A Novel Approach for Constraint Transformation in Petri nets with Uncontrollable Transitions

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***Abstract*—The main contribution of this paper consists in a linear algebraic characterization of the admissible marking set relative to a Petri net with uncontrollable transitions, subject to a linear constraint. In more detail, given a linear constraint that limits the number of tokens in one place, an algorithm is proposed to compute an approximation of the admissible marking set in terms of a disjunction of transformed linear constraints. The optimality of the solution is guaranteed provided that certain conditions are satisfied during the intermediate steps of the iterative approach. In all the other cases the set of markings described by the transformed constraints could be surely contained in the admissible marking set.**

***Index Terms*—Discrete event systems, Petri nets, linear constraints, equivalent transformation.**

I. Introduction

In the last decades the discrete event system (DES) community devoted a lot of efforts to the problem of preventing a DES from reaching some forbidden states [1]-[25], [27]-[33]. Very efficient solutions have been proposed assuming Petri nets (PNs) as the reference model [12], [15], [25]. In particular, it has been proven that, when the set of legal states is expressed in terms of Generalized Mutual Exclusion Constraints (GMEC) it is easy to impose the satisfaction of the constraints by simply adding monitor places. Furthermore, maximal permissiveness (i.e., optimality) of the closed-loop system behavior is guaranteed if all transitions are controllable and observable. On the contrary, when some of the transitions are uncontrollable, it is in general necessary to also forbid some legal markings. Indeed, it could occur that an illegal marking can be reached by a legal one firing an uncontrollable transition, so such a legal marking should be forbidden as well. As a result, even if all transitions are assumed to be observable, the control problem becomes challenging and the main goal becomes that of characterizing (possibly in linear algebraic terms) a set of markings called *admissible marking set*. The admissible marking set is the largest subset of legal markings from which a forbidden marking can never be reached by firing uncontrollable transitions only. The theory of monitor places has been extended in a very elegant and efficient way to handle the problem of imposing linear constraints on PNs with uncontrollable transitions [1], [12], [16], [25]. However, the maximal permissiveness is not guaranteed, i.e., the set of reachable markings of the controlled system is usually a *strict* subset of the admissible marking set.

Recently, great attention has been devoted to the problem of linear constraint transformation [21]-[24], [27]-[29], [31]-[33]. In the case of arbitrary uncontrollable subnets, Luo *et al*. [24] and Wang *et al.* [33] propose approaches to perform the transformation of a GMEC with positive weights. Unfortunately, their approaches do not always lead to an optimal solution [33].

To guarantee optimality of the solution, much works deal with special structures of the uncontrollable subnet [21], [23], [27]-[29], [31], [32] and/or special structures of the constraint(s) [27]-[29]. As an example, an equivalent transformation of a GMEC with positive weights has been provided in [21] and [23] under the assumption that the uncontrollable subnet is *forward concurrent free* (i.e., no transition has two or more input places). The works in [31], [32] deal with uncontrollable subnets that are subclasses of *forward concurrent free nets*. Luo *et al.* in [22] propose a method that simultaneously reduces PNs and linear constraints, which can simplify the control problem in the case of PNs with uncontrollable transitions.

Note that the problem of computing an optimal transformation in the case of uncontrollable subnets that contain both structures of *forward-concurrent* (i.e., a transition has two or more input places) and *forward-conflict* (i.e., a place has two or more output transitions) has not been solved yet. Ma *et al*. [27]-[29] recently studied the constraint transformation in such uncontrollable subnets, which are named uncontrollable subnets with conflicts and synchronizations in [29]. In more detail, the work in [29] shows that the constraint transformation, under such assumption on the uncontrollable subnet, may suffer the "GMEC inflation phenomenon", i.e., the admissible

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marking set cannot be characterized as a finite disjunction of linear constraints thus it may be too complicated to efficiently implement it in a closed-loop form. To avoid this phenomenon, they focus on uncontrollable subnets under more restrictive assumptions and on constraints with a very special structure. In such a case, they propose an algorithm that provides an optimal solution as a disjunction of linear constraints [28], [29]. More specifically, they assume that the constraint imposes an upper bound on the number of tokens in one place only and the uncontrollable subnet is acyclic and *backward conflict free* (BCF), i.e, it has no circuit and each place has at most one input transition. The work in [27] focuses on a subclass of such nets called *assembly flow systems*.

In this work (which is the journal version of [34]), as in [27]-[29], we focus on constraints imposing an upper bound on the number of tokens in one place. Note that the uncontrollable subnets that we handle are more general. In particular, it is only required that the limited place does not belong to any circuit of the uncontrollable subnet. Obviously, the nets we consider may contain both structures of forward-concurrent and forward-conflict. Besides, the proposed approach leads to an optimal solution provided that certain conditions are satisfied during the intermediate steps of the iterative approach. Moreover, it guarantees the optimality of the solution if the uncontrollable subnet is acyclic and BCF as in [28], [29]. In other words, the proposed approach guarantees the optimality of the solution in all those cases where optimality is guaranteed using the approaches in [27]-[29] plus other cases where the approaches in [27]-[29] cannot be applied. Furthermore, the proposed approach also offers an optimal transformation when dealing with cases, for which the approaches in [24], [33] fail.

The remainder of this paper is organized as follows. Section II provides some background on PNs and recalls the notion of admissible marking set. Section III proposes the novel constraint transformation approach. Examples are presented in Section IV. Conclusions are drawn in Section V where we also discuss our future lines of research in this framework.

II. Preliminaries

A. Petri nets

An *ordinary Petri net* (PN) [26] is a 3-tuple $N=(P, T, F)$ where P is the set of *places*, T is the set of *transitions*, and F is called the *flow relation*. P and T are non-empty, finite, and disjoint sets, and $F \subseteq (P \times T) \cup (T \times P)$. Graphically, places and transitions are represented by circles and bars, respectively, and F is a set of directed arcs connecting places and transitions. Given a node $x \in P \cup T$, ${}^\bullet x=\{y \in P \cup T \mid (y, x) \in F\}$ is the set of *inputs* of x, while $x^\bullet=\{y \in P \cup T \mid (x, y) \in F\}$ is the set of *outputs* of x. Furthermore, $\forall X \subseteq P \cup T$, ${}^\bullet X=\bigcup_{x \in X} {}^\bullet x$, and $X^\bullet = \bigcup_{x \in X} x^\bullet$.

The *incidence matrix* of N is $[N]$: $P \times T \rightarrow \{-1, 0, 1\}$ indexed by P and T such that $[N](p, t)=1$ if $p \in t^\bullet \setminus {}^\bullet t$; $[N](p, t)= -1$ if $p \in {}^\bullet t \setminus t^\bullet$; otherwise $[N](p, t)=0$, $\forall p \in P$ and $\forall t \in T$.

A *marking* or *state* of a PN N is a vector m: $P \rightarrow \boldsymbol{N}$ where $\boldsymbol{N}=\{0, 1, 2, \ldots\}$. Generally, m is also denoted by the multi-set notation $\sum_{p \in P} m(p)p$, where $m(p)$ is the number of tokens in place p at m. For instance, $m=[1, 0, 3, 0]^T$ is denoted by $m=p_1+3p_3$. The initial marking of a PN is denoted as m_0 and (N, m_0) is called a net system with initial marking m_0.

A transition t is *enabled* at a marking m, denoted as $m[t\rangle$, if $\forall p \in {}^\bullet t$, $m(p)>0$. t can fire at m if it is enabled at m. If t fires at m, reaching a marking m', we denote this as $m[t\rangle m'$, where $m'(p)=m(p)+[N](p, t)$, $\forall p \in P$. Furthermore, given a sequence of transitions $\alpha=t_{i1}t_{i2}...t_{ik}$, $t_{ij} \in T$, $j=1, 2, \ldots, k$, we say α is enabled at m, denoted as $m[\alpha\rangle$, if $m[t_{i1}\rangle m_1[t_{i2}\rangle m_2[t_{i3}\rangle \ldots m_{k-1}[t_{ik}\rangle$. We denote as $m[\alpha\rangle m_k$ if m_k is reached by firing α at m. We use $R(N, m_0)$ to denote the set of all reachable markings of N from m_0.

