Renzo Caddeo, Irene I. Onnis and Paola Piu, *Loxodromes on invariant surfaces in threemanifolds* in *Mediterranean Journal of* Mathematics, Volume 17 (Feb. 2019), Issue 1, Article number 6.

Loxodromes on invariant surfaces in three-manifolds

Renzo Caddeo, Irene I. Onnis and Paola Piu

Abstract. In this paper, we prove some results concerning the loxodromes on an invariant surface in a three-dimensional Riemannian manifold, a part of which generalizes classical results about loxodromes on rotational surfaces in $\mathbb{R}^3$. In particular, we show how to parametrize a loxodrome on an invariant surface of $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{H}_3$ and we exhibit the loxodromes of some remarkable minimal invariant surfaces of these spaces. In addition, we give an explicit description of the loxodromes on an invariant surface with constant Gauss curvature.

Mathematics Subject Classification (2010). 53A99, 53C22, 53C42, 14M17.

Keywords. Loxodromes, Invariant surfaces, Heisenberg group, Homogeneous spaces, BCV-spaces.

1. Introduction and preliminaries

In this paper, we study the loxodromes on an important family of surfaces in a three-manifold, that of the surfaces which are invariant under the action of a one-parameter group of isometries of the ambient space. Invariant surfaces have been classified by Gaussian or mean curvature in many remarkable three-dimensional spaces (see, for example, [6, 7, 9, 10, 11, 12, 17, 21]). Also, in [13, 14, 20] have been studied two well known types of curves on invariant surfaces:

The second author was supported by grant 2016/24707-4, São Paulo Research Foundation (Fapesp) and by CNPq productivity grant 312700/2017-2. The other two authors were supported by GNSAGA-INdAM, Italy.

the geodesics and the proper biharmonic curves.

A rotational surface of the Euclidean three-space is $SO(2)$-invariant and a loxodrome is a curve on it which meets the meridians at a constant angle. In [15], C.A. Noble obtained the differential equations of a loxodrome on these surfaces and, in particular, he investigated these curves on spheres and spheroids. Recently, the results of [15] have been generalized by Babaarslan and Yayli to the case of helicoidal surfaces in $\mathbb{R}^3$ (see [2]).

As the meridians and the parallels of a rotational surface of $\mathbb{R}^3$ are orthogonal, the loxodromes may also be defined as curves that make a constant angle with the Killing vector field X that is the infinitesimal generator of the one-parameter subgroup of isometries given by $SO(2)$. Therefore, if we denote by G_X the one-parameter subgroup of isometries of the ambient space generated by the Killing vector field X, the loxodromes on a G_X-invariant surface can naturally defined as the curves which make a constant angle with X.

In order to investigate loxodromes on an invariant surface in a three-dimensional Riemannian manifold, we need to recall some basic facts on the geometry of invariant surfaces.

1.1. Equivariant geometry of invariant surfaces

Let (N^3, g) be a three-dimensional connected Riemannian manifold and let X be a Killing vector field on N. Then X generates a one-parameter subgroup G_X of the group of isometries of (N^3, g). From the theory of Riemannian actions (see for example [5] or [1]) we know that:

- The isotropy subgroup G_p, $p \in N$, is compact and the orbit $G(p)$ is diffeomorphic to the quotient space G_X/G_p. We say that $G(q)$ is of *the same type* as $G(p)$ if the isotropy subgroups G_q and G_p are coniugated.
- An orbit $G(p)$ is called *principal* if there exists an open neighborhood $U \subset N$ of p such that all orbits $G(q)$, $q \in U$, are of the same type as $G(p)$. This implies that $G(q)$ is diffeomorphic to $G(p)$. We denote with N_r the *regular part* of N, that is, the subset consisting of points belonging to principal orbits.
- An orbit $G(p)$ is said to be *exceptional* if the dimension of $G(p)$ coincides with the dimension of principal orbits but $G(p)$ is not a principal orbit.

- An orbit $G(p)$ is said to be *singular* if has dimension less than the dimension of the principal orbits.

Consider, for example, the $SO(2)$-action on the sphere $\mathbb{S}^2$. The orbits of the poles are singular since they have dimension less than one, while all other orbits are principal. Notice that as $R(-p) = -R(p)$, for $p \in \mathbb{S}^2$ and $R \in SO(2)$, this action commutes with the antipodal map. This implies that the equivalence relation defined by the antipodal map on $\mathbb{S}^2$ is $SO(2)$-equivariant, inducing an $SO(2)$-action on the real projective space $\mathbb{R}P^2 = \mathbb{S}^2/\{\pm e\}$. Notice that this action is free on the parallels, except for the (projection of the) equator, which goes around twice as we run over $SO(2)$. The isotropy subgroups G_p are as follows:

- $SO(2)$, if p is the class of the north (or south) pole, which is a fixed point;
- $\{e\}$, the subgroup of the identity matrix only, with orbits given by the projections of the parallels, except for the equator. These are the *principal orbits* and the action on this part is free;
- $\{\pm e\}$, if p belongs to the projection of the equator. Notice that the dimension of the orbit is the same as that of a principal orbit; this is an *exceptional orbit* (the only one).

The following theorem (see Theorem 3.1, p. 179 in [5]) is very important for the study of the orbit space of a group action:

Theorem (The Principal Orbit Theorem). *Let G be a compact Lie group acting isometrically on a Riemannian manifold M. The following hold:*

1. *All principal orbits are diffeomorphic,*
2. *N_r is open and dense in N,*
3. *the regular part of the orbit space N_r/G_X is a connected differentiable manifold and the quotient map $\pi : N_r \to N_r/G_X$ is a submersion.*

For a proof one can see also [1] (Theorem 3.82).

Let now $f : M \to (N^3, g)$ be an immersion from a surface M into N^3 and assume that $f(M) \subset N_r$ (the regular part of N, that is, the subset consisting of points belonging to principal orbits). We say that f is a G_X-*equivariant immersion*, and $f(M)$ a G_X-*invariant surface* of N, if there exists an action of G_X on M such that for any $x \in M$ and $a \in G_X$ we have $f(a\,x) = af(x)$. A G_X-equivariant immersion $f : M \to (N^3, g)$ induces on M a Riemannian metric, the pull-back metric, denoted by g_f and called the G_X-*invariant induced metric.*

Let $f : M \to (N^3, g)$ be a G_X-equivariant immersion and let g_f be the G_X-invariant induced metric on M. Assume that N/G_X is connected. Then, f induces an immersion $\tilde{f} : M/G_X \to N_r/G_X$ between the orbit spaces and, moreover, N_r/G_X can be equipped with a Riemannian metric, the *quotient metric*, so that the quotient map $\pi : N_r \to N_r/G_X$ becomes a Riemannian submersion. Thus we have the following diagram

$$\begin{array}{ccc} (M, g_f) & \xrightarrow{\ f\ } & (N_r^3, g) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\ M/G_X & \xrightarrow{\ \tilde{f}\ } & (N_r^3/G_X, \tilde{g}). \end{array}$$

For later use, we describe the quotient metric of the regular part of the orbit space N_r/G_X. It is well known (see, for example, [16]) that N_r/G_X can be locally parametrized by invariant functions of the Killing vector field X. If $\{\xi_1, \xi_2\}$ is a complete set of invariant functions on a G_X-invariant subset of N_r, then the quotient metric is given by $\tilde{g} = \sum_{i,j=1}^{2} h^{ij} d\xi_i \otimes d\xi_j$ where (h^{ij}) is the inverse of the matrix (h_{ij}) with entries $h_{ij} = g(\nabla\xi_i, \nabla\xi_j)$.

