bottom plots: zoom on the initial transient.*

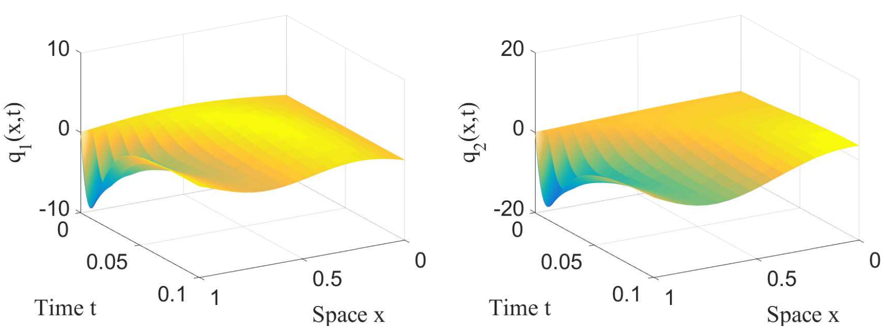

FIG. 8.4. *Spatiotemporal evolution of* $q_1(x,t)$ *(left plot) and* $q_1(x,t)$ *(right plot) with the state-feedback controller* (7.72).

$q_2(x,t)$, which exhibit exponentially vanishing dynamics.

Figure 8.8 shows the temporal evolutions of the state and observation error norms $\|Q(\cdot,t)\|_{H^{2,2}}$ (right plot) and $\|\tilde{Q}(\cdot,t)\|_{H^{2,2}}$ (left plot). Both tend to zero exponentially, thus confirming the correct functioning of the proposed observer-based output-feedback scheme and thereby supporting the theoretical analysis. In addition, it can be noticed that the observation loop has a faster convergence than the control loop, according to the adopted choice of design parameters C and $\bar{C}$.

9. Conclusions. The observer-based output feedback boundary stabilization of some classes of systems of n coupled parabolic linear PDEs has been tackled by exploiting the

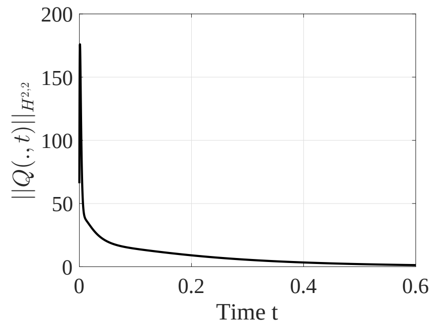

FIG. 8.5. $||Q(\cdot,t)||_{H^{2,2}}$ *norm in the closed-loop test with the state-feedback controller.*

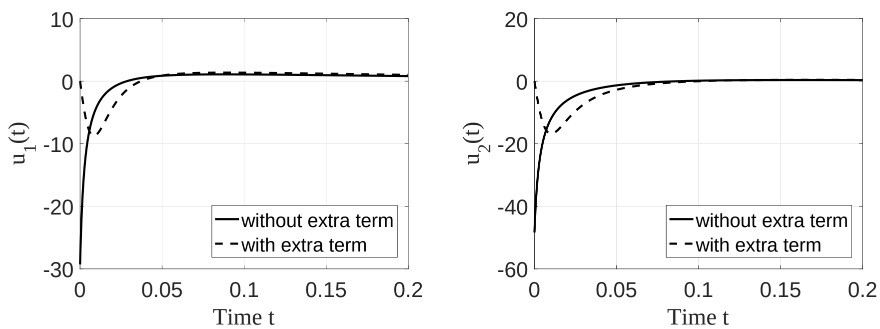

FIG. 8.6. *Boundary controls* $u_1(t)$ *(left plot),* $u_2(t)$ *(right plot) in the closed-loop test with the state-feedback controller.*

backstepping approach. Controllers and observers, the majority of which given in explicit form, have been derived to enforce an arbitrarily fast exponential decay of the state in the space $H^{2,n}$.

Involving spatially and/or temporally dependent parameters into the proposed synthesis and its extension to broader classes of PDEs (e.g., coupled reaction-diffusion-advection PDEs) are of practical interest and among actual challenges, calling for further investigation.

