Tautological entailments and their rivals

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1. Introduction

Tautological entailments were first investigated in the late 1950s by Smiley and by Anderson and Belnap, in the context of their pioneering studies of relevance logics (see e.g. Anderson and Belnap, 1962 and, for an exhaustive survey of these early investigations, Anderson and Belnap, 1975). More precisely, tautological entailments were introduced and motivated as the provable *first-degree entailments* of the relevance logics **E** and **R**, where a first-degree entailment is nothing but a formula of the form $A \rightarrow B$, in which the arrow stands for the relevant implication and both A and B contain no connective but negation ($\neg$) and the classical, extensional connectives $\wedge$ and $\vee$ (no nesting of arrows, therefore, is allowed). Alternatively, one can conceive of tautological entailments as expressing a relevant derivability relation between two classical formulae.

It soon turned out that the system of tautological entailments (henceforth **T**) was an especially well-behaved fragment of the above-mentioned logics: nice normal form, variable sharing and interpolation results were readily proved and a characteristic matrix was provided. This model was subsequently given by Dunn (1976) and Belnap (1977) a perspicuous interpretation in terms of "epistemic truth values", a feature which made **T** particularly appropriate for handling problems of management of inconsistent and incomplete information (see also Routley and Routley, 1972). Since then, tautological entailments - and even more so their generalization known as *Belnap's four-valued logic*, where a more general derivability relation as well as new connectives are introduced - have become a favourite of computer scientists and artificial intelligence experts (see e.g. Levesque, 1984; Ginsberg, 1988; Patel-Schneider, 1989; Fitting, 1990, 1994; Weber, 1998).

In this paper, we shall survey some well-known results about Anderson's and Belnap's tautological entailments. Moreover, we shall try to highlight the connections between this system and other systems of first-degree entailments which share similar properties and are sometimes grounded on closely related intuitive motivations, although such similarities are not always stressed or made clear in the literature. As we shall see, these systems are directly drawn from, or anyway connected to, Lukasiewicz's three-valued logic, Kleene's strong and weak three-valued logics (Kleene, 1952), Priest's "logic of paradox" (Priest, 1979), Halldén's "logic of nonsense" (Halldén, 1949), as well as Epstein's logics of relatedness, equality of content, dependence and dual dependence (Epstein, 1990). Not all of these logics are paraconsistent, and many of them were introduced for quite different purposes than the logical investigation of inconsistency; some, moreover (e.g. Epstein's logics) have been given semantical and proof-theoretical presentations which render a comparison with Anderson's and Belnap's logic somewhat difficult. Therefore, we believe that encompassing them along with tautological entailments in a common framework might help to clarify some relationships or affinities between paraconsistent logics and other non-classical logics of many-valued or broadly relevant provenance.

This paper is structured as follows. After dispatching some preliminaries (§2), in §3 we shall consider four-valued characteristic matrices for tautological entailments and for the first-degree entailments of relatedness logic. From those tables we shall extract three-valued tables for a number of logics; inclusion relationships between such systems will be investigated. In §4 we shall present several uniform proof-theoretic formulations for some of the above systems. General algebraic semantics, relational semantics (in the style of Routley *et al.*, 1982) or set-assignment semantics (in the style of Epstein, 1990) will not be discussed here.

Many results in this paper are not new, except perhaps in their formulations; the appropriate pointers to the literature will be duly provided. However, we shall also prove some original theorems, especially as regards Epstein's logics.

2. Preliminaries

The aim of this section is that of writing down some definitions which will turn out useful in the following, especially in §3.

Definition 1 (first-degree entailment). Let £ be a propositional language containing a denumerable set VAR(£) of variables and the connectives $\neg$ (unary), $\wedge$, $\vee$ (binary). The set FOR(£) of well-formed formulae of £ and the absolutely free algebra $\mathcal{F}$(£) of formulae of £ are constructed as usual. A *first-degree entailment (fde)* is an expression of the form $\phi \to \psi$, where $\phi, \psi \in$ FOR(£). If $\phi \in$ FOR(£), we set

$$\text{Var}(\phi) = \{p : p \in \text{VAR(£)} \ \& \ p \text{ occurs in } \phi\}.$$

Definition 2 (FDE-matrix). A *FDE-matrix* for £ is an ordered pair $\mathcal{M} = \langle \mathcal{A}, R \rangle$, where $\mathcal{A} = \langle A, \neg^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}} \rangle$ is an algebra of type $\langle 1, 2, 2 \rangle$ and $R \subseteq A^2$.

