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Dynamical and Riemannian properties of Engel structures

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To Ramona.

Introduction

Smooth distributions on smooth manifolds, i.e. smooth sub-bundles of the tangent bundle, play a central role in differential geometry. Since the space $\mathrm{Dist}(k, M)$ of distributions of rank k on M is very complicated, it is common to study families of distributions which satisfy some geometric properties. The theory of foliations, for example, studies properties of integrable distributions. We are interested in the study of distributions on the other end of the spectrum: *maximally non-integrable* ones.

A hyperplane distribution (M^n, $\mathcal{H}^{n-1}$) is maximally non-integrable if we can write it locally as the kernel of a 1-form α such that the restriction of $d\alpha$ to $\mathcal{H}$ has maximal rank. If the dimension is odd $n = 2k + 1$, this translates to $\alpha \wedge d\alpha^k \neq 0$ and such distributions are called *contact structures*. If instead $n = 2k+2$ and $\alpha \wedge d\alpha^k \neq 0$, then $\mathcal{H}$ is called *even contact structure*. Notice in the latter that the form $\alpha \wedge d\alpha^k$ is not a top degree form; it is a nowhere vanishing $(n-1)$-form, hence it has a one dimensional kernel $\mathcal{W} = \ker(\alpha \wedge d\alpha^k)$ called *characteristic foliation*. This line field is uniquely determined by the conditions $\mathcal{W} \subset \mathcal{H}$ and $[\mathcal{W}, \mathcal{H}] \subset \mathcal{H}$.

In this thesis we focus on *Engel structures*, which are another family of maximally non-integrable distributions. An Engel structure is a rank 2 distribution $\mathcal{D}$ on a 4-dimensional manifold M which satisfies the conditions

- $\mathcal{E} := [\mathcal{D}, \mathcal{D}]$ is a rank 3 distribution;
- $[\mathcal{D}, \mathcal{E}] = TM$, i.e. $\mathcal{E}$ is even contact.

Let $\mathcal{W}$ denote the characteristic foliation of $\mathcal{E}$. It turns out that these distributions form a flag

$$\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM$$

called the *Engel flag of* $\mathcal{D}$. The existence of this flag gives strong constraints on the topology of M. For instance M has a parallelizable 4-cover. Moreover every $\phi : M \to M$ such that $\phi_*\mathcal{D} = \mathcal{D}$ must verify $\phi_*\mathcal{E} = \mathcal{E}$ and hence $\phi_*\mathcal{W} = \mathcal{W}$. This makes the study of homotopies and symmetries of Engel structures very involved.

We have a Darboux-type theorem for Engel structures, meaning that there exists a *standard model* ($\mathbb{R}^4$, $\mathcal{D}_{st}$) to which every Engel manifold (M, $\mathcal{D}$)

is locally isomorphic. This was proven by Engel in [Eng]. Analogous results hold for contact and even contact structures (see [Car]). In fact the existence of Darboux neighbourhoods, together with maximal non-integrability, characterises them among distributions. More precisely we say that a family of distributions $\mathcal{A} \subset \mathrm{Dist}(k, M)$ is *topologically stable* if it is open and for each point $p \in M$ the germs of diffeomorphisms at p act transitively on the germs of distributions in $\mathcal{A}$. One can rephrase these conditions by saying that there exists a local model $(\mathbb{R}^n, \mathcal{H}_{st})$ to which every distribution $\mathcal{H} \in \mathcal{A}$ is locally diffeomorphic, even after small perturbations. The following result ensures that, together with line fields, (even) contact and Engel structures are the only stable distributions.

Theorem [Car, GeVe]**.** *Given M^n, a family $\mathcal{A} \in \mathrm{Dist}(k, M)$ is topologically stable and maximal if and only if it is one of the following:*

- *n arbitrary and $k = 1$, i.e. line fields;*
- *n odd, $k = n - 1$ and $\mathcal{A}$ is the set of contact structures;*
- *n even, $k = n - 1$ and $\mathcal{A}$ is the set of even contact structures;*
- *$n = 4$, $k = 2$ and $\mathcal{A}$ is the set of Engel structures.*

Line fields [Kat, Sma] and contact structures [Ben, El1, Gei] are extensively studied and stand as active research domains. Even contact structures obey a complete h-principle [McDu] making them very different from contact structures. It is still not known if Engel structures admit a complete h-principle. They appear in many natural constructions dating back to Cartan and more recently many existence results were established [CPPP, PV, Vog3].

In this work we study the relations between Engel structures and other natural geometric structures on M. In particular, we consider compatibility properties with Riemannian metrics and how specific metric properties influence the symmetry of an Engel structure. Moreover, we study dynamical properties of the characteristic foliation $\mathcal{W}$. We are interested in understanding which even contact structures $(\mathcal{W}, \mathcal{E})$ are induced by Engel structures, and we explore some natural conditions on $\mathcal{W}$ which ensure that this happens. We also study the holomorphic analogue of Engel structures. These are holomorphic rank 2 distributions on complex 4-manifolds which satisfy the same non-integrability conditions. We provide examples of such structures on $\mathbb{C}^4$ which are not biholomorphic to the standard one.

Metric properties of Engel structures

The results in Chapter 2 concern Riemannian properties of Engel structures. The main source of inspiration is the theory of contact metric structures and the strategy is to fix some additional structure on $\mathcal{D}$. Namely, two

1-forms α and β satisfying $\mathcal{D} = \ker \alpha \wedge \beta$ and such that

$$\alpha \wedge d\alpha \neq 0, \qquad \alpha \wedge \beta \wedge d\beta \neq 0 \qquad \text{and} \qquad \alpha \wedge d\alpha \wedge \beta = 0,$$

are called *Engel defining forms.* They determine a distribution $\mathcal{R} = \langle T, R \rangle$ transverse to $\mathcal{D}$ via

$$\begin{aligned} i_T(\alpha \wedge d\beta) = 0, \quad \beta(T) = 1, \quad \alpha(T) = 0, \\ i_R(\beta \wedge d\beta) = 0, \quad \beta(R) = 0, \quad \alpha(R) = 1. \end{aligned}$$

This is called the *Reeb distribution associated with* α and β. A natural question to ask is whether we can choose Engel defining forms so that the associated Reeb distribution is integrable. The solution to this problem is not known in general, but it turns out that some metric properties ensure a positive answer in many cases. Since a choice of Engel defining forms α and β determines uniquely the splitting $TM = \mathcal{D} \oplus \mathcal{R}$, it is natural to consider Riemannian metrics g which satisfy $\mathcal{D}^\perp = \mathcal{R}$.

Theorem. *Let $\mathcal{D} = \ker \alpha \wedge \beta$ be Engel and let g be a metric such that $\mathcal{D}^\perp = \mathcal{R}$. If $\mathcal{D}$ is totally geodesic, then there exists a nowhere vanishing function $\mu \in \mathcal{C}^\infty(M)$ such that $\tilde{\beta} = \mu\beta$ satisfies $d\tilde{\beta}^2 = 0$. In particular, $\tilde{\mathcal{R}}$ is integrable.*

The converse of this result is most likely false, but we could not provide a counterexample.

Other conditions for integrability are given by the existence of *Engel vector fields* (i.e. vector fields whose flow preserves $\mathcal{D}$) with some additional properties. A *K-Engel structure* is an Engel structure $(M, \mathcal{D})$ together with a metric g and a vector field Z which is Engel, Killing and orthogonal to $\mathcal{E}$. The existence of such a vector field allows the construction of defining forms α and β satisfying $d\alpha^2 = 0 = d\beta^2$ and $\beta \wedge d\alpha = 0$. This in turn provides a framing $\{W, X, T, R\}$ for TM with $\mathcal{W} = \langle W \rangle$, $\mathcal{D} = \langle W, X \rangle$ and such that $R = Z$ commutes with all other vector fields in the framing. This gives strong constraints on the topology of the manifold and leads to the following classification result.

Theorem. *If $(M, \mathcal{D})$ admits a K-Engel structure, then M is diffeomorphic to one of the following:*

- *T^4;*
- *$L(p, q) \times S^1$ or $S^2 \times T^2$;*
- *a T^2-bundle over a surface;*
- *an S^1-bundle over a 3-manifold.*

Here $L(p,q)$ denotes the (p,q)-lens space, moreover we can very precisely describe the bundle structures involved. It is unclear if each of these manifolds admits a K-Engel structure and this question will be the subject of a future work. The most interesting case here is that of S^1-bundles over 3-manifolds, because all of these examples come from the Engel Boothby-Wang construction. This is the analogue of the Boothby-Wang construction of K-contact manifolds, and it also appeared in [Mit] under the name of prequantum prolongation. A remarkable property of these structures is that they all admit a *contact filling*. This means that they can be realized as boundaries of a contact manifold (X, η) so that a collared neighbourhood of ∂X is isomorphic to the *contactization* of M, i.e. to the contact manifold $(M \times \mathbb{R}, \eta := \beta + s\alpha)$. This is analogous to the notion of symplectic filling of a contact structure, but it is unclear if this is a manifestation of rigidity for Engel structures.

Dynamical properties of Engel structures

The main goal of Chapter 3 is to study the following

Question. If $(M^4, \mathcal{E})$ is even contact, does there exist an Engel structure $\mathcal{D}$ such that $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$?

There are some obvious topological constraints, since a 4-cover of an Engel manifold is parallelizable. For this reason we additionally assume that $\mathcal{E} = \ker \alpha$ and that we have a framing $\mathcal{E} = \langle W, A, B\rangle$ where W spans the characteristic foliation. We consider homotopy classes of plane fields $\mathcal{D}_L = \langle W, L\rangle$ where $L \in \mathcal{E}$, and we look at the action on L of the flow ϕ_t of W. For $p \in M$ and $t \in \mathbb{R}$, we write

$$\Big(T_{\phi_t(p)}\phi_{-t}\Big) L(\phi_t(p)) = \rho(p;t)\Big(\cos\theta(p;t)\ A(p) + \sin\theta(p;t)\ B(p)\Big)$$

modulo $\mathcal{W}(p)$. It is a classical fact that $\mathcal{D}_L$ is Engel if and only if L *rotates without stopping*, i.e. $\partial_t\theta(p;t) > 0$ for a suitable choice of orientation. The idea is then to study the function $\theta(p;t)$ up to homotopy of L as a section of $\mathcal{E}$ never tangent to $\mathcal{W}$.

If $p \in \gamma$ closed orbit of W of period T, we define the *rotation number of* $L(p)$ to be

$$\mathrm{rot}_\gamma(L(p)) = \theta(p;T) - \theta(p;0).$$

The main issue with this definition is that $\mathrm{rot}_\gamma(L(p))$ is not invariant under homotopies of L in the general case. In fact, we need to take into account the *type* of the orbit γ (see Section 1.3.2). Indeed for parabolic and hyperbolic orbits we can have that the rotation number of $L(p)$ is not positive and yet L is homotopic to an Engel structure.

It turns out that we need to consider all possible *initial phases*. Let R_η be the rotation of S^1 of angle η and consider $R_\eta \circ L$, where we rotate the plane $\langle A, B\rangle$. Then we consider the maximum of the rotation number with respect to these rotations $\max(\mathrm{rot}_\gamma(L))$ and we have the following result.

Theorem. *Let $\mathcal{E} = \langle W, A, B\rangle$ be an even contact structure and γ a closed orbit for W. Then L is Engel on γ up to homotopy within $(\mathcal{W}, \mathcal{E})$ if and only if there exists a point $p \in \gamma$ such that* $\max(\mathrm{rot}_\gamma(L)) > 0$.

For elliptic closed orbits γ the rotation number does not depend on the initial phase, hence it is invariant under homotopies of L. We observe that if γ bounds an embedded disc $D^2 \hookrightarrow M$ then the class of vector fields L on the disc is unique up to homotopy, which gives us the following result.

Corollary. *Let $\mathcal{E} = \langle W, A, B\rangle$ and L be as above, and γ an elliptic null-homotopic closed orbit for W. Then L is homotopic to an Engel structure on a neighbourhood of γ if and only if there is a point $p \in \gamma$ such that* $\mathrm{rot}_\gamma(L'(p)) > 0$ *for any choice of non-singular $L' \in \Gamma\langle A, B\rangle$.*

It is very likely that other obstructions of this type arise if we consider more complicated points in the non-wandering set of W. On the other hand, if the dynamics of W are very simple, then positivity of the rotation number on closed orbits is also sufficient for the existence of $\mathcal{D}$. More precisely we consider the case where W is a non-singular Morse-Smale (NMS) vector field (see Section 3.5). The dynamics of these vector fields are completely determined by the behaviour near finitely many closed orbits. This is the reason why the rotation number is so powerful in this context.

Theorem. *Let $(M, \mathcal{E})$ be a closed, oriented even contact 4-manifold such that we have a framing $\mathcal{E} = \langle W, A, B\rangle$ with $\mathcal{W} = \langle W\rangle$. Suppose moreover that W is Morse-Smale. Then there exists a positive Engel structure $\mathcal{D}$ inducing $\mathcal{E}$ if and only if there exists a vector field $L \in \langle A, B\rangle$ such that* $\max(\mathrm{rot}_\gamma(L)) > 0$ *for all γ characteristic closed orbits.*

Remarkably we do not explicitly use the fact that $\mathcal{E}$ is an even contact structure. The important hypothesis here is that $\mathcal{E} = \langle W, A, B\rangle$ and that the flow of W preserves $\mathcal{E}$. For a fixed rank 3 distribution $\mathcal{E}$ on a manifold M^n, we say that a plane field $\mathcal{D} \subset \mathcal{E}$ is *non-integrable within $\mathcal{E}$* if $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$. Exactly the same proof yields the following result.

Theorem. *Let $\mathcal{E} \hookrightarrow TM^n$ be a distribution of rank 3 with the above properties and γ a closed orbit for W. A section L of $\mathcal{E}$ never tangent to $\mathcal{W}$ is maximally non-integrable within $\mathcal{E}$ on γ up to homotopy if and only if* $\max(\mathrm{rot}_\gamma(L)) > 0$.

The special case $n = 3$ and $\mathcal{E} = TM$ furnishes obstructions to the existence of contact structures $\mathcal{D}$ on M for which W is a Legendrian line field. Using the same argument as above we can characterise non-singular Morse-Smale vector fields on 3-manifolds which are Legendrian.

Theorem. *Let M be a closed, oriented 3-manifold such that we have a framing $TM = \langle W, A, B \rangle$. Suppose moreover that W is Morse-Smale. Then there exists a positive contact structure $\mathcal{D}$ such that W is Legendrian if and only if there exists a vector field $L \in \langle A, B \rangle$ such that $\max(\text{rot}_\gamma(L)) > 0$ for all γ closed orbits of W.*

Exotic holomorphic Engel structures on $\mathbb{C}^4$

In Chapter 4 we present some results on the holomorphic analogues of Engel structures. All these results are obtained in collaboration with Rui Coelho and contained in the paper [CoP]. The definitions of contact, even contact and Engel structures still make sense in the holomorphic category. Moreover we have a holomorphic version of the standard models and holomorphic Darboux-type theorems.

There exists a classification of projective contact structures [KPSW] and a partial classification of projective Engel structures [PrSC]. On the other hand, holomorphic contact structures on open manifolds are not well understood. An example of a holomorphic contact structure on $\mathbb{C}^{2n+1}$ which is not standard is given in [For1]. We provide examples of holomorphic Engel structures on $\mathbb{C}^4$ which are not standard.

Given $\mathcal{H} \subset TX$ a holomorphic distribution, an *$\mathcal{H}$-line* is a holomorphic map $f : \mathbb{C} \to X$ which is tangent to $\mathcal{H}$ and not constant. The standard Engel structure $(\mathbb{C}^4, \mathcal{D}_{st})$ admits plenty of such tangent lines. Controlling the geometry of such lines we can exhibit many examples of exotic holomorphic Engel structures on $\mathbb{C}^4$.

Theorem [CoP]**.** *On $\mathbb{C}^4$ there are Engel structures $\mathcal{D}_{\mathcal{E}}$, $\mathcal{D}_{\mathcal{D}}$ and $\mathcal{D}_{\mathcal{W}}$ with the following properties*

1. *$\mathcal{D}_{\mathcal{E}}$ admits no lines tangent to its induced even contact structure;*
2. *$\mathcal{D}_{\mathcal{D}}$ admits no $\mathcal{D}_{\mathcal{D}}$-lines but does admit lines tangent to its induced even contact structure;*
3. *$\mathcal{D}_{\mathcal{W}}$ admits no lines tangent to its characteristic foliation but does admit $\mathcal{D}_{\mathcal{W}}$-lines.*

In particular these Engel structures are pairwise non-isomorphic and not isomorphic to the standard Engel structure $(\mathbb{C}^4, \mathcal{D}_{st})$.

Theorem [CoP]**.** *For every $n \in \mathbb{N} \cup \{\infty\}$ there exists an Engel structure $\mathcal{D}_n$ on $\mathbb{C}^4$ for which the only $\mathcal{D}_n$-lines are tangent to the characteristic foliation $\mathcal{W}_n$, and such that*

$$L_n := \{p \in \mathbb{C}^4 : \exists f : \mathbb{C} \to \mathbb{C}^4 \ \mathcal{D}_n\text{-line with } f(0) = p\}$$

is a proper subset of $\mathbb{C}^4$ which has exactly n connected components for $n \in \mathbb{N}$, and $L_\infty = \mathbb{C}^4$.

Theorem [CoP]**.** *For every* $R \in \mathbb{R} \setminus \{0\}$ *there exists an Engel structure* $\mathcal{D}_R$ *for which the only* $\mathcal{D}_R$*-lines are tangent to the characteristic foliation* $\mathcal{W}_R$*, and such that the set of points which admit such* $\mathcal{W}_R$*-lines is exactly* $\mathbb{C} \times \{0, 1, R\sqrt{-1}\} \times \mathbb{C}^2 \subset \mathbb{C}^4_{(w,x,y,z)}$*. Moreover* $\mathcal{D}_R$ *is isomorphic to* $\mathcal{D}_{R'}$ *if and only if* $R = R'$*.*

Contents

Chapter 1

First results on Engel structures

In this chapter we will introduce the basic notions that are needed throughout the rest of the thesis. In Section 1.1 we present some facts concerning the general theory of distributions on manifolds. In particular we study properties of non-integrable distributions and foundational results such as Chow's Theorem. We introduce the concept of *stable distribution* and present the Darboux models for contact, even contact and Engel structures. The main reference for this section is [Mont1].

In Section 1.2 we present basic results on contact structures. Some aspects of the theory of Engel structures are tightly linked to contact structures, for this reason we will take inspiration from the contact case, especially in Chapter 2. We present local neighbourhood theorems, give an overview on existence results and discuss symplectic fillings. We also give basic definitions in contact metric geometry and introduce K-contact manifolds. The main references are [Gei] for the contact topology part and [Bla] for the contact metric one.

In Section 1.3 we list several properties of even contact structures. After a complete h-principle was established in [McDu], the theory of even contact structures was not developed much further. Here we present the basic theory as well as some results on even contact structures admitting characteristic foliations with special dynamical properties (see also [KV]).

In Sections 1.4 and 1.6 we present the basic theory of Engel structures and the topological constraints that a manifold has to satisfy in order to admit an Engel structure. We will define the Engel flag of $(M, \mathcal{D})$ and discuss its orientability and consequences on the topology of M. In the chapters that follow we will always suppose that M and $\mathcal{D}$ are orientable, so that one can find a framing of $\mathcal{D}$ and extend it to a framing of the whole TM. Section 1.5 furnishes an extensive list of constructions of Engel manifolds. The main references for these sections are [P1, Mont1, Vog1].

Finally in Section 1.7 we give an informal overview of existence results for Engel structures. The goal is to present the main ideas of the works [Vog3], [CPPP] and [PV].

1.1 Stable distributions

All the objects that we will deal with throughout the thesis are assumed to be smooth, if not otherwise specified. The main object of study of this work are *distributions* on manifolds. For a given smooth manifold M a distribution $\mathcal{H}$ of rank k is a subbundle of rank k of the tangent bundle $\mathcal{H} \to TM$. Locally $\mathcal{H}$ is a smooth map to the Grassmannian manifold of k-planes $\mathbb{R}^n \to \mathrm{Gr}(\mathbb{R}^n, k)$. This means that to each point $p \in M$ we smoothly associate a subspace $\mathcal{H}_p \subset T_pM$ of rank k. Equivalently one can think of $\mathcal{H}$ as the sheaf associating to an open neighbourhood $U \subset M$ the collection $\mathcal{H}|_U$ of vector fields defined on U and tangent to some subspace of TM.

For a given distribution $\mathcal{H}$ of rank k, a very natural question to ask is if it can be realized as the tangent bundle of a family of submanifolds of M of dimension k. If this happens we say that $\mathcal{H}$ is *integrable*. Frobenius' Theorem gives an answer in terms of commutators of vector fields tangent to $\mathcal{H}$. We will denote such vector fields by $X \in \Gamma\mathcal{H}$. Thinking of $\mathcal{H}$ as a sheaf we define

$$[\mathcal{H}, \mathcal{H}] = \operatorname{span}\left\{[X, Y] \,\middle|\, X, Y \in \Gamma\mathcal{H}\right\},$$

that is the sheaf spanned by the Lie brackets of vector fields tangent to $\mathcal{H}$. Moreover we define $\mathcal{H}^2 := \mathcal{H} + [\mathcal{H}, \mathcal{H}]$ and inductively $\mathcal{H}^n := \mathcal{H} + [\mathcal{H}, \mathcal{H}^{n-1}]$. It is important to notice that $\mathcal{H}^2$ is only a subsheaf of TM and in general not a distribution, because the rank of $\mathcal{H}^2_p$ may vary with p.

Frobenius' Theorem asserts that $\mathcal{H}$ is integrable if and only if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$ (for a proof see [Lee]). We are interested in distributions satisfying integrability conditions which are, in some sense, on the opposite hand of the spectrum. By definition we have $\mathcal{H} \subset \mathcal{H}^2 \subset \cdots \subset TM$, we say that $\mathcal{H}$ is *bracket-generating* if $\mathcal{H}^N = TM$ for some $N \in \mathbb{N}$. The *growth vector* of a bracket-generating distribution $\mathcal{H}$ at $p \in M$ is the collection of natural numbers

$$\left(\operatorname{rk}\mathcal{H}_p, \operatorname{rk}\mathcal{H}^2_p, \cdots, \operatorname{rk}\mathcal{H}^N_p\right).$$

Hence it keeps track of the rate of growth of $\mathcal{H}$ when we take successive commutators. We say that $\mathcal{H}$ is *strongly bracket-generating* if its growth vector is the maximal possible at every p.

The *curvature* of a distribution $\mathcal{H}$ is the linear bundle maps

$$F : \mathcal{H} \wedge \mathcal{H} \to TM/\mathcal{H} \qquad \text{s.t.} \qquad F(X, Y) = [X, Y] \mod \mathcal{H}.$$

A hyperplane distribution $\mathcal{H}$ on M is *maximally non-integrable* if its curvature is non-degenerate.

Example 1.1.1. Consider the distribution $\mathcal{H} = \langle \partial_x, \partial_y + x^2\partial_z \rangle$ on $\mathbb{R}^3 = \{(x, y, z)\}$. It is easy to see that $\mathcal{H}^2 = \mathcal{H} \oplus \langle x\partial_z \rangle$, which is not a distribution because $\mathcal{H}^2 = T\mathbb{R}^3$ outside the (y, z)-plane π_{yz}, whereas $\mathcal{H}^2 = \mathcal{H}$ on π_{yz}. On the other hand $\mathcal{H}^3 = T\mathbb{R}^3$, so that $(2, 3, 3)$ outside π_{yz} and $(2, 2, 3)$ on π_{yz}. This distribution is hence bracket-generating but not strongly bracket-generating.

Take instead $\mathcal{H} = \langle \partial_x, \partial_y + x\partial_z, \partial_u, \partial_v \rangle$ on $\mathbb{R}^5 = \{(x, y, z, u, v)\}$. Obviously $\mathcal{H}^2 = T\mathbb{R}^5$ so this distribution is strongly bracket-generating but its curvature is degenerate, because ∂_u and ∂_v are contained in its kernel. Hence $\mathcal{H}$ is not maximally non-integrable.

Finally $\mathcal{H} = \langle \partial_x, \partial_y + x\partial_z, \partial_u, \partial_v + u\partial_z \rangle$ on $\mathbb{R}^5 = \{(x, y, z, u, v)\}$ is maximally non-integrable.

One can distinguish between different degrees of non-integrability by studying (families of) submanifolds of M which are $\mathcal{H}$-horizontal, i.e. everywhere tangent to $\mathcal{H}$. It turns out that $\mathcal{H}$-horizontal curves are already very interesting. If $\mathcal{F}$ is a foliation, we can only join to points $p, q \in M$ by a curve tangent to $T\mathcal{F}$ if p, q are in the same leaf of $\mathcal{F}$. Bracket-generating distributions behave very differently as the following classical result ensures (see [Gro1] page 95 for a proof).

Theorem 1.1.2 (Chow's Theorem)**.** *Let M be smooth and $\mathcal{H} \to TM$ a bracket-generating distribution. Any two points $p, q \in M$ can be joined by a $\mathcal{H}$-horizontal path γ. Moreover if $\mathcal{H}$ admits a global framing $\{X_1, ..., X_k\}$, the path γ can be chosen to be a piecewise-smooth glueing of orbits of the X_i.*

Non-integrable distributions are natural in the sense that a generic distribution $\mathcal{H}$ on M is (maximally) non-integrable on a open dense subset of M [MoZhi]. We put the compact-open topology on the space $\mathrm{Dist}(k, M)$ of all distributions of rank k on M^n. This space is a Fréchet space with the structure given by the limit of the $\mathcal{C}^k$-norm topologies (see [Mont1]).

Definition 1.1.3. *A family of distributions $\mathcal{A} \subset \mathrm{Dist}(k, M)$ is* topologically stable *if it is open and the germs of diffeomorphism at $p \in M$ act transitively on the germs of distributions in $\mathcal{A}$.*

A way of rephrasing this is that there exists a model structure $(\mathbb{R}^n, \mathcal{H}_{st})$, called *Darboux model*, such that every distribution in $\mathcal{A}$ is locally diffeomorphic to the Darboux model.

Example 1.1.4. The following are examples of stable distributions, many of them date back to Cartan [Car, Eng].

Line fields: distributions of rank 1. These are automatically integrable and the Flow Box Theorem furnishes the local model

$$\Big(\mathbb{R}^n, \langle \partial_{x_1} \rangle\Big).$$

Contact structures: distributions ξ of corank 1 on odd dimensional manifolds M^{2n+1} which are maximally non-integrable. This means that around each point $p \in M$ there exists a 1-form η such that $\xi = \ker \eta$ and $\eta \wedge d\eta^n \neq 0$. If we choose coordinates $(x_1, y_1, ..., x_n, y_n, z)$ on $\mathbb{R}^{2n+1}$, the Darboux model for ξ is

$$\left(\mathbb{R}^{2n+1}, \xi_{st} = \ker\left(dz - \sum_{i=1}^{n} x_i dy_i\right)\right).$$

Even contact structures: the even-dimensional counterpart of contact structures are distributions $\mathcal{E}$ of corank 1 on even dimensional manifolds M^{2n+2} which are maximally non-integrable. This can be reformulated by saying that around each point $p \in M$ there exists a 1-form α such that $\xi = \ker \alpha$ and $\alpha \wedge d\alpha^n \neq 0$. If we choose coordinates $(w, x_1, y_1, ..., x_n, y_n, z)$ on $\mathbb{R}^{2n+2}$, the Darboux model for $\mathcal{E}$ is

$$\left(\mathbb{R}^{2n+2}, \mathcal{E}_{st} = \ker\left(dz - \sum_{i=1}^{n} x_i dy_i\right)\right).$$

Engel structures: maximally non-integrable rank 2 distributions $\mathcal{D}$ on 4-manifolds, i.e. $[\mathcal{D}, \mathcal{D}] =: \mathcal{E}$ is a rank 3 distribution and $[\mathcal{D}, \mathcal{E}] = TM$. Choosing coordinates (w, x, y, z) on $\mathbb{R}^4$ we have the Darboux model

$$\left(\mathbb{R}^4, \mathcal{D}_{st} = \ker(dy - zdx) \cap \ker(dz - wdx)\right).$$

The following result is a classification of topologically stable distributions.

Theorem 1.1.5 [Car, GeVe]**.** *Given M^n a family $\mathcal{A} \in \mathrm{Dist}(k, M)$ is topologically stable if and only if it is one of the following*

- *n arbitrary and $k = 1$, i.e. $\mathcal{A}$ is the family of line fields;*
- *n odd, $k = n - 1$ and $\mathcal{A}$ is the set of contact structures;*
- *n even, $k = n - 1$ and $\mathcal{A}$ is the set of even contact structures;*
- *$n = 4$, $k = 2$ and $\mathcal{A}$ is the set of Engel structures.*

Notice that, since these families are all open, to show that they are stable it suffices to prove that they admit Darboux models. This is what we refer to as *Darboux-type* theorems. Explicitely for Engel structures this means that every Engel structure $(M, \mathcal{D})$ is locally isomorphic to the local model $(\mathbb{R}^4, \mathcal{D}_{st})$ given in Example 1.1.4.

1.2 Contact structures

Definition 1.2.1. *A contact structure ξ is a maximally non-integrable hyperplane field on an odd dimensional manifold N^{2n+1}. This means that ξ is locally the kernel of a 1-form η satisfying $\eta \wedge d\eta^n \neq 0$.*

A contact structure is *coorientable* if it admits a global defining form η. In what follows we will mostly consider coorientable contact structures ξ. The choice of a defining form η uniquely determines a transverse vector field R_η via

$$i_{R_\eta} d\eta = 0 \qquad \text{and} \qquad \eta(R_\eta) = 1.$$

R_η is called *Reeb vector field* associate to η. A vector field $L \in \mathfrak{X}(N)$ contained in ξ is called *Legendrian vector field*, whereas a vector field $V \in \mathfrak{X}(N)$ whose flow preserves the contact structure is called *contact vector field*. In other words if V is a contact vector field and ϕ_t^V is its flow at time $t \in \mathbb{R}$, then the tangent map to the flow satisfies $(T\phi_t^V)\xi = \xi$.

Remark 1.2.2. Once we fix a defining form $\xi = \ker \eta$ there is a 1-to-1 correspondence between contact vector fields and smooth functions on N. To any contact vector field V we associate a *contact Hamiltonian* by $H = \eta(V)$. Conversely for a contact Hamiltonian H there exists a unique contact vector field V_H satisfying $\eta(V_H) = H$ and

$$i_{V_H} d\eta = (\mathcal{L}_{R_\eta} H)\, \eta - dH.$$

The Reeb vector field is the contact vector field associated with the Hamiltonian $H = 1$.

There is a geometric interpretation of the contact condition in dimension 3. First of all notice that (N^3, ξ) is a contact structure if and only if it is bracket-generating. Moreover this is equivalent to the fact that if we move along a Legendrian line the contact structure *rotates without stopping.* This notion will be made more precise in Chapter 3 and is basically contained in the following result.

Lemma 1.2.3. *Let $(N^{2n+1}, \xi = \ker \eta)$ be a compact contact structure and $L \in \mathfrak{X}(N)$ a non-singular Legendrian vector field, whose flow is denoted by ϕ_t. There exists $\epsilon > 0$ such that $\xi_t = \phi_{t*}\xi$ is transverse to $\xi = \xi_0$ for all $t \in (0, \epsilon)$.*

Proof. Since ξ is coorientable there is another Legendrian vector field $\tilde{L}$ such that $\eta\left([L, \tilde{L}]\right) \neq 0$. Without loss of generality we can suppose $\eta\left([L, \tilde{L}]\right) > 0$. For any $p \in N$ a direct calculation yields

$$0 < \eta\left([L, \tilde{L}]\right)_p = \frac{d}{dt}\bigg|_{t=0} \eta_p\Big(T_{\phi_t(p)}\phi_{-t}\, \tilde{L}\Big).$$

Hence the function $\alpha_p\left(T_{\phi_t(p)}\phi_{-t}\,\tilde{L}\right)$ has positive derivative near $t=0$, so it is increasing and hence positive on $(0,\epsilon)$, for some $\epsilon>0$. Since M is compact, up to taking a smaller ϵ, we can suppose that this holds for all $p\in N$. The hyperplane $\phi_{t*}\xi$ contains the line field $\phi_{t*}\tilde{L}$, hence we conclude. □

As anticipated in the previous section, we have a Darboux-type theorem which ensures that every contact structure (N^{2n+1},ξ^{2n}) is locally isomorphic to the model in Example 1.1.4. The study of deformations of contact structures, i.e. of smooth families ξ_t for $t\in[0,1]$ of contact structures, is very rich and gives information about the topology of the manifold N. The first step in understanding these deformations is the following stability theorem.

Theorem 1.2.4 (Gray's Stability). *Let N^{2n+1} be a closed manifold and ξ_t a smooth family of contact structures for $t\in[0,1]$. Then there exists an isotopy $\psi_t : N\to N$ such that $\xi_t=\psi_t^*\xi_0$ for all $t\in[0,1]$.*

Theorem 1.2.4 can be proven using a method called the *Moser's trick*. This consists in constructing ψ_t as the flow of a time dependent vector field obtained by imposing the condition $\phi_t^*\eta_t=\lambda_t\eta_0$ for η_t defining form for ξ_t. The same technique permits to prove the following generalization of the Darboux theorem referred to as *Contact Weinstein Neighbourhood Theorem*.

Theorem 1.2.5. *Let N^{2n+1} be a smooth manifold (non-necessarily compact) and $M\to N$ a closed embedded submanifold. If η_0 and η_1 are contact forms on N such that*

$$\eta_0|_M=\eta_1|_M \qquad \textit{and} \qquad d\eta_0|_M=d\eta_1|_M\,, \tag{1.1}$$

then there exists a tubular neighbourhood νM and a map $\psi:\nu M\to\nu M$ such that $\psi^*\eta_1=f\eta_0$ on νM for some positive $f:\nu M\to\mathbb{R}$.