A transition is called *controllable* if it can be prevented from firing by a supervisory policy and otherwise it is called *uncontrollable*. The transition set T is accordingly partitioned into two disjoint subsets: T_u is the set of uncontrollable transitions, and T_c is the set of controllable transitions.

A string $x_1x_2\ldots x_n$ is called a *path* of N if $x_{i+1} \in x_i^\bullet$ for all $i=1, \ldots, n-1$, and $x_i \in P \cup T$ for all $i=1, \ldots, n$. A path $x_1x_2\ldots x_n$ is called a *circuit* if $x_1=x_n$. An *uncontrollable path* is a path in which each transition is uncontrollable.

We use $R(N, m_0, u)$ to denote the set of reachable markings in (N, m_0) under the supervision of a policy u. A policy that disables all controllable transitions is called the *least permissive* one, denoted as u_{zero}. Thus, $R(N, m_0, u_{zero})$ is the set of reachable markings in (N, m_0) with all controllable transitions being disabled, i.e., it consists of m_0 and all markings reachable from m_0 by firing uncontrollable transitions. Clearly, $R(N, m_0, u_{zero}) \subseteq R(N, m_0, u)$ for any control policy u. Besides, $R_t(N, m)$ denotes the set of markings (including m) in N reachable from m by firing t once or multiple times.

B. Linear constraints

Using standard notation in the PN literature [15], a *linear constraint* on the marking m of a PN is denoted as (ω, k), i.e., $\omega \cdot m \leq k$, where ω is a weight vector from P to $\boldsymbol{N}$ and k is an integer. The *legal marking set* of (ω, k) is

$$\mathcal{L}_{(\omega,k)} = \{m \in \boldsymbol{N}^{|P|} \mid \omega \cdot m \leq k\},$$

and the *admissible marking set* of (ω, k) is

$$\mathcal{A}_{(\omega,k)} = \{m \in \mathcal{L}_{(\omega,k)} \mid R(N, m, u_{zero}) \subseteq \mathcal{L}_{(\omega,k)}\}.$$

Moreover, a set of linear constraints is denoted as $W=\{(\omega_1, k_1), (\omega_2, k_2), \ldots, (\omega_n, k_n)\}$, where $n \in \boldsymbol{N}^+$. The disjunction of the constraints in W is denoted as $\vee(W)$, i.e., $\vee_{(\omega, k) \in W}\, \omega \cdot m \leq k$. The legal marking set of $\vee(W)$ is

$$\mathcal{L}_{\vee(W)} = \bigcup_{(\omega,k) \in W} \mathcal{L}_{(\omega,k)},$$

and the admissible marking set of $\vee(W)$ is

$$\mathcal{A}_{\vee(W)} = \{m \in \mathcal{L}_{\vee(W)} \mid R(N, m, u_{zero}) \subseteq \mathcal{L}_{\vee(W)}\}.$$

Finally, given two sets of linear constraints W_1 and W_2, we say that they are *equivalent* if $\mathcal{A}_{\vee(W_1)}=\mathcal{A}_{\vee(W_2)}$.

III. Linear Constraint Transformation

As discussed in the Introduction, for a PN system with uncontrollable transitions, to guarantee a legal behavior, its evolution should be limited within its admissible marking set. In this section, we focus on legal marking sets in the form of a linear constraint involving one place only, and consider the problem of characterizing the admissible marking set as a finite disjunction of linear constraints.

Problem 1: Given an ordinary PN subject to a linear constraint (ω_0, k_0): $m(p_0)\le k_0$, compute a finite set of linear constraints W_{OR}, such that $\mathcal{L}_{\vee(W_{OR})}=\mathcal{A}_{(\omega 0,\, k0)}$.

We call the set W_{OR} such that $\mathcal{L}_{\vee(W_{OR})}=\mathcal{A}_{(\omega 0,\, k0)}$ the *optimal transformation* of (ω_0, k_0).

Definition 1 [15]: Consider a PN $N=(P, T, F)$ subject to (ω_0, k_0): $m(p_0)\le k_0$. The *uncontrollable subnet* with respect to (w.r.t) (ω_0, k_0) is the net $N_{\omega 0}=(P_{\omega 0}, T_{\omega 0}, F_{\omega 0})$ where: $P_{\omega 0}$ is the set of places, including p_0, from which p_0 can be reached following an uncontrollable path; $T_{\omega 0}$ is the set of uncontrollable input transitions of places in $P_{\omega 0}$; $F_{\omega 0}$ is the restriction of F to $(P_{\omega 0} \times T_{\omega 0}) \cup (T_{\omega 0} \times P_{\omega 0})$.

Consider the PN in Fig. 1(a) where uncontrollable transitions are denoted with black rectangles, i.e., $T_u=\{t_1\text{-}t_4\}$. The uncontrollable subnet w.r.t (ω_0, k_0): $m(p_0)\le k_0$ is $N_{\omega 0}=(P_{\omega 0}, T_{\omega 0}, F_{\omega 0})$ shown in Fig. 1(b), regardless of k_0.

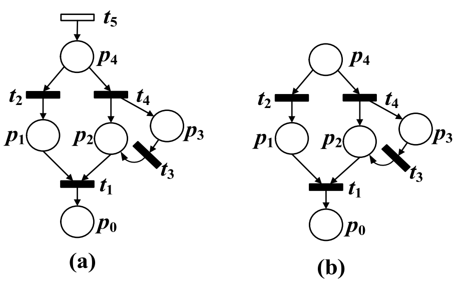

Fig. 1. (a) A PN and (b) its uncontrollable subnet w.r.t (ω_0, k_0): $m(p_0)\le k_0$

In the rest of this paper, the following assumption is made.

***Assumption* 1: The limited place p_0 in (ω_0, k_0): $m(p_0)\le k_0$ does not belong to any circuit of the corresponding uncontrollable subnet.**

A. Transformation of a linear constraint via a transition

Definition 2 [23]: Consider a PN N with incidence matrix $[N]$ and a constraint (ω, k). We denote ϖ the row vector defined as $\varpi=\omega\cdot[N]$.

Clearly, for any t, it holds that $\varpi(t)=\omega\cdot m'-\omega\cdot m$, where m' is the marking reached from m by firing t. It implies that $\varpi(t)$ is a measure of the firing effect of t on (ω, k).

Property 1 [33]: Let N be a PN subject to (ω, k) and $t\in T_u$ such that $\varpi(t)\le 0$. If $m\in\mathcal{L}_{(\omega, k)}$, it is $R_t(N, m)\subseteq\mathcal{L}_{(\omega, k)}$.

Definition 3 [33]: Consider a PN N with set of places P and set of uncontrollable transitions T_u. Let Ω be the set of all possible linear constraints over P. The *uncontrollable transition gain transformation* is a function ρ: $\Omega\times T_u\times P\rightarrow\Omega$ defined as follows:

$\forall(\omega, k)\in\Omega$, $\forall t\in T_u$, $\forall p\in P$, $(\omega', k')=\rho((\omega, k), t, p)$,

where

$$\begin{cases} k'=k \\ \forall p'\in P,\quad \omega'(p')=\begin{cases}\omega(p') & p'\neq p\vee p'\notin{}^{\bullet}t\\ \omega(p')+\varpi(t) & p'=p\wedge p'\in{}^{\bullet}t\setminus t^{\bullet}\\ k+1 & p'=p\wedge p'\in{}^{\bullet}t\cap t^{\bullet}\end{cases}\end{cases}.$$

Definition 4 [33]: Given $(\omega, k)\in\Omega$ and $t\in T_u$, $\varrho((\omega, k), t)$ is defined as

$$\varrho((\omega, k), t)=\begin{cases}\{(\omega,k)\} & \varpi(t)\le 0\\ \bigcup_{p\in{}^{\bullet}t}\{\rho((\omega,k),t,p\} & \varpi(t)>0\end{cases}$$

where ρ follows from Definition 3.