Definition 3 (interpretation). An *interpretation* for £ is an ordered pair $\mathcal{I} = \langle \mathcal{M}, v \rangle$, where $\mathcal{M} = \langle \mathcal{A}, R \rangle$ is a FDE-matrix for £ and v: VAR(£) $\to A$ is a mapping which is extended in the customary way to a homomorphism from $\mathcal{F}$(£) to $\mathcal{A}$ (which we shall denote by v as well, with a notational abuse).

Definition 4 (validity). We say that the fde $\phi \to \psi$ is *true* in the interpretation $\mathcal{I} = \langle \mathcal{M}, v \rangle$, where $\mathcal{M} = \langle \mathcal{A}, R \rangle$, iff $\langle v(\phi), v(\psi) \rangle \in R$; we say that $\phi \to \psi$ is *valid* in the FDE-matrix $\mathcal{M}$ (in symbols: $\mathcal{M} \vDash \phi \to \psi$) iff it is true in every interpretation for £ whose first projection is $\mathcal{M}$.

We now introduce the important concept of *submatrix* of an FDE-matrix.

Definition 5 (submatrix). Let $\mathcal{M} = \langle \mathcal{A}, R \rangle$ be a FDE-matrix. A FDE-matrix $\mathcal{M}' = \langle \mathcal{A}', R' \rangle$ is said to be a *submatrix* of $\mathcal{M}$ iff $\mathcal{A}'$ is a subalgebra of $\mathcal{A}$ and $R' = R\lceil A'$.

It is expedient to recall the following facts:

Lemma 1. *a) If* $\mathcal{M}' = \langle \mathcal{A}', R' \rangle$ *is a submatrix of* $\mathcal{M} = \langle \mathcal{A}, R \rangle$, *then for any* $\phi, \psi \in$ *FOR(£), if* $\mathcal{M} \vDash \phi \to \psi$ *then* $\mathcal{M}' \vDash \phi \to \psi$. *b) If* $\mathcal{M} = \langle \mathcal{A}, R \rangle$, $\mathcal{M}' = \langle \mathcal{A}, R' \rangle$ *and* $R \subseteq R'$, *then for any* $\phi, \psi \in$ *FOR(£), if* $\mathcal{M} \vDash \phi \to \psi$ *then* $\mathcal{M}' \vDash \phi \to \psi$. □

3. Truth tables

We shall start by comparing the system **T** of tautological entailments and a system which we name **E** (after R.L. Epstein), corresponding to the first-degree entailment fragment of symmetric relatedness logic (Epstein, 1990). We introduce them semantically, viz. by means of appropriate FDE-matrices. The one for **T** was first suggested by T. Smiley (as reported in Anderson and Belnap, 1975), while the one for **E** stems from Paoli (1993). Such matrices involve four truth values - t, b, n, f. According to the Dunn-Belnap interpretation one can intuitively think of t as "true only", of b as "both true and false", of n as "neither true nor false" and of f as "false only". These labels must be understood epistemically, not ontologically - moreover, this reading is much more appropriate for the tables of **T** than for those of **E**, whose values still lack a convincing intuitive explanation.

