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Hölder regularity for bounded solutions to a class of anisotropic operators

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Abstract. In this note we show the Hölder regularity for bounded solutions to a class of anisotropic elliptic operators. This result is the dual of the one proved by Liskevich and Skrypnik (Nonlinear Anal 71:1699–1708, 2009).

1. Introduction

In this paper we consider the following class of anisotropic equations

$$\sum_{i=1}^{N-1} \frac{\partial}{\partial x_i} A_{q,i}(x, u, Du) + \frac{\partial}{\partial x_N} A_p(x, u, Du) = 0 \quad \text{in } \Omega, \quad (1)$$

with Ω a regular domain in \mathbb{R}^N , $p > q > 1$. The functions $A_{q,i}(x, u, Du)$ and $A_p(x, u, Du) : \Omega \times \mathbb{R}^{N+1} \rightarrow \mathbb{R}^N$ are assumed to be measurable and satisfying the structure conditions

$$\left\{ \begin{array}{l} i) \quad \sum_{i=1}^{N-1} A_{q,i}(x, u, \eta) \cdot \eta_i \geq C_0 |\eta'|^q \\ ii) \quad A_p(x, u, \eta) \cdot \eta_N \geq C_0 |\eta_N|^p \\ iii) \quad \sum_{i=1}^{N-1} |A_{q,i}(x, u, \eta)| \leq C_1 |\eta'|^{q-1} \\ iv) \quad |A_p(x, u, \eta)| \leq C_1 |\eta_N|^{p-1}, \end{array} \right. \quad (2)$$

where C_0, C_1 are positive constants and for any vector $\eta = (\eta_1, \dots, \eta_N)$ in \mathbb{R}^N ,

$$\eta' = (\eta_1, \dots, \eta_{N-1}). \quad (3)$$

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The prototype of this class of operators is the anisotropic p-Laplacean:

$$\sum_{i=1}^{N-1} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{q-2} \frac{\partial u}{\partial x_i} \right) + \frac{\partial}{\partial x_N} \left(\left| \frac{\partial u}{\partial x_N} \right|^{p-2} \frac{\partial u}{\partial x_N} \right) = 0 \quad \text{in } \Omega.$$

Let us introduce two functional spaces:

$$\begin{aligned} W^{1,[q,p]}(\Omega) &= \left\{ u \in L^1(\Omega) : D_{x_i} u \in L^q(\Omega), i = 1, \dots, N-1; D_{x_N} u \in L^p(\Omega) \right\} \\ W_0^{1,[q,p]}(\Omega) &= W^{1,[q,p]}(\Omega) \cap W_0^{1,1}(\Omega). \end{aligned}$$

Definition. A function $u \in W_{loc}^{1,[q,p]}(\Omega)$ is a local, weak solution to (1) if for every compact set $K \subset \Omega$

$$\int_K \sum_{i=1}^{N-1} A_{q,i}(x, u, Du) \frac{\partial \varphi}{\partial x_i} dx + \int_K A_p(x, u, Du) \frac{\partial \varphi}{\partial x_N} dx = 0, \quad (4)$$

for all test functions $\varphi \in W_0^{1,[q,p]}(\Omega)$, where φ has compact support in K .

In the last decades many papers were devoted to the study on the anisotropic operators (see, for instance [1–3, 17, 18]) but the question of the Hölder regularity of solutions to equations with measurable coefficients was left open (see, for instance, the survey paper [19]).

A first step to face this problem was made by Liskevich and Skrypnik [15] in 2009. They considered the following class of anisotropic quasilinear equations

$$\frac{\partial^2 u}{\partial x_1^2} + \sum_{i=2}^N \frac{\partial}{\partial x_i} A_i(x, u, Du) + b(x, u, Du) = 0 \quad \text{in } \Omega. \quad (5)$$

They assumed $p > 2$, a suitable condition on the lower terms, the following structure conditions

$$\begin{cases} i) & \sum_{i=2}^N A_i(x, u, \eta) \cdot \eta_i \geq C_0 |\bar{\eta}|_{N-1}^p \\ ii) & |A_i(x, u, \eta)| \leq C_1 |\bar{\eta}|_{N-1}^{p-1}, \end{cases}$$

where $\bar{\eta} = (\eta_2, \dots, \eta_N)$ and the following nearness between the exponents holds:

$$2 < p \leq \frac{N\bar{p}}{N-\bar{p}}, \quad \frac{1}{\bar{p}} = \frac{1}{N} \left(\frac{1}{2} + \frac{N-1}{p} \right), \quad \bar{p} < N. \quad (6)$$

We recall that this condition implies the local boundedness of the solutions (see [13, 16]).

More precisely they realized that the parabolic techniques (the so called time expansion of positivity techniques) developed in [7] to study the degenerate equation

$$u_t = \Delta_p u, \quad p > 2,$$

can be adapted to study the elliptic case too, by proving a suitable expansion of positivity in space variables.

In [11], the result proved in [15] was extended to the following class of anisotropic equations

$$\frac{\partial}{\partial x_1} A_p(x, u, Du) + \sum_{i=2}^N \frac{\partial}{\partial x_i} A_{q,i}(x, u, Du) = 0 \quad \text{in } \Omega. \quad (7)$$

The authors assumed $q > p > 1$, the structure conditions

$$\begin{cases} i) & A_p(x, u, \eta) \cdot \eta_1 \geq C_0 |\eta_1|^p \\ ii) & |A_p(x, u, \eta)| \leq C_1 |\eta_1|^{p-1} \\ iii) & \sum_{i=2}^N A_{q,i}(x, u, \eta) \cdot \eta_i \geq C_0 |\bar{\eta}|^q \\ iv) & |A_{q,i}(x, u, \eta)| \leq C_1 |\bar{\eta}|^{q-1} \end{cases}$$

and the local boundness of u . The last assumption is alternative to the assumption of the nearness of p and q .

The approach of the space expansion of positivity seems very promising and it works in different context (see [10, 12]).

More recently, in [9], a stability result was proved.