Let us consider as an example the linear constraint (ω_0, k_0): $m(p_0)\le 1$ and the net in Fig. 1(a). Let us focus on the uncontrollable transition t_1. We want to compute $\varrho((\omega_0, k_0), t_1)$ based on Definition 4. We first observe that it is $\varpi_0(t_1)=1>0$ thus, according to Definition 4, it holds:

$\varrho((\omega_0, k_0), t_1)=\{\rho((\omega_0, k_0), t_1, p_1)\}\cup\{\rho((\omega_0, k_0), t_1, p_2)\}$,

since $\{p_1, p_2\}$ is the set of input places of t_1. Moreover, by Definition 3 it is:

$\rho((\omega_0, k_0), t_1, p_1)=\{(\omega_1, k_1)\}$ where (ω_1, k_1): $m(p_0)+m(p_1)\le 1$, and

$\rho((\omega_0, k_0), t_1, p_2)=\{(\omega_2, k_2)\}$ where (ω_2, k_2): $m(p_0)+m(p_2)\le 1$.

Property 2: Given $(\omega, k)\in\Omega$ and $t\in T_u$, for any $(\omega', k')\in\varrho((\omega, k), t)$, it holds:

1) $\varpi'(t)\le 0$; or

2) $\varpi'(t)>0$ and $\exists p\in{}^{\bullet}t$, such that $\omega(p)=k+1$.

Proof: Follows from Definition 4. ■

We next prove the following proposition, whose significance will be evident in the following.

Proposition 1: Let W be a set of linear constraints. $\mathcal{L}_{\vee(W)}=\mathcal{A}_{\vee(W)}$ if $\forall t\in T_u$ and $\forall(\omega, k)\in W$, one of the following conditions holds:

1) $\varpi(t)\le 0$;

2) $\varpi(t)>0$ and $\exists p\in{}^{\bullet}t$, such that $\omega(p)>k$.

Proof: Consider a linear constraint $(\omega, k)\in W$. Without loss of generality, we suppose that T_u can be partitioned in two sets, denoted as T_{u1} and T_{u2}, such that $\forall t\in T_{u1}$, $\varpi(t)\le 0$ and $\forall t\in T_{u2}$, $\varpi(t)>0$ and $\exists p\in{}^{\bullet}t$, such that $\omega(p)>k$. First, consider $t\in T_{u1}$. It is clear that $\forall m\in\mathcal{L}_{(\omega, k)}$, $R_t(N, m)\subseteq\mathcal{L}_{(\omega, k)}$ by Property 1. Let us now consider $t\in T_{u2}$. Since $\exists p\in{}^{\bullet}t$, such that $\omega(p)>k$, we can conclude that $\forall m\in\mathcal{L}_{(\omega, k)}$, $m(p)=0$. This means that t is not enabled at any $m\in\mathcal{L}_{(\omega, k)}$. Hence, starting from any $m\in\mathcal{L}_{(\omega, k)}$, it is not possible to reach a marking outside of $\mathcal{L}_{(\omega, k)}$ by firing any $t\in T_u$. Since $\mathcal{L}_{\vee(W)}=\cup_{(\omega,\, k)\in W}\mathcal{L}_{(\omega,\, k)}$, it obviously holds that starting from any $m\in\mathcal{L}_{\vee(W)}$, it is not possible to reach a marking outside of $\mathcal{L}_{\vee(W)}$ by firing any $t\in T_u$. Therefore, $\mathcal{L}_{\vee(W)}=\mathcal{A}_{\vee(W)}$. ■

B. Characteristic linear constraints and combination linear constraints

Three notions are introduced in this subsection: *legal linear constraint*, *characteristic linear constraint* and *combination linear constraint* w.r.t. the constraint (ω_0, k_0): $m(p_0)\le k_0$. **Note that in the rest of the paper the constraint (ω_0, k_0) will refer to $m(p_0)\le k_0$ by default.**

Definition 5: A linear constraint (ω, k) is called a *legal linear constraint* w.r.t. (ω_0, k_0) if $\mathcal{L}_{(\omega, k)} \subseteq \mathcal{L}_{(\omega 0, k0)}$.

Consider again the PN in Fig. 1(a). Given the linear constraint (ω_0, k_0): $m(p_0)\leq 2$, it is easy to see that (ω_1, k_1): $2m(p_0)+m(p_1)\leq 5$ is a legal linear constraint w.r.t. (ω_0, k_0) while (ω_2, k_2): $2m(p_0)+m(p_2)\leq 6$ is not a legal constraint w.r.t. (ω_0, k_0).

A necessary and sufficient condition is presented now, under which a linear constraint is legal w.r.t. a given linear constraint.

Property 3: Consider (ω_0, k_0): $m(p_0)\leq k_0$ and a linear constraint (ω, k) with $\omega(p_0)\neq 0$. (ω, k) is a legal linear constraint w.r.t. (ω_0, k_0) *iff*

$$k\leq k^*=(k_0+1)\cdot\omega(p_0)-1.$$

Proof: (=>) For each $m\in\mathcal{L}_{(\omega, k)}$, clearly it is $m\in\mathcal{L}_{(\omega, k^*)}$, i.e.,

$$\omega\cdot m\leq(k_0+1)\cdot\omega(p_0)-1. \tag{1}$$

From the above inequality it follows that $\omega'\cdot m<k_0+1$, where $\omega'=\omega/\omega(p_0)$. Clearly, we have $\omega_0\cdot m \leq \omega'\cdot m$ since $\omega'(p_0)=1$. Hence, $\omega_0\cdot m\leq k_0$ holds. In other words, $\mathcal{L}_{(\omega, k)} \subseteq \mathcal{L}_{(\omega 0, k0)}$ holds. By Definition 5, (ω, k) is a legal linear constraint w.r.t. (ω_0, k_0).

(<=) By contradiction, suppose that $k >k^*$. Since k is an integer, we have $k \geq(k_0+1)\cdot\omega(p_0)$. Clearly, there is a marking $m=(k_0+1)\, p_0$ such that $m\in\mathcal{L}_{(\omega, k)}$ and $m\notin \mathcal{L}_{(\omega 0, k0)}$. In other words, $\mathcal{L}_{(\omega, k)} \not\subset \mathcal{L}_{(\omega 0, k0)}$, which contradicts the assumption that (ω, k) is a legal linear constraint w.r.t. (ω_0, k_0). Hence, it is $k\leq k^*$. ■

Definition 6: A linear constraint (ω, k) is called a *characteristic linear constraint* w.r.t. (ω_0, k_0): $m(p_0)\leq k_0$ if $\omega(p_0)\neq 0$ and $k=k^*$, where $k^*=(k_0+1)\cdot\omega(p_0)-1$.

As an example, let (ω_0, k_0): $m(p_0)\leq 2$ be a linear constraint for the net in Fig. 1(a). We argue that (ω_1, k_1): $2m(p_0)+m(p_1)\leq 5$ is a characteristic linear constraint w.r.t. (ω_0, k_0) since $\omega_1(p_0)\neq 0$ and $k_1=(k_0+1)\cdot\omega_1(p_0)-1$. On the contrary, (ω_2, k_2): $2m(p_0)+m(p_1)\leq 4$ is not a characteristic linear constraint w.r.t. (ω_0, k_0) since $k_2\neq(k_0+1)\cdot\omega_2(p_0)-1$.

Note that (ω_0, k_0) is obviously a characteristic linear constraint w.r.t. itself by Definition 6.

Property 4: Let (ω, k) be a characteristic linear constraint w.r.t. (ω_0, k_0). The constraint (ω, k) is a *legal* linear constraint w.r.t. (ω_0, k_0).

Proof: Straightforward from Definition 6 and Property 3. ■

Property 5: Let (ω, k) be a characteristic linear constraint w.r.t. (ω_0, k_0) and $t\in T_u$. Any constraint $(\omega', k')\in\varrho((\omega, k), t)$ is a characteristic linear constraint w.r.t. (ω_0, k_0).

Proof: By Assumption 1, p_0 does not belong to any circuit of the uncontrollable subnet. Hence, $\omega'(p_0)=\omega(p_0)$ and $k'=k$ for any $(\omega', k')\in\varrho((\omega, k), t)$ by Definition 4. Beside, $k=(k_0+1)\cdot\omega(p_0)-1$ since the linear constraint (ω, k) is a characteristic one w.r.t. (ω_0, k_0). Hence, $k'=(k_0+1)\cdot\omega'(p_0)-1$ holds for any $(\omega', k')\in\varrho((\omega, k), t)$. In other words, $\forall(\omega', k')\in\varrho((\omega, k), t)$, (ω', k') is a characteristic constraint w.r.t. (ω_0, k_0). ■

In what follows, two linear constraints (ω_1, k_1) and (ω_2, k_2) are called *opposite* w.r.t. $t\in T_u$ if either $\varpi_1(t)>0$ and $\varpi_2(t)<0$, or $\varpi_1(t)<0$ and $\varpi_2(t)>0$, i.e., if t has opposite firing effect on the two constraints.

