A Nyström method for mixed boundary value problems on domains with corners*

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Abstract

In this paper we propose a new approach to the numerical solution of the mixed Dirichlet-Neumann boundary value problem for the Laplace equation in planar domains with piecewise smooth boundaries. We consider a perturbed BIE system associated to the problem and present a Nyström method for its numerical solution. As Mellin type integral operators are involved, we need to modify the method close to the corners in order to prove its stability and convergence. Some numerical tests are also given to show the efficiency of the method here described.

Keyword: Mixed Dirichlet-Neumann problem for the Laplacian, boundary integral equations, Nyström method, domains with corners.

AMS Classification 65N35, 65R20, 45F15

1 Introduction

Let us consider the mixed Dirichlet-Neumann boundary value problem for the Laplace equation

$$\begin{cases} \Delta u(P) = 0, & P \in \Omega \\ u(P) = f_D(P), & P \in \Sigma_D \\ \dfrac{\partial u(P)}{\partial n_P} = f_N(P), & P \in \Sigma_N \end{cases} \tag{1}$$

where Ω is a simply connected bounded region in the plane with a piecewise smooth boundary $\Sigma = \overline{\Sigma}_D \cup \overline{\Sigma}_N$, f_D and f_N are given functions on Σ_D and

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Σ_N, respectively, and n_P is the inner normal vector to Σ_N at P. Let P_0 and P_1 be the two interface points of Σ, i.e. $\{P_0, P_1\} = \overline{\Sigma}_D \cap \overline{\Sigma}_N$, and assume that P_0 is the unique corner point of the boundary. Let us remark that the extension to boundary curves with more than one corner is straightforward.

In order to obtain a boundary integral equation (BIE) reformulation of the problem (1), we represent the solution u as the single layer potential

$$u(A) = \int_\Sigma \log|A - Q| \Psi(Q)\, d\Sigma_Q, \quad A \in \Omega \tag{2}$$

where $|A - Q|$ is the Euclidean distance between A and Q, $d\Sigma_Q$ is the arc length element and Ψ denotes the unknown single layer density function. Employing the Dirichlet and Neumann boundary conditions and taking into account the well known jump condition for the normal derivative of the single layer potential at the boundary, we get the following system of boundary integral equations

$$\begin{cases} \displaystyle\int_\Sigma \log|P - Q| \Psi(Q)\, d\Sigma_Q = f_D(P), & P \in \Sigma_D \\ \displaystyle\pi\, \Psi(P) + \int_\Sigma \frac{\partial}{\partial n_P} \log|P - Q| \Psi(Q)\, d\Sigma_Q = f_N(P), & P \in \Sigma_N \end{cases} \tag{3}$$

whose unknown is the density function Ψ on the boundary Σ. Once we solve this system, the solution u of problem (1) can be determined by using (2).

Let us observe that system (3) consists of an integral equation of the second kind and an other one of the first kind. Moreover, it is well known (see [3, 7, 16] and the references therein) that, in the case of domains with piecewise smooth boundaries, even for smooth boundary data f_D and f_N, the single layer density function Ψ could be singular around the corner. Furthermore, the integral operator involved in the second equation is not compact due to the kernel having a fixed singularity at the corner point. All these features of the problem (3) make its numerical treatment rather delicate.

In [7, 16] collocation methods are introduced to solve (3) on non-smooth boundaries. More precisely, in [7] the BIE system is solved by a collocation method using trigonometric cosine functions as approximants. Furthermore, a mesh grading transformation is introduced in order to smooth the singularities of the exact solution and, consequently, to reach fast convergence of the collocation solution. Also in [16] the Author introduces a smoothing transformation and then approximates the smooth transformed solution by Chebyshev polynomial expansions using the zeros of Chebyshev polynomials as collocation nodes. In these papers a complete solvability and stability analysis of the transformed integral equations is carried out in L^2 spaces by using localization and Mellin techniques. Moreover, in both cases, the stability of the collocation method is proved by allowing the possibility of a slight modification near the corners based on a suitable truncation of the approximate solution and an error estimate is provided.

Here, for the numerical approximation of the solution Ψ of (3), we propose a Nyström type method for which we establish stability and convergence results in spaces of continuous functions equipped with the uniform norm. To our knowledge, such approach has never been adopted before for the numerical solution of the BIE system (3) arising from the mixed problem (1). To our aim, first, we convert (3) into a new system of integral equations on the interval $[0, 1]$ whose unknown is smoother than the original solution, by using a proper smoothing change of variable combined with a parametrization of the boundary.
Then, we introduce a small perturbation term in the first equation of the latter system. This leads us to solve a new system of two integral equations of the second kind for which we are able to carry out the study of the solvability in the above function spaces. Note that, under suitable assumptions, the more the perturbation is small the more the solutions of the perturbed and unperturbed systems are close. The introduction of such a perturbation is also crucial in order to prove the properties of the numerical procedure we are going to propose.
In order to approximate the solution of the perturbed system, we apply a new Nyström type method. The method uses a proper combination of the Gauss-Legendre and the Gauss-Radau formulas, as well as of suitable product quadrature rules. In particular, we adopt the Radau formula for the discretization of the Mellin integral operator involved into the system, suitably modified near the singularity point 0 (the parameter value corresponding to the corner point of the boundary). We remark that such a modified discretization, introduced in [8, 11], has never been applied to this kind of problem and is essential to achieve the stability and the convergence of the proposed method. The product quadrature rules are employed for the approximation of the integral operator with the weakly singular logarithmic kernel. We observe, that in addition to the classical formula (8), we also apply the non standard one (9), involving the weights and knots of the Gauss-Radau rule, and the modified moments (7) related to the orthonormal Legendre polynomials. Furthermore, some numerical tricks proposed in [15] are properly adapted to our context in order to address some computational aspects.
Finally, the solution of the linear system which the Nyström method leads to solve is used in the computation of a discrete approximation of the single layer potential (2).

The paper is organized in eight sections. In Section 2 the function spaces are defined and some preliminary results concerning the employed quadrature formulae are given. In Section 3 the system of integral equations on the interval [0, 1] is introduced and the noncompact operator is investigated. Section 4 concerns the perturbed system we propose to solve for which a

Nyström method is developed and analyzed in Section 5. Section 6 is devoted to the approximation of the solution of problem (1) and Section 7 is dedicated to the proofs of the main results. Section 8 contains numerical examples which demonstrate the efficiency of the proposed method.

2 Preliminaries

Let us introduce the Sobolev-type subspaces W_r^p of the spaces $L^p \equiv L^p([0,1])$ and $C \equiv C([0,1])$ endowed with the usual norms, defined as follows

$$W_r^p = \{f \in L^p \mid f^{(r-1)} \in AC((0,1)),\ \|f^{(r)}\varphi^r\|_p < \infty\}, \qquad 1 \leq p < \infty$$

$$W_r^\infty = \{f \in C \mid f^{(r-1)} \in AC((0,1)),\ \|f^{(r)}\varphi^r\|_\infty < \infty\}, \qquad p = \infty,$$

where $r \in \mathbb{N}$, $\varphi(t) = \sqrt{t(1-t)}$ and $AC((0,1))$ denotes the collection of all functions which are absolutely continuous on every closed subset of $(0,1)$. We equip these spaces with the norm

$$\|f\|_{W_r^p} = \|f\|_p + \|f^{(r)}\varphi^r\|_p.$$

We also consider the norms defined on the product spaces $C \times C$ and $W_r^\infty \times W_r^\infty$

$$\|\mathbf{f}\| = \max\{\|f_1\|_\infty,\, \|f_2\|_\infty\}, \quad \mathbf{f} = (f_1,\, f_2) \in C \times C,$$

$$\|\mathbf{f}\|_{r,\infty} = \max\{\|f_1\|_{W_r^\infty},\, \|f_2\|_{W_r^\infty}\}, \quad \mathbf{f} = (f_1,\, f_2) \in W_r^\infty \times W_r^\infty.$$

In the following we will use the symbol $\|\cdot\|$ to denote also its associated operator norm.

We introduce the Gauss-Legendre and the Gauss-Radau formulas on the interval $[0,1]$ (see [5])

$$\int_0^1 f(t)dt = \sum_{k=1}^m \lambda_{m,k}^L f(t_{m,k}^L) + e_m^L(f) \tag{4}$$

$$\int_0^1 f(t)dt = \sum_{k=0}^m \lambda_{m,k}^R f(t_{m,k}^R) + e_m^R(f) \tag{5}$$

where e_m^L and e_m^R denote the respective quadrature errors. Some estimates for such remainder terms, useful in the sequel, can be expressed in terms of the weighted error of best polynomial approximation defined as

$$E_m(f)_{w,p} = \inf_{P_m \in \mathbb{P}_m} \|(f - P_m)\, w\|_p\,,$$

where w is a weight function on the interval $[0,1]$ and $\mathbb{P}_m$ denotes the set of all algebraic polynomials of degree at most m.
In fact, for all $f \in W_r^1$, $r \geq 1$, one has [12, Theorem 5.1.8, p. 338]

$$|e_m^L(f)| \leq \frac{\mathcal{C}}{m^r} E_{2m-1-r}\left(f^{(r)}\right)_{\varphi^r,1}, \quad |e_m^R(f)| \leq \frac{\mathcal{C}}{m^r} E_{2m-r}\left(f^{(r)}\right)_{\varphi^r,1} \tag{6}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.
Here and in the sequel $\mathcal{C}$ denotes a positive constant which may have different values in different formulas. We will write $\mathcal{C}(a, b, \dots)$ to say that $\mathcal{C}$ depends on the parameters $a, b, \dots$ and $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$ to say that $\mathcal{C}$ is independent of the parameters $a, b, \dots$.

Finally, we introduce two product rules for the numerical evaluation of integrals with logarithmic kernel. Denoting by $p_j^{0,0}$ the orthonormal Legendre polynomial of degree j, let

$$c_j(s) = \int_0^1 \log|t-s|\, p_j^{0,0}(t)dt \tag{7}$$

be the corresponding momentum, computable by means of a recurrence formula (see, for instance, [13]). Then, we consider the following quadrature rules

$$\int_0^1 \log|t-s|\, f(t)dt = \sum_{k=1}^{m} w_{m,k}^{P,L}(s) f(t_{m,k}^L) + e_m^{P,L}(f,s), \tag{8}$$

$$\int_0^1 \log|t-s|\, f(t)dt = \sum_{k=0}^{m} w_{m,k}^{P,R}(s) f(t_{m,k}^R) + e_m^{P,R}(f,s), \tag{9}$$

with the weights given by

$$w_{m,k}^{P,L}(s) = \lambda_{m,k}^L \sum_{j=0}^{m-1} c_j(s) p_j^{0,0}(t_{m,k}^L), \quad k = 1, \dots, m,$$

$$w_{m,k}^{P,R}(s) = \lambda_{m,k}^R \sum_{j=0}^{m} c_j(s) p_j^{0,0}(t_{m,k}^R), \quad k = 0, 1, \dots, m.$$

Since the weights satisfy the relations

$$\lim_{m\to\infty} \sum_{k=1}^{m} \left| w_{m,k}^{P,L}(s) \right| = \lim_{m\to\infty} \sum_{k=0}^{m} \left| w_{m,k}^{P,R}(s) \right| = \int_0^1 |\log|t-s||\, dt < \infty,$$

uniformly with respect to s (see [18, 19, 4] and the references therein), for the remainder terms in (8) and (9), for any $f \in C$, we have that

$$|e_m^{P,L}(f,s)| \leq \mathcal{C} E_{m-1}(f)_{v^{0,0},\infty}, \quad |e_m^{P,R}(f,s)| \leq \mathcal{C} E_m(f)_{v^{0,0},\infty}, \tag{10}$$

with $\mathcal{C} \neq \mathcal{C}(s, m, f)$ and $v^{0,0}(x) \equiv 1$.