Let u be a locally bounded weak solution u of the equation

$$\sum_{i=1}^N D_{x_i}(A_{p_i}(x, u, Du)) = 0 \quad \text{in } \Omega.$$

Assume $p_i > 1$, $i = 1, \dots, N$, their harmonic mean \bar{p} less than the dimension of the space N and the following structure conditions:

$$\begin{cases} i) & \sum_{i=1}^N A_{p_i}(x, u, Du) \cdot D_{x_i} u \geq C_0 \sum_{i=1}^N |D_{x_i} u|^{p_i}, \\ ii) & |A_{p_i}(x, u, Du)| \leq C_1 |D_{x_i} u|^{p_i-1}, \quad \forall i = 1, \dots, N. \end{cases}$$

There exists a $q_0 > 1$ such that if

$$p_{\max} - p_{\min} \leq 1/q_0,$$

with $p_{\min} = \min\{p_1, \dots, p_N\}$ and $p_{\max} = \max\{p_1, \dots, p_N\}$, then the solution is locally Hölder continuous in Ω .

The aim of the present paper is to give a new insight of this difficult, challenging and still open problem. Here we prove, in a certain sense, a dual result with respect the one proved in [11, 15]. This should be an important step to prove the regularity in the physical case of 3 spatial dimensions having each direction a different diffusion exponent.

As in [11], the condition on the nearness of the exponents does not appear. Perhaps, such a condition is necessary for the boundedness of the solutions but it does not play any role for the regularity of already bounded solutions.

As in [11, 15], the proof is based on a suitable space expansion of positivity, but, differently from the quoted papers, here one can avoid to work with the transformed equation and this makes simpler the calculations.

In the sequel we say that a constant depends only upon the data if it depends only upon N , C_0 , C_1 , p , q and $\|u\|_{L^\infty(\Omega)}$.

Theorem 1.1. *Let $u(x)$ be a locally bounded weak solution to (1)–(2) with $p > q > 1$, then $u(x)$ is locally Hölder continuous.*

More precisely, in analogy with the parabolic p -Laplacean, it is possible to prove that there are two positive constants $\beta < 1$ and C depending only upon the data such that for any compact K strictly contained in Ω and for any $P_1 = (x', x_N)$, $P_2 = (y', y_N) \in K$

$$|u(P_1) - u(P_2)| \leq C \|u\|_{L^\infty(\Omega)} \left(\frac{|x_N - y_N| + \|u\|_{L^\infty(\Omega)}^{\frac{p-q}{p}} |x' - y'|^{\frac{q}{p}}}{(p, q) - \text{dist}(K, \partial\Omega)} \right)^\beta, \quad (8)$$

where $(p, q) - \text{dist}(K, \partial\Omega)$ is the infimum of $|x_N - y_N| + \|u\|_{L^\infty(\Omega)}^{\frac{p-q}{p}} |x' - y'|^{\frac{q}{p}}$ ranging $(x', x_N) \in K$ and $(y', y_N) \in \partial\Omega$. Such a proof is quite technical and, nowadays, standard. For these reason we omit it and we refer the reader to Chapter 3 of [5].

In our forthcoming researches, as already written, we intend to combine this new approach with the ideas developed in [11, 15] to prove the regularity of the solutions under more general assumptions.

The scheme of this paper is the following: in Sect. 2 we state some preliminary results. In Sect. 3 we begin to prove the main Theorem starting an alternative argument that we conclude in Sect. 4. The proof of Theorem 1.1 will result a direct consequence of the alternative argument.

2. Preliminary results

In this section we collect some results we will use in the sequel.

Let $r > 0$ and define a ball in the last $N - 1$ variables

$$\bar{B}_r = \left\{ (x_2, \dots, x_N) \in \mathbb{R}^{N-1} : \sum_{i=2}^N x_i^2 \leq r^2 \right\},$$

and define the following cylinder in Ω ,

$$Q_{L,r} = \{x \in \mathbb{R}^N : |x_1| \leq L, (x_2, \dots, x_N) \in \bar{B}_r\}.$$

Assume $Q_1 := Q_{L,r}$ contained in Ω . Consider another cylinder $Q_2 := Q_{L_1,r_1}$ with $L_1 < L$ and $r_1 < r$.

Denote with ω the oscillation of u in Ω and let μ_- be the ess inf of u in Ω . Define

$$G(u) := \left[\frac{1}{(u - \mu_- + a\omega H)^{q-1}} - \frac{1}{(\omega H)^{q-1}} \right]_+,$$

where $[\cdot]_+$ denotes the positive part, thus $G(u) = 0$ if $u > \mu_- + (1-a)\omega H$ and $G(u)$ is positive a.e. if $u \leq \mu_- + (1-a)\omega H$. Here $0 < a < 1$, $0 < H < 1$.

Lemma 2.1. *Let u be a locally bounded weak solution of equation (1) and assume that the structure conditions (2) are satisfied. Then there exists a constant $C > 0$ (depending only upon the data) such that*

$$\begin{aligned} \int_{Q_2 \cap A} \left| D_{x_1} \ln_+ \frac{H\omega}{u - \mu_- + a\omega H} \right|^q dx &\leq C \int_{Q_1 \cap A} |D_x \xi|^q dx \\ &+ C \int_{Q_1 \cap A} |D_{x_N} \xi|^p (u - \mu_- + a\omega H)^{p-q} dx \end{aligned} \quad (9)$$

where D_x is the gradient in the first $N-1$ variables, $A = \{x \in \Omega : u < \mu_- + (1-a)\omega H\}$ and $\xi \in C^\infty$ is a function such that $\xi = 1$ in Q_2 , $\xi = 0$ in $\Omega \setminus Q_1$.