Proof. Consider the family of 1-forms $\eta_t=(1-t)\eta_0+t\eta_1$. Equation (1.1) ensures that $\eta_t\wedge d\eta_t^n$ is non-singular on M. Hence there is a neighbourhood νM on which η_t is contact. We use Moser's trick to look for a time dependent vector field $X_t\in\mathfrak{X}(\nu M)$ whose flow ϕ_t satisfies the equation $\phi_t^*\eta_t=\lambda_t\eta_0$. Since the tubular neighbourhood is not compact, we have to make sure that the flow of X_t is defined everywhere for $t=1$. This can be ensured, thanks to Equation (1.1), indeed this allows $X_t=0$ on M.

Derivating $\phi_t^*\eta_t=\lambda_t\eta_0$ yields

$$\dot{\eta}_t+i_{X_t}d\eta_t+d(\eta_t(X_t))=0 \tag{1.2}$$

Since η_t is contact we can write $X_t=L_t+H_tR_t$ where R_t is the Reeb vector field of η_t, L_t is Legendrian and H_t is a time dependent contact Hamiltonian. Now Equation (1.2) rewrites as

$$\eta_1-\eta_0+i_{L_t}d\eta_t+dH_t=0.$$

The contact condition implies that every form λ such that $\lambda(R_t) = 0$ can be written in the form $\lambda = i_{L_t} d\eta_t$ for some Legendrian L_t. This fixes the Legendrian component of X_t. To find the transversal component it now suffices to solve

$$\mathcal{L}_{R_t} H_t = (\eta_0 - \eta_1)(R_t).$$

The previous equation admits local solutions, because R_t is non-singular. Moreover these solutions are unique since the equation is linear, hence they glue together. This gives $X_t = L_t + H_t R_t$ whose flow is the isotopy ψ_t. □

The foundational results [El1] and [BEM] established the existence of a model contact structure (D^{2n+1}, ξ_{ot}) called *overtwisted disc*. A contact structure (N, ξ) is *overtwisted* if there is a contact embedding of the overtwisted disc $(D^{2n+1}, \xi_{ot}) \hookrightarrow (N, \xi)$. These structures are flexible in the sense that they satisfy a complete h-principle. In particular two overtwisted contact structures which are homotopic as plane fields are isotopic as contact structures. What makes the study of contact structures interesting from a topological point of view is the existence of structures (N, ξ) which are not overtwisted, they are called *tight*. This rigidity phenomenon, which appeared for the first time in the seminal paper [Ben], permits use the space of (tight) contact structures on N to study the topology of N. In the next section we will provide some conditions which imply tightness.

1.2.1 Symplectic fillings and metric contact structures

There are many conditions on a contact manifold (N, ξ) which imply tightness [Gei, Ho]. Many of them involve the existence of a *filling* with specific properties.

Definition 1.2.6. *Let $(N, \xi = \ker \eta)$ be a contact structure.*

- *A* weak symplectic filling *of (N, ξ) is a symplectic manifold (W, ω) such that $\partial W = N$ and $\omega|_\xi > 0$;*
- *A* strong symplectic filling *of (N, ξ) is a symplectic manifold (W, ω) such that $\partial W = N$, the symplectic form ω is exact near the boundary and $\omega = d\eta$.*

A strong symplectic filling is also a weak symplectic filling, moreover if (N, ξ) is weakly symplectically fillable then it is tight, but the converse is not true [EH]. Notice that in the definition of strong symplectic filling the condition $\omega = d\eta$ is equivalent to the existence of a Liouville vector field V (i.e. $\mathcal{L}_V \omega = \omega$) transverse to ∂W and pointing outward (with respect to the orientation induced by ω).

Example 1.2.7. An example of fillable contact structure is the standard contact structure ξ_{st} on S^{2n+1}. One way of constructing it is to consider the set of complex tangencies of $S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$, that is $\xi_{st} = TS^{2n+1} \cap iTS^{2n+1}$. It is clear by the construction that (D^{2n+2}, ω_{st}) is a strong symplectic filling (actually it is a Stein filling), where

$$\omega_{st} = \sum_{i=1}^{n+1} dz_i \wedge d\bar{z}_i.$$

Another way of constructing (S^{2n+1}, ξ_{st}) is to consider the Hopf fibration $S^{2n+1} \to S^{2n}$ which has fibre S^1 and then take the hyperplane field orthogonal to the fibres (with respect to the round metric).

The previous example can be generalized by the so called *Boothby-Wang* construction [BW]. Let (M^{2n}, ω) be a symplectic manifold, up to a perturbation we can suppose that $[\omega] \in H^2(M, \mathbb{Z}) \hookrightarrow H^2(M, \mathbb{R})$. Let N be the total space of the S^1-bundle $\pi : N \to M$ with Euler class $[\omega]$. A connection 1-form η is contact because $d\eta = \pi^*\omega$ implies

$$\eta \wedge d\eta^n = \eta \wedge \pi^*\omega^n \neq 0$$

since $\pi^*\omega^n \neq 0$ is non-singular on the horizontal distribution, because ω is symplectic, and η is positive on $\ker \pi_*$.

We say that $(N, \xi = \ker \eta)$ is obtained from (M, ω) via the Boothby-Wang construction.

Remark 1.2.8. Notice that, a priori, the choice of another connection 1-form η' may give a different contact structure. In this case we have a family $\eta_t = (1-t)\eta + t\eta'$ of connection forms and hence a homotopy of contact structures $\xi_t = \ker \eta_t$. Gray's Stability ensures that these contact structures are isotopic, hence there is no ambiguity when referring to *the* contact structure induced by (M, ω).

Lemma 1.2.9. *Let $(N, \xi = \ker \eta)$ be obtained by (M, ω) via the Boothby-Wang construction, then ξ is a strongly simplectically fillable contact structure.*

Proof. The strong symplectic filling of (N, ξ) is given by the disc bundle of the vector bundle with Euler class $[\omega]$. More precisely let $E \to M$ be the plane bundle with Euler class $[\omega]$ and let η be a global angular form on $E \setminus M_0$, where M_0 is the zero section. For any bundle-like Riemannian metric g on E, let r be the radial function on the fibers. The symplectic structure $\Omega = d(r^2\eta) + \pi^*\omega$ on E makes the unit disc bundle into a strong symplectic filling of the unitary bundle (N, ξ). ∎

There are generalizations of the Boothby-Wang construction which apply in the case where M is a symplectic orbifold. This structure arises in the context of metric contact geometry. A contact metric structure (N, η, J) is a contact structure $(N, \ker \eta)$ together with a complex structure J on $\ker \eta$ such that the formula

$$g(A, B) = d\eta(JA, B) + \eta(A)\eta(B)$$

for $A, B \in \mathfrak{X}(N)$ defines a metric. This happens if and only if $J^*d\eta = d\eta$ and $d\eta(J(-), (-))$ is positive definite. A contact metric structure is K-contact if the Reeb vector field R_η is Killing for the metric g. We say that a contact metric structure is quasi-regular if all Reeb orbits are closed. Standard techniques show that, up to a perturbation, every K-contact structure is quasi-regular.

Theorem 1.2.10 (Structural Theorem)**.** *Let (N, η, J) be a quasi-regular K-contact manifold, then the space of orbits of R is a symplectic orbifold and (N, η) is obtained by a orbifold version of the Bootbhy-Wang construction.*

For more details see [BoGa] Theorem 7.1.3. This can be used together with [NiPa] to prove that every K-contact structure is strongly symplectically fillable.

1.3 Even contact structures

Engel structures are tightly linked to even contact structures, which are the even dimensional analogues of contact structures.

Definition 1.3.1. *An* even contact structure *$\mathcal{E}$ is a maximally non-integrable hyperplane field on an even dimensional manifold M^{2n+2}. This means that $\mathcal{E}$ is locally the kernel of a 1-form α satisfying $\alpha \wedge d\alpha^{2n} \neq 0$.*

An even contact structure is co-orientable if there exists a global 1-form α such that $\mathcal{E} = \ker \alpha$. This is equivalent to asking that the line bundle $TM/\mathcal{E}$ is trivial. There exists a unique line field $\mathcal{W} \subset \mathcal{E}$ called *characteristic, kernel or Cauchy line field of $\mathcal{E}$* whose flow preserves $\mathcal{E}$, i.e. $[\mathcal{W}, \mathcal{E}] = \mathcal{E}$. It exists because $\mathcal{E}$ has odd rank, and it is unique because $\mathcal{E}$ is maximally non-integrable. If $\mathcal{E} = \ker \alpha$ then $\alpha \wedge d\alpha^{2n}$ is a nowhere vanishing $(2n+1)$-form in a $(2n+2)$-manifold, hence its kernel is a line field and in fact $\mathcal{W} = \ker \alpha \wedge d\alpha^{2n}$.

The existence of such $\mathcal{W}$ and the role played by its dynamics is a substantial difference with the theory of contact structures. Even contact geometry is tightly linked with contact geometry, as the following result explains.

Proposition 1.3.2 [Mont2]**.** *Let $(M, \mathcal{E})$ be an even contact manifold with characteristic foliation $\mathcal{W}$. Let $N \hookrightarrow M$ be a (possibly open) embedded hypersurface transverse to $\mathcal{W}$, then $\xi = \mathcal{E} \cap TN$ is a contact structure on N.*

Moreover for any fixed section W of $\mathcal{W}$, the corresponding flow ϕ_t acts by contactomorphisms.

If M is closed, the existence of $\mathcal{W}$ implies that the Euler characteristic $\chi(M)$ vanishes. We will see that this is the only restriction to the existence of even contact structures.

The characteristic foliation is also responsible for the infinite dimensional moduli space of deformations of even contact structures. Indeed if a diffeomorphism $\psi : M \to M$ preserves $\mathcal{E}$ then it must also preserve $\mathcal{W}$. If we consider a smooth homotopy of even contact structures $\mathcal{E}_t$ in general $\mathcal{W}_0$ will not be topologically conjugate to $\mathcal{W}_1$. This means that in general there is no isotopy $\psi_t : M \to M$ sending $\mathcal{E}_1$ to $\mathcal{E}_0$. The following result is a version of Gray's Stability for even contact structures.

Theorem 1.3.3. *Let $(M, \mathcal{E}_t)$ be a smooth family of even contact structures for $t \in [0, 1]$, and denote by $\mathcal{W}_t$ the characteristic foliations of $\mathcal{E}_t$. If $\mathcal{W}_t = \mathcal{W}_0$ for all t then there exists an isotopy $\psi_t : M \to M$ such that $\psi_t^* \mathcal{E}_t = \mathcal{E}_0$.*

For a proof see Section 3.6 in [Vog1].

One of the reasons why even contact structures are not extensively studied, is the fact that their geometry is completely determined by the topological information given by the flag of distribution $(\mathcal{W}, \mathcal{E})$.

Definition 1.3.4. *A* formal even contact structure *is a triple $(\tilde{\mathcal{W}}, \tilde{\mathcal{E}}, \tilde{\omega})$ where $\tilde{\mathcal{W}}$ is a line field, $\tilde{\mathcal{E}}$ is a hyperplane distribution satisfying $\tilde{\mathcal{W}} \subset \tilde{\mathcal{E}}$ and $\tilde{\omega}$ is an almost symplectic structure on $\tilde{\mathcal{E}}/\tilde{\mathcal{W}}$.*

In the previous definition $\tilde{\mathcal{E}}$ plays the role of the even contact structure with characteristic foliation $\tilde{\mathcal{W}}$ and $d\alpha$ is represented by $\tilde{\omega}$. The main difference is that there are no differentiable equations relating these objects, as in the case of an even contact structure. There is a canonical map $i : \mathcal{E}(M) \to \mathcal{FE}(M)$ between the space $\mathcal{E}(M)$ of even contact structures and the space $\mathcal{FE}(M)$ of formal even contact structures. The surprising fact is that the formal information is enough to describe the geometric one, in a sense made rigorous by the language of the h-principle (see [Gro2]).

Theorem 1.3.5 [McDu]**.** *Let M be a $(2n+2)$-manifold with vanishing Euler characteristic, then the canonical inclusion $i : \mathcal{E}(M) \to \mathcal{FE}(M)$ is a weak homotopy equivalence.*

The previous theorem says that the geometric information given by the even contact structure does not give any further restriction on the topology of M. Otherwise said even contact structures are classified (up to homotopy) by their formal counter part, which in turn can be classified using the methods of algebraic topology. The main reason for which contact structures are interesting, is that the analogue of Theorem 1.3.5 is false, and one can study the topology of a manifold by understanding its contact geometry.

1.3.1 Even contact vector fields

In this section we study vector fields whose flow preserves an even contact structure. Notice that in general $\mathcal{E}$ does not admit many of these symmetries. In Chapter 2 we will be interested in even contact structures whose characteristic foliation is particularly symmetric. More precisely, we want it to have volume-preserving holonomy, this is equivalent to the existence of a volume form Ω and a section $W \in \Gamma\mathcal{W}$ such that $\mathcal{L}_W\Omega = 0$. The following result provides some equivalent conditions.

Proposition 1.3.6 [KV]**.** *Let $(M^{2n+2}, \mathcal{E} = \ker\alpha)$ be an even contact structure with orientable characteristic foliation $\mathcal{W} = \ker\alpha \wedge d\alpha^n$, then the following are equivalent:*

1. *$\mathcal{W}$ has volume-preserving holonomy;*
2. *$\mathcal{W}$ is the kernel of a closed $(2n+1)$-form;*
3. *α can be chosen so that $d\alpha$ has constant rank $2n$;*
4. *there exists a vector field $Z \in \mathfrak{X}(M)$ transverse to $\mathcal{E}$ whose flow preserves $\mathcal{E}$.*

Proof. $\boxed{1 \Rightarrow 2}$ Let Ω be a volume form and W a non-singular section of $\mathcal{W}$ such that $\mathcal{L}_W\Omega = 0$, then $\omega = i_W\Omega$ is a nowhere vanishing closed $(2n+1)$-form whose kernel is exactly $\mathcal{W}$.

$\boxed{2 \Rightarrow 1}$ Fix a Riemannian metric g and let ρ be the g-dual of W. Let ω be a closed form whose kernel is $\mathcal{W}$ and consider the top degree form $\Omega = \rho \wedge \omega$. This is a volume form because $i_W\Omega = \omega$, moreover

$$\mathcal{L}_W\Omega = di_W\Omega = d(\rho(W)\omega - \rho \wedge i_W\omega) = d\omega = 0.$$

$\boxed{2 \Rightarrow 3}$ It suffices to prove that there exists a choice of a defining form $\tilde{\alpha}$ which satisfies $d\tilde{\alpha}^{n+1} = 0$. Suppose that $\mathcal{W} = \ker\omega$ with ω closed $(2n+1)$-form, then ω must be a multiple of $\alpha \wedge d\alpha^n$, since they have the same kernel. Let $\omega = \mu\alpha \wedge d\alpha^n$ for some $\mu \in \mathcal{C}^\infty(M)$. If we change defining form $\tilde{\alpha} = \lambda\alpha$ where $\lambda \in \mathcal{C}^\infty(M)$ is a nowhere vanishing function, we get

$$\tilde{\alpha} \wedge d\tilde{\alpha}^n = (\lambda\alpha) \wedge (d\lambda \wedge \alpha + \lambda d\alpha) = \lambda^{2n+1}\alpha \wedge d\alpha^n.$$

Choosing $\lambda^{2n+1} = \mu$, we have $d\tilde{\alpha}^{n+1} = d(\tilde{\alpha} \wedge d\tilde{\alpha}^n) = d\omega = 0$.

$\boxed{3 \Rightarrow 4}$ If $d\alpha$ has constant rank $2n$ then $\ker d\alpha$ is a 2-plane field. Now W is in the kernel of $d\alpha$ and $\ker d\alpha$ is trivial as a bundle. Let $\ker d\alpha = \langle W, Z\rangle$. Uniqueness of $\mathcal{W}$ ensures that Z must be transverse to $\mathcal{E}$. The flow of Z preserves $\mathcal{E}$, so that $\mathcal{L}_Z\alpha = \lambda\alpha$ for some $\lambda \in \mathcal{C}^\infty(M)$. Choose $Z \in \ker d\alpha$ such that $\alpha(Z) = 1$, then

$$\mathcal{L}_Z\alpha = i_Z d\alpha + di_Z\alpha = d(\alpha(Z)) = 0.$$

$\boxed{4 \Rightarrow 2}$ By hypothesis, for any defining α we have $\mathcal{L}_Z\alpha = \lambda\alpha$. Pick α so that $\alpha(Z) = 1$. Taking the derivative of this we get

$$0 = \mathcal{L}_Z(\alpha(Z)) = (\mathcal{L}_Z\alpha)(Z) + \alpha(\mathcal{L}_Z Z) = \lambda,$$

hence $i_Z d\alpha = \mathcal{L}_Z\alpha = 0$. This implies that $d\alpha^{n+1} = 0$ because

$$i_Z d\alpha^{n+1} = (n+1)d\alpha^n \wedge i_Z d\alpha = 0,$$

so that $\alpha \wedge d\alpha^n$ is closed. □

Remark 1.3.7. If the flow of a vector field $Z \in \mathfrak{X}(M)$ preserves the even contact structure $\mathcal{E}$, then it must also preserve its characteristic foliation $\mathcal{W}$. If we are in the hypothesis of Proposition 1.3.6 this implies in particular that, if W is a section of $\mathcal{W}$, we have $\mathcal{L}_Z W = aW$ and $\langle W, Z\rangle$ is a foliation. Another way of seeing this is choosing α so that $\ker d\alpha = \langle W, Z\rangle$.

1.3.2 Action of the flow of $\mathcal{W}$

We now consider the action of the flow of $\mathcal{W}$ on $\mathcal{E}/\mathcal{W}$ when M has dimension 4. This is discussed in details in [Mit].

For a fixed section $\mathcal{W} = \langle W\rangle$, we have the action of the associated flow ϕ_t on $\mathcal{E}$ and this induces a map $[T\phi_t] : \mathbb{P}(\mathcal{E}_p/\mathcal{W}_p) \to \mathbb{P}(\mathcal{E}_{\phi_t(p)}/\mathcal{W}_{\phi_t(p)})$. If $p \in M$ is contained in a closed orbit of $\mathcal{W}$ of period T, then $P := [T_p\phi_T] \in \mathrm{PSL}(2,\mathbb{R})$, where we identify $\mathbb{RP}^1 = \mathbb{P}(\mathcal{E}_p/\mathcal{W}_p)$.

Hence the usual classification of such maps we say that a closed orbit γ is

- *Elliptic* if $|\operatorname{tr} P| < 2$ or $P = \pm id$, in this case we can represent P by a matrix of the form
$$P \equiv \pm \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix}$$
for some $\delta \in \mathbb{R}$.
- *Parabolic* if $|\operatorname{tr} P| = 2$ and $P \neq \pm id$, in this case we can represent P by a matrix of the form
$$P \equiv \pm \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}.$$
- *Hyperbolic* if $|\operatorname{tr} P| > 2$, in this case we can represent P by a matrix of the form
$$P \equiv \pm \begin{pmatrix} e^{-\mu} & 0 \\ 0 & e^{\mu} \end{pmatrix}$$
for some $\mu \in \mathbb{R}$.

Notice that parabolic orbit come in two types depending on the sign of entry over the diagonal. We will call *positive parabolic* orbits where this element has sign opposite to the one of the diagonal entries, and negative otherwise. Open orbits γ will in general not have any particular character.

1.4 Engel structures

The last family of stable distributions is the one of Engel structures. The study of these structures is the main topic of this thesis. The main references for the basic theory of Engel structures are [Mont2, P1, Vog1].

Definition 1.4.1. *An* Engel structure $\mathcal{D}$ *is a smooth* 2*-plane field on a smooth* 4*-manifold such that* $\mathcal{E} := \mathcal{D}^2$ *is an even contact structure.*

We can associate to an Engel structure a flag of distributions as stated in the following result.

Lemma 1.4.2. *Let* $(M, \mathcal{D})$ *be an Engel structure, then the characteristic foliation* $\mathcal{W}$ *of* $\mathcal{E}$ *satisfies* $\mathcal{W} \subset \mathcal{D}$*. The flag of distributions* $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$ *is called* Engel flag *of* $\mathcal{D}$*.*

Proof. Recall that $\mathcal{W}$ is defined uniquely by the properties $\mathcal{W} \subset \mathcal{E}$ and $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$. Now suppose that at a point $p \in M$ we have $\mathcal{E}_p = \mathcal{D}_p \oplus \mathcal{W}_p$, then this is true on a neighbourhood U of p. Since $[\mathcal{D}, \mathcal{D}] \subset \mathcal{E}$ and $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$ this implies that $\mathcal{E}$ is integrable on U. □

Deformations of Engel structures $\mathcal{D}_t$ have infinite dimensional moduli spaces [Mont2], but we still have an analogue of Gray's Stability.

Theorem 1.4.3 [Gol]**.** *Let* $(M, \mathcal{D}_t)$ *be a smooth family of Engel structures for* $t \in [0, 1]$*, and denote by* $\mathcal{W}_t$ *the characteristic foliations of* $\mathcal{D}_t$*. If* $\mathcal{W}_t = \mathcal{W}_0$ *for all* t *then there exists an isotopy* $\psi_t : M \to M$ *such that* $\psi_t^* \mathcal{D}_t = \mathcal{D}_0$*.*

For a proof see Section 3.6 in [Vog1].

The orbits of the characteristic foliation of an Engel structure come equipped with a natural projective structure. This means that there are local charts modelled on the geometry $(\mathbb{RP}^1, \mathrm{PSL}(2, \mathbb{R}))$ naturally induced by the Engel structure. For any point $p \in M$ we define a chart for a piece of the orbit of $\mathcal{W}$ through p, by fixing a Darboux chart $(U, \{x, y, z, w\})$ and taking the projection to the w-line at p. The change of coordinates in this case is well-defined because $\mathcal{W}$ is sent to itself, and one can verify that the transformation is via projective maps (see [BrS]).

Another way of specifying a projective structure is by giving the *developing map*. Fix a point $p \in M$ and let γ be the $\mathcal{W}$-orbit through p. We get a map from the universal cover of γ to $\mathbb{RP}^1$ using a section W of $\mathcal{W}$. For this we suppose that $\mathcal{W}$ is trivial, this is always the case if M is orientable (see Lemma 1.6.1). Take the flow of this section $\phi^p : \mathbb{R} \cong \tilde{\gamma} \to M$ such that $t \mapsto \phi_t(p)$. Since this preserves the even contact structure, $(\phi_{t*}\mathcal{D})_p$ is still going to be a plane in $\mathcal{E}_p$ containing $\mathcal{W}_p$. We can consider the line that it spans in the complement of $\mathcal{W}_p$, and we have the developing map

$$\delta_p : \mathbb{R} \cong \tilde{\gamma} \to \mathbb{RP}^1 \cong \mathbb{P}(\mathcal{E}_p/\mathcal{W}_p) \qquad \text{s.t.} \qquad t \mapsto \Big[(\phi_{t*}\mathcal{D})_p\Big].$$

The properties of this map provide very interesting information about the moduli space of $\mathcal{D}$-horizontal curves, i.e. curves tangent to $\mathcal{D}$. In fact if we take a curve $\gamma : I \to M$ which is a piece of a $\mathcal{W}$-orbit and whose developing map is injective, then the only $\mathcal{C}^1$-deformations of γ in the space of $\mathcal{D}$-curves are reparametrizations. This phenomenon of rigidity only happens for such horizontal curves and it is in great contrast to what happens with Legendrian curves in contact manifolds. More details can be found in [BrS]. More recently a complete h-principle has been established for non-rigid Legendrian knots (see [CP])

The classification of projective structures of dimension 1 provides a list of type of closed orbits analogous to the one of Section 1.3.2. For more details see [Mit].

1.5 Simple constructions and examples

We will now describe some standard constructions of Engel structures.

Example 1.5.1 (Cartan prolongation). Let (N^3, ξ) be a contact structure on a 3-manifold, we will construct an Engel structure on the projectivization of ξ. Consider $M = \mathbb{P}\xi$ with the canonical projection $\pi : M \to N$, i.e. the $\mathbb{RP}^1$-bundle over N whose fibre at a point $p \in N$ is the space of lines $l_p \subset \xi_p$. The *canonical distribution* $\mathcal{D}_l$ at a point $l \in M$ is the space of tangent vectors $v \in T_l M$ which project to the line l via the tangent map $T\pi : TM \to TN$. In formulas

$$\mathcal{D} = \Big\{ v \in T_l M \,\Big|\, T_l\pi(v) \in l \Big\}. \tag{1.3}$$

This defines an Engel structure called *Cartan prolongation of* ξ. The induced even contact structure is $(T\pi)^{-1}\xi$, i.e. the space of tangent vectors that project to ξ. This distribution contains the line field tangent to the fibres $\mathcal{W}$, and the holonomy of this line field preserves $\mathcal{E}$. Hence the characteristic foliation of $\mathcal{D}$ is exactly $\mathcal{W}$. Notice that in this case $\mathcal{D}$ is never orientable because it contains the canonical line field $\mathcal{W}$.

Instead of the projectivization of ξ we can take the S^1-bundle of *oriented* lines in ξ. The same construction yields then an Engel manifold which is a fibrewise 2-cover of the projectivization, and for which $\mathcal{D}$ is orientable. If the Euler class $e(\xi)$ vanishes, then the contact structure ξ is trivial as a bundle, say $\xi = \langle C_1, C_2 \rangle$. We can write $\mathcal{D}$ explicitly on $M = N \times S^1$ as $\mathcal{D} = \langle W = \partial_t,\, X = \cos t\, C_1 + \sin t\, C_2 \rangle$. As mentioned above W spans the characteristic foliation.

Using a contact vector field $V \in \mathfrak{X}(N)$ we can modify $\mathcal{D}$ and with it the dynamics of $\mathcal{W}$. More precisely if $\epsilon > 0$ is small the distribution $\mathcal{D}_\epsilon = \langle W + \epsilon V,\, X \rangle$ is Engel and its characteristic foliation is spanned by $W + \epsilon V$. Choosing V carefully one can make sure that $W + \epsilon V$ has no closed orbits. For this it suffices to choose a contact vector field V which has countably many closed orbits and then choose ϵ rationally independent from the periods of the orbits of V (see [P1] Proposition 1.21).

Example 1.5.2 (Lorentz prolongation). Consider a Lorentz 3-manifold (N^3, h), i.e. h is a pseudo-Riemannian metric of signature $(1, 2)$. The light-like cone in the tangent bundle TN is the space $\mathcal{C}(h)$ of tangent vectors $v \in TM$ such that $h(v, v) = 0$. The space of light-like lines M is a 4-dimensional manifold, which carries a canonical distribution $\mathcal{D}$, defined exactly as in Equation (1.3). This is an Engel structure called *Lorentz prolongation of h*. The characteristic foliation is tangent to the restriction of the geodesic flow (see [Mit]).

This structure is easier to describe in the case where every bundle involved is trivial. Suppose that $\langle X_+, Y_-, Z_- \rangle$ is an orthogonal framing of TM such that $h(X_+, X_+) = 1 = -h(Y_-, Y_-) = -h(Z_-, Z_-)$. Then consider $M \times S^1$ with the distribution $\mathcal{D} = \langle \partial_t, X_+ + \cos t\, Y_- + \sin t\, Z_- \rangle$ is the (oriented) Lorentz prolongation of h.

Remark 1.5.3. There is a local construction of Engel distributions which builds a bridge between the two previous examples. On $D^3 \times I$ take coordinates $(x, y, z; t)$ and let $\mathcal{D}$ be an orientable distribution containing ∂_t. This means that we can write $\mathcal{D} = \langle \partial_t, A \rangle$ for $A \in \mathfrak{X}(D^3 \times I)$ tangent to the level sets $D_3 \times \{t\}$. We trivialize the tangent bundle of D^3 via the standard contact framing $X = \partial_x$, $Y = \partial_y + x\partial_z$ and $Z = [X, Y] = \partial_z$. Then at any point $q = (p, t) \in D^3 \times I$ the vector $A(p)$ corresponds to a unique point in the unit sphere in T_pD^3. Using the trivialization $\{X, Y, Z\}$ we can identify all these spheres. Hence fixing such a distribution $\mathcal{D}$ on $D^3 \times I$ is equivalent to giving a D^3-family of curves $A_p : I \to S^2$.

The fact that $\mathcal{D}$ is non-integrable corresponds to $\dot{A}_p := [\partial_t, A_p]$ being not contained in $\mathcal{D}$. Since A_p is a curve on the sphere $S^2 \subset T_pD^3$, its derivative is tangent to S^2 and hence $\dot{A}_p$ is not contained in $\mathcal{D}$ if and only if $A_p : I \to S^2$ is a regular curve.

Now $\mathcal{D}^2$ is non-integrable at $q = (p, t) \in D^3 \times I$ if and only if at least one of the following holds:

1. $\{A_p, \dot{A}_p\}$ is a contact structure near q;
2. $A_p : I \to S^2$ has non-zero curvature at p.

In case 1 we have the framing $\{\partial_t, A, \dot{A}, [A, \dot{A}]\}$, whereas in case 2 we have the framing $\{\partial_t, A, \dot{A}, \ddot{A}\}$. For Cartan prolongations Condition 1 is verified at everypoint, whereas for Lorentz prolongations Condition 2 is true everywhere. The space of Cartan and Lorentz prolongations on a manifold N was studied in [P2].

More details about the next example can be found in [Mit].

Example 1.5.4 (Suspension of a contactomorphism). Let (N, ξ) be a contact, non-necessarily closed 3-manifold and fix a Riemannian metric g on N.

Suppose that $\mathcal{L} \subset \xi$ is a Legendrian line field and $\psi : (N, \xi) \to (N, \xi)$ a contactomorphism such that the angle $\theta(p)$ between $\mathcal{L}(p)$ and $\psi_*\mathcal{L}(p)$ is well-defined, continuous everywhere and bounded. These conditions are automatically verified if ψ is homotopic to the identity and N is compact. Let a, $b : N \to \mathbb{R}^{>0}$ be smooth positive functions and consider the manifold

$$M = \Big\{(p, t) \Big| \, p \in N \text{ and } -a(p) < t < b(p)\Big\},$$

with the canonical projection $\pi : M \to N$.

Since θ is bounded, we can find $k \in \mathbb{N}$ so that $\theta + 2k\pi$ is positive and define a smooth interpolation function $r : M \to \mathbb{R}$ such that $r(p, -a(p)) = 0$ for all $p \in N$ and $r(p, b(p)) = \theta(p) + 2k\pi$ and such that $\partial_t r > 0$. Then we can define an Engel structure on M by $\mathcal{D} = \langle \partial_t, \, R(r)\pi^*\mathcal{L} \rangle$, where $R(r)$ denotes the rotation on ξ of angle r.

1.6 Topology of Engel manifolds

Lemma 1.4.2 gives a flag of distributions $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$ of the Engel structure $\mathcal{D}$. The existence of this flag imposes strong constraints on the topology of the manifold M.

Lemma 1.6.1. *If $(M, \mathcal{D})$ is an Engel structure and $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$ is its induced flag then we have canonical isomorphisms*

$$\det(\mathcal{E}/\mathcal{W}) \cong \det(TM/\mathcal{E}) \qquad \textit{and} \qquad \det(\mathcal{E}/\mathcal{D}) \cong \det(\mathcal{D}). \tag{1.4}$$

In particular

1. *$\mathcal{E}$ is orientable;*
2. *M is orientable if and only if $\mathcal{W}$ is trivial.*

Proof. The isomorphisms essentially come from the Lie brackets pairings

$$\Gamma\mathcal{E} \times \Gamma\mathcal{E} \to \Gamma(TM/\mathcal{E}) \qquad \text{and} \qquad \Gamma\mathcal{D} \times \Gamma\mathcal{D} \to \Gamma(\mathcal{E}/\mathcal{D}),$$

the fact that $\mathcal{D}$ is maximally non-integrable and that the kernel of $\mathcal{E}$ is $\mathcal{W}$.

Claim 1 follows because the isomorphism $\mathcal{E} \cong \mathcal{D} \oplus \mathcal{E}/\mathcal{D}$ and Equation (1.4) readily imply that

$$\det(\mathcal{E}) \cong \det(\mathcal{D}) \otimes \det(\mathcal{E}/\mathcal{D}) \cong \underline{\mathbb{R}}.$$

To prove claim 2 consider $TM \cong \mathcal{E} \oplus TM/\mathcal{E}$. Since $\mathcal{E}$ is always oriented, by point 1, the orientability of M is equivalent to one of $TM/\mathcal{E}$. Equation (1.4) ensures that this is the same as the orientability of $\mathcal{E}/\mathcal{W}$. Using again orientability of $\mathcal{E}$ we have

$$\underline{\mathbb{R}} \cong \det(\mathcal{E}) \cong \det(\mathcal{W}) \otimes \det(\mathcal{E}/\mathcal{W})$$

hence $\mathcal{W}$ is orientable if and only if $\mathcal{E}/\mathcal{W}$ is, which implies the claim. □

Corollary 1.6.2. *If M admits an Engel structure, then it admits a parallelizable 4-cover.*

Proof. Up to passing to a 4-cover we can suppose that both M and $\mathcal{D}$ are orientable. Consider the induced flag $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$, Lemma 1.6.1 ensures that $\mathcal{W}$ and $\mathcal{E}$ are orientable. We conclude that the line bundles $\mathcal{W}$, $\mathcal{D}/\mathcal{W}$, $\mathcal{E}/\mathcal{D}$ and $TM/\mathcal{E}$ are all trivial and hence we get a global framing for $TM \cong \mathcal{W} \oplus \mathcal{D}/\mathcal{W} \oplus \mathcal{E}/\mathcal{D} \oplus TM/\mathcal{E}$. □

In what follows we will always suppose that M is parallelizable. A big family of examples of such manifolds is given by products $M = N \times S^1$, where N is a closed orientable 3-manifold and hence parallelizable. The proof of this fact consists in showing the vanishing of first and second Stiefel-Whitney classes $w_1(N)$ and $w_2(N)$ (for more details see [Gei]). The following is a topological characterisation of parallelizable 4-manifolds.

Theorem 1.6.3 [HiHo]**.** *An orientable 4-manifold is parallelizable if and only if its Euler characteristic $\chi(M)$, second Stiefel-Whitney class $w_2(M)$, and signature $\sigma(M)$ vanish.*

1.7 Existence results for Engel structures

In this section we give an informal overview of existence results for Engel structures.