3 The BIE system

Let us assume that the boundary Σ consists of two smooth arcs Σ_D and Σ_N, not necessarily of the same length. Let P_0, P_1 be the interface points, i.e.

$\{P_0, P_1\} = \overline{\Sigma}_D \cap \overline{\Sigma}_N$, with P_0 the unique corner point of Σ having the interior angle $\alpha = (1-\phi)\pi$, $-1 < \phi < 1$, $\phi \neq 0$. Without loss of generality, from now on we assume that $P_0 = (0,0)$. Moreover, we remark that extensions to more general cases are straightforward.

To transform the system (3) into an equivalent one defined on an interval, we introduce a parametric representation for each portion of Σ. In detail, for $J \in \{D, N\}$, we denote by

$$\sigma_J : s \in [0,1] \to (\xi_J(s), \eta_J(s)) \in \Sigma_J$$

the parametric representation defined on the interval $[0,1]$ for the arc Σ_J. Moreover, we suppose that $\sigma_J \in C^2([0,1])$, $|\sigma_J'(s)| \neq 0$, for each $s \in [0,1]$, and $\sigma_J(0) = P_0$. From now on, in our analysis we shall make the further assumption that the boundary Σ consists of two straight lines in a neighbourhood of P_0. However, using perturbation arguments it should be possible to derive the same results without assuming this restriction (see [7, 16] and the references therein).

It is well-known [3, 7, 16] that, even if the boundary data f_D and f_N are smooth functions, the single layer density function Ψ is not smooth around the corner point P_0. Indeed, we have

$$\Psi(P) = \mathcal{C}(\theta)\rho^\beta + \text{smoother terms}, \quad \beta = \min\left\{\frac{\pi}{2\alpha}, \frac{\pi}{2(2\pi - \alpha)}\right\} - 1, \quad (11)$$

where $P \in \Sigma$ and $(\rho,\, \theta)$ are the polar coordinates centered at P_0.
In order to improve its smoothness properties, we consider a non decreasing smoothing transformation $\gamma(t)$ mapping $(0,1)$ onto $(0,1)$ such that, for some $0 < \varepsilon < 1/2$ and some integer $q \geq 2$,

$$\gamma(t) = t^q + \mathcal{O}\left(t^{q+1}\right), \quad t \in [0, \varepsilon]. \quad (12)$$

Consequently, we define new parameterizations for the arcs Σ_J, $J \in \{D, N\}$,

$$\tilde{\sigma}_J(s) := \sigma_J(\gamma(s)) = (\xi_J(\gamma(s)), \eta_J(\gamma(s))) =: (\tilde{\xi}_J(s), \tilde{\eta}_J(s)). \quad (13)$$

Then, from the BIE system (3) we get the following system of integral equations on $[0,1]$

$$\begin{cases} \displaystyle\int_0^1 \log|\tilde{\sigma}_D(s) - \tilde{\sigma}_D(t)|\tilde{\Psi}_D(t)\,dt - \int_0^1 \log|\tilde{\sigma}_D(s) - \tilde{\sigma}_N(t)|\tilde{\Psi}_N(t)\,dt = \tilde{g}_D(s) \\ \displaystyle|\tilde{\sigma}_N'(s)| \int_0^1 \frac{\partial}{\partial n_s} \log|\tilde{\sigma}_N(s) - \tilde{\sigma}_D(t)|\tilde{\Psi}_D(t)\,dt \\ \displaystyle\qquad + \pi\tilde{\Psi}_N(s) + |\tilde{\sigma}_N'(s)| \int_0^1 \frac{\partial}{\partial n_s} \log|\tilde{\sigma}_N(s) - \tilde{\sigma}_N(t)|\tilde{\Psi}_N(t)\,dt = \tilde{g}_N(s) \end{cases} \quad (14)$$

where $\tilde{\Psi}_D(t) = \Psi(\tilde{\sigma}_D(t))|\tilde{\sigma}_D'(t)|$ and $\tilde{\Psi}_N(t) = \Psi(\tilde{\sigma}_N(t))|\tilde{\sigma}_N'(t)|$ are the unknowns, and $\tilde{g}_D(s) = f_D(\tilde{\sigma}_D(t))$ and $\tilde{g}_N(s) = f_N(\tilde{\sigma}_N(s))|\tilde{\sigma}_N'(s)|$ are the right-hand sides.

Let us note that the new unknowns $\tilde{\Psi}_D$ and $\tilde{\Psi}_N$ are smoother than the previous ones. Indeed, in a neighbourhood of 0 they have the following behaviour

$$\tilde{\Psi}_J(t) = \mathcal{C}_J t^{q(\beta+1)-1} + \text{smoother terms}, \qquad J \in \{D, N\} \tag{15}$$

where $\mathcal{C}_J$ are positive constants and β is defined in (11).

The system (14) can be written in a more compact form as

$$\begin{pmatrix} H_{DD} & -H_{DN} \\ K_{ND} & \pi I - K_{NN} \end{pmatrix} \begin{pmatrix} \tilde{\Psi}_D \\ \tilde{\Psi}_N \end{pmatrix} = \begin{pmatrix} \tilde{g}_D \\ \tilde{g}_N \end{pmatrix} \tag{16}$$

where I denotes the identity operator and, for $J \in \{D, N\}$,

$$(H_{DJ}f)(s) = \int_0^1 h_{DJ}(t,s) f(t) dt, \qquad f \in C \tag{17}$$

with

$$h_{DJ}(t,s) = \log |\tilde{\sigma}_D(s) - \tilde{\sigma}_J(t)|, \tag{18}$$

and

$$(K_{NJ}f)(s) = \int_0^1 k_{NJ}(t,s) f(t) dt, \qquad f \in C \tag{19}$$

with

$$k_{NJ}(t,s) = \begin{cases} \dfrac{\tilde{\eta}'_N(s)[\tilde{\xi}_J(t) - \tilde{\xi}_N(s)] - \tilde{\xi}'_N(s)[\tilde{\eta}_J(t) - \tilde{\eta}_N(s)]}{[\tilde{\xi}_J(t) - \tilde{\xi}_N(s)]^2 + [\tilde{\eta}_J(t) - \tilde{\eta}_N(s)]^2}, & J \neq N \text{ or } t \neq s \\[2ex] \dfrac{1}{2} \dfrac{\tilde{\eta}'_N(t)\tilde{\xi}''_N(t) - \tilde{\xi}'_N(t)\tilde{\eta}''_N(t)}{[\tilde{\xi}'_N(t)]^2 + [\tilde{\eta}'_N(t)]^2}, & J = N \text{ and } t = s. \end{cases} \tag{20}$$

Let us note that the operators H_{DD}, H_{DN} and K_{NN} are compact as maps from C to C since their kernels are weakly singular or continuous on $[0,1] \times [0,1]$. On the contrary, the operator K_{ND} is not compact since its kernel k_{ND} has a fixed singularity in $t = s = 0$.

The following lemma is crucial for the theoretical and numerical analysis of the method we are going to propose. However, in order to state it, we need a preliminary definition.

Let $\chi_0(t)$ be a smooth cut-off function on the interval $[0,1]$ such that

$$\chi_0(t) = 1, \qquad t \in [0, \varepsilon/2], \qquad \text{supp}(\chi_0) \subset [0, \varepsilon] \tag{21}$$

for some $0 < \varepsilon < 1/2$, and $0 \leq \chi_0(t) \leq 1$, for $t \in [0,1]$.
Now, the integral operator K_{ND} can be represented as in the following lemma.

Lemma 3.1. *The integral operator* K_{ND} *defined in* (19) *satisfies the equality*

$$K_{ND} = \chi_0 \, K^\alpha \, \chi_0 + K^1_{ND}, \tag{22}$$

where χ_0 *is the cut-off function given by* (21),

$$(K^\alpha \, f)(s) = \int_0^1 k^\alpha(t,s) f(t) dt \tag{23}$$

with

$$k^\alpha(t,s) = \frac{1}{t} \, \bar{k}^\alpha \left(\frac{s}{t}\right), \qquad \bar{k}^\alpha(z) = \frac{q \, \sin\alpha \, z^{q-1}}{1 - 2z^q \cos\alpha + z^{2q}}, \tag{24}$$

and K^1_{ND} *is a compact operator on* C.

We remark that $k^\alpha(t,s)$ is a Mellin kernel having a fixed singularity at $t = s = 0$. Neverthless, when $q \geq 2$, since (see [9, formula 3.252])

$$\int_0^\infty \frac{\bar{k}^\alpha(z)}{z} dz = \sin(\phi\pi) \int_0^\infty \frac{y^{-1/q} \, dy}{1 + 2y\cos(\phi\pi) + y^2} = \pi \frac{\sin\left(\frac{\phi\pi}{q}\right)}{\sin\left(\frac{\pi}{q}\right)} \tag{25}$$

being $\alpha = (1-\phi)\pi$, the definition of the function $K^\alpha f$ can be extended in the point $s = 0$ as follows (see, for instance, [11, 14])

$$(K^\alpha f)(s) = \begin{cases} \displaystyle\int_0^1 k^\alpha(t,s) f(t) dt, & s \in (0, 1] \\[2ex] \displaystyle f(0) \int_0^\infty \frac{\bar{k}^\alpha(z)}{z} dz, & s = 0. \end{cases} \tag{26}$$

Lemma 3.2. *The integral operator* K^α *defined in* (26) *is a bounded linear operator from* C *to* C *such that*

$$\|K^\alpha\|_{C \to C} < \pi. \tag{27}$$

4 The perturbed BIE system

In the present section we are going to introduce a perturbed BIE system whose solution tends, in some way, to the solution of (16). More precisely, the new BIE system is obtained by introducing in the first equation of (16) a perturbation term $\delta \, \tilde{\Psi}_D$, for a fixed sufficiently small positive $\delta \in \mathbb{R}$. Such a system can be represented in the following form

$$\begin{pmatrix} \delta I + H_{DD} & -H_{DN} \\ K_{ND} & \pi I - K_{NN} \end{pmatrix} \begin{pmatrix} \tilde{\Psi}_{D\delta} \\ \tilde{\Psi}_{N\delta} \end{pmatrix} = \begin{pmatrix} \tilde{g}_D \\ \tilde{g}_N \end{pmatrix} \tag{28}$$

and consists in a pair of Fredholm integral equations of the second kind.
In a more general case, the boundary Σ can be composed by several arcs and have more than one corner point, i.e. $\Sigma = \bigcup_{i=1}^{n_1} \overline{\Sigma}_{D_i} \cup \bigcup_{j=1}^{n_2} \overline{\Sigma}_{N_j}$ with Σ_{D_i} and Σ_{N_j} the sections of the curve over which Dirichlet or Neumann boundary conditions are given, respectively. The resulting BIE system can be represented in a compact form as in (16), but H_{DJ}, K_{NJ}, for $J \in \{D, N\}$, have to be understood as matrices of integral operators of the type in (17) and (19), respectively, I represent a diagonal matrix whose diagonal entries are all equal to the identity operator, $\tilde{\Psi}_D$ and $\tilde{\Psi}_N$ denote the arrays of the unknowns functions, while $\tilde{g}_D$ and $\tilde{g}_N$ are the vectors containing the right hand sides. In this case, fixed a sufficiently small positive δ, the corresponding perturbed system can be represented in a matrix form as in (28).