We now introduce the concept of combination linear constraints of opposite linear constraints w.r.t. an uncontrollable transition.

Definition 7: Let (ω_1, k_1) and (ω_2, k_2) be two linear constraints and $t\in T_u$, such that the two constraints are opposite w.r.t. t. The *combination linear constraint* of (ω_1, k_1) and (ω_2, k_2) w.r.t. t is the linear constraint $(\omega_{1,2}, k_{1,2})=\psi((\omega_1, k_1), (\omega_2, k_2), t)$ defined as follows:

$$\omega_{1,2}=|\varpi_2(t)|\cdot\omega_1+|\varpi_1(t)|\cdot\omega_2,$$
$$k_{1,2}=|\varpi_2(t)|\cdot k_1+|\varpi_1(t)|\cdot k_2+|\varpi_2(t)|\cdot|\varpi_1(t)|.$$

To clarify the above definition, let us consider again the PN in Fig. 1(a), where $T_u=\{t_1\text{-}t_4\}$. Let us consider two linear constraints (ω_1, k_1): $m(p_0)+m(p_2)+m(p_3)\leq 1$ and (ω_2, k_2): $m(p_0)+m(p_1)+m(p_4)\leq 1$, and the uncontrollable transition t_4. The two constraints are clearly opposite w.r.t. t_4 since $\varpi_1(t_4)=2$ and $\varpi_2(t_4)= -1$. According to Definition 7, the combination linear constraint of (ω_1, k_1) and (ω_2, k_2) w.r.t. t_4 is

$$(\omega_{1,2}, k_{1,2}) = \psi((\omega_1, k_1), (\omega_2, k_2), t_4):$$
$$3m(p_0)+2m(p_1)+m(p_2)+m(p_3)+2m(p_4)\leq 5.$$

Property 6: Consider two linear constraints (ω_1, k_1) and (ω_2, k_2) that are opposite w.r.t. $t\in T_u$. Let $(\omega_{1,2}, k_{1,2})$ be the combination linear constraint of (ω_1, k_1) and (ω_2, k_2) w.r.t. t. It holds: $\varpi_{1,2}(t)=0$.

Proof: Follows from the fact that, by Definition 7, it is $\varpi_{1,2}(t)= |\varpi_2(t)|\cdot\varpi_1(t)+|\varpi_1(t)|\cdot\varpi_2(t)$. Since the two constraints are opposite w.r.t t, it is $|\varpi_2(t)|\cdot\varpi_1(t)= -|\varpi_1(t)|\cdot\varpi_2(t)$. ■

Proposition 2: Let $(\omega_{1,2}, k_{1,2})=\psi((\omega_1, k_1), (\omega_2, k_2), t)$ be the combination linear constraint of (ω_1, k_1) and (ω_2, k_2) w.r.t. $t\in T_u$. It holds $k_{1,2} \geq(k_0+1)\cdot\omega_{1,2}(p_0)-1$ if (ω_1, k_1) and (ω_2, k_2) are two characteristic linear constraints w.r.t. (ω_0, k_0): $m(p_0)\leq k_0$.

Proof: According to Definition 7, it is

$$k_{1,2}=|\varpi_2(t)|\cdot k_1+|\varpi_1(t)|\cdot k_2+|\varpi_2(t)|\cdot|\varpi_1(t)|, \text{ and} \tag{2}$$
$$\omega_{1,2}(p_0)=|\varpi_2(t)|\cdot\omega_1(p_0)+|\varpi_1(t)|\cdot\omega_2(p_0) \tag{3}$$

Since (ω_1, k_1) and (ω_2, k_2) are characteristic constraints w.r.t. (ω_0, k_0), by Definition 6 it holds

$$k_1=(k_0+1)\cdot\omega_1(p_0)-1, \text{ and} \tag{4}$$
$$k_2=(k_0+1)\cdot\omega_2(p_0)-1. \tag{5}$$

Now, considering (2)-(5), it is easy to show that

$$k_{1,2}=(k_0+1)\cdot\omega_{1,2}(p_0)-(|\varpi_1(t)|+|\varpi_2(t)| -|\varpi_2(t)|\cdot|\varpi_1(t)|). \tag{6}$$

Since $|\varpi_1(t)|\geq 1$ and $|\varpi_2(t)|\geq 1$, it can be verified that $|\varpi_1(t)|+|\varpi_2(t)| -|\varpi_2(t)|\cdot|\varpi_1(t)|\leq 1$. Hence, $k_{1,2}\geq(k_0+1)\cdot\omega_{1,2}(p_0) -1$. ■

Proposition 2 allows us to draw two important conclusions:

- If we combine two characteristic constraints w.r.t. (ω_0, k_0) according to Definition 7, then the resulting combination linear constraint could be not legal w.r.t. (ω_0, k_0). In particular, this happens if $k_{1,2} > (k_0+1)\cdot\omega_{1,2}(p_0)-1$.
- In the case the combination linear constraint resulting from the above combination is legal w.r.t. (ω_0, k_0), it is a characteristic linear constraint w.r.t. (ω_0, k_0). In particular, this happens if $k_{1,2} = (k_0+1)\cdot\omega_{1,2}(p_0)-1$.

The following proposition provides a necessary and sufficient condition under which the combination linear constraint of two characteristic opposite linear constraints w.r.t. an uncontrollable transition and a given constraint is still a characteristic one w.r.t. the same initial given constraint.

Proposition 3: Let (ω_1, k_1) and (ω_2, k_2) be two opposite characteristic linear constraints w.r.t. (ω_0, k_0) and $t\in T_u$. The combination linear constraint $(\omega_{1,2}, k_{1,2})=\psi((\omega_1, k_1), (\omega_2, k_2), t)$ is a characteristic one w.r.t. (ω_0, k_0) *iff*

$$|\varpi_1(t)|=1 \text{ or } |\varpi_2(t)|=1.$$

Proof: The proof is based on that of Proposition 2. It has been proved that

$$k_{1,2}=(k_0+1)\cdot\omega_{1,2}(p_0)-(|\varpi_1(t)|+|\varpi_2(t)|-|\varpi_2(t)|\cdot|\varpi_1(t)|). \quad (7)$$

(=>) We have $|\varpi_1(t)|+|\varpi_2(t)|-|\varpi_2(t)|\cdot|\varpi_1(t)|=1$ since $|\varpi_1(t)|=1$ or $|\varpi_2(t)|=1$. Clearly, $k_{1,2}=(k_0+1)\cdot\omega_{1,2}(p_0)-1$. Hence, $(\omega_{1,2}, k_{1,2})$ is a characteristic one w.r.t. (ω_0, k_0).

(<=) Since $(\omega_{1,2}, k_{1,2})$ is a characteristic one w.r.t. (ω_0, k_0), then $k_{1,2}=(k_0+1)\cdot\omega_{1,2}(p_0)-1$. Hence, it is known from (7) that $|\varpi_1(t)|+|\varpi_2(t)|-|\varpi_2(t)|\cdot|\varpi_1(t)|=1$. Hence, $|\varpi_1(t)|=1$ or $|\varpi_2(t)|=1$. ■

C. *The proposed algorithm*

In this subsection we present the main result of the paper, namely, a novel iterative algorithm for linear constraint transformation. Before doing that, we provide a further definition and some preliminary results.

Definition 8: Let W_1 and W_2 be two linear constraint sets. We say that W_1 *implies* W_2 (or W_2 *is implied* by W_1), denoted as $W_1\Rightarrow W_2$, if $\mathcal{L}_{\vee(W2)}\subseteq\mathcal{L}_{\vee(W1)}$ and $\mathcal{A}_{\vee(W2)}=\mathcal{A}_{\vee(W1)}$.