Proof. Let us take $G(u)\xi^p$ as a test function. Then using equation (1), we have

$$\int_{\Omega} \sum_{i=1}^{N-1} A_{q,i}(x, u, Du) D_{x_i} (G(u)\xi^p) dx + \int_{\Omega} A_p(x, u, Du) D_{x_N} (G(u)\xi^p) dx = 0.$$

If we use the definition of $G(u)$ and condition (2), then

$$\begin{aligned} &(q-1)C_0 \int_{Q_1 \cap A} |D_x u|^q \frac{1}{(u - \mu_- + a\omega H)^q} \xi^p dx \\ &+ (q-1)C_0 \int_{Q_1 \cap A} |D_{x_N} u|^p \frac{1}{(u - \mu_- + a\omega H)^q} \xi^p dx \\ &\leq pC_1 \int_{Q_1 \cap A} |D_x u|^{q-1} \frac{1}{(u - \mu_- + a\omega H)^{q-1}} \xi^{p-1} |D_x \xi| dx + \\ &+ pC_1 \int_{Q_1 \cap A} |D_{x_N} u|^{p-1} \frac{1}{(u - \mu_- + a\omega H)^{q-1}} \xi^{p-1} |D_{x_N} \xi| dx \end{aligned}$$

The estimate (9) follows by using Young inequality in the right hand side of the previous estimate. \square

Lemma 2.2. (Sobolev–Troisi Inequality [21], see also [20] and [14]) *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and consider $u \in W_0^{1, [p_1, \dots, p_N]}(\Omega)$, $p_i \geq 1$ for all $i = 1, \dots, N$. Let*

$$\frac{1}{\bar{p}} = \frac{1}{N} \sum_{i=1}^N \frac{1}{p_i}, \quad \bar{p}^* = \frac{N\bar{p}}{N - \bar{p}}.$$

Then there exists c depending on N, p_1, \dots, p_N if $\bar{p} < N$ such that

$$\|u\|_{L_{\bar{p}^*}(\Omega)}^N \leq c \prod_{i=1}^N \|D_{x_i} u\|_{L_{p_i}(\Omega)}.$$

Lemma 2.3. (Algebraical lemma [4], see also [8]) *Let $\{Y_m\}$, $m = 0, 1, 2, \dots$, be a sequence of positive numbers, satisfying the recursive inequalities*

$$Y_{m+1} \leq C b^m Y_m^{1+\lambda}$$

where $C, b > 1$ and $\lambda > 0$ are given numbers. If

$$Y_0 \leq C^{\frac{-1}{\lambda}} b^{\frac{-1}{\lambda^2}},$$

then $\{Y_m\}$ converges to zero as $m \rightarrow \infty$.

Lemma 2.4. (Generalised Caccioppoli's Inequality) *Let u be a locally bounded weak solution of (1) and assume that the structure conditions in (2) are satisfied. Then there exists a constant $C > 0$ (depending only upon the data) such that for every test function $\theta \in C_0^1(\Omega)$, we have*

$$\begin{aligned} & \int_{\Omega} \theta^p (|D_x(u-k)_-|^q + |D_{x_N}(u-k)_-|^p) dx \\ & \leq C \int_{\Omega} (\theta^{p-q} |D_x \theta|^q |(u-k)_-|^q + |D_{x_N} \theta|^p |(u-k)_-|^p) dx, \end{aligned}$$

where D_x is the gradient in the first $N - 1$ variables and k any positive constant.

Note that an analogous result also holds if we deal with $(u-k)_+$.

Proof. Take

$$\psi = \psi(x) = -\theta(x)^p (u-k)_-, \quad \theta \in C_0^1(\Omega).$$

as a test function. Note that

$$\begin{aligned} D_x \psi &= p\theta^{p-1}(u-k)_- D_x \theta + \theta^p D_x (u-k)_- \\ D_{x_N} \psi &= p\theta^{p-1}(u-k)_- D_{x_N} \theta + \theta^p D_{x_N} (u-k)_-. \end{aligned}$$

Then using Eq. (1), we have

$$\begin{aligned}
 & \int_{\Omega} p\theta^{p-1}(u-k)_{-} \sum_{i=1}^{N-1} A_{q,i}(x, u, Du) D_{x_i} \theta dx \\
 & + \int_{\Omega} \theta^p \sum_{i=1}^{N-1} A_{q,i}(x, u, Du) D_{x_i} (u-k)_{-} dx \\
 & + \int_{\Omega} p\theta^{p-1}(u-k)_{-} A_p(x, u, Du) D_{x_N} \theta dx \\
 & + \int_{\Omega} \theta^p A_p(x, u, Du) D_{x_N} (u-k)_{-} dx = 0.
 \end{aligned}$$

If we take into account (2), we obtain the following inequality

$$\begin{aligned}
 & C_0 \int_{\Omega} \theta^p (|D_x(u-k)_{-}|^q + |D_{x_N}(u-k)_{-}|^p) dx \\
 & \leq pC_1 \int_{\Omega} (N-1)\theta^{p-1} |(u-k)_{-}| |D_x \theta| |D_x(u-k)_{-}|^{q-1} dx \\
 & + pC_1 \int_{\Omega} \theta^{p-1} |(u-k)_{-}| |D_{x_N} \theta| |D_{x_N}(u-k)_{-}|^{p-1} dx.
 \end{aligned}$$

Again, the statement follows by applying Young inequality. \square

In the next lemma we will consider a DeGiorgi type lemma. It is necessary to introduce the intrinsic geometry induced by the anisotropy of the operator itself (for more details about the intrinsic geometry see [5, 22]).

For a Lebesgue measurable set $E \subset \mathbb{R}^N$ denote by $|E|$ its measure.

Let $R_0, k_0 > 0$ be given numbers and $R_N = R_0^{\frac{q}{p}} k_0^{\frac{p-q}{p}}$. Define for $j = 0, 1, 2, \dots$

$$Q_{R_0, j} := \{x : |x_i| < R_j, \forall i = 1, \dots, N-1, |x_N| < r_j\},$$

where $R_j = \frac{R_0}{2} + \frac{R_0}{2^{j+1}}$ and $r_j = \frac{R_N}{2} + \frac{R_N}{2^{j+1}}$.

Let $c_j = |Q_{R_0, j}|$ the measure of the parallelepiped $Q_{R_0, j}$.

Clearly c_j is a decreasing sequence converging to a strictly positive constant c_{∞} .

Let u be a bounded weak solution of (1) in $Q_{R_0, 0}$, assume that the structure conditions in (2) are satisfied and let $\mu_{+} = \sup_{Q_{R_0, 0}} u$ and $\mu_{-} = \inf_{Q_{R_0, 0}} u$.