Taking into account (22), we rewrite the system (28) in a more compact form as follows

$$(\pi\mathcal{I} + \mathcal{M}_\delta + \mathcal{K})\tilde{\boldsymbol{\Psi}}_\delta = \tilde{\boldsymbol{g}}, \tag{29}$$

where $\tilde{\boldsymbol{\Psi}}_\delta = (\tilde{\Psi}_{D\delta}, \tilde{\Psi}_{N\delta})^T$, $\tilde{\boldsymbol{g}} = (\tilde{g}_D, \tilde{g}_N)^T$, and $\mathcal{I}$, $\mathcal{M}_\delta$ and $\mathcal{K}$ are the operator matrices defined by

$$\mathcal{I} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \mathcal{M}_\delta = \begin{pmatrix} (\delta - \pi)I & 0 \\ \chi_0\, K^\alpha\, \chi_0 & 0 \end{pmatrix}, \quad \mathcal{K} = \begin{pmatrix} H_{DD} & -H_{DN} \\ K^1_{ND} & -K_{NN} \end{pmatrix}. \tag{30}$$

Fixed $\delta > 0$, the next theorem establishes sufficient conditions under which (29) has a unique solution.

Theorem 4.1. *Let δ be a fixed small positive real number and let us assume that* $\mathrm{Ker}(\pi\mathcal{I} + \mathcal{M}_\delta + \mathcal{K}) = \{0\}$ *in $C \times C$. Then, system* (29) *has a unique solution $\tilde{\boldsymbol{\Psi}}_\delta \in C \times C$ for each given right-hand side $\tilde{\boldsymbol{g}} \in C \times C$.*

Theorem 4.2. *Let assume that the unperturbed system* (16) *admits a unique solution $\tilde{\boldsymbol{\Psi}} = (\tilde{\Psi}_D, \tilde{\Psi}_N)^T \in C \times C$ for a given $\tilde{\boldsymbol{g}} = (\tilde{g}_D, \tilde{g}_N)^T \in C \times C$ and that the perturbed system* (29) *is uniquely solvable for all sufficiently small $\delta > 0$; let us say for $0 < \delta \leq \delta_0$. If condition*

$$\left\|(\pi\mathcal{I} + \mathcal{M}_{\delta_0} + \mathcal{K})^{-1}\right\| \delta_0 < 1 \tag{31}$$

is fulfilled, then $\left\|(\pi\mathcal{I} + \mathcal{M}_\delta + \mathcal{K})^{-1}\right\|$ is uniformly bounded for $0 < \delta \leq \delta_0$ and

$$\|\tilde{\boldsymbol{\Psi}} - \tilde{\boldsymbol{\Psi}}_\delta\| = O(\delta), \quad \text{as } \delta \to 0. \tag{32}$$

5 A modified Nyström method

In order to approximate the solution of system (29), first we introduce suitable approximations of all the involved operators, according to the smoothness properties of their kernels.

We begin with the operator K_{NN}. Since it has a continuous kernel we propose to discretize it by means of the following operator (see (4))

$$(K_{NN}^m f)(s) = \sum_{i=1}^{\mu_m} \lambda_{\mu_m,i}^L \, k_{NN}(t_{\mu_m,i}^L, s) f(t_{\mu_m,i}^L), \tag{33}$$

obtained by applying the Gauss-Legendre quadrature rule (4) with μ_m knots.

Concerning the approximation of the Mellin convolution operator K^α defined in (26), we follow an idea proposed in [8, 11]. Thus, defining the following finite rank operator (see (5))

$$(K^{\alpha,m} f)(s) = \sum_{i=0}^{\nu_m} \lambda_{\nu_m,i}^R k^\alpha(t_{\nu_m,i}^R, s) f(t_{\nu_m,i}^R),$$

with $\nu_m + 1$ knots. Then, setting $s_m = \dfrac{c}{m^{2-2\epsilon}}$, for some fixed positive constant c and arbitrarily small positive number ϵ, we introduce the following modified discrete operator

$$(\bar{K}^{\alpha,m} f)(s) = \begin{cases} (K^{\alpha,m} f)(s), & s_m \le s \le 1 \\ \dfrac{1}{s_m} \left[s(K^{\alpha,m} f)(s_m) + (s_m - s)\,(K^\alpha f)(0) \right], & 0 \le s < s_m. \end{cases} \tag{34}$$

In other words, when s is sufficiently far from zero i.e. $s \in [s_m, 1]$, we approximate the integral operator K^α by using the Gauss-Radau quadrature rule (5) with $\nu_m + 1$ nodes, whereas, when s is close to zero i.e. $s \in [0, s_m]$, by the linear polynomial defined by the values $(K^\alpha f)(0)$ and $(K^{\alpha,m} f)(s_m)$ at the interpolation points 0 and s_m, respectively.

In the matter of the approximation of the operator K_{ND}^1 in virtue of the continuity of its kernel k_{ND}^1, we use again formula (5) and get the approximating operator

$$(K_{ND}^{1,m} f)(s) = \sum_{i=0}^{\nu_m} \lambda_{\nu_m,i}^R \, k_{ND}^1(t_{\nu_m,i}^R, s) f(t_{\nu_m,i}^R). \tag{35}$$

From now on, we shall assume $\mu_m = m$ and $\nu_m := \lfloor am \rfloor$, with $a < 1$. In the examples considered in Section 8 we have chosen $a = 1/2$. In fact, numerical evidence shows that this is a good choice.

With regard to the operators H_{DJ}, $J \in \{D, N\}$, defined by (17)-(18), first, we rewrite it in the form

$$(H_{DJ} f)(s) = \int_0^1 \left[\kappa_J(t, s) + \log |t - s|\right] f(t) dt \tag{36}$$

where

$$\kappa_J(t, s) = \log \frac{|\tilde{\sigma}_D(s) - \tilde{\sigma}_J(t)|}{|t - s|}.$$

Then, we propose to discretize the integral operator H_{DD} by means of the operator H_{DD}^m defined as follows

$$(H_{DD}^m f)(s) = \sum_{i=0}^{\nu_m} \lambda_{\nu_m,i}^R \left[\kappa_D(t_{\nu_m,i}^R, s) + \sum_{j=0}^{\nu_m} c_j(s) p_j^{0,0}(t_{\nu_m,i}^R) \right] f(t_{\nu_m,i}^R), \quad (37)$$

and H_{DN} by the operator H_{DN}^m given by

$$(H_{DN}^m f)(s) = \begin{cases} \displaystyle\sum_{i=1}^{\mu_m} \lambda_{\mu_m,i}^L \, h_{DN}(t_{\mu_m,i}^L, s) f(t_{\mu_m,i}^L), & s \neq 0 \,\text{and}\, s \neq 1 \\ \displaystyle\sum_{i=1}^{\mu_m} \lambda_{\mu_m,i}^L \left[\kappa_N(t_{\mu_m,i}^L, s) + \sum_{j=0}^{\mu_m-1} c_j(s) p_j^{0,0}(t_{\mu_m,i}^L) \right] f(t_{\mu_m,i}^L), & \text{otherwise} \end{cases} \quad (38)$$

where the quantities $c_j(s)$ are the modified moments given in (7). Let us observe that the operator H_{DD}^m has been obtained by a proper combination of the Gauss-Radau quadrature formula (5) and the product rule (9), while for defining the operator H_{DN}^m both the Gauss-Legendre rule (4) and the product quadrature formula (8) have been applied.

Let us highlight some computational aspects about formula (37). We note that the computation of the kernel $\kappa_D(t,s)$ suffers from severe loss of accuracy, because of the numerical cancellation, when the distance between t and s is of the order of the machine precision *eps* or less. In order to avoid this pathological situation, we adopt some numerical tricks proposed in [15].

If $|t-s| < eps$, we use the approximation $\kappa_D(t,s) \approx \log|\tilde{\sigma}_D'(t)|$ when $s \neq 0$. Furthermore, taking into account (13), we rewrite $\kappa_D(t,s)$ as

$$\kappa_D(t,s) = \log \frac{|\sigma_D(\gamma(s)) - \sigma_D(\gamma(t))|}{|\gamma(t) - \gamma(s)|} + \log \frac{|\gamma(t) - \gamma(s)|}{|t-s|},$$

and approximate it as $\kappa_D(t,s) \approx \log|\sigma_D'(\gamma(t))| + \log|\gamma'(t)|$ when $s = 0$. In the latter case, according with (12), we further approximate the addendum $\log|\gamma'(t)|$ as $\log|\gamma'(t)| \approx \log q + (q-1)\log t$. Therefore, setting

$$\tilde{\kappa}_D(t,s) = \begin{cases} \log|\tilde{\sigma}_D'(t)|, & s \neq 0 \quad \text{and} \quad |t-s| < eps, \\ \log|\sigma_D'(\gamma(t))| + \log q, & s = 0 \quad \text{and} \quad t < eps, \\ \kappa_D(t,s) & \text{otherwise}, \end{cases}$$

in the practical numerical implementation in place of $(H_{DD}^m f)(s)$ given in (37), we compute

$$(\tilde{H}_{DD}^m f)(s) = \sum_{i=0}^{\nu_m} \lambda_{\nu_m,i}^R \left[\tilde{\kappa}_D(t_{\nu_m,i}^R, s) + \mathcal{A}(t_{\nu_m,i}^R, s) \sum_{j=0}^{\nu_m} c_j(s) p_j^{0,0}(t_{\nu_m,i}^R) \right] f(t_{\nu_m,i}^R),$$

where

$$\mathcal{A}(t,s) = \begin{cases} q, & s=0 \quad \text{and} \quad t < eps \\ 1, & \text{otherwise.} \end{cases} \tag{39}$$

By using similar arguments, when $s = 0$ or $s = 1$, in order to avoid loss of accuracy, we propose to compute instead of $(H_{DN}^m f)(s)$ the quantity

$$(\tilde{H}_{DN}^m f)(s) = \sum_{i=1}^{\mu_m} \lambda_{\mu_m,i}^L \left[\tilde{\kappa}_N(t_{\mu_m,i}^L, s) + \mathcal{A}(t_{\mu_m,i}^L, s) \sum_{j=0}^{\mu_m-1} c_j(s) p_j^{0,0}(t_{\mu_m,i}^L) \right] f(t_{\mu_m,i}^L)$$

with

$$\tilde{\kappa}_N(t,s) = \begin{cases} \log|\tilde{\sigma}_N'(t)|, & s=1 \quad \text{and} \quad 1-t < eps, \\ \log|\sigma_N'(\gamma(t))| + \log q, & s=0 \quad \text{and} \quad t < eps, \\ \kappa_N(t,s), & \text{otherwise,} \end{cases}$$

and $\mathcal{A}(t,s)$ defined as in (39).