The stabilization of coupled PDEs with heterogenous BCs and distinct diffusivity parameters is another open problem to be addressed by attempting to identify a different exponentially stable target system BVP yielding a solvable kernel PDE for the state-feedback design.

Additionally, integration with other design methodologies, such as the sliding mode approach, is due to enhance the underlying robustness features. Particularly, recent investigations [26]-[30] are hoped to complement the present approach in order to control uncertain DPS' governed by perturbed coupled PDEs of parabolic type.

REFERENCES

[1] O.M. AAMO, A. SMYSHLYAEV, AND M. KRSTIC, "Boundary Control Of The Linearized GinzburgLandau Model Of Vortex Shedding" SIAM J. Contr. Optimizat., 43(2005), 6, 1953-1971.
[2] O.M. AAMO, "Disturbance Rejection in 2 $\times$ 2 Linear Hyperbolic Systems" IEEE Trans. Aut. Contr., 58(2013), 5, 1095-1106.

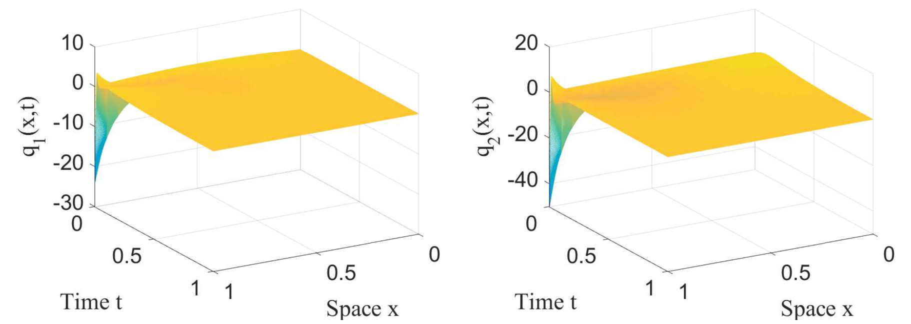

FIG. 8.7. *Spatiotemporal evolution of the state variables in the closed-loop test with the anti-collocated output-feedback stabilizer: $q_1(x,t)$ (left plot), $q_2(x,t)$ (right plot).*

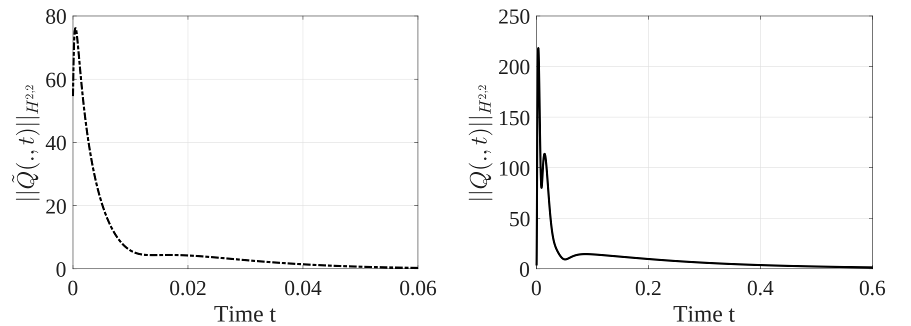

FIG. 8.8. *Temporal evolution of the norms $\|\tilde{Q}(\cdot,t)\|_{H^{2,2}}$ (left plot) and $\|Q(\cdot,t)\|_{H^{2,2}}$ (right plot) with the anti-collocated output-feedback stabilizer.*

[3] A. BACCOLI, Y. ORLOV, A. PISANO, ”On the boundary control of coupled reaction-diffusion equations having the same diffusivity parameters”. Proc. 2014 CDC, pp. 5222-5228, Los Angeles (US).

[4] A. BACCOLI, Y. ORLOV, A. PISANO, “Boundary control of coupled reaction-diffusion processes with constant parameters”. Automatica, 54(2015), 4, 80-90

[5] A. BACCOLI, A. PISANO, “Anticollocated Backstepping Observer Design for a Class of Coupled Reaction-Diffusion PDEs”. Journal of Control Science and Engineering, 164274(2015),

[6] A.G. BUTKOVSKIY “Green’s Functions and Transfer Functions Handbook” Ellis Horwood Lmt., Chichester, 1982.