Define

$$A_{s, j} := \{x \in Q_{R_0, s} : u(x) \leq k_j\}, \quad k_j = \frac{k_0}{2} + \frac{k_0}{2^{j+1}} + \mu_{-}.$$

and $Z_j := \frac{|A_{2j, j}|}{|Q_{R_0, 2j}|}$

Let denote A_∞ as the intersection of the sets $A_{2j,j}$, i.e.,

$$A_\infty = \bigcap_j A_{2j,j} = \{x \in Q_{R_\infty} : u(x) \leq \frac{k_0}{2} + \mu_-\}.$$

Lemma 2.5. (DeGiorgi type lemma) *Let u be a bounded weak solution of (1) in $Q_{R_0,0}$, assume that the structures conditions in (2) are satisfied. There is a number $\nu > 0$ depending only upon the data (but not depending on u , R_0 , and k_0) such that if $Z_0 < \nu$ then $\{Z_j\}$ converges to zero as j goes to infinity.*

Proof. Let \bar{p} and \bar{p}^* be defined as in Sobolev-Troisi Lemma 2.2. If we consider the set $A_{2j+2,j+1}$, we can write

$$\int_{A_{2j+2,j+1}} |u - k_j|^{\bar{p}} dx \geq (k_j - k_{j+1})^{\bar{p}} |A_{2j+2,j+1}|.$$

Then

$$\begin{aligned} |A_{2j+2,j+1}| &\leq \frac{1}{(k_j - k_{j+1})^{\bar{p}}} \int_{A_{2j+2,j+1}} |(u - k_j)_-|^{\bar{p}} dx \\ &\leq \frac{1}{(k_j - k_{j+1})^{\bar{p}}} \int_{A_{2j+1,j}} \theta_{2j+1}^{\bar{p}} |(u - k_j)_-|^{\bar{p}} dx. \end{aligned}$$

Here $\theta_{2j+1} \in C_\infty$ is a function such that $\theta_{2j+1} = 1$ in $Q_{R_0,2j+2}$ and $\theta_{2j+1} = 0$ out of $Q_{R_0,2j+1}$ satisfying $|D_x \theta_{2j+1}(z)| \leq \tilde{c} \frac{4^j}{R_0}$, (where we recall once again that D_x is the gradient with respect the first $N - 1$ variables), $|D_{x_N} \theta_{2j+1}(z)| \leq \tilde{c} \frac{4^j}{R_N}$ for a positive constant \tilde{c} and for all $z \in Q_{R_0,2j+1}$.

If we use Hölder inequality, we obtain

$$|A_{2j+2,j+1}| \leq \frac{1}{(k_j - k_{j+1})^{\bar{p}}} \left[\int_{A_{2j+1,j}} (\theta_{2j+1} |(u - k_j)_-|)^{\bar{p}^*} dx \right]^{\frac{\bar{p}}{\bar{p}^*}} |A_{2j+1,j}|^{1 - \frac{\bar{p}}{\bar{p}^*}},$$

Now by Lemma 2.2, we have

$$\begin{aligned} |A_{2j+2,j+1}| &\leq c \frac{1}{(k_j - k_{j+1})^{\bar{p}}} |A_{2j+1,j}|^{1 - \frac{\bar{p}}{\bar{p}^*}} \\ &\quad \prod_{i=1}^{N-1} \left[\int_{A_{2j+1,j}} \left| D_{x_i} \left(\theta_{2j+1} (u - k_j)_- \right) \right|^q dx \right]^{\frac{\bar{p}}{qN}} \\ &\quad \left[\int_{A_{2j+1,j}} \left| D_{x_N} \left(\theta_{2j+1} (u - k_j)_- \right) \right|^p dx \right]^{\frac{\bar{p}}{pN}} \end{aligned}$$

$$\begin{aligned}
&\leq c \frac{1}{(k_j - k_{j+1})^{\bar{p}}} |A_{2j+1,j}|^{1-\frac{\bar{p}}{\bar{p}^*}} \\
&\quad \left[\int_{A_{2j+1,j}} \left| D_x \left(\theta_{2j+1}(u - k_j)_- \right) \right|^q dx \right]^{\frac{(N-1)\bar{p}}{Nq}} \\
&\quad \left[\int_{A_{2j+1,j}} \left| D_{x_N} \left(\theta_{2j+1}(u - k_j)_- \right) \right|^p dx \right]^{\frac{\bar{p}}{\bar{p}N}} \quad (10)
\end{aligned}$$

Let us estimate $\int_{A_{2j+1,j}} \left| D_x \left(\theta_{2j+1}(u - k_j)_- \right) \right|^q dx$. We have

$$\begin{aligned}
&\int_{A_{2j+1,j}} \left| D_x \left(\theta_{2j+1}(u - k_j)_- \right) \right|^q dx \\
&\leq C \int_{A_{2j+1,j}} \left(\left| D_x(\theta_{2j+1}) \right|^q \left| (u - k_j)_- \right|^q + \theta_{2j+1}^q \left| D_x(u - k_j)_- \right|^q \right) dx.
\end{aligned}$$

Considering the properties of the function θ_{2j+1} , we have

$$\begin{aligned}
\int_{A_{2j+1,j}} \left| D_x(\theta_{2j+1}(u - k_j)_-) \right|^q dx &\leq C \frac{4^{jq}}{R_0^q} \int_{A_{2j+1,j}} \left| (u - k_j)_- \right|^q dx \\
&\quad + C \int_{A_{2j+1,j}} \left| D_x(u - k_j)_- \right|^q dx.
\end{aligned}$$

If we pass to the larger set $A_{2j,j}$ and use the function θ_{2j} with exponent p , we obtain

$$\begin{aligned}
\int_{A_{2j+1,j}} \left| D_x(\theta_{2j+1}(u - k_j)_-) \right|^q dx &\leq C \frac{4^{jq}}{R_0^q} \int_{A_{2j,j}} \left| (u - k_j)_- \right|^q dx \\
&\quad + C \int_{A_{2j,j}} \theta_{2j}^p \left| D_x(u - k_j)_- \right|^q dx.
\end{aligned}$$