Theorem 1 [34]: Let $W=\{(\omega_1, k_1), (\omega_2, k_2)\}$ be a set of linear constraints such that $\{(\omega_0, k_0)\}\Rightarrow W$ and $t\in T_u$ such that $\varpi_1(t)>0$ and $\varpi_2(t)<0$. It holds:

$$\{(\omega_0, k_0)\}\Rightarrow W'=\varrho((\omega_1, k_1), t) \cup \{(\omega_2, k_2), \psi((\omega_1, k_1), (\omega_2, k_2), t)\},$$

if $\psi((\omega_1, k_1), (\omega_2, k_2), t)$ is a characteristic linear constraint w.r.t. (ω_0, k_0).

Theorem 1 claims that a set W of two linear constraints (ω_1, k_1) and (ω_2, k_2) whose union is implied by (ω_0, k_0), can be transformed into an equivalent set W' that is still implied by (ω_0, k_0), via an uncontrollable transition t, whenever the combination of the two constraints (ω_1, k_1) and (ω_2, k_2) w.r.t. t is a characteristic linear constraint w.r.t. (ω_0, k_0). In more detail, the equivalent set W' is obtained as the union of three sets:

- the first one is $\varrho((\omega_1, k_1), t)$ that is computed transforming (ω_1, k_1) in accordance to Definition 4;
- the second set is a singleton containing (ω_2, k_2);
- the third set is still a singleton and contains the combination linear constraint of (ω_1, k_1) and (ω_2, k_2) w.r.t. t that is computed according to Definition 7.

The above constraint transformation can be generalized to sets defined via an arbitrary number n of linear constraints.

Corollary 1: Let $W=\{(\omega_1, k_1), (\omega_2, k_2), \ldots, (\omega_n, k_n)\}$, $n\in\boldsymbol{N}^+$ be a set of linear constraints such that $\{(\omega_0, k_0)\}\Rightarrow W$ and $t\in T_u$. It holds:

$$\{(\omega_0, k_0)\}\Rightarrow W'=\cup_{(\omega, k)\in W}\,\varrho((\omega, k), t)\cup \bigcup_{(i,j)\in E}\{\psi((\omega_i, k_i), (\omega_j, k_j), t)\}$$

if $\forall(i, j)\in E=\{(i, j)|\ i, j\in\{1, 2, \ldots, n\}, \varpi_i(t)>0 \text{ and } \varpi_j(t)<0\}$, $\psi((\omega_i, k_i), (\omega_j, k_j), t)$ is a characteristic linear constraint w.r.t. (ω_0, k_0).

Proof: The result follows from the recursive application of Theorem 1 and the fact that $\varrho((\omega, k), t)=\{(\omega, k)\}$ if $\varpi(t)<0$. ■

Based on Corollary 1, we propose an algorithm to compute a linear constraint transformation of a given constraint (ω_0, k_0) and to evaluate if it is optimal.

Algorithm 1: Linear constraint transformation

Input: A PN N with set of uncontrollable transitions T_u and (ω_0, k_0): $m(p_0)\leq k_0$.

Output: W_{OR} and Flag∈{True, False}. /*Flag=True implies that the linear constraint transformation is optimal, while Flag=False implies that it may be not. */

1. Flag ← True;
2. W_{OR} ← $\{(\omega_0, k_0)\}$;
3. **while** $\exists t\in T_u$ and $\exists(\omega', k')\in W_{OR}$, such that $\varpi'(t)>0$ and $\omega'(p)\leq k'$, $\forall p\in{}^\bullet t$ **do**
4.   **for** each $(\omega', k')\in W_{OR}$ such that $\varpi'(t)>0$ and $\omega'(p)\leq k'$, $\forall p\in{}^\bullet t$ **do**
5. $W_{OR}\leftarrow W_{OR}\setminus\{(\omega', k')\}\cup\varrho((\omega', k'), t)$;
6.   **for** each $(\omega'', k'')\in W_{OR}$ such that $\varpi''(t)<0$ **do**
7.   Compute the combination linear constraint $\psi((\omega', k'), (\omega'', k''), t)$;
8. $(\omega_C, k_C)\leftarrow\psi((\omega', k'), (\omega'', k''), t)$;
9.   **if** (ω_C, k_C) is not a characteristic linear constraint w.r.t. (ω_0, k_0) **then**
10. $(\omega_C, k_C)\leftarrow(\omega_C, k_C^*)$, where $k_C^*=(k_0+1)\cdot\omega_C(p_0)-1$;
11.   Flag ← False;
12.   **end if**
13. $W_{OR}\leftarrow W_{OR}\cup\{(\omega_C, k_C)\}$;
14.   **end for**
15.   **end for**
16. **end while**
17. **Output:** W_{OR} and Flag.
18. **End.**

In the following we briefly explain the above algorithm. We preliminary observe that Flag records if the transformation is optimal. To this aim Flag is initialized at "True", assuming that an optimal transformation is computed. Then we note that the set W_{OR} stores the computed linear constraints and is initialized at $\{(\omega_0, k_0)\}$. By checking if the condition in Step 3 is met, it is decided if the constraints in W_{OR} should be transformed. Specifically, once a transition $t\in T_u$ satisfies the condition in Step 3, the constraints in W_{OR} require to be transformed via t as follows. For each $(\omega', k')\in W_{OR}$ satisfying the two conditions: 1) t has a positive firing effect on (ω', k'); and 2) for each input place of t, its weight in ω' is not bigger than k', we perform the following procedure. First, we replace (ω', k') by some new linear constraints according to Step 5. Next, for each linear constraint in W_{OR} satisfying the condition that t has a negative firing effect on it, we compute a combination linear constraint, as shown in Step 7. Then, in Steps 8-13, we determine if the computed combination linear constraint is a characteristic one. If so, we add it to W_{OR}, otherwise it has to be modified to a characteristic one (to make it legal) before being added to W_{OR}. Note that Flag is changed to "False" if the computed combination linear constraint is modified before being added to W_{OR}, which indicates that the transformation may be not optimal now. The above steps are iterated until the condition in Step 3 is not satisfied and then the final result is outputted.

Lemma 1: Let a PN N and (ω_0, k_0) be the inputs of Algorithm 1. Each linear constraint in W_{OR} computed during the execution of Algorithm 1, is a characteristic constraint w.r.t. (ω_0, k_0).

Proof: Obviously, (ω_0, k_0) is a characteristic constraint w.r.t. itself. Besides, Property 5 shows that the transformation of a characteristic linear constraint via a transition results in characteristic constraints. Moreover, it is clear that (ω_C, k_C) computed in Step 13 is a characteristic linear constraint. As a result, the conclusion holds. ■

The theorem next indicates that the transformation result outputted by Algorithm 1 is optimal if the outputted Flag=True, otherwise it may be not optimal.

Theorem 2: Let a PN N and (ω_0, k_0) be the inputs of Algorithm 1. The linear constraint set W_{OR} outputted by Algorithm 1 satisfies the following two conditions:

1) $\mathcal{L}_{\vee(W_{OR})}=\mathcal{A}_{(\omega 0,k0)}$ if Flag=True; and

2) $\mathcal{L}_{\vee(W_{OR})} \subseteq \mathcal{A}_{(\omega 0,k0)}$ if Flag=False.

Proof: First, we prove that Algorithm 1 ends after a finite number of steps. According to Properties 2 and 6, once W_{OR} is transformed via a transition $t\in T_u$ (i.e., the iteration cycle composed by Steps 3-16 is executed once), another transition is picked for the next transformation. Hence, Algorithm 1 terminates in a finite number of steps for acyclic nets since the number of uncontrollable transitions is finite. As for cyclic nets (i.e., nets with circuits), although there may be a transition $t\in T_u$ whose firing effect is periodically positive on a linear constraint, t can be picked for transformation finite times only, since the input places of t have to meet a condition as shown in Step 3. As a result, Algorithm 1 ends in a finite number of steps.

Next, we prove that $\mathcal{L}_{\vee(W_{OR})}=\mathcal{A}_{(\omega 0,k0)}$ if Flag=True. According to Corollary 1, every time the iteration cycle composed by Steps 3-16 is executed once, the updated W_{OR} satisfies the condition $\mathcal{A}_{\vee(W_{OR})}=\mathcal{A}_{(\omega 0,\, k0)}$. After the iteration cycle is executed a finite number of times, the final W_{OR} does not meet the condition in Step 3. According to Proposition 1, we have $\mathcal{L}_{\vee(W_{OR})}=\mathcal{A}_{\vee(W_{OR})}$. Therefore, it holds $\mathcal{L}_{\vee(W_{OR})}=\mathcal{A}_{(\omega 0,\, k0)}$.