Now, we can give a local description of the G_X-invariant surfaces of N^3. Let $\tilde{\gamma} : (a, b) \subset \mathbb{R} \to (N^3/G_X, \tilde{g})$ be a curve parametrized by arc length and let $\gamma : (a, b) \subset \mathbb{R} \to N^3$ be a lift of $\tilde{\gamma}$, such that $d\pi(\gamma') = \tilde{\gamma}'$. If we denote by $\phi_v,\ v \in (-\epsilon, \epsilon)$, the local flow of the Killing vector field X, then the map

$$\psi : (a, b) \times (-\epsilon, \epsilon) \to N^3\,, \quad \psi(u, v) = \phi_v(\gamma(u)), \tag{1.1}$$

defines a parametrization of a G_X-invariant surface. Conversely, if $f(M)$ is a G_X-invariant immersed surface in N^3, then $\tilde{f}$ defines a curve in $(N^3/G_X, \tilde{g})$ that can be locally parametrized by arc length. The curve $\tilde{\gamma}$ is generally called the *profile curve* of the invariant surface.

Observe that, as the v-coordinate curves are the orbits of the action of the one-parameter group of isometries G_X, the coefficients of the pull-back metric $g_f = E\,du^2 + 2F\,dudv + G\,dv^2$ are function only of u and are given by:

$$\begin{cases} E = g(\psi_u, \psi_u) = g(d\phi_v(\gamma'), d\phi_v(\gamma')) \\ F = g(\psi_u, \psi_v) = g(d\phi_v(\gamma'), X) \\ G = g(\psi_v, \psi_v) = g(X, X). \end{cases}$$

Putting $(\omega(u))^2 := \|X(\gamma(u))\|_g^2 = G$, one has (see [12])

$$EG - F^2 = G = (\omega(u))^2. \tag{1.2}$$

Remark 1.1. When γ is a horizontal lift of $\tilde{\gamma}$, we have that $F = 0$ and $E = 1$. Thus, the equation (1.2) is immediate.

By using (1.2) and Bianchi's formula for the Gauss curvature, we find that

$$K(u) = -\frac{\omega_{uu}(u)}{\omega(u)}. \tag{1.3}$$

Consequently, we have

Theorem 1.2 ([12])**.** *Let $f : M \to (N^3, g)$ be a G_X-equivariant immersion, $\tilde{\gamma} : (a, b) \subset \mathbb{R} \to (N_r^3/G_X, \tilde{g})$ a parametrization by arc length of the profile curve of M and γ a lift of $\tilde{\gamma}$. Then the induced metric g_f is of constant Gauss curvature K if and only if the function $\omega(u)$ satisfies the differential equation*

$$\omega_{uu}(u) + K\omega(u) = 0. \tag{1.4}$$

2. A parametric equation for loxodromes

In this section, we obtain the equation of the loxodromes on an invariant surface which are not orbits in terms of the parameters u and v.

Let M be a G_X-invariant surface of (N^3, g), locally parametrized by (1.1); then the induced metric is given by

$$g_f = E(u)\, du^2 + 2\, F(u)\, du\, dv + (\omega(u))^2\, dv^2.$$

Now, let $\alpha(s) = \psi(u(s), v(s))$ be a loxodrome on M parametrized by arc length, so that

$$1 = g(\alpha', \alpha') = E(u(s))\, u'^2 + 2\, F(u(s))\, u'\, v' + (\omega(u(s)))^2\, v'^2. \tag{2.1}$$

Denoting by $\vartheta_0 \in [0, \pi)$ the constant angle under which the curve α meets the orbits of the Killing vector field X, we have

$$\begin{aligned} \omega(u(s)) \cos\vartheta_0 &= g(\alpha', X) = g(\alpha', \psi_v) \\ &= F(u(s))\, u'(s) + (\omega(u(s)))^2\, v'(s). \end{aligned} \tag{2.2}$$

In the following, we suppose that $\vartheta_0 \neq 0$. Then the loxodrome is not an orbit and

$$\alpha(s) \neq \psi\Big(u_0, \frac{s}{\omega(u_0)}\Big), \qquad u_0 \in (a, b).$$

Remark 2.1. If the loxodrome $\alpha(s)$ is orthogonal to all the orbits that it meets (i.e. $\vartheta_0 = \pi/2$), then it is a geodesic. In fact, we observe that, as α cannot be an orbit, it is a geodesic if satisfies the following system (see [14])

$$\begin{cases} \|\alpha'\| = 1, \\ (F\,u' + \omega^2\,v')' = 0. \end{cases} \tag{2.3}$$

We only have to show that the second equation of (2.3) is satisfied. By using (2.2), from $g(\alpha', X) = 0$ we get $Fu' + \omega^2 v' = 0$.

Lemma 2.2. *Let $M \subset (N^3, g)$ be a G_X-invariant surface locally parametrized by $\psi(u, v)$ given by* (1.1) *and let $\alpha(s) = \psi(u(s), v(s))$ be a loxodrome, parametrized by arc length, which is not an orbit. Then*

$$\begin{cases} F(u(s))\,u' + \omega(u(s))^2\,v' = \omega(u(s))\,\cos\vartheta_0, \\ {u'}^2 = \sin^2\vartheta_0. \end{cases} \tag{2.4}$$

Conversely, if system (2.4) *is satisfied and $u' \neq 0$, then α is a loxodrome parametrized by arc length.*

Proof. If α is a loxodrome parametrized by arc length, then condition (2.2), and therefore the first equation of system (2.4), is satisfied. Also, the vector field $\alpha'(s) = \psi_u\,u'(s) + \psi_v\,v'(s)$ verifies (2.1). Moreover

$$\begin{aligned} 2\,F(u(s))\,u'\,v' + \omega(u(s))^2\,{v'}^2 &= v'(s)\,(F(u(s))\,u' + \omega(u(s))\cos\vartheta_0) \\ &= (F(u(s))\,u' + \omega(u(s))\cos\vartheta_0)\;\frac{\omega(u(s))\cos\vartheta_0 - F(u(s))\,u'}{\omega(u(s))^2} \\ &= \frac{\omega(u(s))^2\,\cos^2\vartheta_0 - F(u(s))^2\,{u'}^2}{\omega(u(s))^2}. \end{aligned} \tag{2.5}$$

Therefore, by substituting (2.5) in (2.1) and using (1.2) we obtain the second equation of (2.4)

$$1 = \frac{E(u(s))\,\omega(u(s))^2 - F(u(s))^2}{\omega(u(s))^2}\,{u'}^2 + \cos^2\vartheta_0 = {u'}^2 + \cos^2\vartheta_0.$$

Conversely, if system (2.4) is satisfied, from the second equations of (2.4), by taking into account (1.2) we have

$$\begin{aligned} 1 = {u'}^2 + \cos^2\vartheta_0 &= \left(\frac{E(u(s))\,\omega(u(s))^2 - F(u(s))^2}{\omega(u(s))^2}\right){u'}^2 + \cos^2\vartheta_0 \\ &= E(u(s))\,{u'}^2 + \frac{\omega(u(s))^2\,\cos^2\vartheta_0 - F(u(s))^2\,{u'}^2}{\omega(u(s))^2}. \end{aligned}$$

Now we use the first equation of (2.4) to obtain

$$\begin{aligned} 1 &= E(u(s))\,{u'}^2 + v'\,[\omega(u(s))\,\cos\vartheta_0 + F(u(s))\,u'] \\ &= E(u(s))\,{u'}^2 + 2\,F(u(s))\,u'\,v' + \omega(u(s))^2\,{v'}^2 \\ &= g(\alpha',\alpha'), \end{aligned} \tag{2.6}$$

so that α has unit speed. Moreover, since α is not an orbit, from the first of equations (2.4) we conclude that α is a loxodrome. □

By integrating system (2.4) we have the following

Theorem 2.3. *A loxodrome on a G_X-invariant surface $M \subset (N^3, g)$, which is not an orbit, can be locally parametrized by $\beta(u) = \psi(u, v(u))$, where*

$$v(u) = \int_{u_0}^{u} \left(\frac{-F}{\omega^2} \pm \frac{\cot\vartheta_0}{\omega} \right) dt. \tag{2.7}$$