Here we can apply Caccioppoli's Inequality of Lemma 2.4 to the second integral on the right hand side and we get,

$$\begin{aligned}
\int_{A_{2j+1,j}} |D_x(\theta_{2j+1}(u - k_j)_-)|^q dx &\leq C \frac{4^{jq}}{R_0^q} \int_{A_{2j,j}} |(u - k_j)_-|^q dx \\
&\quad + C \int_{A_{2j,j}} |D_x \theta_{2j}|^q |(u - k_j)_-|^q dx \\
&\quad + C \int_{A_{2j,j}} |D_{x_N} \theta_{2j}|^p |(u - k_j)_-|^p dx.
\end{aligned}$$

If we use the properties of θ_{2j} ,

$$\begin{aligned}
\int_{A_{2j+1,j}} |D_x(\theta_{2j+1}(u - k_j)_-)|^p dx &\leq C \frac{4^{jq}}{R_0^q} \int_{A_{2j,j}} |(u - k_j)_-|^q dx \\
&\quad + C \frac{4^{jp}}{R_N^p} \int_{A_{2j,j}} |(u - k_j)_-|^p dx.
\end{aligned}$$

Noting that $|u - k_j| \leq 2k_0$ and by the definition of R_N we have

$$\int_{A_{2j+1,j}} |D_x(\theta_{2j+1}(u - k_j)_-)|^p dx \leq C \frac{4^{jp}}{R_0^q} k_0^q |A_{2j,j}|$$

Arguing as above, one obtains a similar estimate also for

$$\int_{A_{2j+1,j}} |D_{x_N}(\theta_{2j+1}(u - k_j)_-)|^p dx,$$

i.e.

$$\begin{aligned}
\int_{A_{2j+1,j}} |D_{x_N}(\theta_{2j+1}(u - k_j)_-)|^p dx &\leq C \frac{4^{jq}}{R_0^q} \int_{A_{2j,j}} |(u - k_j)_-|^q dx \\
&\quad + C \frac{4^{jp}}{R_N^p} \int_{A_{2j,j}} |(u - k_j)_-|^p dx \\
&\leq C \frac{4^{jp}}{R_0^q} k_0^q |A_{2j,j}|.
\end{aligned}$$

Then since $\frac{\bar{p}(N-1)}{qN} + \frac{\bar{p}}{pN} = 1$ we obtain from (10),

$$|A_{2j+2,j+1}| \leq C \frac{1}{(k_j - k_{j+1})^{\bar{p}}} |A_{2j+1,j}|^{1 - \frac{\bar{p}}{p^*}} \left[\frac{4^{jp}}{R_0^q} k_0^q |A_{2j,j}| \right]. \quad (11)$$

Noting that $|A_{2j+1,j}| \leq |A_{2j,j}|$ we get from (11) that,

$$|A_{2j+2,j+1}| \leq \frac{2^{(j+2)\bar{p}}}{k_0^{\bar{p}}} |A_{2j,j}|^{1+(1-\frac{\bar{p}}{p^*})} 4^{(j+1)p} \frac{k_0^q}{R_0^q}.$$

Divide both sides by $|Q_{R_0,2j}|$ to have the following inequality

$$Z_{j+1} \leq C 2^{(j+2)\bar{p}} 4^{(j+1)p} Z_j^{1+\lambda}$$

where $\lambda := (1 - \frac{\bar{p}}{p^*})$.

Then the required statement follows by Lemma 2.3. \square

In the sequel we need to apply the DeGiorgi type lemma in a very special case that allows us to get rid of the anisotropy. More specifically: consider a cube of side $2R_0$, i.e

$$Q_{R_0} := \{x : |x_i| \leq R_0, \forall i = 1, \dots, N\}.$$

Let u be a bounded weak solution of (1) in Q_{R_0} , assume that the structure conditions in (2) are satisfied and let $\mu_+ = \sup_{Q_{R_0}} u$ and $\mu_- = \inf_{Q_{R_0}} u$.

Assume that

$$u(x_1, \dots, x_N) \geq \mu_- + k_0, \quad (12)$$

when $|x_i| \leq R_0$ for $i = 1, \dots, N-1$ and $x_N = \pm R_0$.

Lemma 2.6. (When the anisotropic DeGiorgi Lemma becomes isotropic) *Let u be a bounded weak solution of (1) in Q_{R_0} , assume that conditions (2) and (12) are satisfied. Then there is a number $\nu > 0$ depending only upon the data (but not depending on u , R_0 , and k_0) and a constant γ depending only upon the data and k_0 such that if $Z_0 < \nu$ either $\mu_+ - \mu_- \leq \gamma R_0^N$ or $\{Z_j\}$ converges to zero as j goes to infinity.*

Proof. Define

$$Q_{R_0,j}^* := \{x : |x_i| < R_j, \forall i = 1, \dots, N-1, |x_N| < R_0\},$$

where $R_j = \frac{R_0}{2} + \frac{R_0}{2^{j+1}}$.

Let $\tilde{c}_j = |Q_{R_0,j}^*|$. Clearly \tilde{c}_j is a decreasing sequence converging to a strictly positive constant c_∞ .

Define

$$A_{s,j} := \{x \in Q_{R_0,s}^* : u(x) \leq k_j\}, \quad k_j = \frac{k_0}{2} + \frac{k_0}{2^{j+1}} + \mu_-.$$

and $Z_j := \frac{|A_{2j,j}|}{|Q_{R_0,2j}^*|}$.

In the sets $Q_{R_0,j}^*$, choosing as test function $-\theta^q(u - k_i)_- = -\theta^q(k_i - u)_+$ where θ does not depend on x_N , the Caccioppoli generalised inequality becomes

$$\begin{aligned} & \int_{Q_{R_0,j}^*} \theta^q (|D_x(u - k_i)_-|^q + |D_{x_N}(u - k_i)_-|^p) dx \\ & \leq C \int_{Q_{R_0,j}^*} |D_x \theta|^q |(u - k_i)_-|^q dx, \end{aligned}$$

where with D_x we denoted the gradient with respect the first $N - 1$ variables.