At this point, paraphrasing what we have done in (30), we isolate the discretization of the Mellin convolution operator from those ones of the compact operators. Hence, fixed a sufficiently small $\delta \geq 0$, we introduce the following two matrices

$$\mathcal{M}_\delta^m = \begin{pmatrix} (\delta - \pi) I & 0 \\ \chi_0 \, \bar{K}^{\alpha,m} \, \chi_0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{K}^m = \begin{pmatrix} H_{DD}^m & -H_{DN}^m \\ K_{ND}^{1,m} & -K_{NN}^m \end{pmatrix}. \tag{40}$$

The following theorems establish some properties of such operator matrices which are important for our aims.

Theorem 5.1. *The operators $\mathcal{K}^m : C \times C \to C \times C$ are linear maps such that*

$$\lim_m \|(\mathcal{K}^m - \mathcal{K})\mathbf{f}\| = 0, \quad \forall \mathbf{f} \in C \times C. \tag{41}$$

Moreover, the set $\{\mathcal{K}^m\}_m$ is collectively compact.

Theorem 5.2. *The operators $\mathcal{M}_\delta^m : C \times C \to C \times C$ are linear maps such that, for any $\delta > 0$ sufficiently small we have*

$$\limsup_m \|\mathcal{M}_\delta^m\|_{C\times C \to C\times C} < \pi \tag{42}$$

and

$$\lim_m \|(\mathcal{M}_\delta^m - \mathcal{M}_\delta)\mathbf{f}\| = 0, \quad \forall \mathbf{f} \in C \times C. \tag{43}$$

Moreover, if for some $r \in \mathbb{N}$ and for some $0 < \sigma < q - 1$

$$\mathbf{f} \in W_r^\infty \times W_r^\infty \qquad \text{and} \qquad t^{-\sigma}\mathbf{f}(t) \in C \times C, \tag{44}$$

then the following pointwise error estimate

$$\|(\mathcal{M}_\delta^m - \mathcal{M}_\delta)\mathbf{f}(s)\|_\infty \leq \begin{cases} \mathcal{C} \max\left\{ \dfrac{s}{m^r} s_m^{-r/2-1},\, s s_m^{\sigma-1},\, s^\sigma \right\}, & s \in [0,\, s_m), \\ \dfrac{\mathcal{C}}{m^r} s^{-r/2}, & s \in [s_m,\, 1] \end{cases} \tag{45}$$

holds true with $\mathcal{C} \neq \mathcal{C}(m, s, \delta)$.

Once we have introduced the operators $\mathcal{M}_\delta^m$ and $\mathcal{K}^m$, which discretize the operators $\mathcal{M}_\delta$ and $\mathcal{K}$ appearing in (29), and studied their properties, we are going to describe a "modified" Nyström type method for the numerical solution of system (29). It consists in solving the approximating system

$$(\pi\mathcal{I} + \mathcal{M}_\delta^m + \mathcal{K}^m)\tilde{\boldsymbol{\Psi}}_\delta^m = \tilde{\boldsymbol{g}} \tag{46}$$

whose unknown is the array of functions $\tilde{\boldsymbol{\Psi}}_\delta^m = (\tilde{\Psi}_{D\delta}^m,\, \tilde{\Psi}_{N\delta}^m)^T$.

In order to compute the solution $\tilde{\boldsymbol{\Psi}}_\delta^m$ at the quadrature points, we collocate the first equation of (46) at the points $t_{\nu_m,\iota}^R$, $\iota = 0, \dots, \nu_m$ and the second one at the nodes $t_{\mu_m,\ell}^L$, $\ell = 1, \dots, \mu_m$. In this way, we get the following linear system

$$\begin{cases} \left(\delta\tilde{\Psi}_{D\delta}^m + H_{DD}^m \tilde{\Psi}_{D\delta}^m - H_{DN}^m \tilde{\Psi}_{N\delta}^m\right)(t_{\nu_m,\iota}^R) = \tilde{g}_D(t_{\nu_m,\iota}^R) & \iota = 0, \dots, \nu_m \\ \\ \left(\chi_0 \bar{K}^{\alpha,m} \chi_0 \tilde{\Psi}_{D\delta}^m + K_{ND}^{1,m} \tilde{\Psi}_{D\delta}^m + \pi\tilde{\Psi}_{N\delta}^m - K_{NN}^m \tilde{\Psi}_{N\delta}^m\right)(t_{\mu_m,\ell}^L) = \tilde{g}_N(t_{\mu_m,\ell}^L) & \\ & \ell = 1, \dots, \mu_m \end{cases} \tag{47}$$

of $\nu_m + \mu_m + 1$ equations in the $\nu_m + \mu_m + 1$ unknowns $\tilde{\Psi}_{D\delta}^m(t_{\nu_m,\iota}^R)$, $\iota = 0, \dots, \nu_m$ and $\tilde{\Psi}_{N\delta}^m(t_{\mu_m,\ell}^L)$, $\ell = 1, \dots, \mu_m$.
Let us note that each solution $\tilde{\boldsymbol{\Psi}}_\delta^m$ of system (46) provides a solution of (47) by means of its values at the collocation points. The converse is also true. For each solution of (47), there is a unique solution $\tilde{\boldsymbol{\Psi}}_\delta^m = (\tilde{\Psi}_{D\delta}^m,\, \tilde{\Psi}_{N\delta}^m)^T$ of (46) that agrees with it at the collocation knots.

In what follows we will denote by $\mathcal{M}$ the operator $\mathcal{M}_0$ given by (30) for $\delta = 0$.

Theorem 5.3. *Let* $\Sigma \setminus \{P_0\}$ *be of class* $C^2([0,1])$ *and let us suppose that the unperturbed system* (16) *admits a unique solution* $\tilde{\boldsymbol{\Psi}}$ *for a given right-hand side* $\tilde{\boldsymbol{g}}$. *Let* δ *be a fixed small positive real number and let us assume that* $\mathrm{Ker}(\pi\mathcal{I} + \mathcal{M}_\delta + \mathcal{K}) = \{0\}$ *in* $C \times C$. *Then, for sufficiently large* m, *the operators* $(\pi\mathcal{I} + \mathcal{M}_\delta^m + \mathcal{K}^m) : C \times C \to C \times C$ *are invertible and their inverses are uniformly bounded w.r.t* m.
Moreover, for all $\tilde{\boldsymbol{g}} \in C^p([0,1]) \times C^p([0,1])$ *with* p *large enough, the solution* $\tilde{\boldsymbol{\Psi}}$ *of system* (16) *and the solution* $\tilde{\boldsymbol{\Psi}}_\delta^m$ *of system* (46) *satisfy the following inequality*

$$\|\tilde{\boldsymbol{\Psi}}_\delta^m - \tilde{\boldsymbol{\Psi}}\| \leq \mathcal{C}(\delta)\left(\|(\mathcal{M}_\delta^m - \mathcal{M})\tilde{\boldsymbol{\Psi}}\| + \|(\mathcal{K}^m - \mathcal{K})\tilde{\boldsymbol{\Psi}}\|\right) \tag{48}$$

with $\mathcal{C}(\delta)$ a positive constant independent of m defined by

$$\mathcal{C}(\delta) = \sup_{m \geq m_0} \left\| (\pi \mathcal{I} + \mathcal{M}_\delta^m + \mathcal{K}^m)^{-1} \right\| .$$

Remark We observe that, by proceeding as in the proof of Theorem 4.2, one can prove that if the condition

$$\mathcal{C}(\delta_0)\, \delta_0 < 1 \tag{49}$$

is fulfilled for some small $\delta_0 > 0$, then it follows that

$$\sup_{\delta \leq \delta_0} \mathcal{C}(\delta) < \infty. \tag{50}$$

Moreover, we recall that the numerical results we have obtained (see tables 4, 7 and 10) seem to confirm that bound (50) holds. In this case, the previous theorem would imply the convergence of the proposed Nyström method when $\delta \to 0$ and $m \to \infty$. □

We point out that, in virtue of relations (15) verified by the solution $\tilde{\mathbf{\Psi}}$ of (16), if the exponent q in the smoothing transformation (12) satisfies the condition

$$q \geq \frac{r+2}{2(1+\beta)}, \quad \beta = \min\left\{ \frac{\pi}{2\alpha}, \frac{\pi}{2(2\pi - \alpha)} \right\} - 1, \tag{51}$$

for arbitrarily large $r \in \mathbb{N}$, being α the interior angle at the corner point P_0 of the boundary Σ, the solution $\tilde{\mathbf{\Psi}}$ fulfills (44) with $\sigma = q(\beta+1) - 1$. Consequently, proceeding as in the proof of (45), in correspondence with such value of σ we can estimate $\|(\mathcal{M}_\delta^m - \mathcal{M})\tilde{\mathbf{\Psi}}\|$ as follows

$$\begin{aligned} \|(\mathcal{M}_\delta^m - \mathcal{M})\tilde{\mathbf{\Psi}}\| &= \max\left\{ \|\delta \tilde{\Psi}_D\|_\infty, \left\| (\chi_0(\bar{K}^{\alpha,m} - K^\alpha)\chi_0 \tilde{\Psi}_D \right\|_\infty \right\} \\ &\leq \max\left\{ \delta \|\tilde{\mathbf{\Psi}}\|, \frac{\mathcal{C}}{m^\mu} \right\}, \end{aligned} \tag{52}$$

where $\mu = \min\{r\epsilon, 2(1-\epsilon)\sigma\}$ and $\mathcal{C} \neq \mathcal{C}(m, \delta)$.
Furthermore, the term $\|(\mathcal{K}^m - \mathcal{K})\tilde{\mathbf{\Psi}}\|$, also involved in (48), can be bounded using (6), (10) and [4, Theorem 7]. More precisely, assuming that the curve $\Sigma \setminus \{P_0\}$ is sufficiently smooth and taking into account that, if q satisfies condition (51), $\tilde{\mathbf{\Psi}} \in W_r^\infty \times W_r^\infty$, from (6) one can easily deduce (see the proof of Theorem 3.2 in [11]) that

$$\|(K_{ND}^{1,m} - K_{ND}^1)\tilde{\Psi}_D\| \leq \frac{\mathcal{C}}{m^r} \|\tilde{\Psi}_D\|_{W_r^\infty}, \qquad \mathcal{C} \neq \mathcal{C}(m),$$

$$\|(K_{NN}^m - K_{NN})\tilde{\Psi}_N\| \leq \frac{\mathcal{C}}{m^r} \|\tilde{\Psi}_N\|_{W_r^\infty}, \qquad \mathcal{C} \neq \mathcal{C}(m).$$