Proof. We consider the surface M locally parametrized by $\psi(u, v)$ given by (1.1) and we suppose that $\alpha(s) = \psi(u(s), v(s))$ is a loxodrome on M that is not an orbit parametrized by arc length. As $u' \neq 0$, we can locally invert the function $u = u(s)$ obtaining $s = s(u)$ and we can consider the parametrization of α given by

$$\beta(u) = \alpha(s(u)) = \psi(u, v(u)), \qquad v(u) = v(s(u)).$$

By multiplying by $(ds/du)^2$ the equation

$$E(u(s))\,u'(s)^2 + 2\,F(u(s))\,u'(s)\,v'(s) + \omega(u(s))^2\,v'(s)^2 = g(\alpha',\alpha') = 1$$

we obtain

$$E + 2\,F\,\frac{dv}{du} + \omega^2 \left(\frac{dv}{du}\right)^2 = \left(\frac{ds}{du}\right)^2. \tag{2.8}$$

Moreover, from the second equation in (2.4), we get

$$\left(\frac{ds}{du}\Big|_{u(s)} \right)^2 = \frac{1}{u'(s)^2} = \frac{1}{(\sin\vartheta_0)^2}. \tag{2.9}$$

Substitution of (2.9) in (2.8) gives

$$\omega^2 \left(\frac{dv}{du}\right)^2 + 2\,F\,\frac{dv}{du} + E - \frac{1}{(\sin\vartheta_0)^2} = 0. \tag{2.10}$$

Now, taking into account (1.2), we observe that

$$F^2 - \omega^2 \left[E - \frac{1}{(\sin\vartheta_0)^2} \right] = (\cot\vartheta_0\,\omega)^2.$$

Consequently, from (2.10) we obtain

$$\frac{dv}{du} = \frac{-F \pm \cot\vartheta_0\,\omega}{\omega^2};$$

from this formula we get at once (2.7), which is the equation of a segment of a loxodrome which is not an orbit. □

As immediate consequence of equation (2.9) is given by

Corollary 2.4. *The length of a loxodrome which is not an orbit on a G_X-invariant surface $M \subset (N^3, g)$ between two orbits u_1 and u_2 is given by*

$$s = \frac{u_2 - u_1}{\sin \vartheta_0}. \tag{2.11}$$

2.1. Loxodromes on translational and helicoidal surfaces in $\mathbb{R}^3$

We consider the Euclidean three-dimensional space $\mathbb{R}^3$ with the canonical metric $g = dx^2 + dy^2 + dz^2$. Then the Killing vector fields generate translations and rotations.
In the case of translations along the direction of the unitary Killing vector field X, the quotient space $\mathbb{R}^3/G_X$ is a plane $\mathbb{R}^2$ (orthogonal to X) equipped with the flat metric. The invariant surface is a flat right cylinder over the curve

$$\tilde{\gamma}(u) = (\xi_1(u), \xi_2(u)), \qquad \xi_1'^2 + \xi_2'^2 = 1,$$

and it can be parametrized by

$$\psi(u, v) = \gamma(u) + v\, X,$$

where γ is a horizontal lift of $\tilde{\gamma}$. Therefore, the coefficients of the induced metric are $E = 1$, $F = 0$ and $\omega = 1$ and its loxodromes are parametrized by

$$\beta(u) = \psi(u, v(u)) = \gamma(u) \pm \cot \vartheta_0\, (u - u_0)\, X.$$

So these curves form an angle ϑ_0 with the Killing vector field X, i.e. they are general helices whose axis is X.
In the case of the helicoidal surfaces we can assume, without loss of generality, that the Killing vector field is $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + a\frac{\partial}{\partial z}$, with $a \in \mathbb{R}$. Introducing cylindrical coordinates (r, θ, z), we have

$$X = \frac{\partial}{\partial \theta} + a\, \frac{\partial}{\partial z}, \qquad g = dr^2 + r^2 d\theta^2 + dz^2,$$

and $\{\xi_1 = r, \xi_2 = z - a\,\theta\}$ is a set of independent G_X-invariant functions. Therefore, the regular part of the orbit space is

$$\mathbb{R}^3_r/G_X = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \;:\; \xi_1 > 0\},$$

and, with respect to its orbital metric (see [3],[9])

$$\tilde{g} = d\xi_1^2 + \frac{\xi_1^2}{a^2 + \xi_1^2}\, d\xi_2^2,$$

the projection

$$(r,\theta,z)\overset{\pi}{\mapsto}(r,z-a\,\theta)$$

becomes a Riemannian submersion. Now we suppose that the profile curve of a G_X-invariant surface $\tilde\gamma(u)=(\xi_1(u),\xi_2(u))\in\mathbb{R}^3_r/G_X$, is parametrized by arc length, so that

$$\xi_1'^2(u)+\frac{\xi_1(u)^2}{a^2+\xi_1(u)^2}\,\xi_2'(u)^2=1. \tag{2.12}$$

Then the norm of X restricted to the profile curve is $\omega(u)=\sqrt{r(u)^2+a^2}$ and a lift of $\tilde\gamma$ with respect to π is $\gamma(u)=(\xi_1(u),0,\xi_2(u))$. Therefore, the coefficients of the induced metric of the helicoidal surfaces

$$\psi(u,\theta)=(\xi_1(u)\,\cos\theta,\xi_1(u)\,\sin\theta,\xi_2(u)+a\,\theta)$$

are

$$E=\xi_1'(u)^2+\xi_2'(u)^2,\qquad F=a\,\xi_2'(u),\qquad G=\omega(u)^2. \tag{2.13}$$

From (2.12) we get

$$\xi_2'(u)^2=\frac{\omega(u)^2}{r(u)^2}(1-r'(u)^2) \tag{2.14}$$

and then, from (2.13),

$$\frac{F}{\omega^2}=\frac{a\sqrt{1-\xi_2'(u)^2}}{r(u)\,\sqrt{a^2+r(u)^2}}.$$

By using (2.7) we conclude that the loxodromes on helicoidal surfaces can be parametrized by $\beta(u)=\psi(u,\theta(u))$, where

$$\theta(u)=\int_{u_0}^{u}\Big(-\frac{a\sqrt{1-r'(t)^2}}{r(t)\,\sqrt{a^2+r(t)^2}}\pm\frac{\cot\vartheta_0}{\sqrt{a^2+r(t)^2}}\Big)\,dt.$$

Remark 2.5. If γ is a horizontal lift of $\tilde\gamma$ (i.e. $F=0$ and $E=1$), equation (2.7) reduces to

$$\theta(u)=\pm\int_{u_0}^{u}\frac{\cot\vartheta_0}{\sqrt{a^2+r(t)^2}}\,dt.$$

We observe that, in the case of the surfaces of revolution (i.e. for $a=0$),

$$\beta(u)=(\xi_1(u)\,\cos\theta(u),\xi_1(u)\,\sin\theta(u),\xi_2(u)),$$

with

$$\theta(u)=\pm\cot\vartheta_0\int_{u_0}^{u}\frac{dt}{\xi_1(t)}. \tag{2.15}$$

Example (Sphere). The unit sphere $\mathbb{S}^3$ is obtained by choosing $a = 0$ and $\tilde{\gamma}(u) = (\sin u, \cos u)$, $u \in (0, \pi)$. In this case, equation (2.15) gives

$$\theta(u) = \cot \vartheta_0 \ \ln \left(\tan \left(\frac{u}{2} \right) \right) + c, \qquad c \in \mathbb{R}.$$

Example (Pseudosphere). Choosing

$$\xi_1(u) = e^{-u}, \qquad \xi_2(u) = \operatorname{arctanh}(\sqrt{1 - e^{-2u}}) - \sqrt{1 - e^{-2u}}, \qquad a = 0,$$

we have a pseudosphere. A loxodrome on this surface is parametrized by

$$\theta(u) = \cot \vartheta_0 \, e^u + c, \qquad c \in \mathbb{R}.$$

Example (Twisted sphere). If we consider $a = 1$ and $\xi_1(u) = \sin u$, $u \in (0, \pi)$, the integration of (2.12) leads to $\xi_2(u) = E(u, -1)$, an elliptic integral of the second kind[1]. The corresponding surface is the twisted sphere (see Figure 1).