On the other hand

$$\begin{aligned} & \int_{Q_{R_0,j}^*} \theta^q (|D_x(u - k_i)_-|^q + |D_{x_N}(u - k_i)_-|^p) dx \\ & \leq \int_{Q_{R_0,j}^*} \theta^q (|D_x(u - k_i)_-|^q + |D_{x_N}(u - k_i)_-|^p) dx + |A_{j,i}| \end{aligned}$$

Hence

$$\begin{aligned} & \int_{Q_{R_0,j}^*} \theta^q (|D_x(u - k_i)_-|^q + |D_{x_N}(u - k_i)_-|^p) dx \\ & \leq C \int_{Q_{R_0,j}^*} |D_x \theta|^q |(u - k_i)_-|^q dx + |A_{j,i}| \end{aligned}$$

The statement, now, follows by the classical properties of non homogeneous DeGiorgi classes, by repeating in a straightforward way the proof of Proposition 4.1, Chapter 10 of [6]. \square

Before stating the last preliminary result, let us introduce some notation. For a function v defined in a set E and real numbers $k < l$, put

$$\begin{aligned} [v > l] &= \{x \in E \mid v(x) > l\} \\ [v < k] &= \{x \in E \mid v(x) < k\} \\ [k < v < l] &= \{x \in E \mid k < v(x) < l\}. \end{aligned}$$

For $\rho > 0$ and $y \in \mathbb{R}^N$, denote by $B_\rho(y)$ the ball of radius ρ centered at y , and by $K_\rho(y)$ the cube of edge ρ , centered at y and with faces parallel to the coordinate planes. If y is the origin, let $B_\rho(0) = B_\rho$, and $K_\rho(0) = K_\rho$.

Lemma 2.7. (DeGiorgi [4]) *Let $v \in W^{1,1}(K_\rho(y))$, and let $k < l$ be real numbers. There exists a constant γ depending only on N , p and independent of k, l, v, y, ρ , such that*

$$(l - k)[[v < k]] \leq \gamma \frac{\rho^{N+1}}{[[v > l]]} \int_{[k < v < l]} |Dv| dx. \quad (13)$$

Remark 2.8. The conclusion of the lemma continues to hold for functions $v \in W^{1,1}(E)$ provided E is *convex*; in such a case, instead of ρ^{N+1} , we have d^{N+1} , where d is the diameter of the set E .

3. First alternative

Without loss of generality (modulo suitable standard homothetical transformations), we may assume to work in the unitary cube $Q_{\frac{1}{2}}$; we may also assume that $0 < u < 1$ in $Q_{\frac{1}{2}}$. Consider now a slide S_N of this cube, $S_N = \{(x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N : -\frac{1}{2} < x_i < \frac{1}{2}, \forall 1 \leq i \leq N-1, -\varepsilon < x_N < \varepsilon\}$ with ε a small positive number to be quantified later. By using a suitable homothetical transformation, we can transform S_N into T_N where the set $T_N = \{(x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N : -\frac{1}{2}\theta < x_i < \frac{1}{2}\theta, \forall 1 \leq i \leq N-1, -\frac{1}{2} < x_N < \frac{1}{2}\}$ where θ is a very large positive parameter to be chosen. Note that, when θ is chosen, ε too is quantified. Without loss of generality, we may assume $\theta \in \mathbb{N}$. Now, partition T_N in sub-cubes $Q^{(j)}$ of unitary wedge.

Assume that for each $Q^{(j)}$

$$\left| [x \in Q^{(j)} : u(x) \leq \frac{1}{2}] \right| > \nu |Q^{(j)}| \quad (14)$$

where ν is the quantity claimed in Lemma 2.5.

Lemma 3.1. (The first alternative) *Let u be a bounded weak solution of (1) in $Q_{\frac{1}{2}}$. Assume that the structure conditions (2) are satisfied and that (14) holds for any $Q^{(j)}$. Then there is a $n_0 \in \mathbb{N}$ such that $u \leq 1 - \frac{1}{2^{n_0+1}}$ for a.e. $x \in \frac{1}{2}T_N$.*

Proof. Apply Lemma 2.7 to any $Q^{(j)}$ with

$$l = 1 - \frac{1}{2^n}, \quad k = 1 - \frac{1}{2^{n-1}}, \quad h = l - k = \frac{1}{2^n}, \quad n = 2, \dots, n_0,$$

where n_0 is to be chosen. We obtain

$$\frac{1}{2^n} \left| \left[u > 1 - \frac{1}{2^n} \right] \cap Q^{(j)} \right| \leq \gamma \int_{[k < v < l] \cap Q^{(j)}} |Du| dx.$$

If we sum over all the $Q^{(j)}$, we get

$$\frac{1}{2^n} \left| \left[u > 1 - \frac{1}{2^n} \right] \cap T_N \right| \leq \gamma \int_{[k < v < l] \cap T_N} |Du| dx.$$

Since

$$\int_{[k < v < l] \cap T_N} |Du| dx \leq \int_{[k < v < l] \cap T_N} |D_x u| dx + \int_{[k < v < l] \cap T_N} |D_{x_N} u| dx,$$

(where with D_x we have denoted the gradient with respect the first $N - 1$ variables), denoting $[u > 1 - \frac{1}{2^n}] \cap T_N$ by A_n , we have

$$\begin{aligned} \frac{1}{2^n} |A_n| &\leq \gamma \int_{[k < v < l] \cap T_n} |Du| dx \\ &\leq \gamma \left(\int_{[k < v < l] \cap T_N} |D_x u|^q dx \right)^{\frac{1}{q}} |A_{n-1} \setminus A_n|^{\frac{q-1}{q}} \\ &\quad + \gamma \left(\int_{[k < v < l] \cap T_N} |D_{x_N} u|^p dx \right)^{\frac{1}{p}} |A_{n-1} \setminus A_n|^{\frac{p-1}{p}}. \end{aligned}$$

By generalised Caccioppoli inequality applied to $(u - k)_+$ it follows

$$\begin{aligned} \frac{1}{2^n} |A_n| &\leq \gamma \left[\int_{2T_N} \left[\zeta^{p-q} |D_x \zeta|^q |(u - k)_+|^q \right. \right. \\ &\quad \left. \left. + |D_{x_N} \zeta|^p |(u - k)_+|^p \right] dx \right]^{\frac{1}{q}} \cdot |A_{n-1} \setminus A_n|^{\frac{q-1}{q}} \\ &\quad + \gamma \left[\int_{2T_N} \left[\zeta^{p-q} |D_x \zeta|^q |(u - k)_+|^q \right. \right. \\ &\quad \left. \left. + |D_{x_N} \zeta|^p |(u - k)_+|^p \right] dx \right]^{\frac{1}{p}} \cdot |A_{n-1} \setminus A_n|^{\frac{p-1}{p}}. \end{aligned}$$