[7] R.F. CURTAIN, AND H. ZWART. An introduction to infinite-dimensional linear systems theory. Springer, 1995.

[8] F. DI MEGLIO, R. VAZQUEZ, M. KRSTIC AND N. PETIT, ”Backstepping stabilization of an underactuated 3×3 linear hyperbolic system of fluid flow equations”. Proc. 2012 ACC, Montreal, Canada.

[9] F. DI MEGLIO, R. VAZQUEZ AND M. KRSTIC, “Stabilization of a System of Coupled First-Order Hyperbolic Linear PDEs With a Single Boundary Input” IEEE Trans. Aut. Contr., 58(12)(2013), 3097-3111.

[10] P. GRINDROD, ”Patterns and Waves: The Theory and Applications of Reaction-Diffusion Equations” Clarendon Press, 1991.

[11] B.-Z. GUO, F.-F. JIN “Sliding mode control and active disturbance rejection control to the stabilization of one-dimensional Schrdinger equation subject to boundary control matched disturbance” Int. J. Rob. Nonlin. Contr., 24(16)(2014), 2194-2212

[12] Z. HIDAYAT, R. BABUSKA, B. DE SCHUTTER, AND A. NUNNEZ, “Observers for linear distributed-parameter systems: A survey” Proc. 2011 IEEE International Symposium on Robotic and Sensors Environments (ROSE 2011), Montreal, Canada, pp. 166.171, 2011.

[13] E.E. HOLMES ET AL. ”Partial differential equations in ecology: spatial interactions and population dynamics.” Ecology, 75(1)(1994), 17-29

[14] L. JADACHOWSKI, T. MEURER, AND A. KUGI, “State estimation for parabolic PDEs with varying parameters on 3-dimensional spatial domains”. Proc. 18th IFAC World Congress. Milano, Italy, Au-