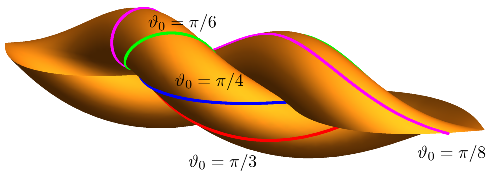

FIGURE 1. Four loxodromes of the twisted sphere in $\mathbb{R}^3$, obtained for $a = 1$.

Example (Twisted pseudosphere). If we consider $a = 1$ and $\xi_1(u) = e^{-u}$, the integration of (2.12) leads to the Gaussian hypergeometric function[2] $F(1/4, 1, 3/4, e^{4u})$ and

$$\xi_2(u) = e^u \sqrt{1 - e^{-4u}} \, (1 - 2 \, F(1/4, 1, 3/4, e^{4u})).$$

[1]The elliptic integral of the second kind is defined by

$$E(\phi, m) = \int_0^{\phi} \sqrt{1 - m \sin^2 \theta} \, d\theta.$$

[2]The Gaussian hypergeometric function $F(\alpha_1, \alpha_2, \beta_1, z)$ is defined for $|z| < 1$ by the Pochhammer power series

$$F(\alpha_1, \alpha_2, \beta_1, z) \overset{not}{=} {}_2F_1[\alpha_1, \alpha_2, \beta_1, z] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \, (\alpha_2)_n}{(\beta_1)_n} \frac{z^n}{n!}.$$

Here $\alpha_1, \alpha_2, \beta_1 \in \mathbb{R}$, $c \notin \mathbb{Z}_{\leq 0}$ and $(\lambda)_n = \lambda(\lambda + 1) \cdots (\lambda + n - 1)$.

The corresponding surface is the twisted pseudosphere (see Figure 2).

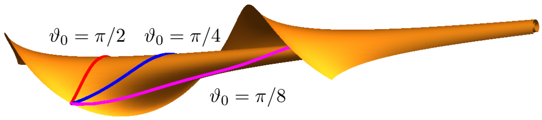

FIGURE 2. Three loxodromes of the twisted pseudosphere in $\mathbb{R}^3$, obtained for $a = 1$.

3. Some remarks about loxodromes and geodesics on invariant surfaces

In this section, we extend a classical result about loxodromes and geodesics on rotational surfaces in the Euclidean three-space to rotational surfaces in the remarkable family of Bianchi-Cartan-Vranceanu spaces. We start by proving the following

Theorem 3.1. *Assume that $\alpha(s) = \psi(u(s), v(s))$ is a loxodrome, parametrized by arc length, on a G_X-invariant surface $M \subset (N^3, g)$ locally defined by the $\psi(u, v)$ in (1.1), with $\vartheta_0 \neq 0, \pi/2$. If α is a geodesic of M, then the surface is flat.*

Proof. First we observe that from (1.3) the Gauss curvature K of an invariant surface depends only on the profile curve, that is K is constant along any orbit. This implies that if the Gauss curvature is constant along a curve, then either the curve is an orbit or the curve lies in a part of the surface where the Gauss curvature is constant. Now, from the Clairaut's Theorem proved in [14], we get that the loxodrome $\alpha(s)$ is a geodesic if, and only if, $\omega(u(s)) \cos\vartheta_0 = c$, $c \in \mathbb{R}$. Therefore, as $\vartheta_0 \neq \pi/2$, it follows that $\omega(u(s))$ is constant. Also, taking into account the fact that α is not an orbit, we conclude that $\omega_{uu}(u(s)) = 0$ and equation (1.3) implies that the Gauss curvature of M is zero along the curve α. Then, from the initial remark, we conclude that α lies in a part of the invariant surface with zero Gauss curvature. □

3.1. Rotational surfaces in the Bianchi-Cartan-Vranceanu spaces

The Bianchi-Cartan-Vranceanu spaces (BCV-spaces) are the three-dimensional Riemannian manifolds endowed with the Riemannian metrics of the following two-parameter family (see [4, 8, 22])

$$g_{\ell,m} = \frac{dx^2+dy^2}{[1+m(x^2+y^2)]^2} + \left(dz + \frac{\ell}{2}\frac{ydx-xdy}{[1+m(x^2+y^2)]}\right)^2, \quad \ell, m \in \mathbb{R}, \tag{3.1}$$

defined on $N=\mathbb{R}^3$ if $m \geq 0$ and on $N=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2 < -1/m\}$ otherwise. Their geometric interest lies in the following fact: *the family of metrics* (3.1) *includes all three-dimensional homogeneous metrics whose group of isometries has dimension* 4 *or* 6*, except for those of constant negative sectional curvature.* The group of isometries of these spaces contains a one-parameter subgroup isomorphic to $SO(2)$, whose infinitesimal generator is the Killing vector field given by

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

The orbits of X are geodesic circles on horizontal planes with centre on the z-axis and the volume function

$$\omega(u) = \sqrt{g_{m,\ell}(X,X)} = \frac{f(u)}{2(1+m\,f(u)^2)}\sqrt{4+\ell^2\,f(u)^2}, \tag{3.2}$$

where $f>0$, represents the Euclidean radius of a principal orbit. Concerning $SO(2)$-invariant surfaces in the BCV-spaces, as a consequence of Theorem 3.1 we have the following result, well known in the space $\mathbb{R}^3$.

Corollary 3.2. *Upon a rotational surface M in a BCV-space a loxodrome which is neither a meridian nor a parallel cannot be a geodesic unless M is a vertical cylinder.*

Proof. We suppose that $\alpha(s)=\psi(u(s),v(s))$ is a loxodrome, parametrized by arc length, that is also a geodesic on the surface M given by

$$\psi(u,v) = (f(u)\,\cos v, f(u)\,\sin v, g(u)).$$

From Theorem 3.1, we have that the surface M is flat. Therefore, by using equation (1.4) one has $\omega(u)=c_1\,u+c_2$, $c_1,c_2\in\mathbb{R}$. Also, as $\omega(u(s)) = \textit{constant}$ (see the proof of Theorem 3.1) and the curve α is not a parallel (i.e. $u'(s)\neq 0$), we have that $\omega_u(u(s))=0$. Then, $c_1=0$ and $\omega(u)=c_2$. Now, from (3.2) we obtain (see [20]) that $\omega'(u)=0$ if, and only if,

$$f'(u)\,[2+(\ell^2-2m)\,f(u)^2]=0. \tag{3.3}$$

Consequently, $f(u)=\textit{constant}$ and M is a vertical cylinder. $\square$

4. Loxodromes on invariant surfaces of $\mathbb{H}^2 \times \mathbb{R}$

Let $\mathbb{H}^2 = \{(x,y) \in \mathbb{R}^2 \,:\, y > 0\}$ be the half plane model of the hyperbolic plane and consider $\mathbb{H}^2 \times \mathbb{R}$ endowed with the product metric

$$g = \frac{dx^2 + dy^2}{y^2} + dz^2. \tag{4.1}$$

The Lie algebra of the infinitesimal isometries of the product $(\mathbb{H}^2 \times \mathbb{R}, g)$ admits the following basis of Killing vector fields [19]

$$\begin{aligned}
X_1 &= \frac{(x^2 - y^2 + 1)}{2}\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},\\
X_2 &= \frac{\partial}{\partial x},\\
X_3 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},\\
X_4 &= \frac{\partial}{\partial z}.
\end{aligned}$$

The class of invariant surfaces in $\mathbb{H}^2 \times \mathbb{R}$ can be divided into three subclasses according to the following

Proposition 4.1 ([17]). *Any surface in $(\mathbb{H}^2 \times \mathbb{R}, g)$ which is invariant under the action of a one-parameter subgroup of isometries G_X generated by a Killing vector field $X = \sum_{i=1}^4 a_i X_i$, $a_i \in \mathbb{R}$, is congruent to a surface invariant under the action of one of the following groups:*

$$G_{14} = G_{X_1 + bX_4}, \qquad G_{24} = G_{aX_2 + bX_4}, \qquad G_{34} = G_{X_3 + bX_4},$$

where $a, b \in \mathbb{R}$.