By choosing a suitable ζ

$$\begin{aligned} \frac{1}{2^n} |A_n| &\leq \gamma \left[h^p \left(\frac{1}{\theta^q h^{p-q}} + 1 \right) |T_N| \right]^{\frac{1}{q}} \cdot |A_{n-1} \setminus A_n|^{\frac{q-1}{q}} \\ &\quad + \gamma \left[h^p \left(\frac{1}{\theta^q h^{p-q}} + 1 \right) |T_N| \right]^{\frac{1}{p}} \cdot |A_{n-1} \setminus A_n|^{\frac{p-1}{p}}, \end{aligned}$$

since by the choice of k and h we have $(u - k)_+ < 1 - (1 - \frac{1}{2^{n-1}}) < 2h$. Moreover

$$\begin{aligned} \frac{1}{2^n} |A_n| &\leq \gamma \left(\frac{1}{2^n} \right)^{\frac{p}{q}} \left[\left(\frac{1}{\theta^q h^{p-q}} + 1 \right) |T_N| \right]^{\frac{1}{q}} \cdot |A_{n-1} \setminus A_n|^{\frac{q-1}{q}} \\ &\quad + \gamma \left(\frac{1}{2^n} \right) \left[\left(\frac{1}{\theta^q h^{p-q}} + 1 \right) |T_N| \right]^{\frac{1}{p}} \cdot |A_{n-1} \setminus A_n|^{\frac{p-1}{p}}. \end{aligned}$$

Suppose that n_o has already been chosen, and let

$$\theta = 2^{n_o \frac{p-q}{q}}$$

As mentioned above, provided n_o is sufficiently large, we can always assume that $n_o \frac{p-q}{q}$ is an integer. Due to the previous choice for θ we derive

$$|A_n| \leq \gamma \left(|T_N|^{\frac{1}{p}} |A_{n-1} \setminus A_n|^{\frac{p-1}{p}} + |T_N|^{\frac{1}{q}} |A_{n-1} \setminus A_n|^{\frac{q-1}{q}} \right).$$

At each step, we have that one of these two inequalities holds:

$$\begin{cases} |A_n|^{\frac{p}{p-1}} \leq \gamma |Q|^{\frac{1}{p-1}} |A_{n-1} \setminus A_n| \\ \text{or} \\ |A_n|^{\frac{q}{q-1}} \leq \gamma |Q|^{\frac{1}{q-1}} |A_{n-1} \setminus A_n|. \end{cases} \quad (15)$$

Once more, suppose that n_o has already been chosen: then, or for p or for q the previous inequality holds at least $\frac{n_o}{2}$ times. Therefore, summing A_n in (15) when it holds for p or q , we conclude that

$$\frac{n_o}{2} |[x \in T_N : u(x) \geq 1 - \frac{1}{2^{n_o}}]| < \gamma |T_N|.$$

The value of n_o is determined in such a way that

$$\frac{2\gamma}{n_o} \leq \nu,$$

where ν is the same parameter as in Lemma 2.5. Notice that T_N is scaled correctly according to the intrinsic geometry considered in Lemma 2.5. Therefore, we can apply such a Lemma, and conclude that

$$u \leq 1 - \frac{1}{2^{n_o+1}}$$

for a.e. $x \in \frac{1}{2}T_N$. □

Hence, if the first alternative occurs, we have reduced the oscillation of the solution u in a quantitative way in S_N .

4. Second alternative and Proof of Theorem 1.1

If the first alternative does not occur, there exists at least one $Q^{(j)} \subset T_N$, centered at $x_0 = (x_1^{(j)}, \dots, x_{N-1}^{(j)}, 0)$, such that

$$\left| \left[x \in Q^{(j)} : u(x) \leq \frac{1}{2} \right] \right| \leq \nu |Q^{(j)}|.$$

Without loss of generality we may assume $x_0 = (x_1^{(j)}, 0, 0, \dots, 0)$. Define Q^0 the cube centered in the origin, so $Q^{(j)} = x_0 + Q^0$.

In the next lemma, we apply the logarithmic estimates to expand the positivity of $Q^{(j)}$ till the origin.

Apply Lemma 2.5 to $Q^{(j)}$ to get, for a.e. $x \in \frac{1}{2}Q^{(j)}$

$$u(x) \geq \frac{1}{4}.$$

Lemma 4.1. (The second alternative) *Let u be a bounded weak solution of (1) in $Q_{\frac{1}{2}}$. Assume that conditions (2) are satisfied and that (14) does not hold for a $Q^{(j)}$. Then for any positive constant $v_0 \in (0, 1)$, there exists a positive integer s_0 such that*

$$\left| \left\{ x \in \frac{1}{2}Q^0 : u(x) \leq e^{-s_0} \right\} \right| \leq v_0 \left| \frac{1}{2}Q^0 \right|. \quad (16)$$

Proof. If we perform the change of variable

$$\begin{aligned} x_1 - x_1^{(j)} &= \frac{y_1}{2} \\ x_i &= \frac{y_i}{2}, \quad \forall i = 2, \dots, N-1 \\ x_N &= \left(\frac{1}{4}\right)^{\frac{p-q}{p}} \frac{y_N}{2} \end{aligned}$$

and calling $v = 4u$, you have that v satisfies an equation of the type (1) satisfying (2). Moreover $v \geq 1$ in the unitary cube centered in the origin. As already noticed, the Lemma is proved if we are able to transport some positivity till to the point $z = (-2x_1^{(j)}, 0, \dots, 0)$.

Let us denote the solution v with u and the coordinates (y_1, \dots, y_N) with (x_1, \dots, x_N) . Without loss of generality assume $-2x_1^{(j)} > 0$, denote with $L = -2x_1^{(j)}$ and assume $L > 1$.