gust/September 2011, pp. 13338-13343, 2011.
[15] L. JADACHOWSKI, T. MEURER, AND A. KUGI, “Backstepping observers for linear PDEs on higher-dimensional spatial domains” Automatica, 51(2015), 85-97.
[16] M.A. KRASNOSELSKII, P.P. ZABREYKO, E.I. PUSTYLNIK, AND P.E. SOBOLEVSKI, “Integral operators in spaces of summable functions” Noordhoff, Leyden, 1976
[17] M. KRSTIC AND A. SMYSHLYAEV, “Boundary Control of PDEs: A Course on Backstepping Designs” SIAM Advances in Design and Control Series, 2008, ISBN 978-0-89871-650-4.
[18] M. KRSTIC, “Adaptive control of an anti-stable wave PDE” Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis., 17(2010), 853-882.
[19] M. KRSTIC, B.-Z. GUO, AND A. SMYSHLYAEV, “Boundary controllers and observers for the linearized Schrodinger equation” SIAM Journal of Control and Optimization, 49(2011), 1479-1497.
[20] T. LASIECKA AND R. TRIGGIANI. Control theory for partial differential equations: continuous and approximation theories, I abstract parabolic systems. Cambridge University Press, 2000.
[21] W. LIU, “Boundary Feedback Stabilization Of An Unstable Heat Equation” SIAM J. Contr. Opt., 42(2003), 3, 1033-1043.
[22] T. MEURER, AND A. KUGI, “Tracking control for boundary controlled parabolic PDEs with varying parameters: combining backstepping with flatness.” Automatica, 45(2009), 5, 1182-1194.
[23] T. MEURER, “Control of Higher-Dimensional PDEs: Flatness and Backstepping Designs.” Communications and Control Engineering Series, Springer-Verlag, 2012.
[24] S. MOURA, J. BENDTSEN, AND V. RUIZ, “Observer Design for Boundary Coupled PDEs: Application to Thermostatically Controlled Loads in Smart Grids”, Proc. 52nd Conference on Decision and Control (CDC 2013), pp. 6286 - 6291, Florence, 2013.
[25] Y. ORLOV, D. DOCHAIN, “Discontinuous feedback stabilization of minimum-phase semilinear infinite-dimensional systems with application to chemical tubular reactor”. IEEE Transactions on Automatic Control, 47(2002), 8, 1293-1304.
[26] Y. ORLOV, A. PISANO AND E. USAI, ”Continuous state feedback tracking of an uncertain heat diffusion process”. Systems and Control Letters, 59(2010), 754-759.
[27] Y. ORLOV, A. PISANO AND E. USAI, ”Tracking Control of the Uncertain Heat and Wave Equation via Power-Fractional and Sliding-Mode Techniques” SIAM J. Contr. Opt., 49(2011), 363-382.
[28] Y. ORLOV, A. PISANO AND E. USAI, ”Exponential stabilization of the uncertain wave equation via distributed dynamic input extension”. IEEE Trans. Aut. Contr., 56(2011), 212-216.
[29] Y. ORLOV, A. PISANO AND E. USAI, ”Boundary control and observer design for an uncertain wave process by second-order sliding-mode technique”. Proc. 2013 CDC, Florence, Italy.
[30] A. PISANO, Y. ORLOV, “Boundary second-order sliding-mode control of an uncertain heat process with unbounded matched perturbation”, Automatica, 48(2012), 1768-1775.
[31] A. SMYSHLYAEV, M. KRSTIC, “Closed-Form Boundary State Feedbacks for a Class of 1-D Partial Integro-Differential Equations” IEEE Trans. Aut. Contr., 49(12)(2004), 2185-2202.
[32] A. SMYSHLYAEV AND M. KRSTIC, “Backstepping observers for a class of parabolic PDEs”, Syst. Contr. Lett., 54(2005), 613-625.
[33] A. SMYSHLYAEV, M. KRSTIC, “Boundary control of an anti-stable wave equation with anti-damping on uncontrolled boundary”, Syst. Contr. Lett., 58(2009), 617-623.
[34] A. SMYSHLYAEV AND M. KRSTIC, “Adaptive Control of Parabolic PDEs” Princeton University Press, 2010, ISBN 978-0691142869.
[35] D. TSUBAKINO, M. KRSTIC, AND Y. YAMASHITA, ”Boundary Control of a Cascade of Two Parabolic PDEs with Different Diffusion Coefficients”. Proc. 2013 CDC, Florence, Italy.
[36] D. TSUBAKINO, AND S. HARA, “Backstepping observer design for parabolic PDEs with measurement of weighted spatial averages, Automatica, Volume 53, March 2015, Pages 179187
[37] A. VANDE WOUWER, AND M. ZEITZ, “State estimation in distributed parameter systems”. In H. Unbehauen (Ed.), Encyclopedia of life support systems (EOLSS). Oxford, UK: EOLSS Publishers, (Chapter) Control systems, robotics and automation, Article No. 6.43.19.3, 2001.
[38] R. VAZQUEZ, Boundary control laws and observer design for convective, turbulent and magnetohydrodynamic flows, Ph.D. thesis, Univ. California, San Diego, CA, USA, 2006.
[39] R. VAZQUEZ, E. SCHUSTER, AND M. KRSTIC, ”Magnetohydrodynamic state estimation with boundary sensors” Automatica, 44(2008), 2517–2527.
[40] R. VAZQUEZ, E. SCHUSTER, AND M. KRSTIC, ”A Closed-Form Full-State Feedback Controller for Stabilization of 3D Magnetohydrodynamic Channel Flow” ASME J. Dyn. Syst. Meas. Trans. Contr., 131(2009), .
[41] R. VAZQUEZ, AND M. KRSTIC, ”Boundary Observer for Output-Feedback Stabilization of Thermal-Fluid Convection Loop” IEEE Trans. Contr. Syst. Tech., 18(4)(2010), 789-797.
[42] R. VAZQUEZ, M. KRSTIC AND J.-M. CORON, ”Backstepping Boundary Stabilization and State Estimation of a 2×2 Linear Hyperbolic System”. Proc. 2011 joint CDC-ECC, Orlando, Florida.
[43] J.-M. CORON, R. VAZQUEZ, M. KRSTIC, AND G. BASTIN, ”Local exponential H^2 stabilization of a 2×2

quasilinear hyperbolic system using backstepping”, SIAM J. Contr. Opt., 51(2013), 2005–2035.