To understand the shape of an invariant surface in $\mathbb{H}^2 \times \mathbb{R}$ we need to describe the orbits of the three groups G_{24}, G_{34} and G_{14}. In [17] it has been shown that the orbit of a point $p_0 = (x_0, y_0, z_0) \in \mathbb{H}^2 \times \mathbb{R}$ is:

• *under the action of* G_{24}, the curve parametrized by

$$(a\,v + x_0, y_0, b\,v + z_0), \quad v \in (-\epsilon, \epsilon), \tag{4.2}$$

which looks like an Euclidean straight line on the plane $y = y_0$;

• *under the action of* G_{34}, the curve parametrized by

$$(e^v x_0, e^v y_0, b\,v + z_0), \quad v \in (-\epsilon, \epsilon), \tag{4.3}$$

which belongs to a vertical plane through the z-axis and looks like a logarithmic curve;

- *under the action of* G_{14}, the curve parametrized by

$$(x(v), y(v), b\,v + z_0), \quad v \in (-\epsilon, \epsilon), \tag{4.4}$$

where

$$\begin{cases} x(v) = \dfrac{(1 - x^2 - y^2)\sin v + 2x\,\cos v}{(1 - x^2 - y^2)\cos v - 2x\sin v + 1 + x^2 + y^2}, \\ y(v) = \dfrac{2y}{(1 - x^2 - y^2)\cos v - 2x\,\sin v + 1 + x^2 + y^2}. \end{cases}$$

Thus,

$$(x(v))^2 + (y(v))^2 - \beta\,y(v) + 1 = 0, \quad \beta = \frac{1 + x_0^2 + y_0^2}{y_0},$$

which looks like an Euclidean helix in a right circular cylinder with Euclidean axis in the plane $x = 0$.

In the following, we describe explicitly how to parametrize an invariant surface in $(\mathbb{H}^2 \times \mathbb{R}, g)$ and we use (2.7) to parametrize the loxodromes on it.

Theorem 4.2. *Let M be a G_X-invariant surface of $(\mathbb{H}^2 \times \mathbb{R}, g)$, and let $\tilde{\gamma}(u) = (\xi_1(u), \xi_2(u))$ be its profile curve in the regular part of the orbit space $(\mathcal{B} = \mathbb{H}^2 \times \mathbb{R}/G_X, \tilde{g})$, which is parametrized by the invariant functions ξ_1 and ξ_2. With respect to the local parametrization $\psi(u, v)$ given by (1.1) we have:*

i) *If $G = G_{24}$ is the group generated by $X_2 + b\,X_4$, $b \in \mathbb{R}$, then the orbit space is $\mathcal{B} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 > 0\}$ and a loxodrome can be parametrized by $\psi(u, v(u))$, where*

$$v(u) = \int_{u_0}^{u} \frac{\left(b\sqrt{\xi_2(t)^2 - \xi_2'(t)^2} \pm \cot\vartheta_0\right)\,\xi_2(t)}{\sqrt{1 + b^2\,\xi_2(t)^2}}\,dt \tag{4.5}$$

and

$$\psi(u, v) = (v, \xi_2(u), b\,v - \xi_1(u)). \tag{4.6}$$

In particular, when $b = 0$ (i.e. $G = G_2$) a loxodrome can be parametrized by

$$\beta(u) = \left(\pm\cot\vartheta_0 \int_{u_0}^{u} \xi_2(t)\,dt, \xi_2(u), -\xi_1(u)\right).$$

ii) *If $G = G_4$, the invariant surface is a right cylinder and its loxodromes are helices that can be parametrized by*

$$\beta(u) = (\xi_1(u), \xi_2(u), \cot\vartheta_0\,(u - u_0)).$$

iii) *If* $G = G_{34}$ *is the group generated by* $X_3 + b\,X_4$, $b \in \mathbb{R}$, *then the orbit space is* $\mathcal{B} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \ : \ \xi_1 \in (0, \pi)\}$ *and a loxodrome can be parametrized by* $\psi(u, v(u))$, *where*

$$v(u) = \int_{u_0}^{u} \frac{\big(- b\sqrt{\sin^2 \xi_1(t) - \xi_1'(t)^2} \pm \cot \vartheta_0 \big) \, \sin \xi_1(t)}{\sqrt{1 + b^2 \, \sin^2 \xi_1(t)}} \, dt \qquad (4.7)$$

and

$$\psi(u, v) = (e^v \, \cos \xi_1(u), e^v \, \sin \xi_1(u), \xi_2(u) + b\,v). \qquad (4.8)$$

iv) *If* $G = G_{14}$, *then the orbit space is* $\mathcal{B} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \ : \xi_1 \geq 2\}$ *and a loxodrome can be parametrized by*

$$v(u) = \int_{u_0}^{u} \frac{-4b\sqrt{\xi_1^2(t) - 4 - \xi_1'(t)^2} \ \pm \ 2(\xi_1^2(t) - 4) \ \cot \vartheta_0}{(\xi_1^2(t) - 4)\sqrt{\xi_1^2(t) - 4\,(b^2 - 1)}} \, dt \quad (4.9)$$

and

$$\psi(u, v) = \Big(\frac{\sqrt{\xi_1^2(u) - 4}\, \sin v}{\sqrt{\xi_1^2(u) - 4}\, \cos v - \xi_1(u)}, \frac{2}{\xi_1(u) - \sqrt{\xi_1^2(u) - 4}\, \cos v}, \xi_2(u) + b\,v \Big). \qquad (4.10)$$

Proof. i) We begin with the calculations for the G_{24}-invariant surfaces. As $X = \frac{\partial}{\partial x} + b\frac{\partial}{\partial z}$, $b \in \mathbb{R}$, a set of two invariant functions is

$$\xi_1 = b\,x - z, \qquad \xi_2 = y > 0.$$

Thus, the orbit space is $\mathcal{B} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \ : \ \xi_2 > 0\}$ and the orbital metric is

$$\tilde{g} = \frac{d\xi_1^2}{1 + b^2 \xi_2^2} + \frac{d\xi_2^2}{\xi_2^2}.$$

A lift of $\tilde{\gamma}$ with respect to π is given by

$$\gamma(u) = (0, \xi_2(u), -\xi_1(u))$$

and, from (1.1) and (4.2), the corresponding G_{24}-invariant surface is parametrized by

$$\psi(u, v) = (v, \xi_2(u), -\xi_1(u) + b\,v).$$

Then

$$F = -b\,\xi_1'(u), \qquad G = \omega^2 = \frac{1}{\xi_2^2(u)} + b^2. \qquad (4.11)$$

Also, as $\tilde{\gamma}(u)$ is parametrized by arc length, it follows that

$$\xi_1'(u) = \omega(u)\sqrt{\xi_2^2(u) - \xi_2'(u)^2}. \qquad (4.12)$$

Then, by substituting (4.11) and (4.12) in equation (2.7), we obtain (4.5).