Define S the slide of the half unitary cube centered in the origin in the $x' = (x_2, \dots, x_N)$ variables, i.e. $\{x \in \mathbb{R}^N : x = (0, x_2, \dots, x_N) : -\frac{1}{4} < x_i < \frac{1}{4}, \forall i = 2, \dots, N\}$.

Let $s_0 > 1$ be an integer, to be fixed later, and define a set A in S so defined:

$$A = \{(0, x') \in S : \exists t \in [0, 2L], u(t, x') \leq e^{-s_0}\}$$

We want to prove that for any positive constant $v_0 \in (0, 1)$, there exists a positive integer s_0 such that

$$\frac{|A|}{|S|} \leq v_0. \quad (17)$$

Let $x' \in A$. Then $\exists t \in [0, 2L]$ such that $u(t, x') \leq e^{-s_0}$.

Therefore,

$$\begin{aligned} s_0 - 1 &\leq \ln_+ \frac{u(0, x') + e^{-s_0}}{u(t, x') + e^{-s_0}} \\ &= \ln_+ \frac{1}{u(t, x') + e^{-s_0}} - \ln_+ \frac{1}{u(0, x') + e^{-s_0}} \\ &= \int_0^t D_{x_1} \left(\ln_+ \frac{1}{u(s, x') + e^{-s_0}} \right) ds \leq \int_0^{2L} \left| D_{x_1} \left(\ln_+ \frac{1}{u(s, x') + e^{-s_0}} \right) \right| ds \end{aligned}$$

Now if we integrate the inequality above over the set A , we obtain

$$(s_0 - 1)|A| \leq \int_S \int_0^{2L} \left| D_{x_1} \left(\ln_+ \frac{1}{u(x) + e^{-s_0}} \right) \right| dx$$

Using Hölder Inequality, we have

$$(s_0 - 1)|A| \leq \left[\int_S \int_0^{2L} \left| D_{x_1} \ln_+ \left(\frac{1}{u(x) + e^{-s_0}} \right) \right|^q dx \right]^{\frac{1}{q}} [2L|S|]^{\frac{q-1}{q}}.$$

Here if we consider Lemma 2.1 choosing $\xi = 1$ in $[0, 2L] \times S$ and $\xi = 0$ out of $[0, 3L] \times 2S$, and putting $H\omega = 1$ and $a = e^{-s_0}$ we get

$$\begin{aligned} (s_0 - 1)|A| &\leq C \left[\int_{2S} \int_0^{3L} \sum_{i=1}^{N-1} |D_{x_i} \xi|^q dx \right. \\ &\quad \left. + \int_{2S} \int_0^{3L} |D_{x_N} \xi|^p e^{-s_0(p-q)} dx \right]^{\frac{1}{q}} [2L|S|]^{\frac{q-1}{q}}. \\ (s_0 - 1)|A| &\leq CL|S| \end{aligned}$$

Dividing by $|S|$ we have

$$\frac{|A|}{|S|} \leq C \frac{L}{s_0 - 1}$$

and (17) follows choosing s_0 large enough, and (17), in turn, implies (16). \square

Thanks to the previous result, we have proved that, around the origin, the solution is far away from zero in most of the cube. However the geometry is not the intrinsic one, so we cannot apply directly the De Giorgi- type lemma. To overcome this difficulty we find two slides, one up and the other one down the origin in the x_N variable, where the solution is far away from zero everywhere. Now we are in the condition to apply Lemma 2.6 and get the positivity everywhere inside the two slides.

Let us prove the main result of this note.

Proof of Theorem 1.1. By using the same notation of Lemma 4.1, choose $v_0 = \frac{1}{4^N} \frac{v}{10}$, where v is the small constant defined in Lemma 2.5, where $R_N = R_0^{\frac{q}{p}} k_0^{\frac{p-q}{p}}$. Let P be the parallelepiped having the first $N - 1$ sides long $\frac{1}{2}$ and the last one $(\frac{1}{2})^{\frac{q}{p}} e^{-s_0 \frac{p-q}{p}}$.

With this choice there exists a $x_{0+}^N \in [\frac{1}{8}, \frac{1}{4}]$ such that in the set

$$R_L := (L, 0, \dots, 0, x_N^{0+}) + P$$

\square

the measure where u is smaller than e^{-s_0} is smaller than $\nu|P|$. If such thing was not occurring, then, in the parallelepiped $P^\diamond = [L - \frac{1}{4}, L + \frac{1}{4}] \times [-\frac{1}{4}, \frac{1}{4}]^{N-2} \times [\frac{1}{8}, \frac{1}{4}]$, the measure where u is smaller than e^{-s_0} would be larger than $\nu|P^\diamond|$ and this is in contradiction with (16) and the choice $\nu_0 = \frac{1}{4^N} \frac{\nu}{10}$. Therefore by Lemma 2.5. in the set $(L, 0, \dots, 0, x_{0+}^N) + \frac{1}{2}P$ we have $u \geq \frac{1}{2}e^{-s_0}$.

Analogously there exists $x_{0-}^N \in [-\frac{1}{4}, -\frac{1}{8}]$ such that in the set $(L, 0, \dots, 0, x_{0-}^N) + \frac{1}{2}P$ we have $u \geq \frac{1}{2}e^{-s_0}$.

Therefore in the parallelepiped $P_1 = \{x \in \mathbb{R}^N : L - \frac{1}{4} < x_1 < L + \frac{1}{4}, -\frac{1}{4} < x_2 < \frac{1}{4}, \dots, -\frac{1}{4} < x_{N-1} < \frac{1}{4}, x_{0-}^N < x_N < x_{0+}^N\}$ we have that on the faces $x_N = x_{0-}^N$ and $x_N = x_{0+}^N$ the solution u is bigger or equal than $\frac{1}{2}e^{-s_0}$. Moreover, by the choice of ν_0 , analogously as above, one can prove that the measure where $u < e^{-s_0}$ is smaller than $\nu|P_1|$. Therefore, the assumptions of Lemma 2.6 are fulfilled and we have either $u \geq \frac{1}{4}e^{-s_0}$ in $\frac{1}{2}P_1$ or the oscillation is quantitatively small.

Hence, also in the case of the second alternative, we reduced the oscillation of the solution u in a quantitative way and this implies the Hölder continuity of the solution.

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