The question about existence of Engel structures on every parallelizable 4-manifolds, was open for a long time. The first solution to the problem is provided by the following result.

Theorem 1.7.1 [Vog3]**.** *Every closed parallelizable 4-manifold M admits an Engel structure.*

The proof makes use of a *round handle decomposition* of M, briefly RHD. This is the analogue of a usual handle decomposition of M into handles $h_k = D^k \times D^{n-k}$, except that one uses pieces of the form $R_k = D^k \times D^{n-k-1} \times S^1$ called *round handles.* We call $\partial_+ R_k = D^k \times \partial D^{n-k-1} \times S^1 = D^k \times S^{n-k-2} \times S^1$ *enter region* or *positive boundary* and $\partial_- R_k = \partial D^k \times D^{n-k-1} \times S^1 = S^{k-1} \times D^{n-k-1} \times S^1$ *exit region* or *negative boundary.*

Not every manifold admits a round handle decomposition. In fact if we have a RHD we can find a line field on M. The idea is to construct a nowhere vanishing vector field on each round handle and then make sure that the model glues through the attaching map. This implies that the Euler characteristic of the manifold must vanish if it admits a RHD. The converse in dimension different then 3 is given by the following result due to Asimov.

Theorem 1.7.2 [Asi]**.** *Let M^n be an orientable manifold with $n \neq 3$ and vanishing Euler characteristic, then M admits a RHD.*

A key technical step in the proof is to realize that a round $(k+1)$-handle can be seen as the union of two regular handles, namely a k-handle and a $(k+1)$-handle. Using this observation and an ordinary handle decomposition, we can obtain a RHD. The hypothesis $\chi(M) = 0$ gives information about the parity of the number of ordinary handles. The result is false in dimension $n = 3$. As we will see in Chapter 3, a RHD is linked to non-singular Morse-Smale flows on M, and only few 3-manifolds admit such flows.

The idea of the proof of Theorem 1.7.1 is to construct explicit Engel structures on the round handles and make sure that they glue together. In this process Proposition 1.3.2 is crucial. Indeed the Engel structures are constructed so that the exit region is transverse to the characteristic foliation $\mathcal{W}$ and the glueing is (in part) done taking advantage of the flexibility of overtwisted contact manifolds.

This existence result does not give information on the homotopy type of framings that can be realized as Engel framings, i.e. induced by the flag $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$. We denote by $\mathcal{FE}ng(M)$ the space of formal Engel structures, that is the space of flags of distributions $\tilde{\mathcal{W}} \subset \tilde{\mathcal{D}} \subset \tilde{\mathcal{E}}$ such that $\det(\tilde{\mathcal{E}}/\tilde{\mathcal{W}}) \cong \det(TM/\tilde{\mathcal{E}})$ and $\det(\tilde{\mathcal{E}}/\tilde{\mathcal{D}}) \cong \det(\tilde{\mathcal{D}})$. Moreover we denote by $\mathcal{E}ng(M)$ the space of Engel structure on M. There is a natural inclusion map $i : \mathcal{E}ng(M) \to \mathcal{FE}ng(M)$. Theorem 1.7.1 ensures that the space $\mathcal{E}ng(M)$ is non-empty. The following result ensures that we can find Engel structures in every homotopy class of framings.

Theorem 1.7.3 [CPPP]**.** *For any parallelizable closed 4-manifold M, the map $i : \mathcal{E}ng(M) \to \mathcal{FE}ng(M)$ is surjective in homotopy.*

The proof of this result is divided in two parts, as many classical h-principle type arguments. We start from a formal Engel structure $\tilde{\mathcal{W}} \subset \tilde{\mathcal{D}} \subset \tilde{\mathcal{E}}$, use general h-principle techniques to reduce it to a local problem around a "hole" of the form $D^3 \times I$ and then use an explicit construction to "fill the hole".

The first part consists in using the complete h-principle for even contact structures (Theorem 1.3.5) to homotope $\tilde{\mathcal{W}} \subset \tilde{\mathcal{E}}$ to an even contact structure. Then we use Thurston's Jiggling Lemma to construct a fine triangulation on M adapted to $\mathcal{W}$. Finally we use Gromov's flexibility results for open manifolds in order to construct an Engel structure on the neighbourhood of the 3-skeleton of the triangulation.

The second part of the proof consists in finding a homotopy on the remaining 4-cells relative to the boundary. Taking a little care in the previous construction, we can make sure that these 4-cells are diffeomorphic to $D^3 \times I$, that $\mathcal{E}$ is standard near the boundary, and that $\tilde{\mathcal{D}}$ is spanned by $\langle \partial_t, X \rangle$ where X is tangent to the level sets $D^3 \times \{t\}$. Here the construction in Remark 1.5.3 is fundamental. In particular we make sure that near the boundary the curves $X_p : I \to S^2$ make at least 3 turns around the equator. Using this extra hypothesis we can homotope any curve X_p to a convex curve.

This translates to a homotopy between $\tilde{\mathcal{D}} = \langle \partial_t, X_p \rangle$ and a genuine Engel structure on $D^3 \times I$.

1.7.1 Flexible Engel structures

After [CPPP] established an existence h-principle the two works [CPP] and [PV] studied some subfamilies of Engel structures satisfying a complete h-principle.

In the work [PV] a family of flexible Engel structures, called overtwisted, is introduced. An Engel structure $(M, \mathcal{D})$ is *overtwisted* if on M there is an embedded copy of a model structure $(D^4, \mathcal{D}_{ot})$, called *Engel overtwisted disc.* This family is flexible in the sense that it satisfies a complete h-principle. In order to describe this model we need to introduce the notion of *Engel-Lutz twist.* In analogy with the contact version of the Lutz twist, the Engel-Lutz twist is a modification of an Engel structure in a neighbourhood of a transverse hypersurface.

If $N^3 \hookrightarrow (M^4, \mathcal{D})$ is transverse to $\mathcal{D}$ then we get a line field $\mathcal{H}_N = TN \cap \mathcal{D}$ and a plane field $\xi_N = TN \cap \mathcal{E}$. On the points where N is transverse to $\mathcal{W}$, Proposition 1.3.2 ensures that ξ_N is contact. We will denote by $N^{+/-}$ the subsets of N where ξ_N is a positive/negative contact structure, moreover we denote by $N^0 = N \setminus (N^+ \cup N^-)$. The idea is to describe the Engel structure on $\mathcal{O}p(N)$ as a family of curves $H_p : I \to S^2$ for $p \in N$, similarly as what happens in Remark 1.5.3. It is possible to find coordinates on a neighbourhood $\mathcal{O}p(N) = N \times (-\epsilon, \epsilon)$ such that $\mathcal{D}(p,t) = \langle \partial_t, X + tY + g(p,t)Z \rangle$ where $\mathcal{H}_N = \langle X \rangle$, $\xi_N = \langle X, Y \rangle$ and $TN = \langle X, Y, Z \rangle$. Notice that the framing of TN is ∂_t-invariant, so that we can use it to interpret $\mathcal{D}$ on $\{p\} \times (-\epsilon, \epsilon)$ as the curve on S^2 cut by the ray $X + tY + g(p,t)Z$. Moreover we can set coordinates so that

- $g(p,t) = 0$ on the complement of a (arbitrarily small) neighbourhood of N^0, this means that ∂_t spans the characteristic foliation $\mathcal{W}$ and ξ_N is contact;
- $g(p,t)$ is convex/concave otherwise, this means that when the curve H_p does not coincide with the equator it is convex/concave.

The idea is now to glue at every $p \in N$ the osculating circle to the curve H_p so that the Engel structure performs an additional turn along $\mathcal{W}$ orbits. More precisely we modify $\mathcal{D}$ on $\mathcal{O}p(N)$ so that where $g(p,t) = 0$ the even contact structure $\mathcal{E}$ is not changed, but the Engel structure performs an additional turn along $\mathcal{W}$-orbits (which is ∂_t in this case). This translates to the fact that H_p makes an additional full turn around the equator. Finally we interpolate this modification on points where $g(p,t) \neq 0$ by adding an osculating circle and then smoothening up the corresponding family of curves. This process yields a new Engel structure $\mathcal{L}(\mathcal{D})$, and is referred to as *adding Engel torsion along N*.

A Lutz twist is the process of adding Engel torsion along a specific type of hypersurfaces. We say that a tubular neighbourhood of a surface $\Sigma \hookrightarrow M$ transverse to $\mathcal{D}$ is *thin* if $\mathcal{O}p(\Sigma) = \Sigma \times D^2$ is such that $\Sigma \times \partial D^2_\epsilon$ is transverse to $\mathcal{D}$ for every $0 < \epsilon \leq 1$. It turns out that every transverse surface has a thin neighbourhood.

Definition 1.7.4. *Let $(M, \mathcal{D})$ be an Engel structure and $T^2 \to M$ a torus transverse to $\mathcal{D}$. The* Engel-Lutz twist along T^2 *is the Engel structure $\mathcal{L}(\mathcal{D})$ obtained by adding Engel torsion along a thin tubular neighbourhood of T^2.*

An overtwisted Engel disc is a (piece of) Engel-Lutz twist performed along a torus with special properties. In order to construct this we need to understand Engel structures near curves transverse to $\mathcal{E}$. If $\gamma : I \to M$ is transverse to $\mathcal{E}$ then there is a tubular neighbourhood $\mathcal{O}p(\gamma) = [0, L] \times D^3$ with coordinates (y, x, z, w) such that we have

$$\mathcal{D}|_{\mathcal{O}p(\gamma)} = \ker(dy - zdx) \cap \ker(dx - wdz).$$

If we take a knot $\eta : S^1 \to D^3$ transverse to the contact structure $\ker(dx - wdz)$, then $\gamma \times \eta$ is going to be a cylinder transverse to $\mathcal{D}$. In order to "move the overtwisted disc in the manifold" we need to be able to shrink this cylinder freely. We consider transformations $\psi_\lambda : [0, L] \times D^3 \to [0, L] \times D^3$ defined by

$$\psi_\lambda(y, x, z, w) = (y,\ \lambda^2(y)x,\ \lambda(y)z,\ \lambda(y)w)$$

for λ positive function. For every knot η there exists a constant $\tau_0 \in \mathbb{R}$ such that for every λ satisfying $|\lambda'| < \tau_0$ we have that the cylinder $\psi_\lambda(I \times \eta)$ is still transverse.

Definition 1.7.5. *With the same notation as above, suppose $L\tau_0 > 2$ and η is the transverse unknot with self-linking number 3, then the Engel-Lutz twist $(I \times D^3, \mathcal{D}_{ot})$ along $I \times \eta$ is the* Engel overtwisted disc.

A fixed family of embeddings of overtwisted discs Δ in M is called *certificate of overtwistedness of $\mathcal{D}$*. Denote by $\mathcal{E}ng_{ot}(M, \Delta)$ the family of overtwisted Engel structures with certificate Δ and by $\mathcal{FE}ng(M, \Delta)$ the space of formal Engel structures admitting certificate Δ.

Theorem 1.7.6 [PV]**.** *The inclusion $\mathcal{E}ng_{ot}(M, \Delta) \to \mathcal{FE}ng(M, \Delta)$ is a weak homotopy equivalence. In particular two Engel structures $\mathcal{D}_0$ and $\mathcal{D}_1$ which are overtwisted and formally homotopic, are homotopic through Engel structures.*

The Engel structures studied in [CPP], and more in detail in [P1], are called *loose* and are defined by a global condition. We will not explain what this condition is, but we want to stress that the fundamental difference from the above discussion is that looseness is, a priori, not described by a local model. A complete h-principle holds for loose Engel structures and it is unclear whether they coincide with overtwisted Engel structure.

Chapter 2

Engel structures and metrics

The goal of this chapter is to study properties of Engel structures which satisfy some natural metric conditions.

In Section 2.1 we introduce Engel defining forms α and β. They first appeared in [Ad1] and were used as a technical tool in [Ad2, Mont2, Vog1]. A choice of α and β permits to define two vector fields T and R which span a transverse distributions $\mathcal{R}$, which we call *associated Reeb distribution.* These vector fields were introduced in [Ad2], but the distribution $\mathcal{R}$ was never studied explicitly. The focus of the section, and of the whole chapter, is to understand geometric properties of α and β and of their associated Reeb distribution.

The end of Section 2.1 as well as Section 2.2 concerns conditions that ensure the existence of defining forms such that the associated Reeb distribution is integrable. Some of these conditions do not give any geometric insight, but are just reformulations which are useful for the remainder of the chapter. We present a condition that is geometric, and that is linked to contact structures. The *contactization* of the Engel structure $(M, \mathcal{D} = \ker\alpha \wedge \beta)$ is the contact manifold $(X = M \times \mathbb{R}, \eta = \beta + s\alpha)$. The existence of an integrable Reeb distribution is linked with how the Reeb vector field of (X, η) intersects graphical hypersurfaces.

Section 2.3 contains some basic facts on metric properties of distributions. We recall the definition of the tensors that ensure that a distribution is totally geodesic (the main reference for this section is [Rov]). In Section 2.4 we study when $\mathcal{D} = \ker\alpha \wedge \beta$ and its associated Reeb distribution $\mathcal{R}$ are totally geodesic. We prove that if g is a metric such that $\mathcal{D}^{\perp} = \mathcal{R}$ then $\mathcal{R}$ cannot be totally geodesic. Moreover if $\mathcal{D}$ is totally geodesic then there exists a multiple $\tilde{\beta}$ of β such that $d\tilde{\beta}^2 = 0$. This is a sufficient condition for the existence of a Reeb distribution which is integrable.

In Section 2.5 we recall the theory of *Engel vector fields Z*, i.e. vector fields satisfying $\mathcal{L}_Z\mathcal{D} \subset \mathcal{D}$. Many of the results we present were developed in [Vog1], but we rephrase them in terms of the framing induced by a pair of defining forms α and β. Moreover we study what happens if Z is also tangent to the Reeb distribution. In Section 2.6 we specialize to the case where Z is also Killing with respect to a metric g that makes it orthogonal to $\mathcal{E}$. If such a Z exists we say that M admits a *K-Engel structure*. Under these hypothesis we can find defining forms verifying $d\alpha^2 = 0 = d\beta^2$ and $d\alpha \wedge \beta = 0$. This allows the construction of a framing $\{W, X, T, R\}$ such that R commutes with every other vector field in the framing.

In Sections 2.7 and 2.8 we analyse the topology of manifolds that admit a K-Engel structure. We use classical theorems of [Mol] and [Pak] to prove Theorem 2.8.5, which completely characterizes K-Engel manifolds. We study more in details the bundle structure of the manifolds that appear in this classification, and we exhibit explicit constructions of K-Engel structures. The most remarkable one is the *Engel Boothby-Wang* construction, the analogue of the one in the K-contact case.

Finally in Section 2.9 we define *contact fillings* of an Engel structure. The construction is inspired by symplectic fillings of a contact structure, but we do not know if it implies any rigidity as in the contact case. We prove that Engel Boothby-Wang manifolds are examples of Engel structures which admit contact fillings.

2.1 Defining forms and the Reeb distribution

Definition 2.1.1. *Two* 1*-forms* α, β *are called* Engel defining forms *if they verify*

$$\alpha \wedge d\alpha \neq 0 \tag{2.1a}$$

$$\alpha \wedge \beta \wedge d\beta \neq 0 \tag{2.1b}$$

$$\alpha \wedge d\alpha \wedge \beta = 0. \tag{2.1c}$$

Equation (2.1a) ensures that $\mathcal{E} = \ker \alpha$ is an even contact structure; denote with $\mathcal{W} = \ker \alpha \wedge d\alpha$ its characteristic foliation. Equation (2.1b) implies that $\ker \beta$ is an even contact structure, and it ensures that its characteristic foliation is transverse to $\mathcal{E}$. Finally Equation (2.1c) implies that $\mathcal{W} \subset \ker \beta$.

Remark 2.1.2. The role of α and β in Definition 2.1.1 is not symmetric. In fact if α and β are Engel defining forms, the distribution $\mathcal{D} = \ker \alpha \cap \ker \beta$ satisfies $\mathcal{D}^2 = \ker \alpha$. On the other hand not all pairs of 1-forms $\tilde{\alpha}$ and $\tilde{\beta}$ which satisfy $\mathcal{D} = \ker \tilde{\alpha} \wedge \tilde{\beta}$ also satisfy $\mathcal{D}^2 = \ker \tilde{\alpha}$.

An Engel structure $\mathcal{D}$ can always be seen as the intersection of two even contact structures $\mathcal{D} = \mathcal{E} \cap \mathcal{E}'$, where the first one is uniquely determined by the condition $\mathcal{D}^2 = \mathcal{E}$. The second one instead is not unique, and must satisfy $\mathcal{W} \subset \mathcal{E}'$ and $\mathcal{W}' \pitchfork \mathcal{E}$. A choice of Engel defining forms α and β corresponds to a co-orientation for $\mathcal{E} = \ker \alpha$ and to a choice of $\mathcal{E}'$ together with a co-orientation $\mathcal{E}' = \ker \beta$.

Proposition 2.1.3. *Let M^4 be parallelizable and let α and β be 1-forms. If α and β are Engel defining forms then the plane field $\mathcal{D} = \ker \alpha \wedge \beta$ is an orientable Engel structure. Conversely if $\mathcal{D}$ is an orientable Engel structure there exist Engel defining forms α and β such that $\mathcal{D} = \ker \alpha \wedge \beta$.*

Proof. If α and β are Engel defining forms then (2.1a) implies that $\mathcal{E} = \ker \alpha$ is even contact. Moreover (2.1b) implies that $d\beta(W, X) \neq 0$ so that $[\mathcal{D}, \mathcal{D}]$ is a rank 3 distribution. Finally (2.1c) ensures that $\mathcal{W} = \ker \alpha \wedge d\alpha \subset \mathcal{D}$, so that $[\mathcal{D}, \mathcal{D}] = \ker \alpha$.

Conversely, suppose that $\mathcal{D}$ is orientable. Since M is orientable then $\mathcal{W}$ is trivial as a line bundle, let W be a non-singular section. Since $\mathcal{D}$ is orientable, it has a non-singular section X nowhere tangent to $\mathcal{W}$. Now $Y = [W, X]$ is a non-singular section of $\mathcal{E}$ nowhere tangent to $\mathcal{D}$, and $Z = [X, Y]$ is transverse to $\mathcal{E}$. Choose α such that $\ker \alpha = \langle W, X, Y \rangle$ and $\alpha(Z) = 1$, and similarly β such that $\ker \beta = \langle W, X, Z \rangle$ and $\beta(Y) = 1$. A simple calculation yields (2.1a), (2.1b) and (2.1c). □

Notice that if α and β are defining forms for $\mathcal{D}$, then all other possible defining forms are given by

$$\tilde{\alpha} = \lambda \alpha, \qquad \tilde{\beta} = \mu \beta + \nu \alpha$$

where $\lambda, \mu, \nu \in \mathcal{C}^\infty(M)$ and λ, μ are nowhere vanishing.

A choice of Engel defining forms determines uniquely a distribution $\mathcal{R}$ transverse to $\mathcal{D}$. Indeed Equation (2.1b) implies that $\alpha \wedge d\beta$ and $\beta \wedge d\beta$ are nowhere vanishing 3-forms, in particular they have a 1-dimensional kernel. Take T nowhere vanishing section of $\ker \alpha \wedge d\beta$, this implies $\alpha(T) = 0$. On the other hand we must have $\beta(T) \neq 0$, because otherwise $\beta \wedge \alpha \wedge d\beta = 0$, which would contradict Equation (2.1b). Hence we can normalize T via $\beta(T) = 1$. Similarly pick R nowhere vanishing section of $\ker \beta \wedge d\beta$ and normalize it via $\alpha(R) = 1$. By construction we have $\beta(R) = 0$.

Definition 2.1.4. *Let $(M, \mathcal{D})$ be an Engel structure and choose Engel defining forms $\mathcal{D} = \ker \alpha \wedge \beta$. The* Reeb distribution *associated with α and β is $\mathcal{R} = \langle T, R \rangle$ where*

$$\begin{aligned} i_T(\alpha \wedge d\beta) = 0, \quad \beta(T) = 1, \quad \alpha(T) = 0, \\ i_R(\beta \wedge d\beta) = 0, \quad \beta(R) = 0, \quad \alpha(R) = 1. \end{aligned}$$

The definition implies $TM = \mathcal{D} \oplus \mathcal{R}$. In what follows we will often denote by $\mathcal{D} = \ker \alpha \wedge \beta$ an Engel structure with a fixed choice of Engel defining forms α and β.

We are interested in understanding geometric properties of $\mathcal{R}$. In general the Reeb distribution is not going to be integrable. The following result gives a necessary and sufficient condition for integrability.

Proposition 2.1.5. *For a given Engel structure* $\mathcal{D} = \ker \alpha \wedge \beta$ *consider the associated Reeb distribution* $\mathcal{R} = \langle T, R \rangle$*, and denote by* $c_{TR} = d\beta(R, T)$*. Then* $\mathcal{R} = \ker(d\beta + c_{TR}\beta \wedge \alpha)$ *and it is integrable if and only if*

$$d(c_{TR}\alpha) \wedge \beta = 0. \tag{2.2}$$

Proof. For the first assertion we notice that $i_R(\beta \wedge d\beta) = 0$ and $\beta(R) = 0$ imply that $i_R d\beta$ is a multiple of β. In fact $i_R d\beta = c_{TR}\beta$, so that $R \in \ker(d\beta + c_{TR}\beta \wedge \alpha)$. Similarly we have have $T \in \ker(d\beta + c_{TR}\beta \wedge \alpha)$ so that $\mathcal{R} \subset \ker(d\beta + c_{TR}\beta \wedge \alpha)$. On the other hand the kernel of this 2-form has rank 2 because

$$\alpha \wedge \beta \wedge (d\beta + c_{TR}\beta \wedge \alpha) = \alpha \wedge \beta \wedge d\beta \neq 0.$$

To prove the last claim choose (locally) two 1-forms ρ, τ such that

$$\rho \wedge \tau = d\beta + c_{TR}\beta \wedge \alpha. \tag{2.3}$$

By Frobenius' Theorem $\mathcal{R}$ is integrable if and only if $\rho \wedge \tau \wedge d\tau = 0$ and $\tau \wedge \rho \wedge d\rho = 0$. Differentiating (2.3) we get

$$d\tau \wedge \rho - \tau \wedge d\rho = c_{TR} d\beta \wedge \alpha - d(c_{TR}\alpha) \wedge \beta = c_{TR}\rho \wedge \tau \wedge \alpha - d(c_{TR}\alpha) \wedge \beta.$$

Hence the integrability condition translates to

$$d(c_{TR}\alpha) \wedge \beta \wedge \tau = 0 \qquad \text{and} \qquad d(c_{TR}\alpha) \wedge \beta \wedge \rho = 0. \tag{2.4}$$

Since $d(c_{TR}\alpha) \wedge \beta \wedge \alpha = c_{TR} d\alpha \wedge \beta \wedge \alpha = 0$ and $d(c_{TR}\alpha) \wedge \beta \wedge \beta = 0$, conditions (2.4) are satisfied if and only if $d(c_{TR}\alpha) \wedge \beta = 0$ □

2.1.1 A useful technical lemma

In this section we list some formulas which describe Lie brackets of a framing induced by $\mathcal{D} = \ker \alpha \wedge \beta$.

Notation 2.1.6. In what follows we will fix a parallelization $\{W, X, T, R\}$ of M and we will use the letters a, b, c and d to denote respectively the W, X, T and R components of vector fields. Moreover we will use lower indices a_{AB}, b_{AB}, c_{AB} and d_{AB} to denote the components of the Lie bracket $[A, B]$, e.g.

$$[T, R] =: a_{TR}W + b_{TR}X + c_{TR}T + d_{TR}R.$$

The definition of T and R provides symmetries in the coefficients of the Lie brackets of the vector fields of the framing $\{W, X, T, R\}$. The following result summarizes the ones that we will use in the rest of the thesis.

Lemma 2.1.7. *Let $\mathcal{D} = \ker\alpha \wedge \beta$ be Engel and fix a framing $\{W,\ X,\ T,\ R\}$ such that $c_{WX} = \beta([W,X]) = 1$ and $d_{XT} = \alpha([X,T]) = 1$.*

We have $c_{WT} = c_{WR} = c_{XT} = c_{XR} = 0$ and $d_{WX} = d_{WT} = 0$. Moreover

$$b_{WX} = d_{WR} \tag{2.5a}$$
$$b_{XT} = -a_{WT} \tag{2.5b}$$
$$d_{TR} = \mathcal{L}_W d_{XR} - \mathcal{L}_X d_{WR} - a_{WX} d_{WR} - d_{XR} d_{WR} \tag{2.5c}$$
$$c_{TR} = a_{WR} + b_{XR} \tag{2.5d}$$
$$b_{WR} = -\mathcal{L}_W d_{TR} + \mathcal{L}_T d_{WR} + a_{WT} d_{WR} + b_{WT} d_{XR} \tag{2.5e}$$
$$b_{TR} = -\mathcal{L}_W c_{TR} + d_{WR} c_{TR} \tag{2.5f}$$
$$c_{TR} = -\mathcal{L}_X d_{TR} + \mathcal{L}_T d_{XR} - a_{WT} d_{XR} + a_{XT} d_{WR} - b_{XR} \tag{2.5g}$$
$$a_{TR} = \mathcal{L}_X c_{TR} - d_{XR} c_{TR}. \tag{2.5h}$$

Proof. Equations $c_{WT} = c_{WR} = c_{XT} = c_{XR} = 0$ are direct consequences of Definition 2.1.4. Moreover $d_{WX} = d_{WT} = 0$ follow from $\mathcal{L}_W \mathcal{E} = \mathcal{E}$. Equations (2.5a)-(2.5h) instead follow from all possible instance of the Jacobi identity. We prove (2.5c) and (2.5d), the other formulas follow similarly

$$\begin{aligned} 0 &= [W,[X,R]] + [R,[W,X]] + [X,[R,W]] \\ &= [W, a_{XR}W + b_{XR}X + d_{XR}R] + [R, a_{WX}W + b_{WX}X + T] \\ &\quad + [X, -a_{WR}W - b_{WR}X - d_{WR}R]. \end{aligned}$$

We continue the calculation mod $\mathcal{D}$

$$\begin{aligned} 0 &= b_{XR}[W,X] + (\mathcal{L}_W d_{XR})R + d_{XR}[W,R] \\ &\quad - a_{WX}[W,R] - b_{WX}[X,R] - [T,R] \\ &\quad + a_{WR}[W,X] - (\mathcal{L}_X d_{WR})R - d_{WR}[X,R] \qquad \text{mod } \mathcal{D} \\ &= \Big(a_{WR} + b_{XR} - c_{TR}\Big)T \\ &\quad + \Big(\mathcal{L}_W d_{XR} - \mathcal{L}_X d_{WR} - a_{WX} d_{WR} - d_{XR} d_{WR} - d_{TR}\Big)R. \end{aligned}$$

□

Notice that Equations (2.5f) and (2.5h) provide another proof of Proposition 2.1.5.

Remark 2.1.8. For a fixed defining form $\mathcal{E} = \ker\alpha$ and any $X \in \Gamma\mathcal{D}$ transverse to $\mathcal{W}$ the form $\beta := -\mathcal{L}_X\alpha = -i_X d\alpha$ is a defining form for $\mathcal{D}$. Indeed, since $i_W d\alpha = 0$ on $\ker\alpha$, we have $\beta(W) = 0$ and, by maximal non-integrability there is a section $Y \in \Gamma\mathcal{E}$ such that $\beta(Y) \neq 0$. Hence $\mathcal{D} = \ker\alpha \wedge \beta$. This choice ensures that $d_{XR} = \alpha([X,R]) = -i_X d\alpha(R) = \beta(R) = 0$. In fact by a change $\beta \mapsto \beta + \nu\alpha$ we have complete freedom on the choice of d_{XR}.

2.2 Existence of integrable $\mathcal{R}$

Understanding whether a given Engel structures admits Engel defining forms that induce an integrable Reeb distribution is a complicated problem. A first step in this direction is to compute the integrability condition (2.2) when we change defining forms.

Lemma 2.2.1. *For a given Engel structure $\mathcal{D} = \ker \alpha \wedge \beta$, choose W and X such that $c_{WX} = 1 = d_{XT}$. For another choice of Engel defining forms $\tilde{\alpha}$ and $\tilde{\beta}$, denote by $\tilde{\mathcal{R}} = \langle \tilde{T}, \tilde{R} \rangle$ the Reeb distribution associated with the new forms and $\tilde{c}_{TR} = d\tilde{\beta}(\tilde{R}, \tilde{T})$. We have*

1. *if $\tilde{\alpha} = \lambda\alpha$ and $\tilde{\beta} = \beta$ for $\lambda \in \mathcal{C}^\infty(M)$ nowhere vanishing then*
$$\tilde{R} = \frac{1}{\lambda} R, \qquad \tilde{T} = T, \qquad \tilde{c}_{TR} = \frac{1}{\lambda} c_{TR};$$

2. *if $\tilde{\alpha} = \alpha$ and $\tilde{\beta} = \mu\beta$ for $\mu \in \mathcal{C}^\infty(M)$ nowhere vanishing then*
$$\begin{aligned} \tilde{R} = R, \quad \tilde{T} &= \frac{1}{\mu}\Big(-\big(\mathcal{L}_X(\ln\mu)\big)W + \big(\mathcal{L}_W(\ln\mu)\big)X + T\Big), \\ \tilde{c}_{TR} &= c_{TR} + \mathcal{L}_R(\ln\mu); \end{aligned}$$

3. *if $\tilde{\alpha} = \alpha$ and $\tilde{\beta} = \nu\alpha + \beta$ for $\nu \in \mathcal{C}^\infty(M)$ then*
$$\begin{aligned} \tilde{R} &= \Big(-\nu^2 - \mathcal{L}_X\nu + \nu d_{XR}\Big)W + \Big(\mathcal{L}_W\nu - \nu d_{WR}\Big)X - \nu T + R, \\ \tilde{T} &= \nu W + T, \\ \tilde{c}_{TR} &= c_{TR} - \nu\mathcal{L}_W\nu - \mathcal{L}_T(\nu) + \nu^2 d_{WR} + \nu d_{TR}. \end{aligned}$$

Proof. Let us prove point 2, the other ones follow from a similar calculation. Since α is not changed and $\tilde{\beta} \wedge d\tilde{\beta} = \mu^2 \beta \wedge d\beta$ we conclude $\tilde{R} = R$. Similarly we must have $\mu\tilde{T} = aW + bX + T$. Imposing $i_{\tilde{T}}(\tilde{\alpha} \wedge d\tilde{\beta}) = 0$ yields the formula for $\tilde{T}$, here we need to use the hypothesis $d\beta(W, X) = -1 = d\alpha(X, T)$. The last formula follows directly from the evaluation $\tilde{c}_{TR} = d\tilde{\beta}(\tilde{T}, \tilde{R})$. □

The previous formulas do not give any geometrical insight in understanding integrability of $\mathcal{R}$. It is more interesting to try to understand when some stronger versions of Equation (2.2) are satisfied. We will now concentrate on the case $c_{TR} = 0$, which happens if and only if $d\beta^2 = 0$.

Lemma 2.2.2. *For a given Engel structure $\mathcal{D} = \ker \alpha \wedge \beta$ there exists a multiple $\tilde{\beta} = \mu\beta$ such that $d\tilde{\beta}^2 = 0$ if and only if for some W and X we have $a_{WR} + b_{XR} = 0$.*

Proof. To verify that $d\tilde{\beta}^2 = 0$ it suffices to prove that $d\tilde{\beta}(\tilde{T}, \tilde{R}) = 0$. Suppose that $a_{WR} + b_{XR} = 0$. Notice that we cannot apply Lemma 2.1.7 because we do not have information on c_{WX} and d_{XT}. Nonetheless a similar calculation yields

$$\begin{aligned}
0 &= \beta\Big([W,[X,R]] + [R,[W,X]] + [X,[R,W]]\Big) \\
&= \beta\Big([W, a_{XR}W + b_{XR}X + d_{XR}R]\Big) + \beta\Big([R, a_{WX}W + b_{WX}X + c_{WX}T]\Big) \\
&+ \beta\Big([X, -a_{WR}W - b_{WR}X - d_{WR}R]\Big) \\
&= \beta\Big(b_{XR}[W,X] + a_{WR}[W,X] + (\mathcal{L}_R c_{WX})T + c_{WX}[R,T]\Big) \\
&= \mathcal{L}_R c_{WX} + c_{WX} d\beta(T,R),
\end{aligned}$$

where in the last equality we used the hypothesis. By maximal non-integrability we must have that $c_{WX} = \beta([W,X])$ is nowhere vanishing, hence we can choose $\mu = c_{WX}^{-1}$. Using the point 2 in Lemma 2.2.1 and the fact that $i_R d\beta$ vanishes on $\mathcal{D}$ we get

$$\begin{aligned}
d(\mu\beta)(\tilde{T}, \tilde{R}) &= \Big(d\mu \wedge \beta + \mu d\beta\Big)\left(\frac{1}{\mu}T, R\right) \\
&= -\frac{1}{\mu}\mathcal{L}_R\mu + d\beta(T,R) = \frac{1}{c_{WX}}\Big(\mathcal{L}_R c_{WX} + c_{WX} d\beta(T,R)\Big) = 0.
\end{aligned}$$

Conversely suppose that $d\tilde{\beta}^2 = 0$ and choose W, X so that $c_{WX} = 1 = d_{XT}$. Lemma 2.2.1 ensures $\tilde{R} = R$ so that $\tilde{a}_{WR} = a_{WR}$ and $\tilde{b}_{XR} = b_{XR}$. We conclude by Equation (2.5d). □

2.2.1 Contactization

We will point out how to construct a contact structure starting from an Engel structure. This construction is well-known to experts, but the author could not find any explicit reference.

Definition 2.2.3. *Let* $(M, \mathcal{D} = \ker\alpha \wedge \beta)$ *be an Engel structure. The* contactization *of* M *is the contact* 5*-manifold* $(X = M \times \mathbb{R},\ \xi = \ker\eta)$ *with* $\eta = \beta + s\alpha$*, where we use* s *for the* $\mathbb{R}$*-coordinate.*

The previous definition depends on the choice of α and β. On the other hand we are interested in properties of ξ which are invariant up to rescaling and translating the $\mathbb{R}$ factor. The following result ensures that ξ is essentially independent of the choice of Engel defining forms.