Moreover, by using both (6) and (10) and a well known estimate for the error of best approximation of functions belonging to the Sobolev spaces W_r^p, $1 \le p \le \infty$ (see, for example, [12, (2.5.22), p. 172]), we get

$$\|(H_{DJ}^m - H_{DJ})\tilde{\Psi}_J\| \le \frac{\mathcal{C}}{m^r}\|\tilde{\Psi}_J\|_{W_r^\infty}, \qquad \mathcal{C} \neq \mathcal{C}(m), \qquad J \in \{D, N\}$$

and then we, finally, conclude

$$\|(\mathcal{K}^m - \mathcal{K})\tilde{\mathbf{\Psi}}\| \le \frac{\mathcal{C}}{m^r}\|\tilde{\mathbf{\Psi}}\|_{r,\infty}, \qquad \mathcal{C} \neq \mathcal{C}(m). \tag{53}$$

6 Approximation of the solution of the mixed problem

The aim of this section is to approximate the solution u of the mixed boundary value problem (1), represented by (2) in the form of a single layer potential.
Using the parameterizations (13) for the arcs Σ_D and Σ_N of the boundary, the harmonic function u in any point $(x, y) \in \Omega$ is given by

$$\begin{aligned} u(x,y) &= \int_0^1 \log\left|(x,y) - (\tilde{\xi}_D(t), \tilde{\eta}_D(t))\right| \tilde{\Psi}_D(t)dt \\ &+ \int_0^1 \log\left|(x,y) - (\tilde{\xi}_N(t), \tilde{\eta}_N(t))\right| \tilde{\Psi}_N(t)dt, \end{aligned} \tag{54}$$

where $\tilde{\Psi}_D(t) = \Psi(\tilde{\sigma}_D(t))|\tilde{\sigma'}_D(t)|$ and $\tilde{\Psi}_N(t) = \Psi(\tilde{\sigma}_N(t))|\tilde{\sigma'}_N(t)|$.
We first approximate the single layer density functions $\tilde{\Psi}_D$ and $\tilde{\Psi}_N$ by the respective approximate densities $\tilde{\Psi}_{D\delta}^m$ and $\tilde{\Psi}_{N\delta}^m$ obtained by the modified Nyström method proposed in Section 5. Then, by using the Gauss-Radau formula (5), with ν_m quadrature knots, for the computation of the first resulting integral and the Gauss-Legendre quadrature formula (4), with μ_m nodes, for the computation of the second one, we obtain the approximating function

$$\begin{aligned} u_\delta^m(x,y) = &\sum_{j=0}^{\nu_m} \lambda_{\nu_m,j}^R \log\left|(x,y) - (\tilde{\xi}_D(t_{\nu_m,j}^R), \tilde{\eta}_D(t_{\nu_m,j}^R))\right| \tilde{\Psi}_{D\delta}^m(t_{\nu_m,j}^R) \\ &+ \sum_{i=1}^{\mu_m} \lambda_{\mu_m,i}^L \log\left|(x,y) - (\tilde{\xi}_N(t_{\mu_m,i}^L), \tilde{\eta}_N(t_{\mu_m,i}^L))\right| \tilde{\Psi}_{N\delta}^m(t_{\mu_m,i}^L). \end{aligned} \tag{55}$$

Let us observe that the values $\tilde{\Psi}_{D\delta}^m(t_{\nu_m,j}^R)$ and $\tilde{\Psi}_{N\delta}^m(t_{\mu_m,i}^L)$ involved in (55) are directly provided by the solution of the linear system (47).

Theorem 6.1. *Assume that q satisfies* (51), *for some $r \in \mathbb{N}$. Let $\tilde{\mathbf{\Psi}} = (\tilde{\Psi}_D, \tilde{\Psi}_N)^T$ be the unique solution of system* (16) *for a given right-hand side*

$\tilde{\boldsymbol{g}} = (\tilde{g}_D, \tilde{g}_N)^T \in C^p([0,1]) \times C^p([0,1])$ *with* p *large enough. Moreover, let us assume that system* (29) *is uniquely solvable in* $C \times C$ *for any sufficiently small* $\delta > 0$, *say* $\delta \leq \delta_0$. *Then, for any* $(x,y) \in \Omega$ *and* $\delta \leq \delta_0$, *the single layer potential* u *given in* (2), *solution of the Dirichlet-Neumann problem* (1), *and the function* u_δ^m *defined in* (55) *satisfy the following estimate*

$$|u(x,y) - u_\delta^m(x,y)| \leq \frac{\mathcal{C}}{d}\left(\frac{1}{m} + \|\tilde{\boldsymbol{\Psi}}_\delta^m - \tilde{\boldsymbol{\Psi}}\|\right) \tag{56}$$

with $d = \min_{J\in\{D,N\}} \min_{0\leq t\leq 1} |(x,y) - (\tilde{\xi}_J(t), \tilde{\eta}_J(t))|$, $\tilde{\boldsymbol{\Psi}}_\delta^m$ *the solution of* (46) *corresponding to* $\tilde{\boldsymbol{g}}$, *and* $\mathcal{C}$ *a positive constant independent of* (x,y), δ, *and* m.

We observe that, taking into account the theoretical error estimates (56) and (48), combined with (52) and (53), and also supported by the numerical evidence, we fix the value $\delta = eps$ as the optimal one and we apply the proposed method for this choice of the perturbation parameter, in order to compute the approximate potential u_δ^m.

7 Proofs

Proof of Lemma 3.1. Under the assumption that each arc of Σ is straight in some neighbourhood of the corner, for the parameterizations of Σ_D and Σ_N we can assume that

$$\sigma_D(s) - \sigma_D(0) = c_D, \quad \sigma_N(s) - \sigma_N(0) = c_N\, e^{\mathrm{i}\alpha} s, \quad s \in [0,\varepsilon]$$

where ε is a sufficiently small positive number, c_D and c_N are complex constants (points in $\mathbb{R}^2$ are here identified with complex numbers), and $\mathrm{i}^2 = -1$. Consequently, taking into account the behaviour of the smoothing transformation $\gamma(t)$ given by (12), for the parameterizations (13) we get for $s \in [0,\varepsilon]$

$$\tilde{\sigma}_D(s) - \tilde{\sigma}_D(0) = c_D\, s^q + \mathcal{O}\left(s^{q+1}\right), \quad \tilde{\sigma}_N(s) - \tilde{\sigma}_N(0) = c_N\, e^{\mathrm{i}\alpha} s^q + \mathcal{O}\left(s^{q+1}\right). \tag{57}$$

Let χ_0 be a smooth cut-off function as in (21). Then, setting

$$\bar{k}_{ND}(t,s) = (1-\chi_0(s))k_{ND}(t,s)\chi_0(t) + k_{ND}(t,s)(1-\chi_0(t)),$$

and taking into account (57), from (19) and (20) we have

$$\begin{aligned}
(K_{ND} f)(s) &= \int_0^\varepsilon [\chi_0(s) k_{ND}(t,s)\chi_0(t) + \bar{k}_{ND}(t,s)] f(t) dt + \int_\varepsilon^1 k_{ND}(t,s) f(t) dt \\
&= \int_0^1 \chi_0(s) \frac{q s^{q-1} t^q \sin\alpha}{s^{2q} - 2 s^q t^q \cos\alpha + t^{2q}} \chi_0(t) f(t) dt + \text{smoother terms} \\
&= (\chi_0 K^\alpha \chi_0 f)(s) + (K_{ND}^1 f)(s)
\end{aligned}$$

where K^α is the operator defined in (23)-(24) and K_{ND}^1 is a compact operator on C. □

Proof of Lemma 3.2. The linearity of the operator K^α is a trivial consequence of definition (26). Now, let us prove that for any $f \in C$, $K^\alpha f \in C$. Since $k^\alpha(t,s)$ is continuous for $t+s>0$, the continuity of $(K^\alpha f)(s)$ is obvious for all $s \in (0,1]$. For $s=0$, it stems from the definition (26), taking also (25) into account. Moreover, by (25) we also get

$$\|K^\alpha\|_{C\to C} = \int_0^\infty \frac{|\bar{k}^\alpha(z)|}{z} dz = \frac{|\sin(\phi\pi)|}{\sin(\phi\pi)} \pi \frac{\sin\left(\frac{\phi\pi}{q}\right)}{\sin\left(\frac{\pi}{q}\right)} < \pi.$$

□

In the sequel, for simplicity of notations, sometimes we shall omit to write the subscript ${}_{C\times C\to C\times C}$ in the symbol $\|\cdot\|_{C\times C\to C\times C}$ used to denote the norm of an operator acting from $C\times C$ to $C\times C$.

Proof of Theorem 4.1. At first we note that $\|\pi\mathcal{I}\| = \pi$ and, in virtue of Lemma 3.2, for any array $\mathbf{f} = (f_1,\, f_2)^T \in C\times C$, we have

$$\|\mathcal{M}_\delta \mathbf{f}\| \le \max\{(\pi-\delta)\|f_1\|_\infty, \|K^\alpha\|\|f_1\|_\infty\} < \pi\,\|\mathbf{f}\|.$$

Being $\|\mathcal{M}_\delta\| < \pi$, from the geometric series theorem, we can deduce that $(\pi\mathcal{I}+\mathcal{M}_\delta)^{-1} : C\times C \to C\times C$ exists and is a bounded operator with

$$\|(\pi\mathcal{I}+\mathcal{M}_\delta)^{-1}\| \le \frac{1}{\pi - \|\mathcal{M}_\delta\|}.$$

Consequently, system (29) is equivalent to the following problem

$$\tilde{\boldsymbol{\Psi}}_\delta + (\pi\mathcal{I}+\mathcal{M}_\delta)^{-1}\mathcal{K}\tilde{\boldsymbol{\Psi}}_\delta = (\pi\mathcal{I}+\mathcal{M}_\delta)^{-1}\tilde{\boldsymbol{g}}. \tag{58}$$

Now, we observe that the operator $(\pi\mathcal{I}+\mathcal{M}_\delta)^{-1}\mathcal{K} : C\times C \to C\times C$ is compact since $\mathcal{K} : C\times C \to C\times C$ is a matrix of compact operators. Thus, for the equation (58) the Fredholm alternative theorem holds true. Hence, by the assumption, we can deduce that (29) has a unique solution for each given right-hand side $\tilde{\boldsymbol{g}} \in C\times C$. □