Finally, we prove that $\mathcal{L}_{\vee(W_{OR})} \subseteq \mathcal{A}_{(\omega 0,\, k0)}$ if Flag=False. Let W_{OR1} and W_{OR2} be the linear constraint sets before and after executing once the iteration cycle composed by Steps 3-16, during which Flag is changed from "True" to "False". Clearly, $\mathcal{A}_{\vee(W_{OR1})}=\mathcal{A}_{(\omega 0,\, k0)}$. Since Flag is changed to "False", at least one combination linear constraint (ω_C, k_C) computed in Step 7 is not a characteristic one w.r.t. (ω_0, k_0), and (ω_C, k_C) is thereby modified as a characteristic one (ω_C, k_C^*). According to Lemma 1, we know that the combination linear constraint (ω_C, k_C) is obtained from two characteristic linear constraints and thus it holds that $\mathcal{L}_{(\omega C,\, kC^*)} \subseteq \mathcal{L}_{(\omega C,\, kC)}$ by Proposition 2. Hence, some admissible markings of W_{OR1} may be removed in this iteration cycle according to the proof of Theorem 1. Hence, we have $\mathcal{A}_{\vee(W_{OR2})} \subseteq \mathcal{A}_{\vee(W_{OR1})}=\mathcal{A}_{(\omega 0,\, k0)}$. As a result, for the final W_{OR}, $\mathcal{L}_{\vee(W_{OR})} \subseteq \mathcal{A}_{(\omega 0,k0)}$. ■

Summarizing, the linear constraint transformation computed via Algorithm 1 is optimal when all the computed combination linear constraints are characteristic ones w.r.t. the initial linear constraint (ω_0, k_0).

D. A special case: acyclic backward conflict free uncontrollable subnets

In this subsection we show that, under the assumption of acyclic BCF nets (namely the assumptions in [28], [29]), regardless of the order in which transitions are considered when implementing Algorithm 1, the optimality of the solution is guaranteed. Before doing that, we present a preliminary result.

Theorem 3: Let a PN N and (ω_0, k_0) be the inputs of Algorithm 1, and W_{OR} be the linear constraint set outputted by Algorithm 1. It holds that $\mathcal{L}_{\vee(W_{OR})}=\mathcal{A}_{(\omega 0,k0)}$ if during transformations, for any couple of opposite linear constraints w.r.t. any transition $t\in T_u$, the absolute value of the firing effect of t is 1 on either of them.

Proof: By Lemma 1, it is clear that any couple of opposite linear constraints computed while executing Algorithm 1 are characteristic ones w.r.t. (ω_0, k_0). Hence, their combination linear constraint is also a characteristic one due to Proposition 3. As a result, Algorithm 1 outputs Flag=True. In other words, the resulting set of constraints W_{OR} is such that $\mathcal{L}_{\vee(W_{OR})}=\mathcal{A}_{(\omega 0,k0)}$. ■

According to Theorem 3, we can decide if the transformation resulted from Algorithm 1 is optimal by simply looking at the firing effect of transitions on constraints during intermediate transformations. Although Theorem 3 does not characterize the PN structure that leads to an optimal solution, it provides an interesting condition, which reveals useful when dealing with some special classes of PNs. In particular, as shown by the following property, it reveals useful when dealing with the class of PNs considered in [28], [29]

Property 7: Consider a PN N and a linear constraint (ω_0, k_0) such that the uncontrollable subnet w.r.t. (ω_0, k_0) is an acyclic BCF net. While performing transformations according to Algorithm 1, for any couple of opposite linear constraints w.r.t. any transition $t\in T_u$, the firing effect of t is 1 on one of them.

Proof: Let us first introduce some definitions and notations. Since the uncontrollable subnet is BCF, we denote as ${}^{u}p$ the single input uncontrollable transition (if any) of place p. Besides, we denote as $P(\omega)=\{p\in P \mid \omega(p)>0\}$ and say that place p is a *head place* w.r.t. (ω, k) if $p\in P(\omega)$ and $\forall p'\in {}^{\bullet}({}^{\bullet}p), p'\notin P(\omega)$. If (ω, k) has only one head place, we denote it p_ω. Finally, we say that (ω, k) is a *good linear constraint* if the following conditions simultaneously hold:

1) it has only one head place p_ω;

2) $\forall p\in P(\omega)\backslash\{p_\omega\}$, $\varpi({}^{u}p)=0$; and 3) $\varpi({}^{u}p_\omega)=1$.

Consider $(\omega', k')\in\varrho((\omega, k), t)$, where (ω, k) is a good constraint. By Definition 4, (ω', k') is still a good constraint since the uncontrollable subnet is acyclic.

Consider a combination linear constraint $(\omega_{1,2}, k_{1,2})=\psi((\omega_1, k_1), (\omega_2, k_2), t)$, where (ω_1, k_1) and (ω_2, k_2) are both good linear constraints, and $t\in T_u$ with $\varpi_1(t)>0$ and $\varpi_2(t)<0$. By Definition 7, it holds $P(\omega_{1,2})=P(\omega_1)\cup P(\omega_2)$ and $\forall t'\in T_u$, $\varpi_{1,2}(t')=|\varpi_2(t)|\cdot\varpi_1(t')$

$+|\varpi_1(t)|\cdot\varpi_2(t')$. Since (ω_1, k_1) is a good constraint and $\varpi_1(t)>0$, it holds $t={}^{u}p_{\omega 1}$ and thus $\varpi_1(t)=1$. Hence, $\forall t'\in T_u$,

$$\varpi_{1,2}(t')=|\varpi_2(t)|\cdot\varpi_1(t')+\varpi_2(t'). \quad (8)$$

Let $p\in P(\omega_{1,2})$. Three cases may occur:

1) $p\in P(\omega_1)\cap P(\omega_2)$: It can be verified that $p\neq p_{\omega 1}$and $p\neq p_{\omega 2}$. Hence, $\varpi_1({}^{u}p)=0$ and $\varpi_2({}^{u}p)=0$ since (ω_1, k_1) and (ω_2, k_2) are both good linear constraints. Therefore, by (8), it is $\varpi_{1,2}({}^{u}p)=0$.

2) $p\in P(\omega_1)\backslash P(\omega_2)$: It is $\varpi_2({}^{u}p)=0$ since $p\notin P(\omega_2)$. Since (ω_1, k_1) is a good constraint, $\varpi_1({}^{u}p)=0$ if $p\in P(\omega_1)\backslash\{p_{\omega 1}\}$. By (8), $\varpi_{1,2}({}^{u}p)=0$ if $p\in P(\omega_1)\backslash(\{p_{\omega 1}\}\cup P(\omega_2))$. Consider now the case $p=p_{\omega 1}$. By Property 6, it is $\varpi_{1,2}({}^{u}p)=0$ since $t={}^{u}p_{\omega 1}$. As a result, $\varpi_{1,2}({}^{u}p)=0$ for any $p\in P(\omega_1)\backslash P(\omega_2)$.

3) $p\in P(\omega_2)\backslash P(\omega_1)$: It is $\varpi_1({}^{u}p)=0$ since $p\notin P(\omega_1)$. By (8), $\varpi_{1,2}({}^{u}p)=0$ if $p\in P(\omega_2)\backslash\{p_{\omega 2}\}$ and $\varpi_{1,2}({}^{u}p)=1$ if $p=p_{\omega 2}$.

Hence, $(\omega_{1,2}, k_{1,2})$ is still a good constraint with $p_{\omega 1,2}= p_{\omega 2}$.

Now, clearly, (ω_0, k_0): $m(p_0)\leq k_0$ is a good constraint. Therefore, based on the above analysis, we can conclude that during the transformations performed by Algorithm 1, each linear constraint (ω, k) is a good linear constraint. In other words, $\forall t\in T_u$, $\varpi(t)=1$ if $\varpi(t)>0$, thus proving the statement. ■

We thereby have the following corollary.

Corollary 2: Let a PN N and (ω_0, k_0) be the inputs of Algorithm 1 and W_{OR} be the linear constraint set outputted by Algorithm 1. It holds: $\mathscr{L}_{\vee(W_{OR})} = \mathcal{A}_{(\omega 0,\, k0)}$, if the uncontrollable subnet w.r.t. (ω_0, k_0) is an acyclic BCF net .