Lemma 2.2.4. *The form* $\eta = \beta + s\alpha$ *defines a contact structure* ξ*. Moreover if we change Engel defining forms* $\alpha \mapsto \tilde{\alpha}$ *and* $\beta \mapsto \tilde{\beta}$*, there is a contactomorphism* $\psi : (X, \ker\eta) \to (X, \ker\tilde{\eta})$ *of the form* $\psi(p, s) = (p, f(p)s + g(p))$ *where* f, $g \in \mathcal{C}^\infty(M)$ *with* f *nowhere vanishing.*

Proof. To verify that η is a contact form we calculate $d\eta = d\beta + sd\alpha + ds \wedge \alpha$ and

$$d\eta^2 = d\beta^2 + s^2 d\alpha^2 + 2sd\alpha \wedge d\beta + 2ds \wedge \alpha \wedge d\beta + 2sds \wedge \alpha \wedge d\alpha. \quad (2.6)$$

This, together with $\alpha \wedge \beta \wedge d\beta \neq 0$ and $\alpha \wedge d\alpha \wedge \beta = 0$, ensures that $\eta \wedge d\eta^2 \neq 0$.

All possible choices of Engel defining forms can be written as $\tilde{\alpha} = \lambda\alpha$ and $\tilde{\beta} = \nu\alpha + \mu\beta$ for λ, ν, $\mu \in \mathcal{C}^\infty(M)$ with λ and μ nowhere vanishing. Hence we have $\tilde{\eta} = \mu\beta + \nu\alpha + s\lambda\alpha$, and if we define

$$\psi(p, s) := \left(p,\ \frac{1}{\lambda(p)}(\mu(p)s - \nu(p))\right)$$

we have $\psi^*\tilde{\eta} = \mu\eta$.

□

Equation (2.6) implies that the Reeb vector field associated with η takes the form (we use the notation introduced in 2.1.6)

$$R_\eta = T + sW + (c_{TR} + sd_{TR} + s^2 d_{WR})\partial_s. \quad (2.7)$$

Since $d\beta^2 = 0$ if and only if $c_{TR} = 0$, Equation (2.7) means that this happens if and only if $M \times \{0\}$ is invariant with respect to the Reeb flow.

Remark 2.2.5. Consider the change of Engel defining forms $\tilde{\alpha} = \alpha$ and $\tilde{\beta} = \nu\alpha + \beta$. Lemma 2.2.4 ensures that there is a hypersurface $\tilde{M} \hookrightarrow X$ graphical on $M \times \{0\}$ such that $(X, \ker\tilde{\eta})$ is the contactization of $(\tilde{M}, \mathcal{D} = \ker\tilde{\alpha} \wedge \tilde{\beta})$, namely the graph of $\nu : M \to \mathbb{R}$.

Conversely take a hypersurface $\tilde{M} \hookrightarrow X$ which is the graph of $g \in \mathcal{C}^\infty(M)$, and define $\psi(p, s) = (p,\ s + g(p))$. The forms $\tilde{\beta} = \eta|_{\tilde{M}} = \beta + g\alpha$ and $\tilde{\alpha} = (\mathcal{L}_{\partial_s}\eta)|_{\tilde{M}} = \alpha$ are defining forms for the Engel structure on $\tilde{M}$ obtained by pushing forward $\ker\alpha \wedge \beta$ via $\psi|_M : M \to \tilde{M}$.

The previous remark, Remark 2.1.2 and Equation (2.7) immediately give the proof of the following result.

Proposition 2.2.6. *Let $(M, \mathcal{D} = \mathcal{E} \cap \mathcal{E}')$ be an Engel manifold, with $\mathcal{E}$ and $\mathcal{E}'$ even contact structures such that $\mathcal{E} = \mathcal{D}^2$, $\mathcal{W} \subset \mathcal{E}$ and $\mathcal{W}' \pitchfork \mathcal{E}$. There is a 1-to-1 correspondence between choices of $\mathcal{E}'$ as above and hypersurfaces of the contactization $M \times \mathbb{R}$ which are graphical over M.*

Moreover, there is a choice of Engel defining forms $\mathcal{D} = \ker\alpha \wedge \beta$ such that $d\beta^2 = 0$ if and only if there is a graphical hypersurface on (X, ξ) invariant with respect to the Reeb flow associated with $\eta = \mu(\beta + s\alpha)$, for some $\mu \in \mathcal{C}^\infty(M)$ nowhere vanishing.

The previous result gives a geometric interpretation of $d\beta^2 = 0$. Unfortunately it is not very useful for practical purposes, because the dynamics of the Reeb vector field associated with η can be very complicated.

2.3 Crash course: totally geodesic distributions

Before continuing the discussion on Engel defining forms, we will recall some basic definitions in Riemannian geometry. The main references here are [Rov] and [Mont1].

In what follows we will always denote by (M, g) a smooth, compact, connected Riemannian manifold, and by ∇ its associated Levi-Civita connection. We are interested in metric properties of distributions $\mathcal{H} \subset TM$. In general there is no natural choice for a complementary bundle, but if a metric is fixed, then we have the orthogonal bundle $\mathcal{V} = \mathcal{H}^\perp$ and the splitting $TM = \mathcal{H} \oplus \mathcal{V}$.

In this setting we will denote, by abuse of notation, the orthogonal projections with the same symbol $\mathcal{H} : \mathcal{H} \oplus \mathcal{V} \to \mathcal{H}$ and $\mathcal{V} : \mathcal{H} \oplus \mathcal{V} \to \mathcal{V}$. The role of $\mathcal{V}$ and $\mathcal{H}$ is interchangeable.

The connection ∇ induces a connection on $\mathcal{V}$ by the formula

$$\nabla^{\mathcal{V}} : \Gamma\mathcal{V} \times \Gamma TM \to \Gamma\mathcal{V} \qquad \text{s.t.} \qquad (V, X) \mapsto \nabla^{\mathcal{V}}_X(V) = \mathcal{V}(\nabla_X V).$$

One can verify that this defines a connection on the vector bundle $\mathcal{V} \to M$. For $V \in \Gamma\mathcal{V}$ we have $\mathcal{H}V = 0$, taking the derivative yields

$$0 = \nabla_X(\mathcal{H}V) = (\nabla_X\mathcal{H})V + \mathcal{H}(\nabla_X V) = (\nabla_X\mathcal{H})(V) + \nabla_X V - \mathcal{V}(\nabla_X V).$$

Hence we have the formula

$$\nabla^{\mathcal{V}}_X(V) = (\nabla_X\mathcal{H})(V) + \nabla_X V. \tag{2.8}$$

Definition 2.3.1. *A* $\mathcal{V}$-geodesic *is a curve $\gamma : I \to M$ tangent to $\mathcal{V}$ and such that $\nabla^{\mathcal{V}}_{\dot\gamma}(\dot\gamma) = 0$.*

Proposition 2.3.2. *Let (M, g) be a Riemannian manifold, let $\mathcal{H}$ be a distribution and let $\mathcal{V}$ be its orthogonal complement $\mathcal{V} = \mathcal{H}^\perp$. Then for any $p \in M$ and $v \in \mathcal{V}_p$ there exists a unique maximal $\mathcal{V}$-geodesic γ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$.*

Proof. If $\mathcal{V}$ has rank k, Equation (2.8) implies that, in local coordinates, the equation $\nabla^{\mathcal{V}}_{\dot\gamma}\dot\gamma = 0$ is a system of k second order differential equations linear in the second order terms. In fact it is a linear perturbation of the usual geodesic equation. Moreover the condition $\dot\gamma \in \mathcal{V}$ is a system of $n-k$ linear differential equations. The initial conditions $\gamma(0) = p$ and $\dot\gamma(0) = v_p \in \mathcal{V}_p$ ensure that we can apply Cauchy-Lipschitz to get local existence and uniqueness. Classical techniques ensure existence and uniqueness of maximal solutions.

□

A foliation is totally geodesic if all its leaves are totally geodesic submanifolds. A way of rephrasing this is the following: every geodesic γ tangent to the foliation at time $t = 0$ is everywhere tangent to it. This interpretation provides the intuition behind the following definition.

Definition 2.3.3. *A distribution $\mathcal{H}$ on a Riemannian manifold (M, g) is* totally geodesic *if for every $p \in M$ and $v_p \in \mathcal{H}_p$ the geodesic through p tangent to v_p is tangent to $\mathcal{H}$ at every point.*

There is a characterization of totally geodesic distributions via the vanishing of a tensor that generalizes the O'Neill tensors of a submersion (see [ON]). Define

$$h^{\mathcal{V}} : \mathcal{V} \times \mathcal{V} \to \mathcal{H} \qquad \text{s.t.} \qquad h^{\mathcal{V}}(V, V') = \frac{1}{2}\mathcal{H}(\nabla_V V' + \nabla_{V'} V).$$

This tensor is symmetric, so we can equivalently take its polarization. The following result ensures that $h^{\mathcal{V}}$ encodes the metric information concerning geodesics tangent to $\mathcal{V}$.

Lemma 2.3.4. *For a given Riemannian manifold (M, g) and a distribution $\mathcal{H}$, let $\mathcal{V} = \mathcal{H}^{\perp}$. Then $\mathcal{V}$ is totally geodesic if and only if $h^{\mathcal{V}} = 0$.*

Proof. By polarization $h^{\mathcal{V}} = 0$ if and only if $h^{\mathcal{V}}(v_p, v_p) = 0$ for all $v_p \in \mathcal{V}_p$. Suppose that $\mathcal{V}$ is totally geodesic, then the unique geodesic γ such that $\gamma(0) = p$ and $\dot\gamma(0) = v_p$ is tangent to $\mathcal{V}$ everywhere. Hence

$$h^{\mathcal{V}}(v_p, v_p) = h^{\mathcal{V}}(\dot\gamma, \dot\gamma)(p) = \mathcal{H}\left(\nabla_{\dot\gamma}\dot\gamma\right)(p) = 0.$$

Conversely suppose $h^{\mathcal{V}} = 0$ and let $v_p \in \mathcal{V}_p$. By Proposition 2.3.2 there exists a (unique) curve γ tangent to $\mathcal{V}$ such that $\dot\gamma(0) = v_p$ and $\nabla^{\mathcal{V}}_{\dot\gamma}\dot\gamma = \mathcal{V}\left(\nabla_{\dot\gamma}\dot\gamma\right) = 0$. This implies

$$\nabla_{\dot\gamma}\dot\gamma = \mathcal{H}\left(\nabla_{\dot\gamma}\dot\gamma\right) = h^{\mathcal{V}}(\dot\gamma, \dot\gamma) = 0,$$

so that γ is a geodesic. We conclude by uniqueness of the geodesic for given initial values.

□

2.4 When are $\mathcal{D}$ and $\mathcal{R}$ totally geodesic?

Since a choice of Engel defining forms α and β determines uniquely the splitting $TM = \mathcal{D} \oplus \mathcal{R}$, it is natural to consider Riemannian metrics g which satisfy $\mathcal{D}^{\perp} = \mathcal{R}$. In this context let A and B denote the g-duals of α and β respectively, i.e.

$$i_A g = \alpha \qquad \text{and} \qquad i_B g = \beta. \tag{2.9}$$

Since α and β are linearly independent, the same is true for A and B. Moreover since $\mathcal{D}^{\perp} = \mathcal{R}$ they must be tangent to $\mathcal{R}$, in particular $\mathcal{R} = \langle A, B\rangle$. Equation (2.9) implies

$$\alpha(A) = \|A\|^2, \quad \beta(B) = \|B\|^2, \quad \alpha(B) = g(A, B) = \beta(A),$$

and using these formulas we get

$$A = g(A, B)T + \|A\|^2 R \qquad \text{and} \qquad B = \|B\|^2 T + g(A, B)R. \tag{2.10}$$

Remark 2.4.1. In this context we have some freedom in the choice of α. Suppose that α, β, and g are as above, and consider the new defining forms $\tilde{\alpha} = \lambda\alpha$ and $\tilde{\beta} = \beta$ for $\lambda \in \mathcal{C}^\infty(M)$ nowhere vanishing. Lemma 2.2.1 ensures that the Reeb distribution $\tilde{\mathcal{R}}$ associated with the new defining forms coincides with $\mathcal{R}$. In particular, g also satisfies $\mathcal{D}^\perp = \tilde{\mathcal{R}}$ and we have the formulas

$$\tilde{A} = \lambda A \qquad \text{and} \qquad \tilde{B} = B. \tag{2.11}$$

2.4.1 $\mathcal{R}$ is never totally geodesic

The goal of this section is to show that there is no metric g such that $\mathcal{D}^\perp = \mathcal{R}$ and $\mathcal{R}$ is totally geodesic.

Lemma 2.4.2. *Let $\mathcal{D} = \ker \alpha \wedge \beta$ be Engel and suppose that g is a metric such that $\mathcal{D}^\perp = \mathcal{R}$, then $\mathcal{R}$ is totally geodesic if and only if for all U, $U' \in \Gamma\mathcal{R}$ and $V \in \Gamma\mathcal{D}$ we have*

$$\mathcal{L}_V(g(U,U')) + g([U',V],U) + g([U,V],U') = 0.$$

Proof. Lemma 2.3.4 ensures that $\mathcal{R}$ is totally geodesic if and only if the following tensor vanishes

$$h^{\mathcal{R}} : \Gamma\mathcal{R} \times \Gamma\mathcal{R} \to \Gamma\mathcal{D} \qquad \text{s.t.} \qquad h^{\mathcal{R}}(U,U') = \frac{1}{2}\mathcal{D}\Big(\nabla_U U' + \nabla_{U'} U\Big).$$

Here $\mathcal{D}$ and $\mathcal{R}$ also denote the orthogonal projections on the respective distributions.

Now $h^{\mathcal{R}}$ is zero if and only if for any $V \in \Gamma\mathcal{D}$ we have $g(h^{\mathcal{R}}(U,U'),V) = 0$. Using Koszul's identity and the fact that $g(U,V) = 0 = g(U',V)$ we have

$$\begin{aligned} 0 =& 4g(h^{\mathcal{R}}(U,U'),V) = 2g(\nabla_U U',V) + 2g(\nabla_{U'}U,V) \\ =& -\mathcal{L}_V(g(U,U')) + g([U,U'],V) - g([U',V],U) + g([V,U],U') + \\ & -\mathcal{L}_V(g(U',U)) + g([U',U],V) - g([U,V],U') + g([V,U'],U) \\ =& -2\Big(\mathcal{L}_V(g(U,U')) + g([U',V],U) + g([U,V],U')\Big). \end{aligned}$$

□

The following result furnishes an obstruction on the metric properties of the Reeb distribution associated with any pair of Engel defining forms, when the metric g makes the splitting $TM = \mathcal{D} \oplus \mathcal{R}$ orthogonal.

Proposition 2.4.3. *Let $\mathcal{D} = \ker \alpha \wedge \beta$ and let g be a metric such that $\mathcal{D}^\perp = \mathcal{R}$, then $\mathcal{R}$ is not totally geodesic.*

Proof. Suppose that $\mathcal{R}$ is totally geodesic and fix a framing $\mathcal{D} = \langle W, X\rangle$ where W spans the characteristic foliation and $c_{WX} = 1 = d_{XT}$. Using Lemma 2.4.2 we get

$$\begin{aligned}
g(h^{\mathcal{R}}(A, A), W) = 0 \;&\Rightarrow\; & \mathcal{L}_W \|A\|^2 + 2\alpha([A, W]) = 0\\
g(h^{\mathcal{R}}(A, A), X) = 0 \;&\Rightarrow\; & \mathcal{L}_X \|A\|^2 + 2\alpha([A, X]) = 0\\
g(h^{\mathcal{R}}(B, B), W) = 0 \;&\Rightarrow\; & \mathcal{L}_W \|B\|^2 + 2\beta([B, W]) = 0\\
g(h^{\mathcal{R}}(B, B), X) = 0 \;&\Rightarrow\; & \mathcal{L}_X \|B\|^2 + 2\beta([B, X]) = 0\\
g(h^{\mathcal{R}}(A, B), W) = 0 \;&\Rightarrow\; & \mathcal{L}_W(g(A, B)) + \alpha([B, W]) + \beta([A, W]) = 0\\
g(h^{\mathcal{R}}(A, B), X) = 0 \;&\Rightarrow\; & \mathcal{L}_X(g(A, B)) + \alpha([B, X]) + \beta([A, X]) = 0
\end{aligned}$$

Equation (2.10) implies

$$\begin{aligned}
[W, A] &= \Big(\mathcal{L}_W(g(A, B))\Big)T + \Big(\mathcal{L}_W \|A\|^2 + \|A\|^2 d_{WR}\Big)R\\
[W, B] &= \Big(\mathcal{L}_W \|B\|^2\Big)T + \Big(\mathcal{L}_W(g(A, B)) + g(A, B)d_{WR}\Big)R\\
[X, A] &= \Big(\mathcal{L}_X(g(A, B))\Big)T + \Big(\mathcal{L}_X \|A\|^2 + g(A, B) + \|A\|^2 d_{XR}\Big)R\\
[X, B] &= \Big(\mathcal{L}_X \|B\|^2\Big)T + \Big(\mathcal{L}_X(g(A, B)) + \|B\|^2 + g(A, B)d_{XR}\Big)R.
\end{aligned}$$

These in turns yield

1. $\mathcal{L}_W \|A\|^2 + 2\|A\|^2 d_{WR} = 0$
2. $\mathcal{L}_X \|A\|^2 + 2g(A, B) + 2\|A\|^2 d_{XR} = 0$
3. $\mathcal{L}_W \|B\|^2 = 0$
4. $\mathcal{L}_X \|B\|^2 = 0$
5. $\mathcal{L}_W(g(A, B)) + g(A, B)d_{WR} = 0$
6. $\mathcal{L}_X(g(A, B)) + \|B\|^2 + g(A, B)d_{XR} = 0$

Now 3 and 4 imply that $\|B\|^2$ is constant on orbits of W and X. Since $\mathcal{D} = \langle W, X\rangle$ is bracket-generating, Chow's Theorem 1.1.2 implies that $\|B\|^2$ is constant. Notice that the same formulas must hold for $\tilde{A}$ and $\tilde{B}$ given by Equation (2.11). Since $\|\tilde{A}\| = |\lambda| \|A\|$, we can suppose that $\|A\|^2$ is also constant, say

$$\|A\| = 1.$$

In particular for some $c \in \mathbb{R}$, we get

1. $d_{WR} = 0$

2. $g(A, B) + d_{XR} = 0$

3./4. $\|B\| = c$

5. $\mathcal{L}_W(g(A, B)) = 0$

6. $\mathcal{L}_X(g(A, B)) + \|B\|^2 + g(A, B)d_{XR} = 0$

Hence (2) and (6) together give

$$\mathcal{L}_X(g(A, B)) = g(A, B)^2 - \|B\|^2 .$$

The Cauchy-Schwarz inequality reads $g(A, B)^2 \leq \|A\|^2 \|B\|^2$ with equality if and only if $\{A, B\}$ is linearly dependent. Since this never happens, the inequality is sharp and

$$\mathcal{L}_X(g(A, B)) < \|A\|^2 \|B\|^2 - \|B\|^2 = 0.$$

This is a contradiction because we must have $\mathcal{L}_X(g(A, B)) = 0$ on the critical points of the function $p \mapsto g(A, B)|_p$. □

The geometric reason for this obstruction is not clear. Notice that the hypothesis on g cannot be relaxed as Example 2.8.6 furnishes an Engel structure $\mathcal{D} = \ker \alpha \wedge \beta$ on T^4 where $\mathcal{R}$ is a totally geodesic foliation. In this case the splitting $TM = \mathcal{D} \oplus \mathcal{R}$ is not orthogonal.

2.4.2 $\mathcal{D}$ totally geodesic implies $\mathcal{R}$ integrable

In this section we study the properties of the Reeb distribution associated with a totally geodesic Engel structure $\mathcal{D} = \ker \alpha \wedge \beta$. Let g be such that $\mathcal{D}^\perp = \mathcal{R}$ and choose a framing $\mathcal{D} = \langle W, X \rangle$ which is orthonormal and such that W spans the characteristic foliation. The proof of the following lemma is exactly the same as the proof of Lemma 2.4.2.

Lemma 2.4.4. *Under the above hypothesis $\mathcal{D}$ is totally geodesic if and only if for all $X_1, X_2 \in \{W, X\}$ and $Y \in \Gamma\mathcal{R}$ we have*

$$g([X_1, Y], X_2) + g([X_2, Y], X_1) = 0$$

The previous result permits us to express the Lie brackets of sections of $\mathcal{D}$ with sections of $\mathcal{R}$ in a simple way.

Corollary 2.4.5. *Under the above hypothesis $\mathcal{D}$ is totally geodesic if and only if*

$$[W, T] = b_{WT}X \qquad [W, R] = b_{WR}X + d_{WR}R$$
$$[X, T] = -b_{WT}W + d_{XT}R \qquad [X, R] = -b_{WR}W + d_{XR}R$$

Proof. Since $i_T d\beta = -c_{TR}\alpha$, all T-components must vanish. Moreover $[W, T] \in \mathcal{E}$ hence $d_{WT} = 0$. Hence the only components left to calculate are the ones in the direction of W and X. Since $\{W, X\}$ is an orthonormal basis and $\mathcal{D}^\perp = \mathcal{R}$ we can calculate them using Lemma 2.4.4:

$$a_{WT} = g([W,T],W) = 0, \quad a_{WR} = g([W,R],W) = 0,$$

similarly $b_{XT} = 0$ and $b_{XR} = 0$. Moreover

$$b_{WT} = g([W,T],X) = -g([X,T],W) = -a_{XT}$$

and $b_{WR} = -a_{XR}$, which concludes the proof. □

The following result links metric properties of $\mathcal{D}$ with integrability properties of $\mathcal{R}$.

Corollary 2.4.6. *Let $\mathcal{D} = \ker \alpha \wedge \beta$ be Engel and let g be a metric such that $\mathcal{D}^\perp = \mathcal{R}$. If $\mathcal{D}$ is totally geodesic, then there exists a nowhere vanishing function $\mu \in \mathcal{C}^\infty(M)$ such that $\tilde{\beta} = \mu\beta$ satisfies $d\tilde{\beta}^2 = 0$. In particular, $\tilde{\mathcal{R}}$ is integrable.*

Proof. The previous corollary says $a_{WR} + b_{XR} = 0$ which is exactly the hypothesis of Lemma 2.2.2. □

The converse of this result is likely false, but we do not have a counterexample. Towards the end of the chapter we will furnish some examples of Engel structures for which $\mathcal{D}$ is totally geodesic.

2.5 Engel vector fields

In this section we will study Engel structures that admit symmetries. The existence of 1-parameter families of contactomorphism for any given contact structure is well-known. For Engel structures on the other hand the existence of such families of symmetries is tightly related to the dynamics of the characteristic foliation.

Definition 2.5.1. *Let $(M, \mathcal{D})$ be an Engel structure an* Engel vector field *is a vector field whose flow preserves $\mathcal{D}$.*

Remark 2.5.2. If Z preserves $\mathcal{D}$ then automatically it must preserve its Engel flag, i.e.

$$\mathcal{L}_Z\mathcal{W} = \mathcal{W}, \quad \mathcal{L}_Z\mathcal{D} = \mathcal{D} \qquad \text{and} \qquad \mathcal{L}_Z\mathcal{E} = \mathcal{E}.$$

In [Mont2] there is an example of Engel structure admitting a unique 1-parameter family of symmetries. The idea of the construction is to make sure that $\mathcal{W}$ itself does not admit many symmetries.

Let $(M, \mathcal{D})$ be an orientable Engel manifold and fix a defining form for the induced even contact structure $\mathcal{E} = \ker \alpha$. Let W be a nowhere vanishing section of the characteristic foliation $\mathcal{W}$, then there exists a function h_W such that $\mathcal{L}_W \alpha = h_W \alpha$. Following [Vog1] we define the space of Engel Hamiltonians as

$$\mathcal{C}^\infty(\alpha) = \left\{ f \in \mathcal{C}^\infty(M) \middle| \mathcal{L}_W f = h_W f \right\}.$$

One can show that $\mathcal{C}^\infty(\lambda\alpha) = \lambda\mathcal{C}^\infty(\alpha)$ for any nowhere vanishing $\lambda \in \mathcal{C}^\infty(M)$.

Lemma 2.5.3 [Vog1]**.** *The space of Engel vector fields is in 1-to-1 correspondence with $\mathcal{C}^\infty(\alpha)$. More precisely for any $f \in \mathcal{C}^\infty(\alpha)$ there exists a unique Engel vector field Z_f such that $\alpha(Z_f) = f$ and conversely if Z is an Engel vector field then $\alpha(Z) \in \mathcal{C}^\infty(\alpha)$.*

Proof. If Z preserves the Engel structure set $f = \alpha(Z)$. Then by Remark 2.5.2, it also preserves the even contact structure. In particular $\mathcal{L}_Z \alpha$ vanishes on W, so that

$$0 = (\mathcal{L}_Z \alpha)(W) = df(W) + d\alpha(Z, W) = \mathcal{L}_W f - h_W f.$$

We will only prove the converse in the case where M and $\mathcal{D}$ are orientable. In this case fix Engel defining forms α and β and a framing $\mathcal{D} = \langle W, X \rangle$ such that $c_{WX} = 1 = d_{XT}$. For $f \in \mathcal{C}^\infty(\alpha)$, we want to find a symmetry Z of the form $Z = wW + xX + tT + fR$.

Imposing the condition $\mathcal{L}_Z \beta = 0$ modulo $\langle \alpha, \beta \rangle$ we get $x = -\mathcal{L}_W t$ and $w = \mathcal{L}_X t$. Now the condition $\mathcal{L}_Z \alpha = 0$ modulo $\langle \alpha \rangle$ translates to

$$\begin{aligned} \mathcal{L}_W f &= h_W f \\ x &= \mathcal{L}_T f + d_{TR} f \\ t &= -\mathcal{L}_X f - d_{XR} f. \end{aligned}$$

The first one is verified since $f \in \mathcal{C}^\infty(\alpha)$ by hypothesis, and the last one determines t. The second one is equivalent to

$$\mathcal{L}_T f + d_{TR} f = \mathcal{L}_W (\mathcal{L}_X f + d_{XR} f),$$

which follows from $f \in \mathcal{C}^\infty(\alpha)$ and Equations (2.5b) and (2.5c).

□

Remark 2.5.4. The proof of the previous result gives us an explicit formula for the Engel vector field Z_f associated with $f \in \mathcal{C}^\infty(\alpha)$. Namely we have $Z_f = wW + xX + tT + fR$ where

$$w = -\mathcal{L}_X t, \qquad x = \mathcal{L}_X t \qquad \text{and} \qquad t = \mathcal{L}_X f + d_{XR} f. \tag{2.12}$$

Equation (2.12) simplifies if we make the choice of defining forms $\beta = i_X d\alpha$, since this ensures $d_{XR} = 0$.

The problem of finding symmetries for a given Engel structure translates then to understanding the space $\mathcal{C}^\infty(\alpha)$, which is tightly linked with the dynamical properties of W. It is still unclear if there are Engel structures which do not admit any 1-parameter families of symmetries.

Equation (2.12) permits to give a very explicit formula for Engel symmetries which are also sections of the Reeb distribution.

Lemma 2.5.5. *Let $\mathcal{D} = \ker \alpha \wedge \beta$ be an Engel structure and denote by $\mathcal{R}$ its Reeb distribution. If Z is an Engel vector field tangent to the Reeb distribution then $Z = kT + fR$ for $k \in \mathbb{R}$ and $f \in \mathcal{C}^\infty(\alpha)$. Moreover if Z is non-singular then the zero set of f is nowhere-dense.*

Proof. Equation (2.12) together with the condition $Z \in \Gamma\mathcal{R}$ implies that $\mathcal{L}_W t = 0$ and $\mathcal{L}_X t = 0$. This means that t is constant on orbits of W and X, hence by Chow's Theorem it is constant.

The last claim follows because if Z is tangent to $\mathcal{E}$ in the neighbourhood of a point then, by uniqueness of $\mathcal{W}$, it must be tangent to W, which would lead to a contradiction. □

The following result furnishes a relation between existence of symmetries of $\mathcal{D}$ and the dynamics of $\mathcal{W}$.

Proposition 2.5.6 [Vog1]**.** *Let $\mathcal{D}$ be an Engel structure and $\mathcal{E}$ its induced even contact structure. There exists an Engel vector field Z transverse to $\mathcal{E}$ if and only if the holonomy of $\mathcal{W}$ is volume-preserving.*

If such a vector field exists then there exists a pair of defining forms $\mathcal{D} = \ker \alpha \wedge \beta$ such that $Z = R$ and $d\alpha^2 = 0$.

Proof. The first claim is contained in Proposition 1.3.6. In order to prove the second statement let Z be an Engel vector field transverse to $\mathcal{E}$ and use Proposition 1.3.6 to find α such that $\alpha(Z) = 1$ and $d\alpha^2 = 0$. We need to find β so that $Z = R$.

By Remark 2.5.2 the flow of Z must preserve the Engel flag $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$. This means that we can choose a framing $\mathcal{E} = \langle W,\, X,\, Y\rangle$ satisfying

$$\mathcal{L}_Z\langle W\rangle = \langle W\rangle, \quad \mathcal{L}_Z\langle X\rangle \subset \langle W, X\rangle, \quad \mathcal{L}_Z\langle Y\rangle \subset \langle W, X, Y\rangle. \tag{2.13}$$

Choose β so that $\ker \beta = \langle W,\, X,\, Z\rangle$ and $\beta(Y) = 1$. Exactly as in the proof of Proposition 2.1.3 the forms α and β are Engel defining forms for $\mathcal{D}$. Now Equation (2.13) implies $[W, Z] = aW$, so that

$$d\beta(Z, W) = \beta([W, Z]) = a\beta(W) = 0.$$

Similarly we have $d\beta(Z, X) = 0$, so that $d\beta = 0$ on $\ker \beta$. This implies $Z \in \ker \beta \wedge d\beta$, and since $\alpha(Z) = 1$ we conclude $Z = R$. □

As we have already seen, even contact structures admitting a defining form $\mathcal{E} = \ker\alpha$ such that $d\alpha^2 = 0$ are very special. If this happens for $\mathcal{E} = \ker\alpha$ induced by the Engel structure $\mathcal{D}$, then there is a special choice of Engel defining forms.

Lemma 2.5.7. *Let $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$ be the Engel flag of $\mathcal{D}$. The following are equivalent*

1. *$\mathcal{W}$ has volume-preserving holonomy;*
2. *there exist α and β such that $\mathcal{L}_R\mathcal{D} \subset \mathcal{D}$;*
3. *there exist α and β such that $\ker d\alpha = \langle W, R\rangle$;*
4. *there exist α and β such that $\beta \wedge d\alpha = 0$;*
5. *there is a choice of α such that the conformal class of $\beta = \mathcal{L}_X\alpha$ does not depend on the choice of $X \in \Gamma\mathcal{D}$ transverse to $\mathcal{W}$.*

Moreover Properties 1 *to* 5 *are all satisfied simultaneously for the same choice of defining forms α and β.*

Proof. $\boxed{1 \Rightarrow 2}$ This was already proven in Lemma 2.5.6.

$\boxed{2 \Rightarrow 3}$ The hypothesis implies that $\mathcal{L}_R\mathcal{E} \subset \mathcal{E}$ so that $\mathcal{L}_R\alpha = i_R d\alpha = \lambda\alpha$ for some $\lambda \in \mathcal{C}^\infty(M)$. Since $i_W(\alpha \wedge d\alpha) = 0$ we have $i_W d\alpha = h_W\alpha$ for some $h_W \in \mathcal{C}^\infty(M)$. Hence $d\alpha^2 = 0$ if and only if $(i_W d\alpha)(R) = 0$, but we have $(i_W d\alpha)(R) = -(i_R d\alpha)(W) = -\lambda\alpha(W) = 0$. This also proves that $\ker d\alpha = \langle W, R\rangle$.

$\boxed{3 \Rightarrow 4}$ This is obvious since both W and R are in the kernel of β and of $d\alpha$.

$\boxed{4 \Rightarrow 5}$ For any choice of X we must have $0 = i_X(\beta \wedge d\alpha) = -\beta \wedge \mathcal{L}_X\alpha$, which is only possible if $\mathcal{L}_X\alpha$ is a multiple of β.

$\boxed{5 \Rightarrow 1}$ By Proposition 1.3.6 it suffices to proof that $i_W d\alpha = 0$. Suppose this is not true, then as in the above proof we must have $i_W d\alpha = h_W\alpha$ with $h_W \in \mathcal{C}^\infty(M)$ not identically zero. For any given $X \in \Gamma\mathcal{D}$ transverse to W, the vector field $\tilde{X} = X + W$ is transverse to W, but $\beta = i_X d\alpha$ is not a multiple of $\tilde{\beta} = i_{\tilde{X}} d\alpha = \beta + h_W\alpha$.

The last statement follows directly from this proof. □

Remark 2.5.8. The above discussion seems to suggest a link between integrability of $\mathcal{R}$ and the existence of special symmetries for $\mathcal{D}$. Proposition 2.1.5 ensures that the Reeb distribution $\mathcal{R}$ associated with the Engel defining forms α and β is integrable if and only if $d(c_{TR}\alpha) = 0$ modulo $\langle\beta\rangle$. This translates to the conditions

$$\mathcal{L}_W c_{TR} = d_{WR} c_{TR} \qquad \text{and} \qquad \mathcal{L}_X c_{TR} = d_{XR} c_{TR}.$$

On the other hand a symmetry $Z \in \Gamma\mathcal{R}$ is completely determined by $f \in \mathcal{C}^\infty(\alpha)$ such that

$$\mathcal{L}_W f = -d_{WR} f \qquad \text{and} \qquad \mathcal{L}_X f = -d_{XR} f + t,$$

where $t \in \mathbb{R}$. These conditions look very similar, we do not know if this is just a coincidence.