Proof of Theorem 4.2. For any $\delta \le \delta_0$ we can write

$$\pi\mathcal{I}+\mathcal{M}_\delta+\mathcal{K} = (\pi\mathcal{I}+\mathcal{M}_{\delta_0}+\mathcal{K})\left[I - (\pi\mathcal{I}+\mathcal{M}_{\delta_0}+\mathcal{K})^{-1}(\mathcal{M}_{\delta_0}-\mathcal{M}_\delta)\right]$$

and then, being by (31)

$$\delta_0 = \sup_{0<\delta\le\delta_0}(\delta_0-\delta) = \sup_{0<\delta\le\delta_0}\|\mathcal{M}_{\delta_0}-\mathcal{M}_\delta\| < \frac{1}{\|(\pi\mathcal{I}+\mathcal{M}_{\delta_0}+\mathcal{K})^{-1}\|},$$

the inverse operators $\left[I - (\pi\mathcal{I} + \mathcal{M}_{\delta_0} + \mathcal{K})^{-1}(\mathcal{M}_{\delta_0} - \mathcal{M}_\delta)\right]^{-1}$ exist and are uniformly bounded, in virtue of the geometric series theorem.
Consequently, for any $\delta \le \delta_0$, we have

$$\left\|(\pi\mathcal{I} + \mathcal{M}_\delta + \mathcal{K})^{-1}\right\| \le \frac{\left\|(\pi\mathcal{I} + \mathcal{M}_{\delta_0} + \mathcal{K})^{-1}\right\|}{1 - \delta_0 \left\|(\pi\mathcal{I} + \mathcal{M}_{\delta_0} + \mathcal{K})^{-1}\right\|} =: M_0. \tag{59}$$

Hence, since it easily seen that

$$\|\tilde{\mathbf{\Psi}} - \tilde{\mathbf{\Psi}}_\delta\| \le \left\|(\pi\mathcal{I} + \mathcal{M}_\delta + \mathcal{K})^{-1}\right\| \; \delta \, \|\tilde{\mathbf{\Psi}}\|, \tag{60}$$

from (59) we deduce that, for $\delta \le \delta_0$,

$$\|\tilde{\mathbf{\Psi}} - \tilde{\mathbf{\Psi}}_\delta\| \le \delta M_0 \|\tilde{\mathbf{\Psi}}\|$$

and the condition (32) follows. □

Proof of Theorem 5.1. By definitions (30) and (40), it results that if

$$\lim_{m\to\infty} \|(H_{DD}^m - H_{DD})f_1\|_\infty = 0, \quad \lim_{m\to\infty} \|(H_{DN}^m - H_{DN})f_2\|_\infty \; = 0, \tag{61}$$

$$\lim_{m\to\infty} \|(K_{ND}^{1,m} - K_{ND}^1)f_1\|_\infty = 0, \quad \lim_{m\to\infty} \|(K_{NN}^m - K_{NN})f_2\|_\infty \; = 0, \tag{62}$$

for any $\mathbf{f} = (f_1,\, f_2)^T \in C \times C$, then (41) follows. Now, taking into account the definitions (33) and (35), the properties of the kernels k_{ND}^1 and k_{NN}, and the convergence of the Radau formula (5) and of the Gauss-Legendre quadrature rule (4), the limit conditions (62) (see, for instance, [2]) can be immediately deduced. In order to prove (61), we recall (36)-(38) and use both the convergence of the formulas and (4) and (5), and the convergence of the product rules (8) and (9). Moreover, the sets of operators $\{K_{ND}^{1,m}\}_m$ and $\{K_{NN}^m\}_m$ are collectively compact (see, for instance, [10, Theorem 12.8] and [1, Theorem 5.1]) as well as the sets $\{H_{DD}^m\}_m$ and $\{H_{DN}^m\}_m$ (see [17, Lemma p. 266 and Theorem 2 p. 269]). Consequently, the sequence of operators $\{\mathcal{K}^m\}_m$ is collectively compact, and this completes the proof. □

In order to prove Theorem 5.2 we need the following lemma.

Lemma 7.1. *Let $k^\alpha(t,s)$ be the kernel defined in* (24)*. Then, for each r such that $1 \le r < 2(q+1)$,*

$$\left\| \frac{\partial^r k^\alpha(\cdot, s)}{\partial t^r} \varphi^r \right\|_1 \le \mathcal{C}\, s^{-\frac{r}{2}}, \qquad \mathcal{C} \ne \mathcal{C}(s), \quad s \in (0,\, 1],$$

where $\varphi(t) = \sqrt{t(1-t)}$.

Proof. By definition, we can write

$$k^{\alpha}(t,s) = \frac{qs^{q-1}t^q \sin\alpha}{s^{2q} - 2t^q s^q \cos\alpha + t^{2q}} = \frac{1}{2\mathrm{i}}\, q\, s^{q-1} \left[\frac{1}{s^q - e^{\mathrm{i}\alpha} t^q} - \frac{1}{s^q - e^{-\mathrm{i}\alpha} t^q}\right]$$

($\mathrm{i}^2 = -1$), from which, for the r-th partial derivative, we can easily obtain

$$\frac{\partial^r k^{\alpha}(t,s)}{\partial t^r} = \frac{1}{2\mathrm{i}} q s^{q-1} \sum_{k=1}^{r} c_k(q) k! t^{kq-r} \left[\frac{(e^{\mathrm{i}\alpha})^k}{\left(s^q - e^{\mathrm{i}\alpha} t^q\right)^{k+1}} - \frac{(e^{-\mathrm{i}\alpha})^k}{\left(s^q - e^{-\mathrm{i}\alpha} t^q\right)^{k+1}}\right],$$

for suitable constants $c_k(q)$, $k = 1, \ldots, r$, depending on the parameter q. Consequently, we deduce

$$\begin{aligned}
\left\| \frac{\partial^r k^{\alpha}(\cdot,s)}{\partial t^r} \varphi^r \right\|_1 &\leq \frac{1}{|2\mathrm{i}|} q s^{q-1} \sum_{k=1}^{r} |c_k(q)| k! \\
&\times \int_0^1 \left| t^{kq-\frac{r}{2}} (1-t)^{\frac{r}{2}} \frac{\sum_{j=0}^{k+1} \binom{k+1}{j} (t^q)^j (s^q)^{k+1-j} \left(e^{\mathrm{i}\alpha(k-j)} - e^{-\mathrm{i}\alpha(k-j)}\right)}{\left(s^{2q} - 2s^q t^q \cos\alpha + t^{2q}\right)^{k+1}} \right| dt \\
&\leq q s^{q-1} \sum_{k=1}^{r} |c_k(q)| k! \left(\int_0^1 t^{kq-\frac{r}{2}} \frac{\left(t^q + s^q\right)^{k+1}}{\left(s^{2q} - 2s^q t^q \cos\alpha + t^{2q}\right)^{k+1}} dt \right),
\end{aligned}$$

from which, setting $t = sz$ and taking into account that $q > \frac{r}{2} - 1$ by assumption, we have

$$\begin{aligned}
\left\| \frac{\partial^r k^{\alpha}(\cdot,s)}{\partial t^r} \varphi^r \right\|_1 &\leq q \sum_{k=1}^{r} |c_k(q)| k! s^{-\frac{r}{2}} \left(\int_0^{\frac{1}{s}} \frac{z^{kq-\frac{r}{2}} (z^q+1)^{k+1}}{(1 - 2z^q \cos\alpha + z^{2q})^{k+1}} dz \right) \\
&\leq q s^{-\frac{r}{2}} \sum_{k=1}^{r} |c_k(q)| k! \left(\int_0^{\infty} \frac{z^{kq-\frac{r}{2}} (z^q+1)^{k+1}}{(1 - 2z^q \cos\alpha + z^{2q})^{k+1}} dz \right) \leq \mathcal{C}(r,q) s^{-\frac{r}{2}}.
\end{aligned}$$

□

Proof of Theorem 5.2. First, we prove (42). For any $\mathbf{f} = (f_1,\, f_2)^T \in C \times C$, by (40) we have

$$\|\mathcal{M}_{\delta}^{m} \mathbf{f}\| = \max\{\|(\delta - \pi) I f_1\|_{\infty},\, \left\|\chi_0 \bar{K}^{\alpha,m} \chi_0 f_1\right\|_{\infty}\}, \tag{63}$$

and for the second term we can write

$$\left\|\chi_0 \bar{K}^{\alpha,m} \chi_0 f_1\right\|_{\infty} = \max\left\{ \sup_{s\in[0,\, s_m]} |(\chi_0 \bar{K}^{\alpha,m} \chi_0 f_1)(s)|,\, \sup_{s\in[s_m,\, 1]} |(\chi_0 \bar{K}^{\alpha,m} \chi_0 f_1)(s)| \right\}. \tag{64}$$

In order to estimate the two terms in the braces in (64), we can proceed as in the proof of Theorem 3.1 in [11]. However, to make the proof self-contained we report the details. By definition (34) of the modified operator $\bar{K}^{\alpha,m}$, for $s \in [s_m, 1]$, using (27) and taking the constant sign of the kernel into account, we can write

$$|(\chi_0 \bar{K}^{\alpha,m} \chi_0 f_1)(s)| \leq \|f_1\|_\infty \left| \sum_{i=0}^{\nu_m} \lambda^R_{\nu_m,i} k^\alpha(t^R_{\nu_m,i}, s) \right|$$
$$\leq \|f_1\|_\infty \left(\int_0^1 |k^\alpha(t,s)|\, dt + |e^R_m(k^\alpha(\cdot,s))| \right) \leq \|f_1\|_\infty \left(\|K^\alpha\| + |e^R_m(k^\alpha(\cdot,s))| \right).$$

Then, since in virtue of (6) and Lemma 7.1, for $s \in [s_m,\, 1]$ we have

$$|e^R_m(k^\alpha(\cdot,s))| \leq \frac{\mathcal{C}}{m^r} \left\| \frac{\partial^r k^\alpha(\cdot,s)}{\partial t^r} \varphi^r \right\|_1 \leq \frac{\mathcal{C}}{m^r} s^{-r/2} \leq \frac{\mathcal{C}}{m^{r\epsilon}},$$

we can deduce that

$$\sup_{s \in [s_m,\, 1]} |(\chi_0 \bar{K}^{\alpha,m} \chi_0 f_1)(s)| \leq \|f_1\|_\infty \left(\|K^\alpha\| + \frac{\mathcal{C}}{m^{r\epsilon}} \right). \tag{65}$$

For $s \in [0, s_m]$, by (34) and by using (65) we can write

$$\begin{aligned} |(\chi_0 \bar{K}^{\alpha,m} \chi_0 f_1)(s)| &\leq \max\{|(K^{\alpha,m} \chi_0 f_1)(s_m)|,\ |(K^\alpha \chi_0 f_1)(0)|\} \\ &\leq \max\left\{ \sup_{s \in [s_m,\, 1]} |(K^{\alpha,m} \chi_0 f_1)(s)|,\ |(K^\alpha \chi_0 f_1)(0)| \right\} \\ &\leq \|f_1\|_\infty \left(\|K^\alpha\| + \frac{\mathcal{C}}{m^{r\epsilon}} \right). \end{aligned} \tag{66}$$