Proof: Straightforward from Property 7 and Theorem 3. ■

IV. Numerical Examples

In this section, two numerical examples are presented to illustrate Algorithm 1.

Example 1: Consider again the PN in Fig. 1(a). Let (ω_0, k_0): $m(p_0)\leq 1$ be the initial linear constraint. The linear constraint transformation of (ω_0, k_0) according to Algorithm 1 is performed as follows.

1) W_{OR} is initialized at $\{(\omega_0, k_0)\}$;

2) We select t_1 for transformation since $\varpi_0(t_1)=1>0$ and $\omega_0(p)\leq k_0$, $\forall p\in{}^{\bullet}t_1$. Thus, (ω_0, k_0) is replaced by $\varrho((\omega_0, k_0), t_1)=\{(\omega_1, k_1), (\omega_2, k_2)\}$, where

(ω_1, k_1): $m(p_0)+m(p_1)\leq 1$ and

(ω_2, k_2): $m(p_0)+m(p_2)\leq 1$.

The updated W_{OR} is $W_{OR}=\{(\omega_1, k_1), (\omega_2, k_2)\}$.

3) We select t_2 for transformation since $\varpi_1(t_2)=1>0$ and $\omega_1(p)\leq k_1$, $\forall p\in{}^{\bullet}t_2$. Thus, (ω_1, k_1) is replaced by $\varrho((\omega_1, k_1), t_2)=\{(\omega_3, k_3)\}$, where

(ω_3, k_3): $m(p_0)+m(p_1)+m(p_4)\leq 1$.

The updated W_{OR} is $W_{OR}=\{(\omega_2, k_2), (\omega_3, k_3)\}$.

4) We select t_3 for transformation since $\varpi_2(t_3)=1>0$ and $\omega_2(p)\leq k_2$, $\forall p\in{}^{\bullet}t_3$. Thus, (ω_2, k_2) is replaced by $\varrho((\omega_2, k_2), t_3)=\{(\omega_4, k_4)\}$, where

(ω_4, k_4): $m(p_0)+m(p_2)+m(p_3)\leq 1$.

The updated W_{OR} is $W_{OR}=\{(\omega_3, k_3), (\omega_4, k_4)\}$.

5) We select t_4 for transformation since $\varpi_4(t_4)=2>0$ and $\omega_4(p)\leq k_4$, $\forall p\in{}^{\bullet}t_4$. Thus, (ω_4, k_4) is replaced by $\varrho((\omega_4, k_4), t_4)=\{(\omega_5, k_5)\}$, where

(ω_5, k_5): $m(p_0)+m(p_2)+m(p_3)+2m(p_4)\leq 1$.

Note that $\varpi_3(t_4)=-1<0$. It means that the combination linear constraint $(\omega_6, k_6)=\psi((\omega_4, k_4), (\omega_3, k_3), t_4)$ has to be computed:

(ω_6, k_6): $3m(p_0)+2m(p_1)+m(p_2)+m(p_3)+2m(p_4)\leq 5$.

Clearly, (ω_6, k_6) is a characteristic one w.r.t. (ω_0, k_0). Hence, it is added to W_{OR} without modification, resulting in $W_{OR}=\{(\omega_3, k_3), (\omega_5, k_5), (\omega_6, k_6)\}$. Finally, such a set W_{OR} is the output of Algorithm 1 since $\varpi(t)\leq 0$ for any $t\in T_u$ and any $(\omega, k)\in W_{OR}$.

It can be easily verified that the resulting W_{OR} is an optimal transformation of (ω_0, k_0) since the only combination linear constraint (ω_6, k_6) is a characteristic one w.r.t. (ω_0, k_0).

Note that the approaches in [27]-[29] cannot be applied to this example since the net in Fig. 1(a) is not an acyclic BCF net.

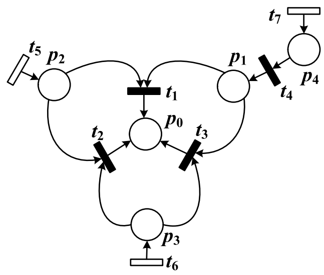

Fig. 2. The PN considered in Example 2

Example 2: Consider the PN N in Fig. 2 with $T_u=\{t_1-t_4\}$ and let (ω_0, k_0): $m(p_0)\leq 1$ be the initial linear constraint. The linear constraint transformation of (ω_0, k_0) according to Algorithm 1 is performed as follows.

1) W_{OR} is initialized at $\{(\omega_0, k_0)\}$;

2) t_1 is selected for transformation, resulting in $W_{OR}=\{(\omega_1, k_1), (\omega_2, k_2)\}$, where

(ω_1, k_1): $m(p_0)+m(p_1)\leq 1$, and

(ω_2, k_2): $m(p_0)+m(p_2)\leq 1$;

3) t_2 is selected for transformation, resulting in $W_{OR}=\{(\omega_3, k_3), (\omega_4, k_4), (\omega_2, k_2)\}$, where

(ω_3, k_3): $m(p_0)+m(p_1)+m(p_2)\leq 1$, and

(ω_4, k_4): $m(p_0)+m(p_1)+m(p_3)\leq 1$;

4) t_3 is selected for transformation. Note that $\varpi_2(t_3)=1>0$ and $\varpi_4(t_3)=-1<0$. This means that a combination linear constraint should be computed for (ω_2, k_2) and (ω_4, k_4) via t_3 and it is definitely a characteristic one due to Proposition 3. Hence, we have $W_{OR}=\{(\omega_3, k_3), (\omega_4, k_4), (\omega_5, k_5), (\omega_6, k_6), (\omega_7, k_7)\}$, where

(ω_5, k_5): $m(p_0)+m(p_2)+m(p_1)\leq 1$,

(ω_6, k_6): $m(p_0)+m(p_2)+m(p_3)\leq 1$, and

$(\omega_7, k_7)= \psi((\omega_2, k_2), (\omega_4, k_4), t_3)$:

$2m(p_0) +m(p_1)+m(p_2)+m(p_3)\leq 3$.

5) t_4 is selected for transformation, resulting in $W_{OR}=\{(\omega_3', k_3), (\omega_4', k_4), (\omega_6, k_6), (\omega_7', k_7)\}$, where

(ω_3', k_3): $m(p_0)+m(p_1)+m(p_2)+m(p_4)\leq 1$,

(ω_4', k_4): $m(p_0)+m(p_1)+m(p_3)+m(p_4)\leq 1$, and

(ω_7', k_7): $2m(p_0) +m(p_1)+m(p_2)+m(p_3)+m(p_4)\leq 3$.

Now, W_{OR} is the final transformation since $\varpi(t)\leq 0$ for any $t\in T_u$ and any $(\omega, k)\in W_{OR}$.

Clearly, the final W_{OR} is the optimal transformation of (ω_0, k_0) since the only combination linear constraint (ω_7, k_7) is a characteristic one w.r.t. (ω_0, k_0).

Note that the approach in [33] when applied to Example 2 does not provide an optimal transformation. This is because a non-empty complementary marking set [33] that appears after $t_1t_2t_3$ being picked for transformation has to be neglected to perform the follow-up transformation via t_4.

V. Conclusions and Future Work

A novel approach is provided in this work to perform constraint transformation in PNs with uncontrollable transitions. In particular, it deals with a linear constraint that limits the number of tokens in one place and is applicable to any net provided that the limited place in the constraint does not belong to any circuit of the uncontrollable subnet. The solution is in the form of a disjunction of linear constraints and its optimality is guaranteed if all the combination linear constraints computed during transformations are characteristic ones w.r.t. the initial linear constraint.

Our future work will be devoted to two main goals:

1) characterizing the structures of PNs for which it is known a priori that our proposed algorithm guarantees an optimal solution; and

2) improving the proposed algorithm, making it applicable to more general problems of constraint transformation.

References

[1] F. Basile, P. Chiacchio, and A. Giua, “Suboptimal supervisory control of Petri nets in presence of uncontrollable transitions via monitor places,” *Automatica*, vol. 42, no. 6, pp. 995-1004, 2006.