2.6 Engel Killing vector fields

In this section we will study Engel structures admitting transverse Killing symmetries.

Proposition 2.6.1. *Let $(M, \mathcal{D})$ be an Engel structure and let g be a Riemannian metric. Suppose that $Z \in \mathfrak{X}(M)$ is Engel, Killing and orthogonal to $\mathcal{E}$, then there exists a choice of defining forms α and β such that $d\alpha^2 = 0 = d\beta^2$ and $\beta = -\mathcal{L}_X\alpha$ for some $X \in \Gamma\mathcal{D}$ transverse to $\mathcal{W}$.*

Proof. Since $Z \perp \ker\alpha$ it must be in particular transverse to it, hence Proposition 2.5.6 implies the existence of α and β such that $R = Z$ and $d\alpha^2 = 0$. Up to rescaling g we can suppose that $\|R\| = 1$. Fix an orthonormal basis $\mathcal{D} = \langle W, X\rangle$ and complete it with a vector field Y to an orthonormal basis of $\mathcal{E}$. This implies that $\{W, X, Y, R\}$ is an orthonormal framing.

Since R is Engel, as in the proof of Proposition 1.3.6, we have

$$\begin{aligned}
[W, R] &= a_{WR} W \\
[X, R] &= a_{XR} W + b_{XR} X \\
[Y, R] &= a_{YR} W + b_{YR} X + c_{YR} Y.
\end{aligned}$$

Since R is Killing we have

$$0 = (\mathcal{L}_R g)(W, W) = \mathcal{L}_R(g(W, W)) - 2g(\mathcal{L}_R W, W) = 2a_{WR}$$

so that $[W, R] = 0$. Similarly $b_{XR} = c_{YR} = 0$. Moreover

$$0 = (\mathcal{L}_R g)(W, X) = \mathcal{L}_R(g(W, X)) - g(\mathcal{L}_R W, X) - g(W, \mathcal{L}_R X) = a_{XR}$$

so that $[X, R] = 0$. Similarly $a_{YR} = 0$ and $b_{YR} = 0$. Hence we have $[W, R] = [X, R] = [Y, R] = 0$.

Rescale β so that $\beta(Y) = 1$, by Lemma 2.2.1 this does not change R. We already have $d\alpha^2 = 0$, moreover $(\mathcal{L}_R\beta)(W) = \mathcal{L}_R(\beta(W)) - \beta(\mathcal{L}_R W) = 0$ and similarly for the other vector fields in the framing, so that $\mathcal{L}_R\beta = 0$. This implies that $d\beta^2 = 0$ and $\beta \wedge d\alpha = 0$, so that $\beta = -\mathcal{L}_X\alpha$ follows from Proposition 2.5.6. ◻

Corollary 2.6.2. *Let $(M, \mathcal{D})$ be an Engel structure and let g be a Riemannian metric. Suppose that $Z \in \mathfrak{X}(M)$ is Engel, Killing and orthogonal to $\mathcal{E}$, then there exists a framing $TM = \langle W, X, T, R\rangle$ such that $\mathcal{D} = \langle W, X\rangle = \ker \alpha \wedge \beta$ for some defining forms whose associated Reeb distribution is $\mathcal{R} = \langle T, R\rangle$ with $Z = R$. Moreover we have*

$$\begin{aligned}
[W, R] &= [X, R] = [T, R] = 0\\
[W, X] &= a_{WX} W + T\\
[W, T] &= a_{WT} W + b_{WT} X\\
[X, T] &= a_{XT} W - a_{WT} X + R
\end{aligned}$$

where the functions a_{WX}, a_{WT}, b_{WT} and a_{XT} are constant on the orbits of R.

Proof. The proof of Proposition 2.6.1 implies the existence of a framing $\{W, X, Y, R\}$ such that R commutes with every vector field in the framing. Now $\alpha \wedge d\beta = 0$ implies $d\alpha \wedge d\beta = 0$, which in turn implies $[T, R] = 0$.

We need to rescale W and X so that $c_{WX} = 1 = d_{XT}$. This is possible because the Jacobi identity implies

$$\begin{aligned}
\mathcal{L}_R\Big(\beta([W, X])\Big) &= (\mathcal{L}_R\beta)\Big([W, X]\Big) - \beta\Big([R, [W, X]]\Big)\\
&= \beta\Big([X, [R, W]] + [W, [X, R]]\Big) = 0,
\end{aligned}$$

and similarly $\mathcal{L}_R(\alpha([X, T])) = 0$. Hence we can rescale W and X as follows

$$W \mapsto \frac{\alpha([X, Y])}{\beta([W, X])} W, \qquad X \mapsto \frac{1}{\alpha([X, Y])} X$$

to get a new framing of $\mathcal{D}$ satisfying all previous conditions and additionally $c_{WX} = 1 = d_{XT}$. We have

$$\begin{aligned}
[W, R] &= [X, R] = [T, R] = 0\\
[W, X] &= a_{WX} W + b_{WX} X + T\\
[W, T] &= a_{WT} W + b_{WT} X\\
[X, T] &= a_{XT} W + b_{XT} X + R.
\end{aligned}$$

Equation (2.5a) implies $b_{WX} = d_{WR} = 0$ and Equation (2.5b) implies $b_{XT} = -a_{WT}$. Notice that we can rescale g on $\mathcal{D}$ to make this basis orthonormal. □

The converse of Proposition 2.6.1 is not true. The existence of defining forms satisfying $d\alpha^2 = 0 = d\beta^2$ and $d\alpha \wedge \beta = 0$ only ensures that R is a Killing vector field if $\mathcal{L}_R$ acts in a diagonalizable way on $\mathcal{D}$.

Proposition 2.6.3. *Let $\mathcal{D} = \ker \alpha \wedge \beta$ be an Engel structure such that $d\alpha^2 = 0 = d\beta^2$ and $d\alpha \wedge \beta = 0$. Suppose that there exists $X \in \Gamma\mathcal{D}$ transverse to $\mathcal{W}$ and such that $\mathcal{L}_R X = bX$ for $b \in \mathcal{C}^\infty(M)$. Then there exists a framing*

$TM = \langle W, X, T, R\rangle$ such that R commutes with any vector field in the framing. In particular R is Killing for the metric g making this framing orthonormal.

Proof. First of all notice that Lemma 2.5.7 ensures that the flow of R preserves $\mathcal{D}$ and its Engel flag. In particular for any framing $\mathcal{D} = \langle W, X\rangle$ we must have that $\mathcal{L}_R W = aW$ and $\mathcal{L}_R X = bX + cW$ for some smooth functions a, b, and c. The hypothesis ensures that we can choose X such that $c = 0$. Up to rescaling we can choose W and X multiples of these such that $c_{WX} = 1 = d_{XT}$.

The condition $d\beta^2 = 0$ implies that $[T, R]$ is a multiple of R. Since $d\alpha \wedge \beta = 0$ implies $d\alpha \wedge d\beta = 0$, we must have $[T, R] = 0$. Using $\mathcal{L}_R\beta = 0$ we get

$$\begin{aligned} 0 = \mathcal{L}_R\Big(\beta([W, X])\Big) &= -\beta\Big([R, [W, X]]\Big) \\ &= \beta\Big([X, [R, W]] + [W, [X, R]]\Big) = -a - b. \end{aligned}$$

Similarly $\mathcal{L}_R\alpha = 0$ and $[T, R] = 0$ imply $0 = \mathcal{L}_R\Big(\alpha([X, T])\Big) = -b$. Hence the claim. □

Remark 2.6.4. There is an alternative way of proving $[W, R] = 0$ once we know $[T, R] = 0$. Take the family of forms $\beta_t = \beta + t\alpha$ for $t \in \mathbb{R}$, it is easy to verify that $d\beta_t^2 = 0$ and $d\alpha \wedge d\beta_t = 0$ for all $t \in \mathbb{R}$. This means that the associated Reeb distribution $\mathcal{R}_t = \langle T_t, R_t\rangle$, also satisfies $[T_t, R_t] = 0$. Consider $A_1 = T + tW$ and $A_2 = R$. Since we chose W such that $\beta([W, X]) = 1$, we have $A_i \in \Gamma\mathcal{R}_t$ for all $t \in \mathbb{R}$. Moreover $\alpha(A_i)$, $\beta_t(A_i)$ are constants for $i = 1, 2$, thus they must be linear combinations of T_t and R_t with real coefficients. In particular $0 = [A_1, A_2] = t[W, R]$ for all $t \in \mathbb{R}$ so that W and R must commute.

Definition 2.6.5. *A* K-Engel structure *is a triple $(\mathcal{D}, g, Z)$ where $\mathcal{D}$ is an Engel structure, g is a metric and Z is a vector field which is Engel, Killing and orthogonal to $\mathcal{E}$.*

Moreover the Engel defining forms α and β satisfying $d\alpha^2 = 0 = d\beta^2$ and $d\alpha \wedge \beta = 0$ are called *K-Engel forms.* We will often denote only by $\mathcal{D} = \ker\alpha \wedge \beta$ the K-Engel structure ($\mathcal{D} = \ker\alpha \wedge \beta$, g, $Z = R$), if we do not want to put an accent on the metric g.

2.7 Some remarks on the dual of W

Before continuing with the discussion on general K-Engel structures we make some observations about some special cases. If $(M, \mathcal{D})$ is Engel and its flag is orientable, a choice of a framing of $\mathcal{D} = \langle W, X\rangle$, with $\mathcal{W} = \langle W\rangle$

and of defining forms α and β yields the framing $\{W, X, T, R\}$. We are interested in the properties of the dual coframing $\{\rho, \tau, \beta, \alpha\}$ in the case where $d\alpha^2 = 0$ and $\beta = -\mathcal{L}_X\alpha$. Under these hypothesis we can determine whether α and β are K-Engel by looking at ρ.

Lemma 2.7.1. *Suppose $\mathcal{D} = \ker\alpha \wedge \beta$ satisfies $d\alpha^2 = 0$ and $\beta = -\mathcal{L}_X\alpha$. Then α and β are K-Engel if and only if there is a choice of W and X such that $\mathcal{L}_R\rho = 0$ modulo $\langle\beta\rangle$.*

Proof. The key observation is that $\mathcal{L}_R\rho = -a_{WR}\rho - a_{XR}\tau$ modulo $\langle\beta\rangle$. If α and β are K-Engel then Corollary 2.6.2 ensures that there is a choice of W and X such that they commute with R. This implies in particular $a_{WR} = 0$ and $a_{XR} = 0$.

Conversely suppose we have such a framing. Up to rescaling X, we have $c_{WX} = 1$ and this does not change ρ and R. Notice that

$$d_{XT} = d\alpha(T, X) = \beta(T) = 1.$$

Moreover the hypothesis $d\alpha^2 = 0$ and $\beta = -\mathcal{L}_X\alpha$ imply $d_{WR} = d_{XR} = d_{TR} = 0$. These together with Equation (2.5g) imply $c_{TR} = -b_{XR}$. Equation (2.5d) reads $c_{TR} = a_{WR} + b_{XR}$, so using $\mathcal{L}_R\rho \wedge \beta = 0$ yields

$$2c_{TR} = a_{WR} = 0,$$

which translates to $d\beta^2 = 0$. Now all hypothesis of Proposition 2.6.3 except possibly for $\mathcal{L}_RX = bX$, but this is again a consequence of $\mathcal{L}_R\rho \wedge \beta = 0$, since this means $a_{XR} = 0$.

□

Notice that the hypothesis of the previous lemma is verified if $d\rho = 0$. Many aspects of the theory of K-Engel structures such that $d\rho = 0$ resemble the theory of Sasakian manifolds (see [Bla] Section 6.8). The proof of the following is borrowed from the analogous one in the Sasakian case. I wish to thank Giovanni Placini for explaining it to me.

Theorem 2.7.2. *Let $\mathcal{D} = \ker\alpha \wedge \beta$ be K-Engel, fix the induced K-Engel framing $\{W, X, T, R\}$ and its dual coframing $\{\rho, \tau, \beta, \alpha\}$. If $d\rho = 0$ then the cup-length of M is smaller than 4.*

Proof. The idea is to prove that R is contained in the kernel of any harmonic 1-form. In this case for any $a_1, ..., a_4 \in H^1(M, \mathbb{R})$ we pick harmonic representatives $\theta_1, ..., \theta_4$ and the cup product will be the class of $\theta_1 \wedge \cdots \wedge \theta_4$, but this is zero because R is in its kernel.

Since R is a Killing vector field, its flow acts by isometries, hence it sends harmonic forms to harmonic forms. Moreover it acts trivially in cohomology. These two facts imply $\mathcal{L}_R\theta = 0$ for all θ harmonic 1-form. Write $\theta = \eta + f\alpha$

where $\eta(R) = 0$ and $f = \theta(R)$, we want to prove that $f = 0$. Using the fact that θ is closed we have

$$0 = \mathcal{L}_R\theta = d(\theta(R)) = df.$$

Hence f is constant and $0 = d\theta = d\eta + f d\alpha$. A simple calculation yields

$$d(\rho \wedge \eta \wedge \alpha) = d\rho \wedge \eta \wedge \alpha - \rho \wedge d\eta \wedge \alpha + \rho \wedge \eta \wedge d\alpha = -f\rho \wedge \alpha \wedge d\alpha,$$

where we used $d\rho = 0$ and the fact that $\rho \wedge \eta \wedge d\alpha = 0$ because R is in the kernel of all the forms (see Lemma 2.5.7). Now $\rho \wedge \alpha \wedge d\alpha$ is a volume form because $\rho(W) = 1$, hence if $f \neq 0$ this would imply that it is also exact, which is impossible since M is closed.

□

In the next section we will see that there are K-Engel structures on T^4, so the hypothesis $d\rho = 0$ in the previous lemma is crucial.

2.8 Topology of K-Engel manifolds

We will now focus on topological obstructions to the existence of a K-Engel structure. Corollary 2.6.2 ensures that there exists a framing $\{W, X, T, R\}$ such that R commutes with all vector fields in the framing. This fact has strong consequences for the topology of M, which come from the theory of transverse structures to a foliation. For more details see [Mol].

Definition 2.8.1. *Let $(M, \mathcal{F})$ be a foliation, we say that $\mathcal{F}$ is* transversally parallelizable *if there exists a global framing for the normal bundle $\nu\mathcal{F}$ which is invariant under the holonomy of $\mathcal{F}$.*

A more sophisticated way to rephrase this is given by transverse G-structures. Let $Q = B^1_T(M, \mathcal{F})$ be the transverse framing bundle, i.e. the principal $GL(\mathbb{R}, q)$-bundle of framings transverse to $T\mathcal{F}$, here $q = \operatorname{codim} \mathcal{F}$. A transverse G-structure on $\mathcal{F}$ is a reduction of the structure group of Q to $G < GL(\mathbb{R}, q)$. Hence a foliation is transversally parallelizable if and only if it admits an e-structure, where e is the trivial group.

There is a more "hands on" way of verifying if a foliation is transversally parallelizable, which involves foliate vector fields. A vector field $V \in \mathfrak{X}(M)$ is called *foliate vector field for* $\mathcal{F}$ if for every tangent vector field $Y \in \Gamma T\mathcal{F}$ we have that the commutator is again tangent, i.e. $\mathcal{L}_V Y \in \Gamma T\mathcal{F}$. This implies that V is preserved by the holonomy of $\mathcal{F}$. Hence a foliation is transversally parallelizable if and only if there exists a transverse framing $\{V_1, ..., V_q\}$ of foliate vector fields.

Let $(M, \mathcal{D} = \ker \alpha \wedge \beta)$ be K-Engel and consider the foliation $\mathcal{F}$ induced by the flow of R, then $\{W, X, T\}$ is a transveral framing by foliate vector fields. This means that $\mathcal{F}$ is transversally parallelizable. The fact that $R^\perp = \ker \alpha$ is bracket-generating implies more.

Lemma 2.8.2. *If $(M, \mathcal{D} = \ker\alpha \wedge \beta)$ is K-Engel and $p, q \in M$ then there exists an isotopy $\psi_t : M \to M$ which commutes with the flow of R and such that $\psi_1(p) = q$. In particular all R orbits are isotopic on M.*

Proof. By hypothesis R is transverse to $\ker\alpha$ and we have a framing $\{W, X, T\}$ of vector fields commuting with R. This means that if we can join p and q with a piecewise smooth path γ obtained by glueing together pieces of orbits of W, X and T we will obtain the isotopy ψ_t. The existence of such a path is exactly the statement of Chow's Theorem 1.1.2. □

The existence of a transversally parallelizable foliation restricts the topology of M. For a proof of the following see [Mol] Theorem 4.2 page 118.

Theorem 2.8.3 [Mol]**.** *Let $(M, \mathcal{F})$ be a transversally parallelizable foliation on a compact connected manifold M. Then the closure of the leaves of $\mathcal{F}$ are the fibres of a locally trivial fibration $\pi : M \to W$ called* basic fibration *on the* basic manifold *W.*

We apply the previous result to $\mathcal{F}$ induced by R. This ensures that, depending on the dimension of the closure of the leaves of $\mathcal{F}$, we have a fibre bundle on a manifold W whose fibres have dimension $k \in \{1, 2, 3, 4\}$. The case $k = 1$ means that R is totally periodic and it will be treated in Section 2.8.2. If $k = 2$ then the leaves must be tori, because R is a nowhere vanishing vector field tangent to them. This case will be treated in Section 2.8.1.

For the final cases we use standard arguments of the theory of Killing vector fields. Since R is Killing, its flow ϕ_t acts by isometries on M, hence it is a 1-parameter subgroup of $\mathrm{Isom}(M, g)$. Since the isometry group is a Lie group, the closure of ϕ_t is an Abelian subgroup and hence a torus $T^m \subset \mathrm{Isom}(M, g)$. This is a closed subgroup and hence it is embedded, so that T^m acts effectively on M. The dimension k of the fibres must satisfy $k \leq m$, so that if $k = 3$ or $k = 4$ we have an effective T^3 action on M. The topology of the manifolds admitting such actions was completely classified by Pak.

Theorem 2.8.4 [Pak]**.** *If M^4 admits an effective T^3-action then M is diffeomorphic to T^4, $S^2 \times T^2$ or $L(p, q) \times S^1$, where $L(p, q)$ denotes the (p, q)-lens space.*

The previous discussion is a proof of the following result.

Theorem 2.8.5. *If $(M, \mathcal{D})$ admits a K-Engel structure, then M is diffeomorphic to one of the following:*

- *T^4;*
- *$L(p, q) \times S^1$ or $S^2 \times T^2$;*
- *a T^2-bundle over a surface;*

- *an S^1-bundle over a 3-manifold.*

It is unclear which of these manifolds admit K-Engel structures. We end this section with the construction of a family of K-Engel structures on T^4 providing examples for which the dimension of the closure of the orbits of R varies.

Example 2.8.6. On $\mathbb{R}^4$ with coordinates (t, x, y, z) consider the distribution given by

$$\mathcal{D} = \Big\langle W = \cos(2\pi t)\partial_x + \sin(2\pi t)\partial_y + \partial_z,\ X = \partial_t \Big\rangle.$$

This defines an Engel structure, in fact it is the Lorentz prolongation of the standard Lorentz structure on $\mathbb{R}_{2,1}$. We choose defining forms

$$\alpha = dz - \cos(2\pi t)dx - \sin(2\pi t)dy, \qquad \beta = -\sin(2\pi t)dx + \cos(2\pi t)dy.$$

An explicit calculation yields

$$R = \partial_z \qquad \text{and} \qquad T = -\sin(2\pi t)\partial_x + \cos(2\pi t)\partial_y.$$

Since R is Engel and Killing for the metric making $\{W, X, T, R\}$ orthonormal, this is a K-Engel structure.

Up to choosing a lattice Λ in $\mathbb{R}^4$ in the right way we can make sure that this structure passes to the quotient $T^4 = \mathbb{R}^4/\Lambda$. Moreover we can control the dimension of the closure of the orbits of R. The only condition on $\Lambda = \langle e_1, ..., e_4\rangle$ is that the t-components of the vectors must be integers (otherwise $\cos(2\pi t)$ and $\sin(2\pi t)$ will not pass to the quotient). This allows the choice $e_4 = (0, 0, 0, 1)$ and leaves complete freedom for e_1, e_2 and e_3 in the orthogonal space to e_4. Since $R = \partial_z$ we can make sure that the closure of its orbits in the quotient is S^1, T^2, T^3.

Notice that it is not possible to have dense orbits. In this case indeed Corollary 2.6.2 ensures that a_{WX}, a_{WT}, b_{WT} and a_{XT} are constant on the leaves of R and hence on T^4. This means that the framing $\{W, X, T, R\}$ satisfies $[W, X] = T$ and $[X, T] = R$ and all other brackets vanish. In turn this means that $d\rho = 0$ so that Theorem 2.7.2 would imply that the cup-length of T^4 is smaller than 4.

It is unclear if there are examples of K-Engel structures such that all orbits of R are dense.

2.8.1 K-Engel T^2-bundles

In this section we will study K-Engel structures on T^2-bundles over a surface Σ_g of genus g. These bundles are classified up to isomorphism by the *monodromy* and the *Euler class* (see [Wal]).

The monodromy $\rho : \pi_1(\Sigma_g) \to \pi_0(\mathrm{Diff}^+(T^2)) \cong SL(2,\mathbb{Z})$ is a representation which reveals the structure of the bundle over the 1-skeleton. Take the natural projection $\pi_0 : \mathrm{Diff}^+(T^2) \to \pi_0(\mathrm{Diff}^+(T^2))$. Recall that $\pi_0(\mathrm{Diff}^+(T^2)) \cong SL(2,\mathbb{Z})$ and consider the induced map on classifying spaces $B\pi_0 : B\,\mathrm{Diff}^+(T^2) \to BSL(2,\mathbb{Z})$. For any T^2-bundle $M \to \Sigma_g$ take the classifying map $f_M : \Sigma_g \to B\,\mathrm{Diff}^+(T^2)$. The monodromy morphism is the map induced on π_1 by the composition $B\pi_0 \circ f_M : \Sigma_g \to BSL(2,\mathbb{Z})$, i.e. $\rho = \pi_1(B\pi_0 \circ f_M) : \pi_1(\Sigma_g) \to \pi_1(BSL(2,\mathbb{Z})) = SL(2,\mathbb{Z})$.

For any given representation $\rho : \pi_1(\Sigma_g) \to SL(2,\mathbb{Z})$ one can explicitely construct a bundle $M \to \Sigma_g$ having ρ as monodromy homomorphism. Consider the manifold $\tilde{M} = \tilde{\Sigma}_g \times T^2$, where $\tilde{\Sigma}_g$ is the universal cover of Σ_g. The fundamental group $\pi_1(\Sigma_g)$ acts properly discontinuosly on $\tilde{M}$ via $g \cdot (p, q) = (p \cdot g^{-1}, \rho(g)q)$, where the action on $\tilde{\Sigma}_g$ is by deck transformations. Hence the quotient space $M = \tilde{M}/\pi_1(\Sigma_g) =: \Sigma_g \times_\rho T^2$ is a manifold which in fact is the total space of a T^2-fibration whose monodromy is ρ.

The Euler class is a pair of integers (m, n) which represents an obstruction to the existence of a section for $\pi : M \to \Sigma_g$. For a given T^2-bundle consider the evaluation at a point $ev : \mathrm{Diff}^+(T^2) \to T^2$ and the corresponding map $Bev : B\,\mathrm{Diff}^+(T^2) \to BT^2 \cong \mathbb{CP}^\infty \times \mathbb{CP}^\infty$. The composition with the classifying map $Bev \circ f_M$ defines a pair of integers (m, n) which is the Euler class. We say that a bundle with Euler class $(0, 0)$ is *flat*.

There is a way to construct a bundle with Euler class (m, n) starting from a flat one $\pi : M \to \Sigma_g$. Take a disc $D^2 \to \Sigma_g$ and make a surgery on the torus $\pi^{-1}D^2$ via the glueing map $T^2 \times \partial D^2 \to T^2 \times \partial D^2$ given by

$$(u,\, v,\, \theta) \mapsto \left(u + \frac{1}{2\pi} m\theta,\, v + \frac{1}{2\pi} n\theta,\, \theta\right). \tag{2.14}$$

The following result gives some more information about the structure of the T^2-bundles that arise as K-Engel structures.

Lemma 2.8.7. *If $(M,\, \mathcal{D} = \ker \alpha \wedge \beta)$ is a K-Engel structure such that the closures of the leaves of R are 2-tori then there are two commuting, totally periodic, linearly independent vector fields R_1 and R_2 tangent to the fibres of the fibration $M \to \Sigma_g$.*

Proof. We use a classical argument for Killing vector fields to find R_1, R_2 commuting, Killing, totally periodic vector fields on M such that $R = r_1R_1 + r_2R_2$ for r_1, $r_2 \in \mathbb{R}$.

Let ϕ_t^R be the flow of R, this is a 1-parameter subgroup of the compact Lie group of isometries of (M, g). Since it is Abelian, its closure is a torus T^m. Moreover, since the closure of the leaves of R are 2-dimensional by hypothesis, and using the proof of Theorem 2.8.4 (see [Pak] end of page 671) we conclude that $m = 2$. Fix a basis $\langle A_1,\, A_2\rangle$ for the Lie algebra of T^2 and consider the vector fields $R_i = \exp(tA_i)$ for $i = 1$, 2. By construction $[R_i, R] = 0$ for $i = 1$, 2.

The only thing left to prove is that $\{R_1, R_2\}$ is linearly independent. By contradiction suppose this does not happen at a point $p \in M$. Without loss of generality we have $R_1(p) = \lambda R_2(p)$ for some $\lambda \in \mathbb{R}$. Since R_1 and R_2 commute with R, the orbit of R at p coincides with the orbit of R_2. Since the latter is totally periodic this yields to a contradiction, because Lemma 2.8.2 ensures that the closure of every orbit of R is 2-dimensional. □

The previous result implies a strong constraint on the bundle structure.

Corollary 2.8.8. *If $(M, \mathcal{D} = \ker \alpha \wedge \beta)$ is a K-Engel structure such that the closures of the leaves of R are 2-tori then $M \to \Sigma_g$ is a principal T^2-bundle and α and β are invariant forms.*

Proof. The first claim follows directly from Lemma 2.8.7. Moreover its proof shows that R_1 and R_2 are Killing and Engel, and this proves the claim on α and β. □

Remark 2.8.9. Notice that Lemma 2.8.7 does not imply that every vector field tangent to the fibres of $M \to \Sigma_g$ is Engel. Indeed already non-constant rescalings of R_1 and R_2 are not Engel in general. Moreover it is a priori possible that neither of them is transverse to $\mathcal{E}$.

It is unclear which principal bundles admit K-Engel structures such that the closures of the orbits of R are 2-dimensional. We present some constructions of K-Engel structures on T^2-bundles where R is totally periodic.

Example 2.8.10. Consider the surface Σ_g equipped with a metric h of constant scalar curvature k and construct the unit circle bundle $S^1 \to N \to \Sigma_g$. Denote by $A \in \mathfrak{X}(N)$ a unit vector field tangent to the fibres. The Levi-Civita connection induces a choice of horizontal bundle on N and we denote by B a tautological vector field, i.e. for $l \in T_pN$ of unit norm we want $\pi_*(B(p, l)) \in \mathbb{R}l$. Finally we choose C so that $\{B, C\}$ is an orthonormal basis for the horizontal bundle of N. It is a classical result that

$$[A, B] = C, \qquad [B, C] = kA \qquad \text{and} \qquad [C, A] = B.$$

This means that the dual basis $\{a, b, c\} \in T^*N$ satisfies

$$da = -kb \wedge c, \qquad db = -c \wedge a \qquad \text{and} \qquad dc = -a \wedge b.$$

Using these formulas one can verify that on $M = N \times S^1$ the forms

$$\alpha = a - \cos t\ b - \sin t\ c \qquad \text{and} \qquad \beta = \sin t\ b + \cos t\ c$$

are K-Engel and the induced framing is

$$\begin{aligned} W &= A + \cos t\ B + \sin t\ C - (k+1)\partial_t \\ X &= \partial_t \\ T &= -\sin t\ B + \cos t\ C \\ R &= A - \partial_t. \end{aligned}$$

Example 2.8.11. If a T^2-bundle admits an S^1-action tangent to the fibres then the monodromy ρ has the form

$$\rho(\gamma) = \begin{pmatrix} 1 & \lambda(\gamma) \\ 0 & 1 \end{pmatrix}$$

for $\lambda(\gamma) \in \mathbb{Z}$ (see Proposition 4.4 in [Wal]).

Take the flat T^2-bundle over T^2 with monodromy given by $\rho(a_i) = A_i$ for $\pi_1(T^2) = \langle a_1, a_2 \rangle$, where

$$A_i = \begin{pmatrix} 1 & \lambda_i \\ 0 & 1 \end{pmatrix}, \qquad \text{with } \lambda_i \in \mathbb{Z},\ i = 1, 2.$$

Take coordinates $\tilde{M} = \mathbb{R}^2 \times T^2 = \{(x, y, u, v)\}$, then the 1-forms

$$a = du + (\lambda_1 x + \lambda_2 y)dv \qquad \text{and} \qquad b = dv$$

are invariant with respect to the transformations

$$\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} x-1 \\ y \end{pmatrix}, A_1 \begin{pmatrix} u \\ v \end{pmatrix} \right)$$

and

$$\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right) \mapsto \left(\begin{pmatrix} x \\ y-1 \end{pmatrix}, A_2 \begin{pmatrix} u \\ v \end{pmatrix} \right)$$

hence they define 1-forms on M. Similarly the formulas

$$U = \partial_u \qquad \text{and} \qquad V = \partial_v - (\lambda_1 x + \lambda_2 y)\partial_u$$

define nowhere-vanishing vector fields tangent to the fibres of M. The forms

$$\alpha = a - \cos v\, dz - \sin v\, dy \qquad \text{and} \qquad \beta = -\sin v\, dx + \sin v\, dy$$

are K-Engel forms with $R = U$.

2.8.2 Engel Boothby-Wang

The construction in this section is motivated by the properties that K-Engel structures with totally periodic R satisfy. The same construction appeared in the work of Mitsumatsu [Mit] under the name of *prequantum prolongation*.

Consider a K-Engel structure $\mathcal{D} = \ker \alpha \wedge \beta$ such that R is totally periodic. The results in Section 2.8 ensure that M is an S^1-bundle over a 3-manifold N with R tangent to fibres. Denote the projection by $\pi : M \to N$.

Since $\mathcal{L}_R \beta = 0$, we get a 1-form on N by $\lambda = \pi_* \beta$. Since $\ker \beta \wedge d\beta = \langle R \rangle$, the form λ is a contact form. Similarly W descends to a Legendrian vector field $L = \pi_* W$. Finally α is a connection form because $\mathcal{L}_R \alpha = 0$, $\alpha(R) = 1$, and $d\alpha$ descends to a closed 2-form $\pi^* \omega = d\alpha$, satisfying $i_L \omega = 0$. This implies that $[\omega] \in H^2(N, \mathbb{Z})$, because this must be the Euler class of the bundle M.

Remark 2.8.12. Let (N, λ) be a contact 3-manifold and let L be a non-singular Legendrian vector field. Then L is the kernel of a closed non-singular 2-form ω if and only if a rescaling of L preserves the contact volume $\lambda \wedge d\lambda$.

Indeed suppose that some rescaling $\tilde{L}$ preserves the contact volume, then form $\omega = i_{\tilde{L}}(\lambda \wedge d\lambda)$ is closed. Conversely suppose that we have a closed 2-form ω whose kernel is spanned by L. Then $\tilde{\omega} = i_L(\lambda \wedge d\lambda)$ is also non-singular and its kernel is spanned by L. This readily implies that $\tilde{\omega}$ and ω are proportional, so that up to rescaling L, we have the claim.

Reversing the previous construction we obtain the Engel version of the Boothby-Wang construction [BW].

Proposition 2.8.13. *Let (N^3, λ) be a contact structure and suppose that L is a Legendrian vector field such that $\omega = i_L(\lambda \wedge d\lambda)$ is an integral closed 2-form. Then the principal S^1-bundle $\pi : M \to N$ with Euler class $[\omega]$ admits a K-Engel structure $\mathcal{D} = \ker \alpha \wedge \beta$, where α is a connection form and $\beta = \pi^* \lambda$.*

Proof. We need to verify that α and β are K-Engel defining forms. First of all α defines an even contact structure because by definition $d\alpha = \pi^*\omega$, so that $\alpha \wedge d\alpha \neq 0$. The choice of the connection form is not unique, and different choices yield homotopic Engel structures (see Section 1.4 in [Mit] for more details).

Since λ is contact β is even contact. Moreover $\ker \beta \wedge d\beta$ is spanned by the vector field R tangent to the fibres and normalized by α. This implies in turn that $\alpha \wedge \beta \wedge d\beta \neq 0$. Finally $\alpha \wedge d\alpha \wedge \beta = 0$, since already $\beta \wedge d\alpha = \pi^*(\lambda \wedge \omega) = 0$ because $i_L(\lambda \wedge \omega) = 0$. So $\mathcal{D}$ is an Engel structure.

Now for dimensional reasons $d\alpha^2 = 0 = d\beta^2$. As we have already seen $d\alpha \wedge \beta = 0$ and R acts on $\mathcal{D}$ in a diagonalizable way, as can be seen by choosing $X = \pi^* \tilde{L}$ where $\tilde{L}$ is a Legendrian line field nowhere tangent to L. Proposition 2.6.3 implies that $\mathcal{D}$ is K-Engel. □

2.9 Contact fillings

There is a way to see Engel structures as special submanifolds of contact 5-dimensional manifolds. The K-Engel structures coming from the Engel Boothby-Wang construction are examples of such submanifolds in compact contact 5-manifolds.

Let $(X^5, \xi = \ker \eta)$ be a contact structure and let M be an orientable embedded hypersurface $M^4 \to X^5$ transverse to ξ. This means that (locally) we can find a Legendrian vector field $L \in \mathfrak{X}(X)$ transverse to M. This data permits to define two 1-forms on M as follows

$$\beta := \eta|_M \qquad \text{and} \qquad \alpha = (\mathcal{L}_L \eta)|_M \,. \tag{2.15}$$

We look for conditions such that a neighbourhood of M is contactomorphic to the contactization of the Engel structure having defining forms α and β. We have the following consequence of the Contact Weinstein's Neighbourhood Theorem 1.2.5.