Hence, combining (64), (65), and (66) with (63), we have

$$\|\mathcal{M}^m_\delta \mathbf{f}\| \leq \|\mathbf{f}\| \max\left\{ \pi - \delta, \|K^\alpha\| + \frac{\mathcal{C}}{m^{r\epsilon}} \right\},$$

from which, recalling (27), (42) immediately follows.
The thesis (43) can be proved by observing that, in virtue of (40) and (30), we have

$$\|(\mathcal{M}^m_\delta - \mathcal{M}_\delta)\mathbf{f}\| = \left\|(\chi_0(\bar{K}^{\alpha,m} - K^\alpha)\chi_0 f_1\right\|_\infty \leq \left\|(\bar{K}^{\alpha,m} - K^\alpha)\chi_0 f_1\right\|_\infty$$

and, then, by proceeding as in the proof of Theorem 3.1 in [11]. Estimate (45) can be deduced from Theorem 3.2 in [11]. □

Proof of Theorem 5.3. In virtue of Theorem 5.2, we can state that the operators $\pi\mathcal{I} + \mathcal{M}^m_\delta : C \times C \to C \times C$ are bounded and pointwise convergent to $\pi\mathcal{I} + \mathcal{M}_\delta$. Moreover, since $\|\pi\mathcal{I}\| = \pi$, taking into account (42), we can apply the Neumann series Theorem and deduce that, for sufficiently large m, say $m \geq m_0$, the operators $(\pi\mathcal{I} + \mathcal{M}^m_\delta)^{-1} : C \times C \to C \times C$ exist and are uniformly bounded with

$$\left\|(\pi\mathcal{I}+\mathcal{M}_\delta^m)^{-1}\right\| \le \frac{1}{\pi - \sup\limits_{n\ge m_0}\|\mathcal{M}_\delta^n\|}.$$

Then, using also Theorem 5.1 and [10, Theorem 10.8 and Problem 10.3], we can claim that, the operators $(\pi\mathcal{I}+\mathcal{M}_\delta^m+\mathcal{K}^m)^{-1}: C\times C \to C\times C$ exist and are uniformly bounded for $m \ge m_0$, i.e. the method is stable. Moreover, by standard arguments, we get

$$\tilde{\mathbf{\Psi}}_\delta^m - \tilde{\mathbf{\Psi}} = (\pi\mathcal{I}+\mathcal{M}_\delta^m+\mathcal{K}^m)^{-1}\left[(\mathcal{M}_\delta^m-\mathcal{M})\tilde{\mathbf{\Psi}} + (\mathcal{K}^m-\mathcal{K})\tilde{\mathbf{\Psi}}\right]$$

from which (48) immediately follows. □

Proof of Theorem 6.1. Setting $L_J(x,y,t) = \log|(x,y)-(\tilde{\xi}_J(t),\,\tilde{\eta}_J(t))|$, $J \in \{D,N\}$, for any fixed point $(x,y)\in\Omega$, by (54) and (55) we have

$$\begin{aligned}
&|u(x,y)-u_\delta^m(x,y)|\\
&\le \left|\int_0^1 L_D(x,y,t)\tilde{\Psi}_D(t)\,dt - \sum_{j=0}^{\nu_m}\lambda^R_{\nu_m,j}L_D(x,y,t^R_{\nu_m,j})\tilde{\Psi}^m_{D\delta}(t^R_{\nu_m,j})\right|\\
&+ \left|\int_0^1 L_N(x,y,t)\tilde{\Psi}_N(t)\,dt - \sum_{i=1}^{\mu_m}\lambda^L_{\mu_m,i}L_N(x,y,t^L_{\mu_m,i})\tilde{\Psi}^m_{N\delta}(t^L_{\mu_m,i})\right|\\
&=: A+B. \qquad (67)
\end{aligned}$$

Now let us estimate the first term of (67). The second one can be estimated in a similar way. We can write

$$\begin{aligned}
A &\le \left|\int_0^1 L_D(x,y,t)\tilde{\Psi}_D(t)\,dt - \sum_{j=0}^{\nu_m}\lambda^R_{\nu_m,j}L_D(x,y,t^R_{\nu_m,j})\tilde{\Psi}_D(t^R_{\nu_m,j})\right|\\
&+\left|\sum_{j=0}^{\nu_m}\lambda^R_{\nu_m,j}L_D(x,y,t^R_{\nu_m,j})\left[\tilde{\Psi}_D(t^R_{\nu_m,j})-\tilde{\Psi}^m_{D\delta}(t^R_{\nu_m,j})\right]\right| =: A_1+A_2.
\end{aligned}$$

In virtue of (6) and taking into account (15), we have

$$\begin{aligned}
A_1 &\le \frac{\mathcal{C}}{m}E_{2m-1}\left(\left[L_D(x,y,\cdot)\tilde{\Psi}_D\right]'\right)_{\varphi,1} \le \frac{\mathcal{C}}{m}\left\|\left[L_D(x,y,\cdot)\tilde{\Psi}_D(\cdot)\right]'\varphi\right\|_1\\
&\le \frac{\mathcal{C}}{m}\int_0^1\left[\sum_{j=0}^{1}L_D^{(j)}(x,y,t)\tilde{\Psi}_D^{(1-j)}(t)\right]t^{\frac{1}{2}}\,dt \le \frac{\mathcal{C}}{m}\sum_{j=0}^{1}\|[L_D^{(j)}(x,y,\cdot)]\|_\infty.
\end{aligned}$$

Then, since the estimates

$$\|L_J(x,y,\cdot)\|_\infty \le \frac{\mathcal{C}}{d}, \qquad \|L'_J(x,y,\cdot)\|_\infty \le \frac{\mathcal{C}}{d}, \qquad \mathcal{C}\ne\mathcal{C}((x,y)),$$

hold true with $d = \min_{J\in\{D,N\}} \min_{0\le t\le 1} |(x,y) - (\tilde{\xi}_J(t),\, \tilde{\eta}_J(t))|$ we can deduce that $A_1 \le \dfrac{\mathcal{C}}{md}$. Moreover, we have

$$
\begin{aligned}
A_2 &\le \|L_D(x,y,\cdot)\|_\infty \|\tilde{\Psi}_D - \tilde{\Psi}^m_{D\delta}\|_\infty \sum_{j=0}^{\nu_m} \lambda^R_{\nu_m,j} \\
&\le \frac{\mathcal{C}}{d} \|\tilde{\Psi}_D - \tilde{\Psi}^m_{D\delta}\|_\infty.
\end{aligned}
$$

Consequently, estimating the term A in (67) by using the obtained estimates of A_1 and A_2 and proceeding in an analogous way in order to estimate also the term B, we get the thesis (56). □

8 Numerical tests

In this section we show the performance of our method by some numerical examples. In order to give the boundary conditions f_D and f_N, in all the tests we choose a test harmonic function u and we combine the parametric representation of the considered boundary Σ with the smoothing transformation satisfying (12) given by (see [6])

$$
\gamma(t) = \frac{t^q}{t^q + (1-t^q)}, \quad 0 \le t \le 1.
$$

We define the following absolute errors

$$
\varepsilon^m_\delta(x,y) = |u(x,y) - u^m_\delta(x,y)|, \qquad (x,y) \in \Omega
$$

$$
e^m_\delta = \max_{j=1,\dots,1000} \left\| \bar{\mathbf{\Psi}}^m_\delta(s_j) - \bar{\mathbf{\Psi}}^{256}_{eps}(s_j) \right\|_\infty \tag{68}
$$

and

$$
err^m_\delta = \max_{j=1,\dots,1000} \left\| \bar{\mathbf{\Psi}}^m_\delta(s_j) - \bar{\mathbf{\Psi}}^m_{eps}(s_j) \right\|_\infty ,
$$

where u^m_δ is given by (55), $\bar{\mathbf{\Psi}}^m_\delta = (\bar{\Psi}^m_{D\delta},\, \bar{\Psi}^m_{N\delta})^T$, with $\bar{\Psi}^m_{D\delta}$ the Lagrange polynomial interpolating $\tilde{\Psi}^m_{D\delta}$ in the Radau quadrature nodes $t^R_{\nu_m,\iota}$, $\iota = 0,\dots,\nu_m$ and $\bar{\Psi}^m_{N\delta}$ the Lagrange polynomial interpolating $\tilde{\Psi}^m_{N\delta}$ in the μ_m Legendre knots $t^L_{\mu_m,\ell}$, $\ell = 1,\dots,\mu_m$, $s_1,\dots,s_{1000}$ are equispaced points in the interval $(0,1)$, and eps is the machine precision. Concerning the errors err^m_δ, we also consider the estimated order of convergence when m is fixed and $\delta \to 0$ given by

$$
EOC^m = \frac{\log\left(err^m_{\delta_1}/err^m_{\delta_2}\right)}{\log\left(\delta_1/\delta_2\right)}.
$$

The values of EOC^m reported in tables 2, 6 and 10 show that $err^m_\delta = \mathcal{O}(\delta)$ as $\delta \to 0$, for each sufficiently large m. The numerical evidence seems to

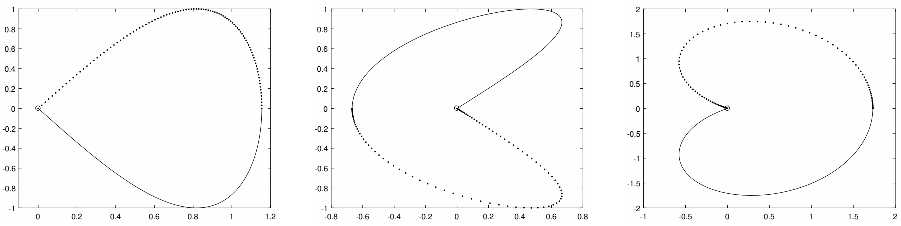

Figure 1: From left to right, the domains of Example 8.1, Example 8.2 and Example 8.3 whose arcs Σ_D and Σ_N are represented by the solid and dashed line, respectively.

support the conjecture that condition (32) is verified. Furthermore, in all our tests we have also verified that the value of δ, when $\delta \to 0$, doesn't have any effect on the conditioning of the final linear system (47) (see, for instance, Table 3) as well as the constant $\mathcal{C}(\delta)$ involved in the estimate (48) seems to be uniformly bounded for sufficiently small δ (see tables 4, 7 and 10).

Then, both the theoretical error estimate and the numerical evidence have suggested us to choose $\delta = eps$ as optimal value for the computation of the approximate potential u_δ^m.
Moreover, in all the examples we have taken $\mu_m = m$ and $\nu_m = \frac{m}{2}$ since the numerical results show that this is a good choice.