iii) We consider the case of G_{34}-invariant surfaces in $\mathbb{H}^2 \times \mathbb{R}$, where we consider a system of cylindrical coordinates (r, θ, z). The orbit space is given by

$$\mathcal{B} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \, : \, \xi_1 \in (0, \pi)\},$$

where $\{\xi_1 = \theta, \xi_2 = z - b \ln r\}$ are G_{34}-invariant functions. As we have seen, endowing the orbit space with the quotient metric

$$\tilde{g} = \frac{d\xi_1^2}{\sin^2 \xi_1} + \frac{d\xi_2^2}{1 + b^2 \sin^2 \xi_1},$$

the projection

$$(r, \theta, z) \overset{\pi}{\mapsto} (\theta, z - b \ln r)$$

becomes a Riemannian submersion. Using cylindrical coordinates, a lift of $\tilde{\gamma}$ with respect to π is given by

$$\gamma(u) = (1, \xi_1(u), \xi_2(u))$$

and, from (1.1) and (4.4) it follows that the corresponding invariant surface is given by

$$\psi(u, v) = (e^v \, \cos \xi_1(u), e^v \, \sin \xi_1(u), \xi_2(u) + b\, v).$$

Thus we obtain that

$$F = b\, \xi_2'(u), \qquad G = \omega^2 = \frac{1 + b^2 \, \sin^2 \xi_1(u)}{\sin^2 \xi_1(u)}. \tag{4.13}$$

Also, as $\tilde{\gamma}(u)$ is parametrized by arc length, we get

$$\xi_2'(u) = \omega(u) \sqrt{\sin^2 \xi_1(u) - \xi_1'(u)^2}. \tag{4.14}$$

Substituting (4.13) and (4.14) in equation (2.7), we obtain (4.7).

iv) We consider the case of G_{14}-invariant surfaces in $\mathbb{H}^2 \times \mathbb{R}$, again with respect to a system of cylindrical coordinates (r, θ, z). The orbit space is given by

$$\mathcal{B} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \, : \, \xi_2 \geq 2\},$$

where

$$\xi_1 = \frac{r^2 + 1}{r \sin \theta}, \qquad \xi_2 = z - b \arctan\left(\frac{2r \, \cos \theta}{r^2 - 1}\right), \qquad b \in \mathbb{R},$$

are G_{14}-invariant functions. As we have seen, endowing the orbit space with the quotient metric

$$\tilde{g} = \frac{d\xi_1^2}{\xi_1^2 - 4} + \frac{\xi_1^2 - 4}{\xi_1^2 + 4(b^2 - 1)}\, d\xi_2^2,$$

the projection

$$(r, \theta, z) \overset{\pi}{\mapsto} \Big(\frac{r^2 + 1}{r\,\sin\theta}, z - b\,\arctan\Big(\frac{2r\,\cos\theta}{r^2 - 1}\Big)\Big)$$

becomes a Riemannian submersion. By using cylindrical coordinates, a lift of $\tilde{\gamma}$ with respect to π is given by

$$\gamma(u) = \Big(\frac{\xi_1(u) + \sqrt{\xi_1^2(u) - 4}}{2}, \frac{\pi}{2}, \xi_2(u)\Big).$$

The corresponding invariant surface is parametrized by

$$\psi(u,v) = \Big(\frac{\sqrt{\xi_1^2(u) - 4}\,\sin v}{\sqrt{\xi_1^2(u) - 4}\,\cos v - \xi_1(u)}, \frac{2}{\xi_1(u) - \sqrt{\xi_1^2(u) - 4}\,\cos v}, \xi_2(u) + b\,v\Big).$$

Then we find that

$$G = \omega^2 = \frac{\xi^2(u) - 4}{4} + b^2, \qquad F = b + \xi_2'(u). \tag{4.15}$$

Also, as $\tilde{\gamma}(u)$ is parametrized by arc length, we have

$$\xi_2'(u) = \frac{2\,\omega(u)}{\sqrt{\xi_1^2(u) - 4}}\sqrt{1 - \frac{\xi_1'(u)^2}{\xi_1^2(u) - 4}}. \tag{4.16}$$

Now we use (4.15) and (4.16) in (2.7) to get (4.9). □

Example. We consider the minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ given by the graph of the function $z = -\ln y$ (see [17]). This surface is G_2-invariant and its profile curve is given by

$$\tilde{\gamma}(u) = (u/\sqrt{2}, e^{u/\sqrt{2}}).$$

By using (4.6), the surface can be parametrized by

$$\psi(u,v) = (v, e^{u/\sqrt{2}}, -u/\sqrt{2}).$$

Therefore from (4.5) it comes out that the loxodromes which are not orbits are given by

$$\beta(u) = \psi(u, \sqrt{2}\cot\vartheta_0\, e^{u/\sqrt{2}}).$$

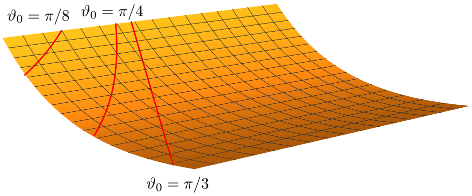

FIGURE 3. Three loxodromes of the G_2-invariant minimal graph in $\mathbb{H}^2 \times \mathbb{R}$ given by $z = -\ln y$.

Example (The funnel surface). The *funnel surface* is a complete minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ and it is the graph of the function $z = \ln(\sqrt{x^2 + y^2})$ (see [17]). This surface is G_{34}-invariant and it can be obtained starting from the simplest curve in $\mathbb{H}^2 \times \mathbb{R}/G_{34}$, choosing $b = 1$ and $\xi_2 = 0$, whose a parametrization by arc length is $\tilde{\gamma}(u) = (2 \operatorname{arccot} e^{-u}, 0)$. Using cylindrical coordinates, a lift of $\tilde{\gamma}$ with respect to π is given by

$$\gamma(u) = (1, 2 \operatorname{arccot} e^{-u}, 0)$$

and, by means of (4.8), the funnel surface can be parametrized by

$$\psi(u, v) = (-e^v \tanh u, e^v \operatorname{sech} u, v).$$

Taking (4.7) into account, we find that the loxodromes which are not orbits can be parametrized by

$$\beta(u) = \psi\left(u, \int_{u_0}^{u} \frac{\pm \cot \vartheta_0}{\sqrt{1 + \cosh^2 t}}\, dt\right).$$

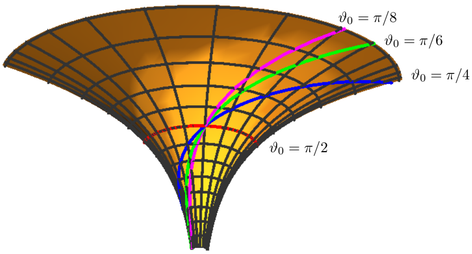

FIGURE 4. Four loxodromes of the funnel surface through the point $(0, 1, 0)$ as seen from the point $(1, 10, -4)$.

5. Loxodromes on invariant surfaces of the Heisenberg group $\mathbb{H}_3$

We consider on the three-dimensional Heisenberg space $\mathbb{H}_3$, represented in $Gl_3(\mathbb{R})$ by

$$\begin{bmatrix} 1 & x & z+\frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \qquad x, y, z \in \mathbb{R},$$

the left-invariant metric

$$g = dx^2 + dy^2 + \left(\frac{1}{2}y\,dx - \frac{1}{2}x\,dy + dz\right)^2.$$

The isometry group of $(\mathbb{H}_3, g)$ has dimension 4, which is the maximal one for a non constant curvature three-manifold. In this case we have the following

Proposition 5.1. *([19]) The Lie algebra of the infinitesimal isometries of $(\mathbb{H}_3, g)$ admits the following basis of Killing vector fields*

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial z}, \\ X_2 &= \frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial z}, \\ X_3 &= \frac{\partial}{\partial z}, \\ X_4 &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \end{aligned}$$

Also, the one-dimensional subgroups of the isometry group $Isom(\mathbb{H}_3, g)$ belong to one of the two following families (see [9]):

1) the one-parameter subgroups generated by linear combinations

$$a_1X_1 + a_2X_2 + a_3X_3 + bX_4,$$

with $b \neq 0$, that are called subgroups of *helicoidal type*. If $a_i = 0$, for $i \in \{1, 2, 3\}$, we obtain the group $SO(2)$ generated by X_4;

2) the one-parameter subgroups generated by linear combinations of X_1, X_2 and X_3, that are called of *translational type*.