Lemma 2.9.1. *Let (X, η, L, M) be as above and define α and β as in Equation (2.15). Then there is an open neighbourhood $\mathcal{O}p(M) = M \times (-\epsilon, \epsilon)$ and a function $f : M \times (-\epsilon, \epsilon) \to \mathbb{R}$ such that*

$$f\eta = \beta + s\alpha$$

on $\mathcal{O}p(M)$. Here we denote by s the coordinate along $(-\epsilon, \epsilon)$.

Proof. Use the flow ϕ_t of L to construct an embedding of a tubular neighbourhood $\psi : M \times (-\epsilon, \epsilon) \to X$ such that $\psi(p, s) = \phi_s(p)$. We identify $M \equiv M \times \{0\}$, $\alpha \equiv \psi^*\alpha$ and $\beta \equiv \psi^*\beta$, and we define $\eta_1 = \beta + s\alpha$. We have $L = \partial_s$ and $\eta_0 = \psi^*\eta$ is a contact form on $M \times (-\epsilon, \epsilon)$. We want to prove that η_1 is also a contact form on $\mathcal{O}p(M)$ and that

$$\eta_0|_M = \eta_1|_M \qquad \text{and} \qquad d\eta_0|_M = d\eta_1|_M \tag{2.16}$$

so that Theorem 1.2.5 gives us a map $\tilde{\psi} : M \times (-\epsilon, \epsilon) \to M \times (-\epsilon, \epsilon)$ which satisfies $\tilde{\psi}^*\eta_1 = f\eta$.

Since $\phi_0 = id$, Equation (2.15) implies Equation (2.16). A direct calculation yields

$$\eta_1 \wedge d\eta_1^2 = 2\Big(ds \wedge \alpha \wedge \beta \wedge d\beta + s ds \wedge \alpha \wedge \beta \wedge d\alpha\Big),$$

hence if $\alpha \wedge \beta \wedge d\beta \neq 0$ on M we conclude that η_1 is contact on a (possibly smaller) tubular neighbourhood of M. Plugging L into $\eta \wedge d\eta^2 \neq 0$ we obtain $\eta \wedge i_L d\eta \wedge d\eta \neq 0$. Since the kernel of $i_L(\eta \wedge d\eta^2)$ is L, and this is transverse to M, we have

$$0 \neq \Big(i_L(\eta \wedge d\eta^2)\Big)\Big|_M = -2\,\Big(\eta \wedge d\eta \wedge i_L d\eta\Big)\Big|_M = -2\beta \wedge d\beta \wedge \alpha.$$

□

Remark 2.9.2. In the previous theorem we cannot ensure in general that $\tilde{\psi}_* L = \partial_s$. On the other hand we will only be interested in the quantities $\mathcal{L}_L\eta$ and $\mathcal{L}_L d\eta$ on M, and the formula $f\eta = \beta + s\alpha$ implies that $\mathcal{L}_{\partial_s}(f\eta) = \alpha = \mathcal{L}_L\eta$ and $\mathcal{L}_{\partial_s} d(f\eta) = d\alpha = \mathcal{L}_L d\eta$ on M.

We say that L *preserves the contact volume on* M if

$$\mathcal{L}_L\left(\eta \wedge d\eta^2\right)\Big|_p = 0 \qquad \forall p \in M.$$

Remark 2.9.3. For any given (X, η, L, M) as above, we can rescale η so that L preserves the contact volume on M. Let $g : X \to \mathbb{R}$ be such that $\mathcal{L}_L(\eta \wedge d\eta^2) = g\,\mathrm{vol}_X$. Since L is transverse to M, there is function λ satisfying $3\,\mathcal{L}_L\lambda|_M = -\,g|_M$. For every $p \in M$ we have

$$\begin{aligned}\mathcal{L}_L\left(e^{\lambda}\eta \wedge d(e^{\lambda}\eta)^2\right)\Big|_p &= \mathcal{L}_L\left(e^{3\lambda}\eta \wedge d\eta^2\right)\Big|_p \\ &= (3\mathcal{L}_L\lambda)e^{3\lambda}\left(\eta \wedge d\eta^2\right)\Big|_p + e^{3\lambda}\,\mathcal{L}_L\left(\eta \wedge d\eta^2\right)\Big|_p = 0.\end{aligned}$$

Definition 2.9.4. *Let $(X^5, \xi = \ker\eta)$ be a contact manifold and $M \to N$ an embedded hypersurface. Let L be a Legendrian line field transverse to M which preserves the contact volume on M. We say that M is an* Engel-type hypersurface *if $\alpha = \mathcal{L}_L\eta|_M$ is an even contact structure on M.*

The previous definition is justified by the following result.

Lemma 2.9.5. *Let $(X^5, \ker\eta)$ be a contact structure and M be an Engel-type hypersurface, then $\alpha = \mathcal{L}_L\eta|_M$ and $\beta = \eta|_M$ are Engel defining forms for $\mathcal{D} = \ker\alpha \wedge \beta$. Moreover there exists a neighbourhood of M in X contactomorphic to the contactization of $(M, \mathcal{D})$.*

Proof. Throughout the proof we will use Remark 2.9.2 to identify $\alpha = \mathcal{L}_{\partial_s}(f\eta) = \mathcal{L}_L\eta$ on M. By assumption we have $\alpha \wedge d\alpha \neq 0$. Lemma 2.9.1 ensures that we have a neighbourhood of M such that $f\eta = \beta + s\alpha$. Moreover the proof of the lemma ensures that $\alpha \wedge \beta \wedge d\beta \neq 0$.

The formula

$$f\eta \wedge d(f\eta)^2 = 2\Big(ds \wedge \alpha \wedge \beta \wedge d\beta + s ds \wedge \alpha \wedge \beta \wedge d\alpha\Big)$$

ensures that

$$\alpha \wedge \beta \wedge d\alpha = i_{\partial_s}\,\mathcal{L}_{\partial_s}\left(f\eta \wedge d(f\eta)^2\right)\Big|_M.$$

Now since $f\eta|_M = \eta|_M$ and $\mathcal{L}_{\partial_s}(f\eta)|_M = \mathcal{L}_L\eta|_M$, by Remark 2.9.2 we have $d(f\eta)|_M = d\eta|_M$ and $\mathcal{L}_{\partial_s}(d(f\eta))|_M = \mathcal{L}_L d\eta|_M$, hence

$$\alpha \wedge \beta \wedge d\alpha = i_{\partial_s}\mathcal{L}_{\partial_s}\,(f\eta \wedge d(f\eta))|_M = i_{\partial_s}\mathcal{L}_L\,(\eta \wedge d\eta)|_M = 0.$$

□

Example 2.9.6. All Engel manifolds $(M, \mathcal{D})$ appear as Engel-type hypersurfaces of their contactization. An example of Engel-type hypersurface on a compact contact manifold is provided by the Engel structure on $S^3 \times S^1$ given by the Boothby-Wang construction on $(S^3, \eta = a, \omega = db)$. Here a is the standard contact form on S^3 and b is the 1-form obtained from it by multipling by the quaternion j. In standard coordinates $\mathbb{C}^2 = (z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$ we have

$$\begin{aligned} a &= -y_1 dx_1 + x_1 dy_1 - y_2 dx_2 + x_2 dy_2 \\ b &= y_2 dx_1 + x_2 dy_1 - y_1 dx_2 - x_1 dy_2. \end{aligned}$$

The Engel Boothby-Wang construction yields an Engel structure on $M = S^3 \times S^1$ with defining forms $\alpha = dt + b$ and $\beta = a$. We have a contact structure on $X = N \times D^2$ which is given by the kernel $\tilde{\eta} = \beta + r^2\alpha$ where r is the radial coordinate. Then $M = \partial X$ is an Engel-type hypersurface with $L = \partial_r$.

The previous example and the analogous definitions for the symplectic case motivates the following definition.

Definition 2.9.7. *A* contact filling *of an Engel structure* $(M, \mathcal{D})$ *is a contact manifold* (X, η) *such that* $M = \partial X$ *is an Engel-type hypersurface.*

It is unclear which Engel structures admit a contact filling. The following result ensures that Engel Boothby-Wang manifolds are fillable.

Theorem 2.9.8. *Let* $(M, \mathcal{D})$ *be obtained from* (N, λ, ω) *via the Boothby-Wang construction. Then* M *admits a contact filling.*

Proof. By hypothesis $\pi : M \to N$ is an S^1-bundle and the Engel defining forms are a connection form α and $\beta = \pi^*\lambda$. Consider now the disc bundle $X \to N$ of M and denote by r the radial coordinate on each fibre. The form $\eta = \beta + r^2\alpha$ is contact and $M = \partial X$ is an Engel-type hypersurface, with $rL = \partial_r$. □

Chapter 3

Engel Dynamics

The main goal of this chapter is to understand which even contact structures $(M, \mathcal{E})$ are induced by Engel structures $\mathcal{D}$, i.e. $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$. The intuition driving the definition of the rotation number was pointed out to me during the problem sessions at the AIM workshop *Engel structures* in San José, April 2017. For this I am particularly thankful to Prof. Yakov Eliashberg.

In Section 3.1 we introduce the notation and point out some topological obstructions. For instance the existence of $\mathcal{D}$ implies that M has a parallelizable 4-cover. For this reason we restrict the attention to even contact structures $\mathcal{E} = \langle W, A, B \rangle$ which are trivial as bundles and coorientable, where $\mathcal{W} = \langle W \rangle$. We look for vector fields $L \in \mathcal{E}$ such that $\langle W, L \rangle$ is Engel. In Section 3.2 we consider points $p \in M$ contained in a closed orbit γ of $\mathcal{W}$. We introduce the notion of rotation number $\mathrm{rot}_\gamma(L(p))$ of $L(p)$ around γ and we point out how this is related to the existence of an Engel structure compatible with $\mathcal{E}$. The behaviour of the rotation number under homotopies depends on the type of the closed orbit γ. This is the content of Section 3.3. If γ is elliptic, the rotation number is invariant under homotopies of L. In this case there exists a vector field L_1 homotopic to L through sections of $\mathcal{E}$ and such that $\langle W, L_1 \rangle$ is Engel on a neighbourhood of γ if and only if $\mathrm{rot}_\gamma(L(p)) > 0$. This condition is very easy to verify if γ bounds an embedded disc.

In Section 3.4 we point out that the previous construction works in a more general setting. As an application of this we find a necessary condition that a vector field on a 3-manifold has to verify in order to be Legendrian for some contact structure.

The necessary condition given by the rotation number is likely far from being also sufficient in the general case. On the other hand, if we only consider vector fields W whose dynamics very simple, then positivity of the rotation number around each closed orbit is also sufficient for the existence of $\mathcal{D}$. By "very simple" we mean non-singular Morse-Smale (NMS), the theory of these flows is recalled in Section 3.5. In Section 3.6, resp. 3.7, we study which NMS vector fields are Legendrian, resp. the characteristic vector field of an Engel structure respectively.

Finally Section 3.8 is devoted to a different approach which is inspired by contact-type forms introduced in [McDu]. Here the main technical tool is the theory of Sullivan currents, but the geometric interpretation of the results is not clear.

3.1 Setting and notation

Let $\mathcal{E}$ be an even contact structures on a 4-manifold M, we say that an Engel structure $\mathcal{D}$ on M is *compatible* with $\mathcal{E}$ if $\mathcal{D}^2 = \mathcal{E}$. Otherwise said $\mathcal{D}$ is compatible with $\mathcal{E}$ if the latter is its induced even contact structure. We will investigate the following question: when does $(M, \mathcal{E})$ even contact 4-manifold admit a compatible Engel structure?

There are some obvious topological obstructions to a positive answer, as we have seen in Chapter 1. Namely if $(M, \mathcal{E})$ does not admit a parallelizable 4-cover then it does not admit an Engel structure. In order to avoid these cases we suppose that $\mathcal{E}$ is trivial as a bundle and admits a global framing $\mathcal{E} = \langle W, A, B\rangle$, where W spans the characteristic foliation $\mathcal{W}$. Finally suppose $\mathcal{E}$ is coorientable, so that there exists a 1-form α such that $\mathcal{E} = \ker \alpha$, and choose it so that $d\alpha(A, B) = 1$.

Fix the endomorphism $J \in \operatorname{End}\mathcal{E}$ given by $JW = 0$, $JA = B$ and $JB = -A$. By construction we have

$$d\alpha(JX, JY) = d\alpha(X, Y) \tag{3.1}$$

for all $X, Y \in \Gamma\mathcal{E}$ and $d\alpha(-, J-)$ is positive definite on $\langle A, B\rangle$.

Let $L \in \Gamma\mathcal{E}$ never tangent to $\mathcal{W}$, we want to determine when the distribution $\mathcal{D}_L := \langle W, L\rangle$ is homotopic within $\mathcal{E}$ to an Engel structure compatible with $\mathcal{E}$.

Definition 3.1.1. *Let $L \in \Gamma\mathcal{E}$ never tangent to $\mathcal{W}$ and $K \subset M$. We say that $\mathcal{D}_L$ is a* (positive) Engel structure on K up to homotopy within $(\mathcal{W}, \mathcal{E})$ *if there exists a smooth family of vector fields $L_\tau \in \Gamma\mathcal{E}$ for $\tau \in [0, 1]$ such that L_τ is never tangent to $\mathcal{W}$ for all $\tau \in [0, 1]$, $L_0 = L$ and $\mathcal{D}_{L_1}$ is an Engel structure compatible with the orientation of $\mathcal{E}$ on a neighbourhood of K.*

Notice that the framing $\{A, B\}$ allows us to identify L with a map $L : M \to S^1$. It will come in handy to lift this map to the universal cover $\tilde{M}$ so that the vector field takes the form $L = \cos\psi\ A + \sin\psi\ B$ for some function $\psi : \tilde{M} \to \mathbb{R}$ making the following diagram commute

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\psi} & \mathbb{R} \\ \downarrow & & \downarrow \\ M & \xrightarrow{L} & S^1. \end{array}$$

Similarly, we can write homotopies in the form $L_s = \cos\psi_s\ A + \sin\psi_s\ B$ for $\psi_s : \tilde{M} \to \mathbb{R}$ smooth family of functions $s \in [0,1]$.

Denote the flow of W at time t by ϕ_t, and its tangent map at $p \in M$ by $T_p\phi_t : T_pM \to T_{\phi_t(p)}M$. Since by Proposition 1.3.2 the flow of W preserves $\mathcal{E}$, we can choose α so that we have a positive function λ which satisfies

$$(\phi_s^*\alpha)_p = \alpha_p \circ \Big(T_{\phi_s(p)}\phi_{-s}\Big) = \lambda(p;s)\alpha_p. \tag{3.2}$$

We want to understand the pull-back of L with respect to the flow of W. Denote by

$$\tilde{L}(p;t) := \Big(T_{\phi_t(p)}\phi_{-t}\Big)L(\phi_t(p)) = a(p;t)\ A(p) + b(p;t)\ B(p) + c(p;t)\ W(p).$$

Since L is never tangent to $\mathcal{W}$ we have that $\rho(p;t) := \sqrt{a^2(p;t)+b^2(p;t)}$ is everywhere positive. Moreover there exists a unique function $\theta(p;t)$ smooth in t, such that $\theta(p;0) \in [0,2\pi)$ and

$$\tilde{L}(p;t) = \rho(p;t)\Big(\cos\theta(p;t)\ A(p) + \sin\theta(p;t)\ B(p)\Big) + c(p;t)\ W(p). \tag{3.3}$$

3.1.1 What happens if we change time?

We want to understand how θ varies if we change t. A straightforward calculation gives (we denote the derivative with respect to t with a dot)

$$\frac{d}{dt}\tilde{L}(p;t) = \frac{\dot{\rho}(p;t)}{\rho(p;t)}\tilde{L}(p;t) + \dot{\theta}(p;t)J_p\tilde{L}(p;t) \qquad \mod \mathcal{W}(p).$$

Since $\mathcal{W} = \ker\alpha \wedge d\alpha$ implies $i_W d\alpha|_{\ker\alpha} = 0$, we get

$$\begin{aligned} \rho^2(p;t)\dot{\theta}(p,t) &= \dot{\theta}(p,t)\ d\alpha_p\Big(\tilde{L}(p;t),\ J_p\tilde{L}(p;t)\Big) \\ &= d\alpha_p\left(\tilde{L}(p;t),\ \frac{d}{dt}\tilde{L}(p;t)\right). \end{aligned} \tag{3.4}$$

Using the properties of the flow ϕ_t we have

$$\begin{aligned} \tilde{L}(p;t+s) &= \Big(T_{\phi_{t+s}(p)}\phi_{-(t+s)}\Big)\Big(L(\phi_{t+s}(p))\Big) \\ &= \Big(T_{\phi_s(p)}\phi_{-s}\Big) \circ \Big(T_{\phi_t(\phi_s(p))}\phi_{-t}\Big)\Big(L(\phi_t(\phi_s(p)))\Big) \\ &= \Big(T_{\phi_s(p)}\phi_{-s}\Big)\Big(\tilde{L}(\phi_s(p),t)\Big). \end{aligned} \tag{3.5}$$

Putting together (3.2), (3.4) and (3.5) we get

$$\begin{aligned}
\rho(p;t+s)^2\dot\theta(p;t+s) &= d\alpha_p\left(\tilde L(p;t+s),\ \frac{d}{dt}\tilde L(p;t+s)\right)\\
&= d\alpha_p\left(\left(T_{\phi_s(p)}\phi_{-s}\right)\tilde L(\phi_s(p);t),\ \left(T_{\phi_s(p)}\phi_{-s}\right)\frac{d}{dt}\tilde L(\phi_s(p);t)\right)\\
&= \left(\phi_s^* d\alpha_p\right)\left(\tilde L(\phi_s(p);t),\ \frac{d}{dt}\tilde L(\phi_s(p);t)\right)\\
&= \lambda(p;s) d\alpha_p\left(\tilde L(\phi_s(p);t),\ \frac{d}{dt}\tilde L(\phi_s(p);t)\right),
\end{aligned}$$

where in the last step we used the fact that both $\tilde L$ and its derivative are in the kernel of α. Finally we get

$$\rho(p;t+s)^2\dot\theta(p;t+s) = \lambda(p;s)\rho(\phi_s(p);t)^2\dot\theta(\phi_s(p);t). \tag{3.6}$$

This equation can be used to prove the following well-known fact.

Proposition 3.1.2. *The distribution $\langle W, L\rangle$ is Engel on the orbit of p if and only if $\dot\theta(p;s) \neq 0$ for all $s \in \mathbb{R}$.*

Proof. The distribution is Engel at p if and only if $\langle W,\, L,\, [W,L]\rangle_p = \mathcal{E}_p$ which happens if and only if $\{L_p,\, [W,L]_p\}$ is linearly independent. We write

$$\begin{aligned}
[W,L]_p = (\mathcal{L}_W L)_p = \left.\frac{d}{dt}\right|_{t=0}\left(T_{\phi_t(p)}\phi_{-t}\right)L(\phi_t(p)) = \left.\frac{d}{dt}\right|_{t=0}\tilde L(p;t)\\
= \dot\rho(p;0)L(p) + \dot\theta(p;0)J_pL(p) \qquad \mod \mathcal{W}_p.
\end{aligned}$$

Hence $\langle W, L\rangle$ is Engel at p if and only if $\dot\theta(p;0) \neq 0$. Using (3.6) we conclude that

$$\dot\theta(\phi_s(p);0) = \frac{\rho(t;s)^2}{\lambda(p;s)}\dot\theta(p;s),$$

which means that $\langle W, L\rangle$ is Engel at $\phi_s(p)$ if and only if $\dot\theta(p;s) \neq 0$. □

Remark 3.1.3. Notice that the sign of $\dot\theta(p;s)$ determines if the orientation of $\mathcal{E}$ induced by the Engel structure $\langle W, L\rangle$ is the same as the one induced by $\langle W,\, A,\, B\rangle$.

3.2 Rotation number

We can interpret Proposition 3.1.2 geometrically by saying that L *rotates without stopping* along orbits of W. With this idea in mind we give the following

Definition 3.2.1. *Let $\gamma \subset M$ be a closed orbit of W of period T and $p \in \gamma$, we call* rotation number of $L(p)$ around γ *the quantity*

$$\mathrm{rot}_\gamma(L(p)) = \theta(p;T) - \theta(p;0).$$

Notice that in general this number is not an integer and the following example shows that it is not invariant under homotopies of L.

Example 3.2.2. Consider the geometric even contact structure $\langle W, X, Y\rangle$ on Sol_1^4/Γ defined in [Vog2]. By construction $[W,X]=-X$ and $[W,Y]=Y$ so that the flow of W exponentially contracts X. This means that every time that Γ is such that W admits a closed orbit γ, we have $\mathrm{rot}_\gamma(X(p))=0$ since θ is constant. On the other hand $X_s = X+sY$ is a homotopy between X and $X+Y$, but the latter spans an Engel structure together with W. In particular $X+Y$ has positive rotation number by Proposition 3.1.2.

We have invariance under a more restricted family of homotopies of L.

Lemma 3.2.3. *Let L_τ for $\tau\in[0,1]$ be a smooth family of vector fields tangent to $\mathcal{E}$ and nowhere tangent to $\mathcal{W}$. If $L_\tau(p)=L_0(p)$ for all $\tau\in[0,1]$ then*

$$\mathrm{rot}_\gamma(L_1(p)) = \mathrm{rot}_\gamma(L_0(p)).$$

Proof. Using (3.5) with $t=0$ and $s=T$ we get

$$\tilde{L}_\tau(p;T) = \Big(T_{\phi_T(p)}\phi_{-T}\Big)\tilde{L}_\tau(\phi_T(p);0) = \Big(T_p\phi_{-T}\Big)L_\tau(p).$$

By the same calculations done in (3.6), we get

$$\begin{aligned}\rho_\tau(p;T)^2\frac{d}{d\tau}\theta_\tau(p;T) &= d\alpha\left(\tilde{L}_\tau(p;T),\ \frac{d}{d\tau}\tilde{L}_\tau(p;T)\right)\\ &= \phi_T^* d\alpha\left(L_\tau(p),\ \frac{d}{d\tau}L_\tau(p)\right)\\ &= \lambda(p;T)\frac{d}{d\tau}\theta_\tau(p;0) = 0.\end{aligned}$$

This readily implies that $\mathrm{rot}_\gamma(L_\tau(p))$ is constant in τ. □

The previous result suggests to take into account all possible *initial phases*. More precisely for $\eta\in\mathbb{R}$ and $L: M\to S^1$, we consider the rotation R_η of S^1 of angle η and define $R(L,\eta)=R_\eta\circ L$. Otherwise said if $L=\cos\psi\ A+\sin\psi\ B$, we consider the family of vector fields $R(L,\eta)=\cos(\psi+\eta)\ A+\sin(\psi+\eta)\ B$. We define

$$\Phi^L_{p,\gamma}:\mathbb{R}\to\mathbb{R} \qquad \text{s.t.} \qquad \eta\mapsto \mathrm{rot}_\gamma\Big(\ R(L,\eta)|_p\Big). \tag{3.7}$$

Since this function is 2π-periodic we can consider the quantity

$$\max(\mathrm{rot}_\gamma(L)) = \max_\eta(\Phi^L_{p,\gamma}).$$

Using this object we can give a necessary and sufficient condition for $\mathcal{D}_L$ to be Engel on γ up to homotopy within $(\mathcal{W},\mathcal{E})$.

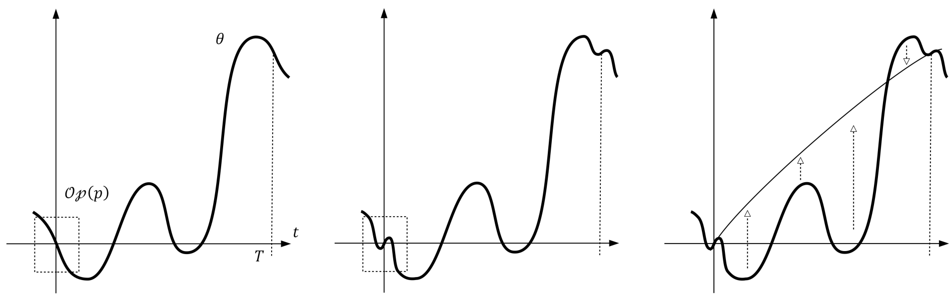

Figure 3.1: Homotopy of θ when the rotation number is positive.

Theorem 3.2.4. *Let $\mathcal{E} = \langle W,\, A,\, B\rangle$ be an even contact structure and γ a closed orbit for W. Then L is Engel on γ up to homotopy within $(\mathcal{W},\, \mathcal{E})$ if and only if there exists a point $p \in \gamma$ such that* $\max(\mathrm{rot}_\gamma(L)) > 0$.

Proof. Suppose first that L is Engel up to homotopy on γ, and let L_τ for $\tau \in [0, 1]$ be such that $\langle W,\, L_1\rangle$ is Engel. We need to show that for a given $p \in \gamma$, there is a homotopy relative to $L_1(p)$ between L_1 and $R(L, \eta)$ for some $\eta \in \mathbb{R}$. This implies indeed by Lemma 3.2.3 and Proposition 3.1.2 that

$$\Phi^L_{p,\gamma}(\eta) = \mathrm{rot}_\gamma \left(L_1(p)\right) > 0.$$

For a fixed $p \in M$ there is an angle η such that $R(L, \eta)(p) = L_1(p)$. On $\mathcal{O}p(p)$ we can homotope $R(L, \eta)$ to L_1 relative to p. Since both $R(L, \eta)$ and L_1 are homotopic to L, they must be homotopic to each other, and since the fundamental group of S^1 is Abelian, there is a homotopy relative to p.

Conversely suppose $\max(\mathrm{rot}_\gamma(L)) > 0$, without loss of generality we can suppose that $\mathrm{rot}_\gamma(L(p)) > 0$. First consider a small neighbourhood of p and homotope L relative to $\{p\}$ and to the boundary of $\mathcal{O}p(p)$ to an Engel structure near p. Since this homotopy was performed relative to $\{p\}$, the rotation number at $L(p)$ has not changed by Lemma 3.2.3.

For $\epsilon > 0$ small, take a disc $D^3 \hookrightarrow M$ centered at $\phi_\epsilon(p)$ and everywhere transverse to $\mathcal{W}$. Up to shrinking disc D^3, we can suppose that the map $F : D^3 \times [\epsilon, T - \epsilon] \to M$ given by the flow $(q, t) \mapsto \phi_t(q)$ is an embedding and hence a flow box for W. In this chart we have

$$F^*\tilde{L}(q;t) = \rho(q;t)\Big(\cos\psi(q;t)\ F^*A(p) + \sin\psi(q;t)\ F^*B(p)\Big)$$

with $\psi(0;t) = \theta(\phi_\epsilon(p);t)$ on γ. Up to choosing $\epsilon > 0$ small enough we can suppose that $\psi(0;T-\epsilon) - \psi(0;\epsilon) > 0$; here we use $\mathrm{rot}_\gamma(L(p)) > 0$. Hence there exists a homotopy $\psi_\tau : D^3 \times [\epsilon, T-\epsilon] \to \mathbb{R}$ such that $\psi_0 = \psi$, the restriction of ψ_τ to the boundary $\partial(D^3 \times [\epsilon, T-\epsilon])$ is ψ and

$$\psi_1(0;t) = h(t)(\psi(0;T-\epsilon) - \psi(0;\epsilon)) + \psi(0;\epsilon)$$

for a smooth step function h (see Figure 3.1). Then $\mathcal{D}_{L_1}$ is Engel on a (possibly smaller) neighbourhood of γ. $\square$

3.3 Character of closed orbits of $\mathcal{W}$

Following the ideas of Section 1.3.2 we now study the behaviour of closed orbits according to how ϕ_t acts on $\mathcal{E}/\mathcal{W}$. The following result tells us how $\Phi^L_{p,\gamma}(\eta)$ changes when we change η.

Proposition 3.3.1. *Let $\mathcal{E} = \langle W,\, A,\, B\rangle$ be an even contact structure and γ a closed orbit for W.*

1. *If γ is hyperbolic, then for every $\eta \in \mathbb{R}$ we have*

$$\left|\Phi^L_{p,\gamma}(\eta) - \mathrm{rot}_\gamma(L(p))\right| < \frac{\pi}{2}.$$

   Moreover there exists a constant $c \in (0,\, \pi/2)$ such that L is Engel on γ up to homotopy if and only if $\mathrm{rot}_\gamma(L(p)) > -c$;

2. *if γ is positive parabolic, then for every $\eta \in \mathbb{R}$ we have*

$$\left|\Phi^L_{p,\gamma}(\eta) - \mathrm{rot}_\gamma(L(p))\right| < \pi.$$

   Moreover there exists a constant $c \in (0,\, \pi)$ such that L is Engel on γ up to homotopy if and only if $\mathrm{rot}_\gamma(L(p)) > -c$;

3. *if γ is negative parabolic then L is Engel on γ up to homotopy if and only if $\mathrm{rot}_\gamma(L(p)) > 0$;*

4. *if γ is elliptic then $\Phi^L_{p,\gamma}(\eta)$ is constant in η.*

Proof. Use the notation

$$\Big(T_{\phi_t(p)}\phi_{-t}\Big)\Big(R_\eta \circ L(\phi_t(p))\Big) = \rho_\eta(p;t)\Big(\cos\theta_\eta(p;t)\ A(p) + \sin\theta_\eta(p;t)\ B(p)\Big)$$

modulo $\mathcal{W}(p)$ where $\theta_\eta(p;0) = \theta(p;0) + \eta$. We need to determine $\theta_\eta(p;T)$. Set

$$M_\eta(t) := \Big(T_{\phi_t(p)}\phi_{-t}\Big)R_\eta\Big(T_{\phi_t(p)}\phi_{-t}\Big)^{-1},$$

so that

$$\begin{aligned}\tilde{L}_\eta(p;T) &= \Big(T_p\phi_{-T}\Big)\Big(R_\eta \circ L(p)\Big) = M_\eta(T)\Big(T_p\phi_{-T}(L(p))\Big)\\ &= \rho(p;T)M_\eta(T)\Big(\cos\theta(p;T)\,A(p) + \sin\theta(p;T)\,B(p)\Big)\end{aligned}$$

modulo $\mathcal{W}(p)$. There is a function $r = r(\eta,\theta)$ which depends on $M_\eta(T)$ and on the angle $\theta(p;T)$ such that

$$\tilde{L}_\eta(p;T) = \tilde{\rho}(p;T)\Big(\cos(\theta(p;T) + r)\ A(p) + \sin(\theta(p;T) + r)\ B(p)\Big)$$

modulo $\mathcal{W}(p)$. Moreover since $M_\eta(0) = R_\eta$ and $M_\eta(t)$ is continuous we conclude that $\theta_\eta(p;T) = \theta(p;T) + r$. We will discuss what the angular displacement r is in the various cases.

Consider $P = rT_p\phi_{-T}$ where r is the square root of the norm of the determinant of $T_p\phi_{-T}$. This is enough because the term in Equation 3.3 that is interesting for calculating the rotation number is θ and not ρ.

If γ is hyperbolic, without loss of generality we can suppose that $P(A(p)) = e^{-\mu}A(p)$ and $P(B(p)) = e^{\mu}B(p)$, so that in this basis

$$P = \begin{pmatrix} e^{-\mu} & 0 \\ 0 & e^{\mu} \end{pmatrix}.$$

Moreover up changing initial phase $L(p) = A(p)$. This means that $r = \eta + \epsilon$ where $|\epsilon| < \pi/2$ since P displaces a point $q \in S^1$ of an angle of at most $\pi/4$. Now

$$\Phi^L_{p,\gamma}(\eta) = \theta_\eta(p;T) - \theta_\eta(p;0) = \theta(p;T) + \eta + \epsilon - \theta(p;0) - \eta = \mathrm{rot}_\gamma(L(p)) + \epsilon$$

and it suffices to set $c = |\min_\eta(\epsilon(\eta))|$.

If γ is positive parabolic, we can suppose as above that $L(p) = A(p)$ and that P in the basis $\{A(p), B(p)\}$ writes as

$$P = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

As above $r = \eta + \epsilon$ with $0 \leq \epsilon < \pi$ and again we set $c = |\min_\eta(\epsilon(\eta))|$. On the other hand if γ is negative parabolic then by changing η we can only decrease the rotation number, i.e. $r = \eta - \epsilon$ with ϵ as above.

Finally if γ is elliptic we have

$$P = \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{pmatrix},$$

so that $M_\eta(T) = R_\delta R_\eta R_\delta^{-1} = R_\eta$ and $\theta_\eta(p;T) = \theta(p;T) + \eta$.

□

Remark 3.3.2. The previous result ensures that the only cases where it can happen that $\mathrm{rot}_\gamma(L(p)) \leq 0$ and nonetheless L is homotopic to an Engel structure on γ occur when γ is hyperbolic or positive parabolic. Moreover in these cases $\mathrm{rot}_\gamma(L(p))$ is not allowed to be ”too negative”. In order to overcome this subtlety we will consider the quantity $\max(\mathrm{rot}_\gamma(L))$ in what follows.

The previous theorem gives crucial information in the case of a null-homotopic closed orbit.

Corollary 3.3.3. *Let $\mathcal{E} = \langle W, A, B\rangle$ and L be as above, and let γ be an elliptic null-homotopic closed orbit for W. Then L is homotopic to an Engel structure on a neighbourhood of γ if and only if there is a point $p \in \gamma$ such that* $\mathrm{rot}_\gamma(L'(p)) > 0$ *for any choice of non-singular $L' \in \Gamma\langle A, B\rangle$.*

Proof. Suppose that there exists L_1 such that $\langle W, L_1\rangle$ is Engel on $\mathcal{O}p(\gamma)$. For any $p \in \gamma$ and L there is a phase η such that $L_1(p) = R(L, \eta)(p)$. Since γ is null-homotopic, and $\dim M = 4$ there is an embedded disc D^2 such that $\partial D^2 = \gamma$. Hence there exists a homotopy relative to $L(p)$ between L_1 and $R(L, \eta)$, and by Theorem 3.2.4 we must have $\Phi^L_{p,\gamma}(\eta) > 0$. □

In particular under the hypothesis of Corollary 3.3.3 it does not matter which vector field L we choose to calculate the rotation number.