Example 8.1. *Let us consider the problem* (1) *defined in the teardrop domain* Ω *(see Figure 1) bounded by the curve* Σ *having the following parametric representation*

$$\sigma(t) = \left(\frac{2}{\sqrt{3}} \sin \pi t, - \sin 2\pi t \right), \qquad t \in [0, 1].$$

The domain has a single outward-pointing corner $P_0 = (0, 0)$ *with interior angle* $\alpha = \frac{2}{3}\pi$. *Then, to give a realistic behavior of the solution* u *at the corner point, we consider problem* (1) *having as solution the harmonic function* $u(x, y) = \rho^{\frac{3}{4}} \cos\left(\frac{3\theta}{4}\right)$, *where* (ρ, θ) *are the polar coordinates centered at* P_0. *According with* (51), *taking the smoothing parameter* $q = 4$ *one has that* $\tilde{\boldsymbol{\Psi}} \in W_r^\infty \times W_r^\infty$ *with* $r = 1$.
Table 1 contains the errors $\varepsilon_{eps}^m(x, y)$ *and* e_{eps}^m *for sufficiently large* m, *Table 2 contains the errors* err_δ^m *and the estimated orders of convergence* EOC^m *we get for different values of* m. *Finally, we report the condition numbers (CN) in infinity norm of the matrix associated to the linear system* (47) *in Table 3 and the errors* e_δ^m *in Table 4, obtained for fixed* m *and different small values of the parameter* δ.

Table 1: $\varepsilon^m_{eps}(x,y)$ and e^m_{eps} for Example 8.1 with $q=4$, $c=10$ and $\epsilon=1e{-}03$

m	$\varepsilon^m_{eps}(0.01,0)$	$\varepsilon^m_{eps}(0.1,0)$	$\varepsilon^m_{eps}(0.3,0.1)$	$\varepsilon^m_{eps}(0.8,0.4)$	e^m_{eps}
16	1.10e-02	7.14e-03	3.60e-03	2.45e-02	2.98e+00
32	2.99e-05	2.78e-05	3.13e-04	1.05e-03	3.65e-01
64	1.92e-06	8.17e-06	2.71e-07	9.77e-08	1.68e-03
128	1.06e-09	1.58e-09	1.81e-11	2.21e-11	1.87e-06
256	1.11e-16	9.44e-16	3.50e-15	1.20e-14	-

Table 2: err^m_δ and EOC^m for Example 8.1 with $q=4$, $c=10$ and $\epsilon=1e{-}03$

m	δ	err^m_δ	EOC^m	m	δ	err^m_δ	EOC^m
16	10^{-10}	2.73e-09	1.00e+00	32	10^{-10}	2.81e-09	1.00e+00
	10^{-12}	2.73e-11	1.00e+00		10^{-12}	2.81e-11	1.00e+00
	10^{-14}	2.67e-13	1.01e+00		10^{-14}	2.76e-13	1.00e+00
64	10^{-10}	2.85e-09	1.00e+00	128	10^{-10}	2.86e-09	1.00e+00
	10^{-12}	2.85e-11	1.00e+00		10^{-12}	2.86e-11	1.00e+00
	10^{-14}	2.82e-13	1.00e+00		10^{-14}	2.90e-13	9.96e-01

Example 8.2. *Let Ω be the boomerang domain represented in Figure 1 whose contour Σ is parametrized by*

$$\sigma(t) = \left(\frac{2}{3}\sin(3\pi t), \sin(2\pi t)\right), \qquad t\in[0,1].$$

The curve Σ has a single corner point at $P_0 = (0,\,0)$ with interior angle $\alpha = \frac{3}{2}\pi$. We test our method choosing as exact solution $u(x,y) = \rho^{\frac{1}{3}}\cos\left(\frac{\theta}{3}\right)$ with (ρ,θ) the polar coordinates centered at P_0. According with (51), taking the smoothing parameter $q=4$ one has that $\tilde{\boldsymbol{\Psi}} \in W^\infty_r \times W^\infty_r$ with $r=1$. Tables 5, 6 and 7 report the numerical results we get.

Table 3: Condition numbers for Example 8.1 with $q=4$, $c=10$ and $\epsilon = 1e-03$

m	δ	CN	m	δ	CN
32	10^{-10}	7.40e+02	64	10^{-10}	3.03e+03
	10^{-12}	7.40e+02		10^{-12}	3.03e+03
	10^{-14}	7.40e+02		10^{-14}	3.03e+03
	eps	7.40e+02		*eps*	3.03e+03
128	10^{-10}	1.23e+04	256	10^{-10}	5.71e+04
	10^{-12}	1.23e+04		10^{-12}	5.71e+04
	10^{-14}	1.23e+04		10^{-14}	5.71e+04
	eps	1.23e+04		*eps*	5.71e+04

Table 4: e_δ^m for Example 8.1 with $q = 4$, $c = 10$ and $\epsilon = 1e-03$

m	δ	e_δ^m	m	δ	e_δ^m	m	δ	e_δ^m
32	10^{-10}	3.65e-01	64	10^{-10}	1.68e-03	128	10^{-10}	1.87e-06
	10^{-12}	3.65e-01		10^{-12}	1.68e-03		10^{-12}	1.87e-06
	10^{-14}	3.65e-01		10^{-14}	1.68e-03		10^{-14}	1.87e-06

Table 5: $\varepsilon_{eps}^m(x,y)$ and e_{eps}^m for Example 8.2 with $q = 4$, $c = 10$ and $\epsilon = 1e-03$

m	$\varepsilon_{eps}^m(0.01,0)$	$\varepsilon_{eps}^m(0.1,0)$	$\varepsilon_{eps}^m(0.3,0.1)$	$\varepsilon_{eps}^m(0.8,0.4)$	e_{eps}^m
16	4.47e-03	2.77e-02	3.72e-02	8.63e-02	4.25e+00
32	2.44e-04	2.56e-03	3.58e-03	9.57e-03	3.03e-01
64	2.08e-06	9.80e-07	7.18e-05	2.35e-06	3.43e-02
128	6.51e-08	1.67e-09	3.31e-08	7.45e-08	1.15e-02
256	4.76e-09	6.61e-11	1.04e-10	3.11e-11	

Example 8.3. *Let us consider the heart-shaped domain (see Figure 1) bounded by the curve Σ given by the following parametric equation*

$$\sigma(t) = \begin{pmatrix} \cos(1+\frac{\alpha}{\pi})\pi t - \sin(1+\frac{\alpha}{\pi})\pi t \\ \sin(1+\frac{\alpha}{\pi})\pi t + \cos(1+\frac{\alpha}{\pi})\pi t \end{pmatrix} \begin{pmatrix} \tan\frac{\alpha}{2} \\ 1 \end{pmatrix} - \begin{pmatrix} \tan\frac{\alpha}{2} \\ \cos\pi t \end{pmatrix}, \; t \in [0,1],$$

where $\alpha \in (\pi, 2\pi)$ is the interior angle of the single corner $P_0 = (0,0)$.
In Tables 8, 9 and 10 we show the results obtained by fixing $\alpha = \frac{5}{3}\pi$ and by choosing as boundary data those corresponding to the exact solution $u(x,y) = e^x \cos y$. In this case, since the potential $u(x,y)$ is a very smooth function, we can conjecture that the solution $\tilde{\mathbf{\Psi}}$ is very smooth, too, i.e, $\tilde{\mathbf{\Psi}} \in W_r^\infty \times W_r^\infty$ with r large enough.

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Table 6: err_δ^m and EOC^m for Example 8.2 with $q = 4$, $c = 10$ and $\epsilon = 1e{-}03$

m	δ	err_δ^m	EOC^m	m	δ	err_δ^m	EOC^m
32	10^{-10}	5.37e-09	1.00e+00	64	10^{-10}	5.48e-09	1.00e+00
	10^{-12}	5.38e-11	1.00e+00		10^{-12}	5.48e-11	1.00e+00
	10^{-14}	5.28e-13	1.00e+00		10^{-14}	5.42e-13	1.00e+00
128	10^{-10}	5.48e-09	1.00e+00	256	10^{-10}	5.50e-09	1.00e+00
	10^{-12}	5.48e-11	1.00e+00		10^{-12}	5.49e-11	1.00e+00
	10^{-14}	5.47e-13	1.00e+00		10^{-14}	5.44e-13	9.79e-01

Table 7: e_δ^m for Example 8.2 with $q = 4$, $c = 10$ and $\epsilon = 1e - 03$

m	δ	e_δ^m	m	δ	e_δ^m	m	δ	e_δ^m
32	10^{-10}	3.03-01	64	10^{-10}	3.43-02	128	10^{-10}	1.15-02
	10^{-12}	3.03-01		10^{-12}	3.43-02		10^{-12}	1.15-02
	10^{-14}	3.03-01		10^{-14}	3.43-02		10^{-14}	1.15-02

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Table 8: $\varepsilon_{eps}^m(x, y)$ and e_{eps}^m for Example 8.3 with $q = 4$, $c = 10$ and $\epsilon = 1e{-}03$

m	$\varepsilon_{eps}^m(0.01, 0)$	$\varepsilon_{eps}^m(0.1, 0)$	$\varepsilon_{eps}^m(0.3, 0.1)$	$\varepsilon_{eps}^m(0.8, 0.4)$	e_{eps}^m
16	1.88e-01	7.81e-01	3.00e-01	8.76e-01	1.09e+01
32	1.14e-02	1.56e-02	2.88e-02	3.74e-02	5.29e+00
64	3.36e-05	5.84e-04	8.43e-06	3.12e-05	4.60e-01
128	4.36e-11	1.17e-08	3.55e-10	1.85e-12	2.23e-03
256	9.33e-15	2.66e-13	4.17e-14	5.40e-14	

Table 9: err_δ^m and EOC^m for Example 8.3 with $q = 4$, $c = 10$ and $\epsilon = 1e{-}03$

m	δ	err_δ^m	EOC^m	m	δ	err_δ^m	EOC^m
16	10^{-10}	3.91e-09	1.00e+00	32	10^{-10}	6.70e-09	1.00e+00
	10^{-12}	3.91e-11	1.00e+00		10^{-12}	6.69e-11	1.00e+00
	10^{-14}	3.77e-13	1.01e+00		10^{-14}	6.55e-13	1.00e+00
64	10^{-10}	1.08e-08	1.00e+00	128	10^{-10}	1.12e-08	1.00e+00
	10^{-12}	1.08e-10	1.00e+00		10^{-12}	1.12e-10	1.00e+00
	10^{-14}	1.06e-12	1.00e+00		10^{-14}	1.13e-12	9.99e-01

Table 10: e_δ^m for Example 8.3 with $q = 4$, $c = 10$ and $\epsilon = 1e - 03$

m	δ	e_δ^m	m	δ	e_δ^m
64	10^{-10}	6.74e-01	128	10^{-10}	3.31e-03
	10^{-12}	6.74e-01		10^{-12}	3.31e-03
	10^{-14}	6.74e-01		10^{-14}	3.31e-03

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