[2] F. Basile, R. Cordone, and L. Piroddi, “Integrated design of optimal supervisors for the enforcement of static and behavioral specifications in Petri net models,” *Automatica*, vol. 49, pp. 3432–3439, 2013.

[3] O. T. Baruwa, M. A. Piera, A. Guasch, “Deadlock-free scheduling method for flexible manufacturing systems based on timed colored Petri nets and anytime heuristic search,” *IEEE Trans. Systems Man & Cybernetics: Systems*, vol. 45, no. 5, pp. 831-846, 2015.

[4] Y. F. Chen, Z. W. Li and K. Barkaoui, “New Petri net structure and its application to optimal supervisory control: Interval inhibitor arcs,” *IEEE Trans. Syst. Man Cybernetics: Systems*, vol. 44, no. 10, pp. 1384-1400, 2014.

[5] Y. F. Chen, Z. W. Li, K. Barkaoui, and A. Giua, “On the enforcement of a class of nonlinear constraints on Petri nets,” *Automatica*, vol. 55, pp. 116-124, 2015.

[6] Y. F. Chen, Z. W. Li, K. Barkaoui, N. Q. Wu and M. C. Zhou, “Compact supervisory control of discrete event systems by Petri nets with data inhibitor arcs,” *IEEE Trans. Systems, Man, and Cybernetics: Systems*, vol. 47, no. 2, pp. 364–379, Feb. 2017.

[7] R. Cordone and L. Piroddi, “Parsimonious monitor control of Petri net models of FMS,” *IEEE Trans. Systems, Man, and Cybernetics Part A: Systems and Humans*, vol. 43, no. 1, pp. 215–221, 2013.

[8] A. Dideban, H. Alla, “Reduction of constraints for controller synthesis based on safe Petri nets,” *Automatica*, vol. 44, no. 7, pp.1697-1706, 2008.

[9] A. Dideban, M. Zareiee, and H. Alla. “Controller synthesis with highly simplified linear constraints in PN model,” *Asian Journal of Control*, vol.15, no. 1, pp. 80-94, 2013.

[10] M. P. Fanti, A. M. Mangini, G. Pedroncelli, and W. Ukovich, “Decentralized deadlock-free control for AGV systems”, in *Proceedings of the American Control Conference*, 2015, pp. 2414-2419.

[11] M. P. Fanti, and M. C. Zhou, “Deadlock control methods in automated manufacturing systems”, *IEEE Trans. Systems, Man, and Cybernetics, Part A*, vol. 34, no. 1, pp. 5-22, 2004.

[12] A. Giua, F. DiCesare, and M. Silva, “Generalized mutual exclusion constraints for Petri nets with uncontrollable transitions,” in *Proc. IEEE Int. Conf. on Systems, Man, and Cybernetics*, USA, 1992, pp. 947–949.

[13] A. Ghaffari, N. Rezg, and X. Xie, “Design of a live and maximally permissive Petri net controller using the theory of region,” *IEEE Trans. Robot. Autom.*, vol. 19, no. 1, pp. 137–142, Jan. 2003.

[14] B. Huang, M. Zhou, G. Zhang, *et al*., “Lexicographic multiobjective integer programming for optimal and structurally minimal Petri net supervisors of automated manufacturing systems,” *IEEE Trans. Systems, Man, and Cybernetics: Systems*, vol. 45, no. 11, pp. 1459–1470, 2015.

[15] L. E. Holloway, B. H. Krogh, and A. Giua, “A survey of Petri net methods for controlled discrete event systems,” *Discrete Event Dynamic Systems: Theory and Application*, vol. 7, pp. 151-190, 1997.

[16] M. V. Iordache and P. J. Antsaklis, “Synthesis of supervisors enforcing general linear constraints in Petri nets,” *IEEE Trans. Automatic Control*, vol. 48, no. 11, pp. 2036-2039, 2003.

[17] M. Iordache and P. Antsaklis, “Petri net supervisors for disjunctive constraints,” in *Proc. American Control Conference*, 2007, pp. 4951–4956.

[18] Z. W. Li, G. Y. Liu, M-H. Hanisch, and M. C. Zhou, “Deadlock prevention based on structure reuse of Petri net supervisors for flexible manufacturing systems,” *IEEE Trans. Systems, Man and Cybernetics, Part A: Systems and Humans*, vol. 42, no. 1, pp.178-191, 2012.

[19] Z. W. Li and M. Zhao, “On controllability of dependent siphons for deadlock prevention in generalized Petri nets,” *IEEE Trans. Systems, Man, and Cybernetics, Part A*, vol. 38, no. 2, pp. 369-384, 2008.

[20] H. X. Liu, K. Y. Xing, W. M. Wu, M. C. Zhou, and H. L. Zou, “Deadlock prevention for flexible manufacturing systems via controllable siphon basis of Petri nets. *IEEE Trans. Systems, Man, and Cybernetics: Systems*, vol. 45, no. 3, pp. 519-529, 2015.

[21] J. Luo and K. Nonami, “Approach for transforming linear constraints on Petri nets,” *IEEE Trans. Automatic Control*, vol. 56, no.11, pp. 2751–2765, Nov. 2011.

[22] J. Luo, H. Ni, W. Wu, S. Wang, and M. Zhou, “Simultaneous reduction of Petri nets and linear constraints for efficient supervisor synthesis,” *IEEE Trans. Automatic Control*, vol. 60, pp. 88–103, 2015.

[23] J. L. Luo, W. M. Wu, H. Y. Su, and J. Chu, “Supervisor synthesis for enforcing a class of generalized mutual exclusion constraints on Petri nets”, *IEEE Trans. Systems, Man, and Cybernetics, Part A: Systems and Humans*, vol. 39, no. 6, pp. 1237-1246, 2009.

[24] J. L. Luo, H. Shao, K. Nonami, and F. J. Jin, “Maximally permissive supervisor synthesis based on a new constraint transformation method”, *Automatica*, vol. 48, no. 6, pp. 1097-1101, 2012.

[25] J. Moody and P. Antsaklis, “Petri net supervisors for DES with uncontrollable and unobservable transitions”, *IEEE Trans. Automatic Control*, vol. 45, no. 3, pp. 462-476, 2000.

[26] T. Murata, “Petri nets: Properties, analysis and applications,” *Proceedings of the IEEE*, vol. 77, no. 4, pp. 541-580, 1989.

[27] Z. Ma, Z. Li, A. Giua, “A constraint transformation technique for Petri nets with certain uncontrollable structures,” in *Proc. IEEE International Workshop on Discrete Event Systems*, Cachan, France, 2014, pp. 66–72.

[28] Z. Ma, Z. Li, A. Giua, “A constraint transformation technique in Petri nets with backward-conflict-free uncontrollable structures,” in *Proc. 2015 American Control Conference*, Chicago, USA, 2015, pp.365-370.

[29] Z. Y. Ma, Z.W. Li, A. Giua, "Characterization of admissible marking sets in Petri nets with conflicts and synchronizations," *IEEE Trans. on Automatic Control*, vol. 62, no. 8, 2017. DOI: 10.1109/TAC.2016.2585647

[30] M. Uzam, “On suboptimal supervisory control of Petri nets in the presence of uncontrollable transitions via monitor places”, *International Journal of Advanced Manufacturing Technology*, vol. 47, no.5-8, pp. 567-579, 2010.

[31] S. G. Wang, C. Y. Wang, and M. C. Zhou, “Design of optimal monitor-based supervisors for a class of Petri nets with uncontrollable transitions”, *IEEE Trans. Systems, Man, and Cybernetics: Systems*, vol. 43, no. 5, pp. 1248-1255, 2013.

[32] S. G. Wang, D. You, and C. Y. Wang, “Optimal supervisor synthesis for Petri nets with uncontrollable transitions: a bottom-up algorithm”, *Information Sciences*, vol. 363, pp. 261–273, 2016.

[33] S. G. Wang, D. You, M. C. Zhou, and C. Seatzu, “Characterization of admissible marking sets in Petri nets with uncontrollable transitions”, *IEEE Trans. Automatic Control*, vol. 61, no. 7, pp. 1953-1958, 2016.

[34] D. You, S. G. Wang, and C. Seatzu, “A new linear constraint transformation approach for Petri nets with uncontrollable transitions”, *13th Int. Workshop on Discrete Event Systems*, 2016, Xi’an, China, 2016, pp. 258-263.