3.4 Non-integrable rank 3 distributions

The calculations that we carried out in the previous sections do not use the fact that we are considering an even contact structure on a 4-manifold (except for Corollary 3.3.3). The crucial hypothesis is instead the existence of a plane field $\mathcal{E}$ of rank 3 such that

1. $\mathcal{E}$ admits a global framing $\{W, A, B\}$;
2. the flow of W preserves $\mathcal{E}$.

We will extend the results of the previous section to the more general setting of a subbundle $\mathcal{E} \hookrightarrow TM$ of rank 3 on an n-dimensional manifold satisfying 1 and 2.

In what follows we will denote by ϕ_t the flow of W at time t, and by L a section of $\mathcal{E}$ which is never tangent to $\langle W\rangle$. Moreover we denote by $\mathcal{D}_L$ the distribution $\mathcal{D}_L = \langle W, L\rangle$. If $\mathcal{D}_L^2 = \mathcal{E}$ we say that $\mathcal{D}_L$ is *maximally non-integrable within $\mathcal{E}$*. Finally for $K \subset M$ we say that L_0 is *maximally non-integrable within $\mathcal{E}$ up to homotopy on K* if there exists a smooth homotopy L_τ for $\tau \in [0, 1]$ such that each L_τ is never tangent to W, and $\mathcal{D}_{L_1}$ is maximally non-integrable within $\mathcal{E}$ on $\mathcal{O}p(K)$.

In Section 3.1 we used the existence of a defining form α for $\mathcal{E}$ whose differential has some special properties. In order to extend this we introduce a 2-form ω such that $\omega(A, B) = 1$ and $i_W\omega = 0$. If we define $J \in \operatorname{End}\mathcal{E}$ by $JW = 0,\ JA = B$ and $JB = -A$, we get

$$\omega(JX, JY) = \omega(X, Y)$$

for all X, Y sections of $\mathcal{E}$, which is the analogue of (3.1). Similarly, since the flow of W preserves $\mathcal{E}$ and $i_W\omega = 0$ we have that $\phi_t^*\omega|_{\mathcal{E}} = \lambda\omega|_{\mathcal{E}}$ for some function λ. This can be used to prove Formula (3.6), which is the crucial result in all other calculations. The definition of the rotation number copies in this setting.

For a fixed closed orbit γ of W the map $P = r\, T\phi_t|_{\mathcal{E}/\mathcal{W}}$, where r is the square root of the norm of the determinant of $T\phi_t|_{\mathcal{E}/\mathcal{W}}$, as in Section 1.3.2. Using the same classification we talk of hyperbolic, positive/negative parabolic and elliptic closed orbits of W. We are ready to give the above-mentioned generalization of Theorem 3.2.4. The proof of the following result is exactly the same as above.

Theorem 3.4.1. *Let $\mathcal{E} \hookrightarrow TM^n$ be a distribution of rank 3 as above and γ a closed orbit of W. Then for a section $L \in \Gamma\mathcal{E}$ never tangent to $\mathcal{W}$, the distribution $\mathcal{D}_L$ is maximally non-integrable within $\mathcal{E}$ on γ up to homotopy if and only if* $\max(\mathrm{rot}_\gamma(L)) > 0$.

If $n = 3$ then $\mathcal{E} = TM$ and the previous theorem provides a necessary and sufficient condition for $\mathcal{D}_L$ to be a contact structure.

Corollary 3.4.2. *Let N be a closed orientable 3-manifold, W a non-singular vector field on N and γ a closed orbit of W. For a vector field L never tangent to W the plane field $\mathcal{D}_L = \langle W, L\rangle$ is a contact structure on γ up to homotopy if and only if* $\max(\mathrm{rot}_\gamma(L)) > 0$.

Notice that the proof of Corollary 3.3.3 also copies if $n > 3$. In the case of a 3-dimensional manifold the concepts of contractible and unknotted (i.e. bounding an embedded disc) do not coincide. With this difference in mind we have the following generalization of Corollary 3.3.3.

Corollary 3.4.3. *Let $\mathcal{E} = \langle W, A, B\rangle \hookrightarrow TM^n$ and L be as above and γ an elliptic unknotted closed orbit for W. Then $\mathcal{D}_L$ is non-integrable within $\mathcal{E}$ on a neighbourhood of γ up to homotopy if and only if there is a point $p \in \gamma$ such that* $\mathrm{rot}_\gamma(L'(p)) > 0$ *for any choice of non-singular $L' \in \Gamma\langle A, B\rangle$.*

3.5 Crash course: Morse-Smale vector fields

The construction of the previous section works well with vector fields W whose limit sets are closed orbits. The easiest example of such dynamics is given by Morse-Smale vector fields. We will provide here the basic facts from the theory of differentiable flows on manifolds needed for the understanding of the rest of the chapter. References for this section are [Irw, Kat].

In what follows we will consider a smooth flow $\phi_t : M \to M$ on a closed smooth manifold and indicate with W its tangent vector field. Moreover we will fix an auxiliary Riemannian metric g and consider the associated distance d. We will be only interested in non-singular vector fields, hence $W(p) \neq 0$ for all $p \in M$. A closed orbit γ of W is *non-degenerate* if the derivative of its Poincaré map does not have eigenvalues in the unit circle $S^1 \hookrightarrow \mathbb{C}$. Otherwise said for all eigenvalues $\lambda = e^{a+ib} \in \mathbb{C}^*$ we ask $a \neq 0$.

Theorem 3.5.1. *Let γ be a non-degenerate closed orbit of ϕ_t, then the sets*

$$W^s(\gamma) = \{p \in M | \, d(\phi_t(p), \gamma) \to 0 \text{ for } t \to +\infty\}$$
$$W^u(\gamma) = \{p \in M | \, d(\phi_t(p), \gamma) \to 0 \text{ for } t \to -\infty\}$$

are immersed submanifolds called respectively stable *and* unstable *submanifold. Moreover they are homeomorphic to $\mathbb{R}^k$ and $\mathbb{R}^h$ respectively, for some $h, k \in \mathbb{N}$ such that* $\dim M + 1 = k + h$.

We are interested in the asymptotic behaviour of orbits of ϕ_t.

Definition 3.5.2. *The* non-wandering set Ω *of the flow $\phi_t : M \to M$ is the set of points $p \in M$ such that for all open neighbourhoods $p \in U \subset M$ and for all $t \in \mathbb{R}$ there exists $T > t$ such that $\phi_T(U) \cap U \neq \emptyset$.*

Otherwise said, although $p \in \Omega$ may be sent *far away* by ϕ_t, points near p (i.e. in U) always come back near p. K. Sigmund suggested the name *nostalgic* points instead of non-wandering, because "though p it self may never come back, its thoughts keep coming back" (see page 47 in [Irw]). It is easy to check that Ω is a closed, ϕ_t-invariant and non-empty when M is compact.

We are now ready to introduce Morse-Smale flows. Since we are only interested in non-singular vector fields, we will not give the definition in its full generality.

Definition 3.5.3. *A* non-singular Morse-Smale vector field (NMS) *W on a manifold M is a non-singular vector field which satisfies the following conditions*

1. *W has finitely many closed orbits $\gamma_1, ..., \gamma_k$ and they are all non-degenerate;*

2. *the non-wandering set is the union of the closed orbits $\Omega = \gamma_1 \cup \cdots \cup \gamma_k$;*

3. *for every $i, j \in \{1, ..., k\}$ the stable manifold $W^s(\gamma_i)$ and the unstable manifold $W^u(\gamma_j)$ intersect transversely.*

Point 3 of the previous definition means that either $W^s(\gamma_i) \cap W^u(\gamma_j) = \emptyset$ or the sum of their tangent spaces spans TM on the intersection.

The main reason why we are interested in Morse-Smale vector fields is that their dynamical properties are, in a sense, completely determined by what happens near the closed orbits and by how these are *linked* together.

Theorem 3.5.4 [Morg]**.** *Let W be a non-singular Morse-Smale vector field on M. Then M admits a round-handle decomposition $M_0 \subset M_1 \subset \cdots \subset M_k = M$ such that every handle R is a neighbourhood of closed orbits γ of W and the index of R (as a handle) is the index of γ (as a closed orbit). Moreover the attaching procedure is performed using the flow and W which is transverse to every M_i.*

The idea of the proof is to order the closed orbits of M via $\gamma_i \leq \gamma_j$ if $W^u(\gamma_i) \cap W^s(\gamma_j) \neq 0$. Otherwise said $\gamma_i \leq \gamma_j$ if there is a orbit whose α-limit is γ_i and whose ω-limit is γ_j. The following result ensures that one can choose a total ordering of $\{\gamma_1, ..., \gamma_k\}$.

Theorem 3.5.5 [Sma] (No cycle condition)**.** *Let $\{\gamma_1, ..., \gamma_k\}$ be the set of closed orbits of a non-singular Morse-Smale vector field with the ordering defined above. Then there exists no non-trivial sequence $\gamma_{i_1} \leq \gamma_{i_2} \leq \cdots \leq \gamma_{i_1}$.*

In order to construct the RHD of M one starts by attaching the source orbits by disjoint union, then any point in $M \setminus \{\gamma_1, ..., \gamma_k\}$ has to have one of the source orbits as ω-limit. Suppose that we have constructed inductively M_i such that

- $\gamma_1, ..., \gamma_i \in M_i$;
- $\gamma_l \cap M_i = \emptyset$ for $j > i$;
- the flow is transverse pointing outward on ∂M_i.

We take a neighbourhood R_{i+1} of γ_{i+1}, the construction of the ordering ensures that points in $R_{i+1} \setminus \gamma_i$ have ω-limit in M_i. We attach R_{i+1} using all flow lines of W that have ω-limit in M_i. The problem with this procedure is that it may introduce corners. Moreover the boundary of M_{i+1} will not be transverse to W. The solution is to smoothen the corners as illustrated in Figure 3.2.

3.6 Morse-Smale Legendrian vector fields

The goal of this section is to understand which non-singular Morse-Smale vector fields on 3-manifolds are Legendrian. Since the dynamics of these vector fields can be described once we understand neighbourhoods of the closed orbits, it is reasonable to expect that the rotation number will play a central role. In fact we will prove that it provides the only obstructions to the existence of ξ contact, such that $L \in \xi$. The case of Morse-Smale gradient vector fields was studied in [EG].

It is important to notice that only very few 3-manifolds admit NMS vector fields, nonetheless S^3 is a very interesting example that does admit NMS flows. The following result classifies completely the topology of such manifolds.

Theorem 3.6.1 [Morg]**.** *Let N be an orientable, prime 3-manifold with boundary such that the Euler characteristic of every boundary component vanishes. Let $\partial_- N$ be an arbitrary union of these components. Suppose N is not $S^1 \times D^2$. The pair $(N, \partial_- N)$ admits a non-singular Morse-Smale flow if and only if N is a union of non-trivial Seifert spaces attached to one another along components of their boundaries.*

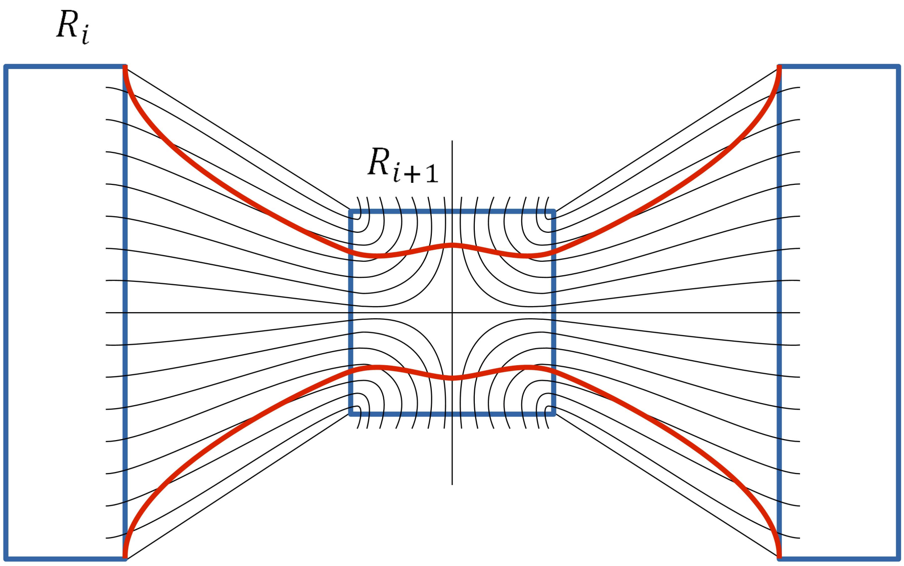

Figure 3.2: Smoothen the corners.

The following result furnishes a necessary and sufficient condition for a NMS flow to be Legendrian for some contact structure.

Theorem 3.6.2. *Let N be a closed, oriented 3-manifold, and let W be a NMS vector field such that we have a framing $TN = \langle W, A, B\rangle$. There exists a positive contact structure ξ for which W is Legendrian if and only if there exists a vector field $L \in \langle A, B\rangle$ such that $\max(\mathrm{rot}_\gamma(L)) > 0$ for all γ closed orbits.*

Proof. If such a ξ exists then we can take L to be any vector complementary to W within ξ and the claim follows by Proposition 3.1.2. Conversely suppose that L satisfies the above properties. The idea is to construct the contact structure ξ inductively using the decomposition provided by Theorem 3.5.4.

The first step of the induction is to construct a contact structure in the neighbourhood of the sources. This is possible thanks to Theorem 3.4.1. This procedure yields a contact structure ξ homotopic to $\langle W, L\rangle$. Notice that we can make sure that the boundary of the (possibly disconnected) manifold that we obtain with this procedure is transverse to W.

For the inductive step suppose that we have attached $k-1$ handles to obtain N_{k-1}, and that we want to attach the k-th handle R_k. Theorem 3.5.4 ensures that R_k is a neighbourhood of γ_k, and that the attaching procedure happens via the flow of W. We first construct a contact structure on R_k using Theorem 3.4.1, this is possible because of the hypothesis on L. The existence of L also ensures that the contact structure on M_{k-1} extends to an almost contact structure on M_k which is contact on a neighbourhood of M_{k-1} and of γ_k.

In general we cannot homotope this almost contact structure to a contact structure on M_k. The problem is that the attaching region is of the form $R_k^+ \times I$ where $R_k^+ \times \{1\}$ is the subset of R_k where W points inwards, and W is tangent to the I factor on $R_k^+ \times I$. This means that the restriction of L to $R_k^+ \times I$ writes as $L = \cos f_t\, A + \sin f_t\, B$, where $f_t : \tilde{R}_k^+ \times \{t\} \to S^1$ is a I-family of angle functions . Hence we can homotope L transversely to ∂_t so that $\langle \partial_t, L \rangle$ is contact if and only if $f_1(p) > f_0(p)$. There is no reason for this to happen in general.

We are not interested in contact structures homotopic to $\langle W, L \rangle$, we just need a contact structure for which W is Legendrian. Hence instead of glueing R_k directly, we first make sure to increase $f_1(p)$ using the fact that W is transverse to ∂N_{k-1}. We take a collared neighbourhood of the boundary where L rotates positively "a bit" and we substitute it with one where L rotates "massively".

More precisely let $K = \max\{f_1(p) - f_0(p) | p \in \tilde{R}_k\}$. For any $p \in \partial N_{k-1} \times (-\epsilon, \epsilon)$ the vector field L can be described by a map $h_p : (-\epsilon, \epsilon) \to S^1$ which is a small embedding. We substitute it with $\tilde{h}_p : (-\epsilon, \epsilon) \to S^1$ which coincides with h on $\mathcal{O}p(\{-\epsilon, \epsilon\})$, and such that it makes a number of turns around S^1 bigger than K. The net effect of this is that we have changed the homotopy type of L, but the difference between the new angle functions f_1 and f_0 is positive. This ensures that we can homotope L to a contact structure on the attaching region.

It only remains to round the edges of N_k in case we are attaching an index 1 round handle. This can be done exactly via the procedure described in the proof of Theorem 3.5.4. Indeed this procedure "digs" the new N_k inside the manifold with corners that we have just constructed. After this process the contact structure will be the restriction of the previously constructed one.

□

It is interesting to know when a given vector field $L \in \mathfrak{X}(N)$ is transverse to a contact structure. This question was already studied in [Gir] for the case where L is tangent to the fibres of a S^1- fibre bundle over a surface, and in [LiM] for the case L tangent to the fibres of a Seifert fibration.

Notice that if L is Legendrian for some contact structure ξ then there is a contact structure $\tilde{\xi}$ transverse to L. Choose $\tilde{L}$ complementary to L in ξ and flow ξ in the direction of $\tilde{L}$ for small time. Lemma 1.2.3 ensures that $\tilde{\xi}$ is transverse to ξ. Since it has to contain $\tilde{L}$, it must be transverse to L.

With the techniques developed in this Chapter we can present an example of a vector field which is transverse to a contact structure but never Legendrian. Namely take a sink and source with trivial monodromy on $S^1 \times D^2$ and we glue them by the $(1,1)$-map. Up to small perturbations we get a

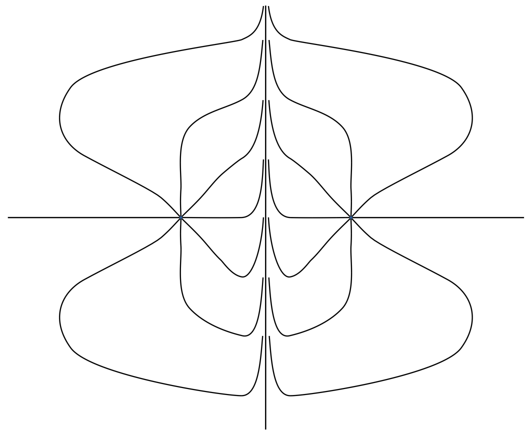

Figure 3.3: Phase portrait of L on see $\mathbb{R}^3 = S^3 \setminus \{N\}$. The picture is understood via rotational symmetry around the vertical axis.

NMS vector field L on S^3 (see Figure 3.3). Another way of constructing L is to consider the canonical Reeb foliation on these round handles and take the vector field normal to the leaves. Thanks to the following result we can $\mathcal{C}^0$-deform the tangent bundle of the Reeb foliation to get a contact structure.

Theorem 3.6.3. *Given a $\mathcal{C}^0$-foliation $\mathcal{F}$ one can find a smooth family ξ_t of confoliations which continuously depends on $t \in [0,1]$ and such that $\xi_0 = T\mathcal{F}$, ξ_1 is contact and ξ_t is $\mathcal{C}^0$-small.*

Recall that a *confoliation* is a plane distribution $\xi = \ker \eta$ on a 3-manifold such that $\eta \wedge d\eta \geq 0$ (see [ET]). For a proof of Theorem 3.6.3 see Chapter 2 of [ET].

On the other hand L has two unknotted closed orbits which have trivial monodromy, which means that they are elliptic and have rotation number 0. This obstructs the existence of a contact structure for which L is Legendrian.

3.7 Morse-Smale even contact structures

We now turn the attention to even contact structures whose characteristic foliation is Morse-Smale, we call these *Morse-Smale even contact structures*. The analogue of Theorem 3.6.2 still holds and the proof copies mutatis mutandi.

Theorem 3.7.1. *Let $(M, \mathcal{E})$ be a closed, oriented Morse-Smale even contact 4-manifold. Suppose that we have a framing $\mathcal{E} = \langle W, A, B\rangle$ with $\mathcal{W} = \langle W\rangle$. Then there exists a positive Engel structure $\mathcal{D}$ inducing $\mathcal{E}$ if and only if there* *exists a vector field* $L \in \langle A, B\rangle$ *such that* $\max(\mathrm{rot}_\gamma(L)) > 0$ *for all* γ *closed characteristic orbits.*

It is not so clear if such even contact structures exist. In fact one can show that many NMS on 4-manifolds do not have the right monodromy for being the characteristic foliation of an even contact structure.

On the other hand if we allow $\mathcal{C}^0$-perturbations of W then we can always suppose that the closed orbits have tubular neighbourhoods $\nu\gamma = S^1 \times D^3$ where W writes as

$$W|_{\nu\gamma} = \partial_\theta + 2\epsilon_1 x\partial_x + 2\epsilon_2 y\partial_y + 4\epsilon_3 z\partial_z,$$

where $\epsilon_i = \pm 1$ depending on the index of γ. These models always permit to construct an even contact form α on $\nu\gamma$ such that W spans the characteristic foliation. If the ϵ_i are all equal then W is Liouville for (a multiple of) the symplectic form $\omega = dx \wedge dy + dz \wedge d\theta$, so that we take $\alpha = i_W\omega$. If the ϵ_i are not all equal, then the vector field $V = 2\epsilon_1 x\partial_x + 2\epsilon_2 y\partial_y + 4\epsilon_3 z\partial_z$ is contact with respect to $\alpha = dz - xdy + ydx$ on D^3, so that νM can be seen as the suspension of the time 1 flow of V.

Example 3.7.2. Morse-Smale even contact structures can be obtained by suspension of a Morse-Smale contactomorphism whose non-wandering set only consists of fixed points. A way of constructing such contactomorphism is to look for contact vector fields which are Morse-Smale and do not have closed orbits.

An explicit example is given by the contact vector field V on (S^3, α_{st}) associated with the contact Hamiltonian $h(x_1, y_1, x_2, y_2) = y_1/2$. One can verify that V take the form

$$V(x_1, y_1, x_2, y_2) = \frac{1}{2}\left((1 - x_1^2)\partial_{x_1} + y_1\partial_{y_1} + x_2\partial_{x_2} + y_2\partial_{y_2}\right).$$

We get an even contact structure on $M = S^3 \times S^1$ whose characteristic foliation only has 2 closed orbits, namely a source and a sink. This is induced by an Engel structure since M is obtained as a suspension of a contactomorphism isotopic to the identity (see Example 1.5.4).

It is unclear if every homotopy class of even contact structures admits a Morse-Smale even contact structure.

3.8 Aside: Sullivan currents and Engel structures

In the previous discussion the non-wandering set Ω of W plays a central role in understanding whether or not an even contact structure $\mathcal{E}$ is induced by an Engel structure. The main interest of this Chapter so far has been the study of the easiest non-wandering points possible: the periodic ones. It remains open to understand if more general points $p \in \Omega$ may furnish other obstructions. The main difficulty in answering this question is the fact that open orbits do not admit a character (as defined in Section 1.3.2) in general. In this last section we will use the techniques of *foliation currents*, developed in [Sul] and [Schw], to get some partial results in this direction.

3.8.1 What is a foliation current?

This section provides the basic definitions and results of the theory of currents on manifolds. For a more detailed exposition see [Sul] and [CC].

On a smooth, closed orientable manifold M we denote by $\Omega^q(M)$ the space of q-forms. Equipped with the $\mathcal{C}^\infty$-topology, this is a Fréchet space. A *q-current* is an element of the topological (strong) dual $\Omega_q(M)$, i.e. the space of continuous functionals $c : \Omega^q(M) \to \mathbb{R}$. We will use the notation $\langle c, \omega\rangle = c(\omega)$ for the evaluation of a current c on a form ω.

Example 3.8.1. For a point $p \in M$ and a q-vector $v_p \in \Lambda^q T_pM$, define the functional $\delta_{v_p}(\omega) = \omega_p(v_p)$ for $\omega \in \Omega^q(M)$. One can verify that this is continuous so it is a q-current; we refer to it as *Dirac current* of v_p.

Taking the adjoint of the differential $d_q : \Omega^q(M) \to \Omega^{q+1}(M)$ we get a boundary operator $\delta_q : \Omega_{q+1}(M) \to \Omega_q(M)$. Currents in the image $\mathcal{B}_q := \operatorname{Im}\delta_q$ are called *q-boundaries* and elements in the kernel $\mathcal{Z}_q = \ker\delta_{q-1}$ are called *q-cycles*. It turns out that $\mathcal{B} \subset \mathcal{Z}$ and the homology of this complex is isomorphic to $H_*(M, \mathbb{R})$.

If we have a foliation $\mathcal{F}$ of dimension q on M, we can construct a cone $\mathcal{C}_\mathcal{F}$ of q-currents by taking the cone generated by Dirac currents of q-vectors tangent to $\mathcal{F}$. Currents in $\mathcal{C}_\mathcal{F}$ are called *foliation currents*. This cone always has a compact basis and studying how it intersects the spaces $\mathcal{B}_q$ and $\mathcal{Z}_q$ provides information about the geometry of $\mathcal{F}$. These conditions are particularly simple when $\mathcal{F}$ has dimension 1. In this case $\mathcal{C}_\mathcal{F}$ always intersects $\mathcal{Z}$ and many dynamical properties of $\mathcal{F}$ such as having volume-preserving holonomy or being totally geodesic, are related to properties of $\mathcal{C}_\mathcal{F}$. The following result is an example of the power of this theory.

Theorem 3.8.2. *If $\mathcal{F}$ is a foliation by curves, the following are equivalent*

1. *$\mathcal{F}$ admits no non-trivial foliation boundaries, i.e. $\mathcal{C}_\mathcal{F} \cap \mathcal{B} = \{0\}$;*
2. *there is a closed 1-form transverse to $\mathcal{F}$;*
3. *$\mathcal{F}$ is transverse to the fibres of a smooth fibration $\pi : M \to S^1$;*
4. *$\mathcal{F}$ admits a global section.*

For a proof refer to Chapter 10 of [CC].

Remark 3.8.3. If $\mathcal{E}$ is an even contact structure such that $\mathcal{W}$ admits a global section then the construction in Example 1.5.4 provides and Engel structure inducing $\mathcal{E}$. This together with Theorem 3.8.2 provides another hint that $\mathcal{C}_{\mathcal{W}}$ contains information regarding whether $\mathcal{E}$ is induced by an Engel structure or not.

3.8.2 Engel structures and currents

McDuff applied the theory of currents to contact geometry (see [McDu]). Let M^{2n+1} be a smooth, closed, orientable manifold and η a 1-form such that $d\eta^n \neq 0$. We say that η is of *contact-type* if there exists a 1-form $\tilde{\eta}$ which is contact and such that $d\eta = d\tilde{\eta}$. The idea to determine whether or not η is of contact type is to consider the foliation by curves $\mathcal{F} = \ker d\eta^n$. If we have $\tilde{\eta} = \eta + \theta$ with θ closed and $\tilde{\eta}$ contact, then in particular $\tilde{\eta}(R) \neq 0$ for every non-singular $R \in \Gamma\mathcal{F}$. This means that for every foliation boundary $c \in \mathcal{C}_{\mathcal{F}} \setminus \{0\}$ we must have

$$0 \neq c(\tilde{\eta}) = c(\eta + \theta) = c(\eta) + c(\theta) = c(\eta),$$

since $c = \delta b$ means $c(\theta) = \delta b(\theta) = b(d\theta) = 0$. Hence it is necessary that $c(\eta) \neq 0$ for any c non-zero foliation boundary. It turns out that the converse is also true.

Theorem 3.8.4 [McDu]**.** *Let M^{2n+1} be smooth, closed and orientable, and let $\eta \in \Omega^1(M)$ such that $d\eta^n \neq 0$. Set $\mathcal{F} = \ker d\eta^n$, then η is of contact-type if and only if $\langle c, \eta \rangle \neq 0$ for all $c \in \mathcal{C}_{\mathcal{F}} \cap \mathcal{B} \setminus \{0\}$.*

We want to establish an analogous result for Engel structures. Let $(M, \ker\alpha)$ be an even contact manifold. Fix the following notation

- A p-form $\omega \in \Omega^p(M)$ is *α-closed* if $d\omega \wedge \alpha = 0$;
- A 1-form $\beta \in \Omega^1(M)$ is *α-contact* if $\alpha \wedge \beta \wedge d\beta$ is a volume form;
- Denote by $A(\mathcal{W})$ the space of 1-forms η such that $\eta(W) = 0$, where W spans the characteristic foliation $\mathcal{W}$.

Remark 3.8.5. Since $A(\mathcal{W})$ is the kernel of the contraction by W, it is a closed subset of $\Omega^1(M)$. This implies that it is a Fréchet space, and since $\Omega^1(M)$ is reflexive, $A(\mathcal{W})$ has the same property (see [Scha]). This allows us to use the isomorphism between $A(\mathcal{W})$ and $A^*(\mathcal{W}) = \{c : A(\mathcal{W}) \to \mathbb{R} \text{ bounded}\}$ and to use Hahn-Banach theorem.

An (orientable) Engel structure $\mathcal{D}$ whose induced even contact structure is $\mathcal{E} = \ker\alpha$ is the kernel of $\alpha \wedge \beta$ for an α-contact form $\beta \in A(\mathcal{W})$.

Consider a form $\beta \in A(\mathcal{W})$ and suppose that $\alpha \wedge d\beta$ is a non-singular 3-form. In particular its kernel is a line field, denote by T a non-singular section. We say that β is of *Engel-type* if there exists $\beta' \in A(\mathcal{W})$ such that $\alpha \wedge d\beta' = \alpha \wedge d\beta$ and β' is α-contact. Notice that this happens if and only if $\beta'(T) \neq 0$.

Recall that $\eta \in \Omega^1(M)$ is closed if and only if for all $b \in \mathcal{B}$ boundary we have $\langle b, \eta \rangle = 0$. We need the analogue of this for α-closed forms.

Lemma 3.8.6. *Consider the space*

$$\mathcal{A} = \Big\{ c : \Omega^2(M) \to \mathbb{R} \Big| \langle c, \omega \rangle = 0 \quad \forall \omega \in \Omega^2(M) \quad \text{s.t.} \quad \alpha \wedge \omega = 0 \Big\}$$

and let $\mathcal{B}_\alpha = \delta(\mathcal{A})$ be the image of $\mathcal{A}$ under the coboundary operator. Then η is α-closed if and only if $\langle \mathcal{B}_\alpha, \eta \rangle = 0$.

Proof. Since $\langle \delta c, \eta \rangle = \langle c, d\eta \rangle$, it suffices to prove that $\omega \in \Omega^2(M)$ satisfies $\omega \wedge \alpha = 0$ if and only if $\langle \mathcal{A}, \omega \rangle = 0$. First of all notice that for every X, $Y \in \mathfrak{X}(M)$ tangent to $\ker \alpha$, and ω such that $\omega \wedge \alpha = 0$, we must have $\omega(X, Y) = 0$. This means that every Dirac 2-current which is an evaluation on $u_p, v_p \in \ker \alpha_p$ is in $\mathcal{A}$.

Take ω such that $\langle \mathcal{A}, \omega \rangle = 0$, this means in particular that for every $X \in \mathfrak{X}(M)$ tangent to $\ker \alpha$, the 1-form $i_X \omega$ is a multiple of α. Indeed it is zero on $\ker \alpha$, because all Dirac currents which are evaluations on $X(p)$, $v_p \in \ker \alpha_p$ are in $\mathcal{A}$. This readily implies that the form $\omega \wedge \alpha$ has a kernel of dimension at least 3, because this kernel has to contain $\ker \alpha$. This is only possible if $\omega \wedge \alpha = 0$. The other direction is obvious. □

Elements of $\mathcal{B}_\alpha$ are called *α-boundaries.* We denote with $\mathcal{C}_T$ the compactly supported convex cone of currents associated with the kernel of $\alpha \wedge d\beta$. We are now ready to prove the main result.

Theorem 3.8.7. *A form $\beta \in A(\mathcal{W})$ such that $\alpha \wedge d\beta \neq 0$ is of Engel-type if and only if*

$$\langle b, \beta \rangle \neq 0 \tag{3.8}$$

for all non-zero foliation α-boundaries $b \in \mathcal{C}_T \cap \mathcal{B}_\alpha \setminus \{0\}$.

Proof. Suppose β is of Engel type, then we have $\beta' \in A(\mathcal{W})$ as above. This means in particular that $\beta' = \beta + \eta$ where η is α-closed in $A(\mathcal{W})$, because $d\beta \wedge \alpha = d\beta' \wedge \alpha$. For every such η and $b \in \mathcal{C}_T \cap \mathcal{B}_\alpha \setminus \{0\}$, let $c \in \mathcal{A}$ with $\delta c = b$, then

$$0 \neq \langle b, \beta' \rangle = \langle b, \beta + \eta \rangle = \langle b, \beta \rangle + \langle \delta c, \eta \rangle = \langle b, \beta \rangle + \langle c, d\eta \rangle = \langle b, \beta \rangle$$

where the first inequality follows by $\beta'(T) \neq 0$ and we used Lemma 3.8.6 to conclude $\langle c, d\eta \rangle = 0$ from the fact that η is α-closed.

Conversely suppose that (3.8) is verified and consider the hyperplane

$$H_\beta = \Big\{ b \in A^*(\mathcal{W}) \Big| \langle b, \beta \rangle = 0 \Big\}.$$

By hypothesis, the compact set $\bar{\mathcal{C}}_T \cap H_\beta$ does not intersect the closed subspace $\mathcal{B}_\alpha$, where $\bar{\mathcal{C}}_T$ is a compact base for $\mathcal{C}_T$. Using Hahn-Banach theorem, we get another hyperplane H which contains $\mathcal{B}_\alpha$ and does not intersect $\bar{\mathcal{C}}_T \cap H_\beta$.

Using reflexivity of $A(\mathcal{W})$, we identify a functional $\kappa : A^*(\mathcal{W}) \to \mathbb{R}$ whose kernel is exactly H with an element $\kappa \in A(\mathcal{W})$. Since $\mathcal{B}_\alpha \subset H$, using Lemma 3.8.6, we conclude that κ is α-closed. Consider the quotient $A^*(\mathcal{W})/H \cap H_\beta$, this is a 2 dimensional vector space whose points are separated by the maps β and κ. Moreover the image of $\bar{\mathcal{C}}_T$ is a compact subset which does not intersect 0, since $H \cap H_\beta \cap \mathcal{C}_T = \emptyset$ by construction. Therefore there is a linear combination $\lambda_1\beta + \lambda_2\kappa$ which is always positive on $\bar{\mathcal{C}}_T$. Since $\bar{\mathcal{C}}_T$ is compact we may assume that $\lambda_1 \neq 0$ so that we can choose $\eta = \lambda_2/\lambda_1\kappa$. This leads to $\beta' = \beta + \eta$ which satisfies $\langle b, \beta' \rangle \neq 0$ for all $b \in \mathcal{C}_T$ which is what we wanted. $\square$

One of the motivations for studying contact-type forms is to understand which hypersurfaces in a symplectic manifold come from contact structures, or equivalently which Hamiltonian vector fields are Reeb vector fields on hypersurfaces. In a similar fashion we can consider even contact hypersurfaces in contact 5-manifolds $(M, \alpha) \hookrightarrow (X, \eta)$ and suppose that $\beta = \eta|_M$ satisfies $\beta \in \mathcal{A}(W)$ and $\alpha \wedge d\beta \neq 0$. In this setting Theorem 3.8.7 gives a necessary and sufficient condition for the existence of an Engel structure $\mathcal{D} = \ker \alpha \wedge \beta'$ with $\alpha \wedge d\beta = \alpha \wedge d\beta'$. This might give some hints for understanding which hypersurfaces in a contact 5-manifold are Engel-type in the sense of Section 2.9.