Therefore, a surface in the space $(\mathbb{H}_3, g)$ is called *helicoidal* (respectively, *translational*) if it's invariant under the action of a helicoidal (respectively, a translational) one-parameter subgroup of isometries. In [9] one finds the following

Proposition 5.2. *A surface in $(\mathbb{H}_3, g)$ which is invariant under the action of a one-parameter subgroup of isometries G_X generated by a*

Killing vector field $X = \sum_i a_i X_i$, $a_i \in \mathbb{R}$, *is congruent to a surface invariant under the action of one of the following subgroups:*

$$G_1, \qquad G_3, \qquad G_{43}.$$

Theorem 5.3. *Let* M *be a* G_X*-invariant surface of the Heisenberg group* $(\mathbb{H}_3, g)$*, and let* $\tilde{\gamma}(u) = (\xi_1(u), \xi_2(u))$ *be its profile curve in the regular part of the orbit space* $(\mathcal{B} = \mathbb{H}_3/G_X, \tilde{g})$*, which is parametrized by the invariant functions* ξ_1 *and* ξ_2*. With respect to the local parametrization* $\psi(u, v)$ *given by* (1.1)*, we have:*

i) *if* $G = G_1$*, then the orbit space is* $\mathcal{B} = \mathbb{R}^2$ *and a loxodrome can be parametrized by* $\psi(u, v(u))$*, where*

$$v(u) = \int_{u_0}^{u} \frac{\xi_1(t)\sqrt{1 - \xi_1'(t)^2} \pm \cot \vartheta_0}{\sqrt{1 + \xi_1(t)^2}} \, dt \tag{5.1}$$

and

$$\psi(u, v) = \Big(v, \xi_1(u), \frac{\xi_1(u)}{2} v - \xi_2(u)\Big). \tag{5.2}$$

ii) *if* $G = G_3$*, then the orbit space is* $\mathcal{B} = \mathbb{R}^2$ *and a loxodrome can be parametrized by* $\psi(u, v(u))$*, where*

$$v(u) = -\frac{\xi_1(u)}{2} \int_{u_0}^{u} \sqrt{1 - \xi_1'(t)^2 \, dt} + \int_{u_0}^{u} \xi_1(t) \sqrt{1 - \xi_1'(t)^2} \, dt \pm \cot \vartheta_0 \, (u - u_0) \tag{5.3}$$

and

$$\psi(u, v) = (\xi_1(u), \xi_2(u), v). \tag{5.4}$$

These curves are general helices with axis X_3*.*

iii) *If* $G = G_{43}$*, then the orbit space is* $\mathcal{B} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \; : \; \xi_1 \geq 0\}$ *and a loxodrome can be parametrized by* $\psi(u, v(u))$*, where*

$$v(u) = \int_{u_0}^{u} \frac{(\xi_1(t)^2 - 2a) \sqrt{1 - \xi_1'(t)^2} \pm 2 \cot \vartheta_0 \, \xi_1(t)}{\xi_1(t) \sqrt{4\xi_1(t)^2 + (\xi_1(t)^2 - 2a)^2}} \, dt \tag{5.5}$$

and

$$\psi(u, v) = (\xi_1(u) \cos v, \xi_1(u) \sin v, \xi_2(u) + a \, v). \tag{5.6}$$

Proof. i) We start considering the case of G_1-invariant surfaces. As

$$\xi_1 = y, \qquad \xi_2 = \frac{xy}{2} - z$$

are X_1-invariant functions, then the orbit space is given by $\mathcal{B} = \mathbb{R}^2$, equipped with the metric

$$\tilde{g} = d\xi_1^2 + \frac{d\xi_2^2}{1 + \xi_1^2}.$$

Taking the lift of $\tilde{\gamma}$ given by

$$\gamma(u) = (0, \xi_1(u), -\xi_2(u)),$$

the corresponding invariant surface is parametrized as in (5.2). Thus, we have that

$$F = -\xi_1(u)\,\xi_2'(u), \qquad G = 1 + \xi_1(u)^2. \tag{5.7}$$

Also, as $\tilde{\gamma}(u)$ is parametrized by arc length, it turns out

$$\xi_2'(u) = \omega(u)\sqrt{1 - \xi_1'(u)^2}. \tag{5.8}$$

Finally, we substitute (5.7) and (5.8) in (2.7) and we obtain (5.1).

ii) If $G = G_3$, the orbit space is $(\mathbb{R}^2, \tilde{g})$, where $\tilde{g} = du^2 + dv^2$. Also, a G_3-invariant vertical cylinder can be parametrized by (5.4) and

$$F = \frac{\xi_1'(u)\,\xi_2(u) - \xi_2'(u)\,\xi_1(u)}{2}, \qquad G = 1.$$

Moreover, as $\tilde{\gamma}(u)$ is parametrized by arc length, one gets

$$\xi_2'(u) = \sqrt{1 - \xi_1'(u)^2}. \tag{5.9}$$

Then we use (5.9) to find

$$\begin{aligned}\int_{u_0}^{u} \frac{F}{\omega^2}\,dt &= \int_{u_0}^{u} F\,dt \\ &= \frac{1}{2}\xi_1(u)\,\xi_2(u) - \int_{u_0}^{u} \xi_1(ut)\,\xi_2'(ut)\,dt \\ &= \frac{1}{2}\xi_1(u)\int_{u_0}^{u}\sqrt{1 - \xi_1'(t)^2}\,dt - \int_{u_0}^{u}\xi_1(t)\sqrt{1 - \xi_1'(t)^2}\,dt.\end{aligned} \tag{5.10}$$

Finally, substituting (5.10) in (2.7), we obtain (5.3).

iii) If $X = X_4 + a\,X_3$ and we use cylindrical coordinates (r, θ, z), we have that $\{\xi_1 = r, \xi_2 = z - a\,\theta\}$ is a set of independent invariant functions. Therefore, on the regular part of the orbit space

$$\mathcal{B}_r = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \ : \ \xi_1 > 0\},$$

we take the orbital metric

$$\tilde{g} = d\xi_1^2 + \frac{4\,\xi_1^2\,d\xi_2^2}{4\xi_1^2 + (\xi_1^2 - 2a)^2}.$$

Then, as $\tilde{\gamma}$ is parametrized by arc length (with respect to the metric $\tilde{g}$), we get

$$\xi_2'(u) = \omega(u)\,\frac{\sqrt{1 - \xi_1'(u)^2}}{\xi_1(u)}. \tag{5.11}$$

Choosing $\tilde{\gamma}(u) = (\xi_1(u), 0, \xi_2(u))$, the helicoidal surface can be parametrided by (5.6) and thus

$$F = \frac{\xi_2'(u)\,(2a - \xi_1(u)^2)}{2}, \qquad G = \frac{4\xi_1(u)^2 + (\xi_1(u)^2 - 2a)^2}{4}. \tag{5.12}$$

Making use of (5.11) and (5.12) in (2.7), we obtain (5.5). □

Example (The helicoidal catenoid). The *helicoidal catenoid* is a helicoidal minimal surface in the Heisenberg group (see [9]), that is obtained choosing $a = 1/2$ and the profile curve given by

$$\tilde{\gamma}(u) = (\sqrt{u^2+1}, (u - \operatorname{arccot} u)/2).$$

By using (5.6), this surface can be parametrized by

$$\psi(u,v) = \left(\sqrt{u^2+1}\,\cos v, \sqrt{u^2+1}\,\sin v, \frac{u + v - \operatorname{arccot} u}{2}\right)$$

and from (5.5) it follows that the loxodromes which are not orbits can be parametrized by

$$v(u) = \sqrt{2}\,(1 \pm \cot\vartheta_0)\,\arctan\left(\frac{u}{\sqrt{2}}\right) - \arctan u.$$

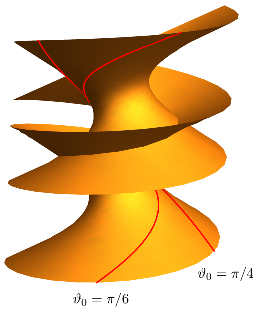

FIGURE 5. Two loxodromes of the helicoidal catenoid in $\mathbb{H}_3$.