Chapter 4

Holomorphic Engel structures

In this chapter we will present some results on holomorphic Engel structure which were obtained in a joint work with Rui Coelho and published in [CoP].

The object of study are the holomorphic analogue of Engel structures, i.e. maximally non-integrable holomorphic rank 2 distribution on complex 4-manifolds. For these structures Darboux type theorems still hold. The theory of holomorphic Engel structures on closed manifolds is very rigid and not much is known about open manifolds.

We construct infinite families of holomorphic Engel structures on $\mathbb{C}^4$ which are not biholomorphic to the standard one. The construction was inspired by the analogous result for holomorphic contact structures in [For1]. The idea is to consider the space of non-constant holomorphic maps $f : \mathbb{C} \to \mathbb{C}^4$ tangent to $\mathcal{D}$ at every point, these are called *$\mathcal{D}$-lines*. The fact that Engel structures induce flags of distributions $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$ allows to control the topology and the geometry of the space of points $p \in \mathbb{C}^4$ admitting such $\mathcal{D}$-lines. Exploiting this we can construct infinite families of examples.

4.1 Introduction and definitions

In this chapter X^n will denote a complex manifold of complex dimension n and $\mathcal{H}$ a holomorphic subbundle of TX of complex rank k. We consider the holomorphic analogues of the distributions we introduced in Chapter 1.

Definition 4.1.1. *Let $(X, \mathcal{H})$ be as above:*

- *if n is odd, $k = n - 1$ and $\mathcal{H}$ is locally defined as the kernel of a holomorphic 1-form α, satisfying $\alpha \wedge d\alpha^n \neq 0$, then $\mathcal{H}$ is a* holomorphic contact structure*;*

- *if n is even, $k = n - 1$ and $\mathcal{H}$ is locally defined as the kernel of a holomorphic* 1*-form α, satisfying $\alpha \wedge d\alpha^n \neq 0$, then $\mathcal{H}$ is a* holomorphic even contact structure*;*
- *if $n = 2$, $k = 4$ and $[\mathcal{H}, \mathcal{H}]$ is a holomorphic even contact structure, then $\mathcal{H}$ is a* holomorphic Engel structure*.*

The definitions given in the real case for the characteristic line field $\mathcal{W}$ and the Engel flag $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$ (Lemma 1.4.2) copy in the holomorphic case. The formulas defining the standard structures in the real case also work in the complex case.

Example 4.1.2 (Standard structures)*.* Consider $X = \mathbb{C}^{2n+1}$ with holomorphic coordinates $(x_1, y_1, ..., x_n, y_n, z)$ then the form

$$\eta_{st} = dz - \sum_{i=1}^{n} x_i dy_i$$

defines a holomorphic contact structure $\xi_{st} = \ker \eta_{st}$ called *standard contact structure.*

If $X = \mathbb{C}^{2n+2}$ with holomorphic coordinates $(w, x_1, y_1, ..., x_n, y_n, z)$ then the form

$$\alpha_{st} = dz - \sum_{i=1}^{n} x_i dy_i$$

defines a holomorphic even contact structure $\mathcal{E}_{st} = \ker \alpha_{st}$ called *standard holomorphic even contact structure.* Its characteristic foliation is $\mathcal{W} = \langle \partial_w \rangle$.

Finally if $X = \mathbb{C}^4$ with holomorphic coordinates (w, x, y, z) then the forms

$$\alpha_{st} = dy - zdx, \qquad \beta_{st} = dz - wdy$$

define a holomorphic Engel structure $\mathcal{D}_{st} = \ker \alpha_{st} \cap \ker \beta_{st}$ called *standard holomorphic Engel structure*, and whose characteristic foliation $\mathcal{W} = \langle \partial_w \rangle$.

One can prove Darboux-type theorems for each of these distributions. The study of closed holomorphic contact and holomorphic Engel structures is very rigid. One can define the prolongation of a holomorphic contact structure in analogy to what happens in the real case and similarly we have Lorentz prolongations. The following partial classification suggests that these should be the only possible holomorphic Engel structures on projective manifolds. Recall that a line bundle $L \to X$ is called *pseudo-effective* if its first Chern class $c_1(L)$ is contained in the cone of effective divisors in $H^{1,1}(X, \mathbb{R})$.

Theorem 4.1.3 [PrSC]**.** *Let X be a closed projective Engel manifold with its flag $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$. Then we only have two possibilities*

1. *$\mathcal{W}^{-1}$ is not pseudo-effective, in this case $(X, \mathcal{D})$ is the Cartan prolongation of a projective contact 3-manifold;*

2. *$(\mathcal{D}/\mathcal{W})^{-1}$ is not pseudo-effective. In this case, if we further assume that $\mathcal{D} \cong \mathcal{W} \oplus \mathcal{D}/\mathcal{W}$, then X is the Lorentz prolongation of a conformal structure on a complex 3-manifold.*

Moreover the two classes have a unique common element which is the universal family of lines contained in a quadric hypersurface in $\mathbb{CP}^4$.

4.2 Infinite families of examples

We are interested in the case of open manifolds.

Theorem 4.2.1. *For every $n \in \mathbb{N}$ there exists a holomorphic contact form η on $\mathbb{C}^{2n+1}$ such that any holomorphic map $f : \mathbb{C} \to \mathbb{C}^{2n+1}$ satisfying $f^*\eta = 0$ is constant. In particular the complex contact manifold $(\mathbb{C}^{2n+1}, \ker \eta)$ is not contactomorphic $(\mathbb{C}^{2n+1}, \ker \eta_{st})$.*

Notice that the standard contact structure on $\mathbb{C}^{2n+1}$ admits plenty of maps $f : \mathbb{C} \to \mathbb{C}^{2n+1}$ satisfying $f^*\eta_{st} = 0$ which are not constant. The proof consists in finding a Fatou-Bieberbach domain in a (directed) Kobayashi hyperbolic subset of $(\mathbb{C}^{2n+1}, \eta_{st})$. We need to introduce these some standard notions (see [For1]).

Let X be a complex manifold X and $\mathcal{H} \subset TX$ a holomorphic distribution, we say that a holomorphic map from the disc $f : D \to X$ is *horizontal*, if it is tangent to $\mathcal{H}$. We denote such maps by $f : (D, T_D) \to (X, \mathcal{H})$. For a fixed $p \in X$ we define the *Finsler pseudo-length* of a vector $v_p \in \mathcal{H}_p$ as follows

$$F_{\mathcal{H}}(v_p) = \inf \left\{ \frac{1}{|\lambda|} \Big| \exists f : (D, T_D) \to (X, \mathcal{H}),\ f(0) = p,\ f'(0) = \lambda v_p \right\}. \quad (4.1)$$

Denote with $\Omega_{\mathcal{H}}(p, q)$ the space of curves $\gamma : I \to X$ which are piecewise smooth tangent to $\mathcal{H}$ and $\gamma(0) = p$, $\gamma(1) = q$. Using this we can define the *Kobayashi directed pseudo-length* between two point $p, q \in X$ as follows

$$d_{\mathcal{H}}(p, q) = \inf \left\{ \int_0^1 F_{\mathcal{H}}(\gamma'(t))dt \Big| \gamma \in \Omega_{\mathcal{H}}(p, q) \right\}. \quad (4.2)$$

Notice that in the case $\mathcal{H} = TX$ these coincide with the classical notions of Finsler pseudo-length and Kobayashi pseudo-distance (see [Kob1, Kob2]). We say that $(X, \mathcal{H})$ is Kobayashi hyperbolic if $d_{\mathcal{H}}$ is a metric.

Definition 4.2.2. *Let $(X, \mathcal{H})$ be as above, a holomorphic map $f : \mathbb{C} \to X$ is a $\mathcal{H}$-*line *at $p \in X$ or a* horizontal line *at p if $f(0) = p$, f is not constant and it is tangent to $\mathcal{H}$ at every point.*

Remark 4.2.3. If a given $(X, \mathcal{H})$ admits a $\mathcal{H}$-line at some point $p \in X$ then the Finsler pseudo-length of $v_p = f'(0) \in \mathcal{H}_p$ must be zero. Indeed for any $\lambda \in \mathbb{R}$ we have that $f_\lambda(z) = f(\lambda z)$ is a holomorphic horizontal map which satisfies $f(0) = p$ and $f'(0) = \lambda v_p$. By (4.1) this means exactly $F_{\mathcal{H}}(v_p) = 0$.

The idea in Forstnerič's proof of Theorem 4.2.1 is to construct a proper subset $U \subset \mathbb{C}^{2n+1}$ such that $(U, \ker \eta_{st}|_U)$ is Kobayashi hyperbolic. Then he finds a subset $\Omega \subset U$ which is biholomorphic to $\mathbb{C}^{2n+1}$ via $\Phi : \mathbb{C}^{2n+1} \to \Omega$. Finally he pulls back $\eta_{st}|_U$ via Φ to obtain the contact form η, whose kernel is then Kobayashi hyperbolic on $\mathbb{C}^{2n+1}$. A proper subset $\Omega \subset \mathbb{C}^n$ which is biholomorphic to $\mathbb{C}^n$ is called *Fatou-Bieberbach domain.*

Our results in the Engel case were inspired by Theorem 4.2.1. Here we have a greater flexibility given by the existence of the flag $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$.

Theorem 4.2.4. *On $\mathbb{C}^4$ there are Engel structures $\mathcal{D}_\mathcal{E}$, $\mathcal{D}_\mathcal{D}$ and $\mathcal{D}_\mathcal{W}$ with the following properties*

1. *$\mathcal{D}_\mathcal{E}$ admits no lines tangent to its induced even contact structure;*
2. *$\mathcal{D}_\mathcal{D}$ admits no $\mathcal{D}_\mathcal{D}$-lines but does admit lines tangent to its induced even contact structure;*
3. *$\mathcal{D}_\mathcal{W}$ admits no lines tangent to its characteristic foliation but does admit $\mathcal{D}_\mathcal{W}$-lines.*

In particular these Engel structures are pairwise non-isomorphic and not isomorphic to the standard Engel structure $(\mathbb{C}^4, \mathcal{D}_{st})$.

The standard Engel structure admits many $\mathcal{D}_{st}$-lines (see Example 4.3.1), including many tangent to the characteristic foliation. Controlling the geometry of the characteristic foliation, we are able to construct infinite families of non-isomorphic holomorphic Engel structures.

Theorem 4.2.5. *For every $n \in \mathbb{N} \cup \{\infty\}$ there exists an Engel structure $\mathcal{D}_n$ on $\mathbb{C}^4$ for which the only $\mathcal{D}_n$-lines are tangent to the characteristic foliation $\mathcal{W}_n$, and such that*

$$L_n := \left\{ p \in \mathbb{C}^4 \middle| \exists f : \mathbb{C} \to \mathbb{C}^4 \ \mathcal{D}_n\text{-line with } f(0) = p \right\}$$

is a proper subset of $\mathbb{C}^4$ which has exactly n connected components for $n \in \mathbb{N}$, and $L_\infty = \mathbb{C}^4$.

We first construct $\mathcal{D}_\infty$ using an open set in the Cartan prolongation of a Kobayashi hyperbolic contact structure in $\mathbb{C}^3$. This will admit very few $\mathcal{D}_\infty$-lines by construction. Then we use a result, due to Buzzard and Fornæss that allows one to control the set of points in $\mathbb{C}^4$ which admit such horizontal lines. A more careful analysis leads to the following result.

Theorem 4.2.6. *For every $R \in \mathbb{R} \setminus \{0\}$ there exists an Engel structure $\mathcal{D}_R$ for which the only $\mathcal{D}_R$-lines are tangent to the characteristic foliation $\mathcal{W}_R$, and such that the set of points which admit such $\mathcal{W}_R$-lines is exactly $\mathbb{C} \times \{0, 1, R\sqrt{-1}\} \times \mathbb{C}^2 \subset \mathbb{C}^4_{(w,x,y,z)}$. Moreover $\mathcal{D}_R$ is isomorphic to $\mathcal{D}_{R'}$ if and only if $R = R'$.*

4.3 Technical lemmata

Let us first see that the standard contact structure and Engel structure admit many tangent lines.

Example 4.3.1. Given a point $p = (\bar{x}_1, \bar{y}_1, ..., \bar{x}_n, \bar{y}_n, \bar{z})$ in $(\mathbb{C}^{2n+1}, \ker \alpha_{st})$ and a vector $v = (u_1, v_1, ..., v_n, u_n, w) \in \xi_p$ the map $f : \mathbb{C} \to \mathbb{C}^{2n+1}$ given by $f(\zeta) = (x_1(\zeta), y_1(\zeta), ..., x_n(\zeta), y_n(\zeta), z(\zeta))$ with

$$x_i(\zeta) = \bar{x}_i + u_i\zeta, \quad y_i(\zeta) = \bar{y}_i + v_i\zeta, \quad z(\zeta) = \bar{z} + \sum_{i=1}^{n} \left(\bar{x}_i v_i \zeta + u_i v_i \frac{\zeta^2}{2} \right)$$

is a horizontal line with $f(0) = p$ and $f'(0) = v$.

The leaves of the characteristic foliation $\mathcal{W}$ of $\mathcal{D}_{st}$ provide examples of lines tangent to the standard Engel structure. In fact, given a point $p = (w_0, x_0, y_0, z_0) \in \mathbb{C}^4$ and a vector $v = (v_w, v_x, v_y, v_z) \in \mathcal{D}_p$ (hence $v_z = w_0 v_x$ and $v_w = z_0 v_x$) the map $f : \mathbb{C} \to \mathbb{C}^4$ such that $f(\zeta)$ is give by

$$\left(w_0 + v_w\zeta,\, x_0 + v_x\zeta,\, y_0 + v_y\zeta + v_x v_z \frac{\zeta^2}{2} + v_x^2 v_w \frac{\zeta^3}{6},\, z_0 + v_z\zeta + v_x v_w \frac{\zeta^2}{2} \right)$$

is a horizontal line with $f(0) = p$ and $f'(0) = v$.

Following [For1] we let $\{c_i\}_{i\in\mathbb{N}}$ be a positive diverging monotonic sequence. Denote with D_z the unit disc in the z-direction, with $\overline{D}_z$ the closed unit disc and with $D^{2n}_{(x,y)} \subset \mathbb{C}^{2n}$ the unit polydisc in the (x, y)-space and with $\partial D^{2n}_{(x,y)}$ its boundary. Let

$$K = \bigcup_{i=1}^{\infty} 2^{i-1} \partial D^{2n}_{(x,y)} \times c_i \overline{D}_z. \tag{4.3}$$

Lemma 4.3.2 [For1]**.** *Assume $c_i \geq 2^{3i+1}$ for all $i \in \mathbb{N}$ and let K be given by (4.3). For every holomorphic disc $f : D \to \mathbb{C}^{2n+1}$ which is tangent to the standard holomorphic contact structure with $f(0) \in 2^{N_0} D^{2n+1}$ for some $N_0 \in \mathbb{N}$ we have*

$$|x_i'(0)| < 2^{N_0+1}, \quad |y_i'(0)| < 2^{N_0+1}, \quad |z'(0)| < 2^{2N_0+1} \tag{4.4}$$

for all $i = 1, ..., n$.

This is Lemma 2.1 in [For1] and we will give a proof of the analogue result in the Engel case (Lemma 4.3.3). The estimates (4.4) give a lower bound for the Finsler pseudo-length of any vector tangent to the standard contact structure restricted to $\mathbb{C}^{2n+1} \setminus K$. This means that the Kobayashi pseudo-metric is in fact positive definite and hence $\ker \eta_{st}|_{\mathbb{C}^{2n+1}\setminus K}$ is Kobayashi hyperbolic.

Now consider $\mathbb{C}^4$ with the standard Engel structure. let $\{c_i\}_{i\in\mathbb{N}}$ $\{d_i\}_{i\in\mathbb{N}}$ and $\{e_i\}_{i\in\mathbb{N}}$ be positive diverging monotonic sequences. With the same notation as before let

$$A = \bigcup_{i=1}^{\infty} 2^{i-1}\partial D^3_{(w,x,z)} \times c_i\overline{D}_y. \tag{4.5}$$

$$B = \bigcup_{i=1}^{\infty} 2^{i-1}\partial D^2_{(w,x)} \times d_i\overline{D}_y \times e_i\overline{D}_z. \tag{4.6}$$

By a direct adaptation of Lemma 2.1 in [For1], we can prove the following result.

Lemma 4.3.3. *Assume $d_i \geq 2^{5i+2}$ and $e_i \geq 2^{3i+1}$ for every $i \in \mathbb{N}$. Let $N_0 \in \mathbb{N}$ and $f : D \to \mathbb{C}^4 \setminus B$ be a $\mathcal{D}_{st}$-horizontal embedding of a disc with $f(0) \in 2^{N_0}D^4$. Then we have the estimates*

$$|w'(0)| < 2^{N_0+1},\ |x'(0)| < 2^{N_0+1},\ |y'(0)| < 2^{3N_0+2},\ |z'(0)| < 2^{2N_0+1}.$$

Proof. We may assume without loss of generality that f is holomorphic on $\overline{D}$ (otherwise replace f by $\zeta \mapsto f(r\zeta)$ for some $r < 1$). This gives $N \in \mathbb{N}$ such that $|x(\zeta)| < 2^N$ and $|w(\zeta)| < 2^N$ for all $\zeta \in \overline{D}$. The Cauchy integral formula for a circle centred at $\zeta = 0$ of ray $r = 1 - 2^{-N}$ gives

$$|x'(\zeta)| < 2^{2N} \quad \text{and} \quad |w(\zeta)x'(\zeta)| < 2^{3N}$$

for $|\zeta| \leq r$. Since f is horizontal, we have the conditions

$$y'(\zeta) = z(\zeta)x'(\zeta) \quad \text{and} \quad z'(\zeta) = w(\zeta)x'(\zeta), \tag{4.7}$$

which in turn give

$$|z(\zeta)| \leq |z(0)| + \left|\int_0^\zeta w dx\right| < 2^{N_0} + 2^{3N} < 2^{3N+1} \leq d_N$$
$$|y(\zeta)| \leq |y(0)| + \left|\int_0^\zeta z dx\right| < 2^{N_0} + 2^{5N+1} < 2^{5N+2} \leq c_N$$

for $|\zeta| \leq r$. From these estimates, the definition of B, and the fact that $f(D)$ does not intersect B, it follows that $(w(\zeta), x(\zeta))$ does not intersect $2^{N-1}\partial D^2$ for $|\zeta| \leq r$. Since $2^{N-1}\partial D^2$ disconnects $2^N D^2$ and $(w(0), x(0)) \in 2^{N_0}D^2 \subset 2^{N-1}D^2$, we conclude that

$$(w(\zeta), x(\zeta)) \in 2^{N-1}D^2 \quad \text{for} \quad |\zeta| \leq 1 - 2^{-N}.$$

If $N - 1 > N_0$, we can repeat the same argument to get

$$(w(\zeta), x(\zeta)) \in 2^{N-2}D^2 \quad \text{for} \quad |\zeta| \leq 1 - 2^{-N} - 2^{-(N-1)},$$

and after finitely many repetitions

$$(w(\zeta), x(\zeta)) \in 2^{N_0} D^2 \quad \text{for} \quad |\zeta| \le 1 - 2^{-N} - \ldots - 2^{-(N_0+1)} \le \frac{1}{2}.$$

Applying the Cauchy estimate now gives $|x'(0)| \le 2^{N_0+1}$ and $|w'(0)| \le 2^{N_0+1}$, while using equation (4.7) we get

$$|z'(0)| = |w(0)x'(0)| \le 2^{2N_0+1} \quad \text{and} \quad |y'(0)| = |z(0)x'(0)| \le 2^{3N_0+2},$$

completing the proof of the lemma. □

The following lemma has a completely analogous proof.

Lemma 4.3.4. *Assume $c_i \ge 2^{3i+1}$ for every $i \in \mathbb{N}$. Let $N_0 \in \mathbb{N}$ and $f : D \to \mathbb{C}^4 \setminus A$ be a $\mathcal{E}_{st}$-horizontal embedding of a disc with $f(0) \in 2^{N_0} D^4$. Then we have the estimates*

$$|w'(0)| < 2^{N_0+1},\ |x'(0)| < 2^{N_0+1},\ |y'(0)| < 2^{2N_0+1},\ |z'(0)| < 2^{N_0+1}.$$

Lemma 4.3.3 and Lemma 4.3.4 furnish the proper subsets of $\mathbb{C}^4$ on which $\mathcal{D}_{st}$ and $\mathcal{E}_{st}$ are Kobayashi hyperbolic. In order to find Fatou-Bieberbach domains inside these subsets we will need the following result, which was proven by Forstnerič.

Theorem 4.3.5 [For1]**.** *Let $0 < a_1 < b_1 < a_2 < b_2 < \ldots$ and $c_i > 0$ be sequences of real numbers such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = +\infty$. Let $n > 1$ be an integer and*

$$K = \bigcup_{i=1}^{\infty} \left(b_i \overline{D}^{n-1} \setminus a_i D^{n-1} \right) \times c_i \overline{D} \subset \mathbb{C}^n. \tag{4.8}$$

Then there exists a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^n \setminus K$.

4.4 Proof of the results

4.4.1 Proof of Theorem 4.2.4

In what follows fix $0 < \varepsilon < 1$ and consider the real sequences

$$a_i = 2^{i-1} - \varepsilon \qquad \text{and} \qquad b_i = 2^{i-1} + \varepsilon.$$

To construct $\mathcal{D}_{\mathcal{E}}$ we fix $c_i = 2^{3i+1}$ and let A be the set determined by c_i according to (4.5). Lemma 4.3.4 ensures that $(\mathbb{C}^4 \setminus A, \mathcal{E}_{st})$ is hyperbolic, moreover Lemma 4.3.5 gives a Fatou-Bieberbach map $\Phi : \mathbb{C}^4 \to \Omega \subset \mathbb{C}^4 \setminus A$. We set $\mathcal{D}_{\mathcal{E}} := \Phi^* \mathcal{D}_{st}$ so that its induced even contact structure is $\Phi^* \mathcal{E}_{st}$. Lemma 4.3.4 furnishes a lower bound for the Finsler metric It follows that the $\Phi^*\mathcal{E}_{st}$-directed Kobayashi pseudo-distance on Ω is a genuine distance, i.e. the restriction of the standard even contact structure to Ω is hyperbolic.

To construct $\mathcal{D}_{\mathcal{D}}$ we fix $d_i = 2^{5i+2}$ and $e_i = 2^{3i+1}$ and let K be the set determined by $n = 3, a_i, b_i$ and $c_i = d_i$ according to (4.3). Let B be the set determined by d_i and e_i according to (4.6), and notice that $B \subset K \times \mathbb{C}$. By Lemma 4.3.5 there exists a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^3 \setminus K$. Define $\Xi = \Omega \times \mathbb{C}$. The subset $\Xi \subset \mathbb{C}^4$ is a Fatou-Bieberbach domain in $\mathbb{C}^4$ which fulfils $\Xi \cap (K \times \mathbb{C}) = \emptyset$; in particular, $\Xi \cap B = \emptyset$. Let $\Phi : \mathbb{C}^4 \to \Xi$ be the Fatou-Bieberbach map. We define $\mathcal{D}_{\mathcal{D}} = \Phi^*(\mathcal{D}_{st})$. Lemma 4.3.3 furnishes a lower bound for the Finsler metric. Again it follows that the $\mathcal{D}_{st}$-directed Kobayashi pseudo-distance on Ξ is a genuine distance, i.e. the restriction of the standard Engel structure to Ξ is hyperbolic. Notice that in this construction the induced even contact structure $\mathcal{E}_{\mathcal{D}}$ is not hyperbolic. Indeed we have many $\mathcal{E}_{st}$-lines $f : \mathbb{C} \to \Xi$ of the form

$$f(\zeta) = (w_0, x_0, y_0, \zeta)$$

where (w_0, x_0, y_0) is not contained in A, which can be pulled-back via Φ.

To construct $\mathcal{D}_{\mathcal{W}}$ consider the set

$$K = \bigcup_{i=1}^{\infty} 2^{i-1}\partial D^2_{(w,y)} \times 2^i \overline{D}_z$$

contained in the (w, y, z)-plane in $\mathbb{C}^4$. All $\mathcal{W}$-horizontal holomorphic copies of $\mathbb{C}$ are of the form $f(\zeta) = (w(\zeta), x_0, y_0, z_0)$ for some w holomorphic and hence they will intersect K for some ζ. Indeed if $N_0 \in \mathbb{N}$ is such that $|z_0| < d_{N_0}$ then f does not intersect K only if $|w(\zeta)| < 2^{N_0-1}$ for all $\zeta \in \mathbb{C}$, which is not true. Theorem 4.3.5 ensures the existence of a Fatou-Bieberbach map $\tilde{\Phi} : \mathbb{C}^3 \to \Omega \subset \mathbb{C}^3 \setminus K$ so that also $\Phi = \tilde{\Phi} \times id : \mathbb{C}^4 \to \Omega \times \mathbb{C} \subset \mathbb{C}^4$ is a Fatou-Bieberbach map. By the above discussion there are no copies of $\mathbb{C}$ tangent to the characteristic foliation of the standard Engel structure restricted to Ω. We then define $\mathcal{D}_{\mathcal{W}} := \Phi^*\mathcal{D}_{st}$, this structure does not have lines tangent to the characteristic foliation, nevertheless $\mathbb{C}^4$ is not $\mathcal{D}_{\mathcal{W}}$-hyperbolic, since the pull-back of the $\mathcal{D}_{st}$-line

$$f : \mathbb{C} \hookrightarrow \mathbb{C}^4 \qquad \text{s.t.} \qquad f(\zeta) = (0, \zeta, 0, 0)$$

is a $\mathcal{D}_{\mathcal{W}}$-line.

4.4.2 Proof of Theorem 4.2.5

We use Forstnerič's hyperbolic contact structure on $\mathbb{C}^3$, which is the pull-back $\eta = \Phi^*\eta_{st}$ of the restriction of the standard contact structure on a hyperbolic Fatou-Bieberbach domain in $\mathbb{C}^3 \setminus K$. Consider the Cartan prolongation $M = \mathbb{P}(\xi_h)$ of $\xi_h = \ker \eta$ with its Engel structure $\mathcal{D}(\xi_h)$. Since

$\ker \eta_{st}$ is trivial as a holomorphic bundle, M is biholomorphic to $\mathbb{C}^3 \times \mathbb{CP}^1$. Given $p \in \mathbb{CP}^1$, consider in M the open set $\mathbb{C}^4 = \mathbb{C}^3 \times \mathbb{CP}^1 \setminus (\mathbb{C}^3 \times \{p\})$ and the restriction of the Engel structure $\mathcal{D}_\infty = \mathcal{D}(\xi_h)|_{\mathbb{C}^4}$. We claim that this structure has the properties stated in Theorem 4.2.5.

Indeed suppose that $f : \mathbb{C} \to \mathbb{C}^4$ is a $\mathcal{D}_\infty$-line. Then if we denote by $\pi : M \to \mathbb{C}^3$ the canonical projection of the projectivization, the composition $\pi \circ f$ is tangent to ξ_h in $\mathbb{C}^3$. Since $(\mathbb{C}^3, \xi_h)$ is hyperbolic, $\pi \circ f$ must be constant, so f is tangent to the fibres. This proves that the only $\mathcal{D}_\infty$-lines are tangent to the characteristic foliation $\mathcal{W}_\infty$.

Fix $n \in \mathbb{N}$. In order to construct $\mathcal{D}_n$, we use the following result

Theorem 4.4.1 [BuFo]**.** *Let L be a closed, 1-dimensional, complex subvariety of $\mathbb{C}^2$, and B_0 a ball with $\overline{B}_0 \cap L = \emptyset$. Then there exists a Fatou-Bieberbach domain $\Omega \subset \mathbb{C}^2 \setminus \overline{B}_0$ with $L \subset \Omega$ and a biholomorphic map Ψ from Ω onto $\mathbb{C}^2$ such that $\mathbb{C}^2 \setminus \Psi(L)$ is Kobayashi hyperbolic. Moreover, all non-constant images of $\mathbb{C}$ in $\mathbb{C}^2$ intersect $\Psi(L)$ in infinitely many points.*

Now we choose

$$\tilde{L}_n = \bigcup_{k=1}^{n} \mathbb{C} \times \{k\} \subset \mathbb{C}^2_{(w,x)} .$$

Theorem 4.4.1 gives a Fatou-Bieberbach map $\Phi_n : \mathbb{C}^2 \to \Omega_n \subset \mathbb{C}^2$ such that $\Omega_n \setminus \tilde{L}_n$ is Kobayashi hyperbolic and the w-curves $f_i : \mathbb{C} \to \mathbb{C}^2$ s.t. $\zeta \mapsto (\zeta, i)$ are still contained in Ω_n. Now take the Fatou-Bieberbach map $\Psi_n = \Phi_n \times id : \mathbb{C}^4 \to \Omega_n \times \mathbb{C}^2 \subset \mathbb{C}^4$ and the Engel structure $\mathcal{D}_n = \Psi_n^* \mathcal{D}_\infty$. By construction $\mathcal{D}_n$ only admits $\mathcal{D}_n$-lines on the points

$$L_n = \tilde{L}_n \times \mathbb{C}^2 = \Big\{(w,x,y,z) \in \mathbb{C}^4 \Big| \, x \in \{1,...,n\}\Big\}$$

hence completing the proof of Theorem 4.2.5.

4.4.3 Proof of Theorem 4.2.6

For some $R \in \mathbb{R} \setminus \{0\}$, we will consider the subvariety

$$C_R = \Big(\mathbb{C} \times \{0, 1, R\sqrt{-1}\}\Big) \cup (\{0\} \times \mathbb{C}) \subset \mathbb{C}^2.$$

By Theorem 4.4.1, there exists a Fatou-Bieberbach domain $\Omega_R \subset \mathbb{C}^2$ which contains C_R, and such that the complement $\Omega_R \setminus C_R$ is Kobayashi hyperbolic. Moreover, any curve $\mathbb{C} \to \Omega_R$ intersects C_R an infinite number of times. Denote by $\mathcal{W}_R$, resp. $\mathcal{W}_{R'}$, the 1-foliation on $\Omega_R \times \mathbb{C}^2$, resp. $\Omega'_R \times \mathbb{C}^2$, determined by the projections $p : \Omega_R \times \mathbb{C}^2 \to \mathbb{C}^3$, resp. $p' : \Omega_{R'} \times \mathbb{C}^2 \to \mathbb{C}^3$, given by $(w,x,y,z) \mapsto (x,y,z)$. We introduce also the projections $\pi : \Omega_R \times \mathbb{C}^2 \to \mathbb{C}$ and $\pi' : \Omega_{R'} \times \mathbb{C}^2 \to \mathbb{C}$ given by $(w,x,y,z) \mapsto x$ and the notation $V_R = \pi^{-1}\{0, 1, R\sqrt{-1}\}$ and $V'_{R'} = {\pi'}^{-1}\{0, 1, R'\sqrt{-1}\}$. Notice that V_R (resp. $V'_{R'}$) consists exactly of the points of Ω_R (resp. $\Omega_{R'}$) through which a $\mathcal{W}_R$-line (resp. $\mathcal{W}_{R'}$-line) passes.

Lemma 4.4.2. *Suppose that $R, R' \in \mathbf{R} \setminus \{0\}$ and $R \neq R'$. Then there exists no biholomorphic map $\Phi : \Omega_R \times \mathbb{C}^2 \to \Omega_{R'} \times \mathbb{C}^2$ such that $\Phi_*(\mathcal{W}_R) = \mathcal{W}_{R'}$.*

Proof. Suppose such a Φ exists and consider the map $h : \mathbb{C} \to \mathbb{C}$ given by $h = \pi' \circ \Phi \circ \iota$, where ι is the inclusion $\iota(\zeta) = (0, \zeta, 0, 0) \in \Omega_R \times \mathbb{C}^2$. Notice that horizontal curves in $\mathcal{W}_R$ must map to horizontal curves in $\mathcal{W}_{R'}$. Moreover, we have $h^{-1}\{0, 1, R'\sqrt{-1}\} = \{0, 1, R\sqrt{-1}\}$. It follows that we have a biholomorphic map $\Phi|_{V_R} : V_R \to V'_{R'}$. This implies in particular that the restriction $h : \{0, 1, R\sqrt{-1}\} \to \{0, 1, R'\sqrt{-1}\}$ is bijective. Since h is non-constant, it either has an essential singularity or a pole at infinity.

If h has an essential singularity at infinity, then by the big Picard theorem h takes every value in $\mathbb{C}$ infinitely many times, with one possible exception. This contradicts the fact that $h^{-1}\{0, 1, R'\sqrt{-1}\} = \{0, 1, R\sqrt{-1}\}$.

Otherwise, h is a polynomial with exactly one zero, so it must be linear. On the other hand, $h(\{0, 1, R\sqrt{-1}\}) = \{0, 1, R'\sqrt{-1}\}$, which is impossible for $R \neq R'$. □

Now given the Fatou-Bieberbach map $\Phi_R : \mathbb{C}^4 \to \Omega_R \times \mathbb{C}^2 \subset \mathbb{C}^4$ we define $\mathcal{D}_R := \Phi_R^* \mathcal{D}_{st}$ and Theorem 4.2.6 is a direct consequence of Lemma 4.4.2.

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