6. Loxodromes on invariant surfaces with constant Gauss curvature

In this section we consider the case of a G_X-invariant surface $M \subset (N^3, g)$ such that the induced metric is of constant Gauss curvature. For this case we shall restrict our investigation to the case when the lift γ, used to construct the parametrization of the surface (1.1), is horizontal. With this assumption, the equation (2.7) can be integrated.

Proposition 6.1 (Positive curvature). *Let $M \subset (N^3, g)$ be a G_X-invariant surface of constant positive Gauss curvature $K = 1/R^2$, locally parametrized by $\psi(u,v)$ given by (1.1) with γ horizontal lift. Then a loxodrome on M which is not an orbit can be parametrized by*

$$v(u) = \mp \frac{R\, \cot \vartheta_0}{\sqrt{a}}\, \operatorname{arcsinh}\left(R\, \frac{\omega_u(u)}{\omega(u)}\right) + b, \qquad a, b \in \mathbb{R},\ a > 0. \tag{6.1}$$

Proof. Firstly, as $K = 1/R^2$, from (1.4) we obtain

$$R^2 \omega_{uu}(u) + \omega(u) = 0. \tag{6.2}$$

Also, from (6.2) follows that

$$\frac{d}{du}\left((\omega(u))^2 + R^2\,(\omega_u(u))^2\right) = 2\,\omega_u(u)\,(\omega(u) + R^2\,\omega_{uu}(u)) = 0,$$

which implies that there exists a constant $a \in \mathbb{R}$, $a > 0$, such that

$$(\omega(u))^2 + R^2\,(\omega_u(u))^2 = a. \tag{6.3}$$

Combining (6.2) and (6.3), we find

$$\omega_{uu}\,\omega - \omega_u^2 = -\frac{a}{R^2}. \tag{6.4}$$

Considering the new variable

$$\eta(u) = R\,\frac{\omega_u(u)}{\omega(u)},$$

and taking into account (6.4), we get

$$d\eta = -\frac{a}{R\,\omega^2}\,du, \tag{6.5}$$

and also, by means of (6.3),

$$\sqrt{1+\eta^2} = \frac{\sqrt{a}}{\omega}. \tag{6.6}$$

Finally, integrating (2.7) we have

$$\begin{aligned} v(u) &= \pm \cot\vartheta_0 \int \frac{1}{\omega}\,du = \mp \frac{R\,\cot\vartheta_0}{\sqrt{a}} \int \frac{d\eta}{\sqrt{1+\eta^2}} \\ &= \mp \frac{R\,\cot\vartheta_0}{\sqrt{a}}\,\operatorname{arcsinh}\eta + b \qquad (6.7) \\ &= \mp \frac{R\,\cot\vartheta_0}{\sqrt{a}}\,\operatorname{arcsinh}\left(\frac{R\,\omega_u(u)}{\omega(u)}\right) + b, \quad b \in \mathbb{R}. \end{aligned}$$

□

Proposition 6.2 (Negative curvature). *Let $M \subset (N^3, g)$ be a G_X-invariant surface of constant negative Gauss curvature $K = -1/R^2$, locally parametrized by $\psi(u,v)$ given by (1.1), with γ horizontal lift. Then a loxodrome on M which is not an orbit can be parametrized by:*

$$v(u) = \mp \frac{\cot \vartheta_0}{\omega_u(u)} + b, \qquad \textit{if} \quad a = 0;$$

$$v(u) = \mp \frac{R \cot \vartheta_0}{\sqrt{-a}} \ln \left(\frac{R\,\omega_u(u) + \sqrt{-a}}{\omega(u)} \right) + b, \quad \textit{if} \quad a < 0;$$

$$v(u) = \pm \frac{R \cot \vartheta_0}{\sqrt{a}} \arcsin \left(\frac{R\,\omega_u(u)}{\omega(u)} \right) + b, \qquad \textit{if} \quad a > 0,$$

where $b \in \mathbb{R}$.

Proof. As $K = -1/R^2$, equation (1.4) becomes

$$\omega(u) - R^2 \omega_{uu}(u) = 0.$$

This implies

$$\omega(u)^2 - R^2\, \omega_u(u)^2 = a, \quad a \in \mathbb{R}. \tag{6.8}$$

The last two conditions imply

$$\omega_{uu}(u)\,\omega(u) - \omega_u(u)^2 = \frac{a}{R^2}. \tag{6.9}$$

In this case, the constant a can be any real number. Performing changes of variables similar to the case of constant positive curvature, equation (2.7) can be integrated.
When $a \neq 0$, making the change

$$\eta(u) = R\,\frac{\omega_u(u)}{\omega(u)}$$

and taking into account (6.9), we get

$$d\eta = \frac{a}{R\,\omega^2(u)}\, du.$$

- For $a < 0$, from (6.8) we get

$$\sqrt{\eta^2 - 1} = \frac{\sqrt{-a}}{\omega}$$

and then

$$
\begin{aligned}
v(u) &= \pm \cot\vartheta_0 \int \frac{1}{\omega}\, du = \mp \frac{R\,\cot\vartheta_0}{\sqrt{-a}} \int \frac{d\eta}{\sqrt{\eta^2-1}} \\
&= \mp \frac{R\,\cot\vartheta_0}{\sqrt{-a}}\, \ln(\eta + \sqrt{\eta^2-1}) + b \\
&= \mp \frac{R\,\cot\vartheta_0}{\sqrt{-a}}\, \ln\Big(\frac{R\,\omega_u(u) + \sqrt{-a}}{\omega(u)}\Big) + b, \qquad b \in \mathbb{R}.
\end{aligned}
$$

• If $a > 0$, from (6.9) we obtain

$$
\sqrt{1-\eta^2} = \frac{\sqrt{a}}{\omega}.
$$

Consequently,

$$
\begin{aligned}
v(u) &= \pm \cot\vartheta_0 \int \frac{1}{\omega}\, du = \pm \frac{R\,\cot\vartheta_0}{\sqrt{a}} \int \frac{d\eta}{\sqrt{1-\eta^2}} \\
&= \pm \frac{R\,\cot\vartheta_0}{\sqrt{a}}\, \arcsin\eta + b \\
&= \pm \frac{R\,\cot\vartheta_0}{\sqrt{a}}\, \arcsin\Big(\frac{R\,\omega_u(u)}{\omega(u)}\Big) + b, \qquad b \in \mathbb{R}.
\end{aligned}
$$

• If $a = 0$, the result follows by observing that, as $\omega > 0$, from (6.8) we have that $\omega_u \neq 0$; moreover from (6.9) it follows

$$
\frac{1}{\omega(u)} = \frac{d}{du}\Big(\frac{1}{\omega(u)_u}\Big).
$$

□

When the Gauss curvature is zero, one has $\omega_u = a \in \mathbb{R}$ and the following

Proposition 6.3 (Flat case). *Let $M \subset (N^3, g)$ be a flat G_X-invariant surface, locally parametrized by $\psi(u,v)$ given by* (1.1) *with γ horizontal lift. Then a loxodrome on M which is not an orbit can be parametrized by*

$$
v(u) = \pm \frac{\cot\vartheta_0}{a}\, \ln\omega(u) + b, \quad \textit{if} \quad a \neq 0,
$$

$$
v(u) = \pm \frac{\cot\vartheta_0}{c}\, u + b, \qquad \textit{if} \quad a = 0,
$$

where $b, c \in \mathbb{R}$.

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