Definition 6 (the algebras $\mathcal{A}^4_{\mathbf{T}}$ and $\mathcal{A}^4_{\mathbf{E}}$). Let $V = \{t, b, n, f\}$. We set $\mathcal{A}^4_{\mathbf{T}} = \langle V, \neg^4_{\mathbf{T}}, \wedge^4_{\mathbf{T}}, \vee^4_{\mathbf{T}} \rangle$, where the operations are defined by the following tables:

$\neg^4_{\mathbf{T}}$	
f	t
n	n
b	b
t	f

$\wedge^4_{\mathbf{T}}$	f	n	b	t
f	f	f	f	f
n	f	n	f	n
b	f	f	b	b
t	f	n	b	t

$\vee^4_{\mathbf{T}}$	f	n	b	t
f	f	n	b	t
n	n	n	t	t
b	b	t	b	t
t	t	t	t	t

We also set $\mathcal{A}^4_{\mathbf{E}} = \langle V, \neg^4_{\mathbf{E}}, \wedge^4_{\mathbf{E}}, \vee^4_{\mathbf{E}} \rangle$, where the operations are defined by the following tables:

$\neg^4_{\mathbf{E}}$	
f	t
n	n
b	b
t	f

$\wedge^4_{\mathbf{E}}$	f	n	b	t
f	f	b	b	f
n	b	n	b	b
b	b	b	b	b
t	f	b	b	t

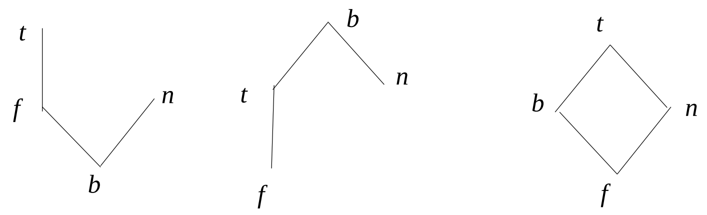
$\vee^4_{\mathbf{E}}$	f	n	b	t
f	f	b	b	t
n	b	n	b	b
b	b	b	b	b
t	t	b	b	t

Definition 7 (relations arising out of the tables). Let $\mathbf{S} \in \{\mathbf{T}, \mathbf{E}\}$. We define the following binary relations on V:

$\leq^{4l}_{\mathbf{S}} = \{\langle x, y \rangle : x \wedge^4_{\mathbf{S}} y = x\}$;
$\leq^{4u}_{\mathbf{S}} = \{\langle x, y \rangle : x \vee^4_{\mathbf{S}} y = y\}$;
$\leq^4_{\mathbf{S}} = \leq^{4l}_{\mathbf{S}} \cup \leq^{4u}_{\mathbf{S}}$.

Lemma 2. *a)* $\leq_{\boldsymbol{T}}^{4l}$, $\leq_{\boldsymbol{T}}^{4u}$, $\leq_{\boldsymbol{E}}^{4l}$, $\leq_{\boldsymbol{E}}^{4u}$ *are semilattice orderings on* V*; b)* $\leq_{\boldsymbol{T}}^{4l} = \leq_{\boldsymbol{T}}^{4u} = \leq_{\boldsymbol{T}}^{4}$*; c)* $\leq_{\boldsymbol{T}}^{4}$ *is a lattice ordering on* V.

Proof. By inspection. Hasse diagrams for $\leq_{\mathbf{E}}^{4l}$, $\leq_{\mathbf{E}}^{4u}$, $\leq_{\mathbf{T}}^{4}$ are provided below.



$\leq_{\mathbf{E}}^{4l}$ $\leq_{\mathbf{E}}^{4u}$ $\leq_{\mathbf{T}}^{4}$ □

Remark that $\leq_{\mathbf{E}}^{4}$ is not even a preordering on V, since it is not transitive.

Definition 8 (the FDE-matrices $\mathcal{M}_{\mathbf{T}}$, $\mathcal{M}_{\mathbf{E}}$). Let $\mathbf{S} \in \{\mathbf{T}, \mathbf{E}\}$. We set $\mathcal{M}_{\mathbf{S}} = \langle \mathcal{A}_{\mathbf{S}}^{4}, \leq_{\mathbf{S}}^{4} \rangle$.

Let us now consider more closely the four-valued tables for **T** and **E**. It is interesting to remark that, if we restrict ourselves to the values *t*, *b*, and *f*, we respectively obtain the *strong* and the *weak* Kleene tables for $\neg$, $\wedge$, $\vee$ (Kleene, 1952; weak Kleene tables are also known as *Bochvar* tables). As far as **T** is concerned, restriction to $\{t, n, f\}$ would lead exactly to the same result; yet not for **E** - since the indicated subset would not even be a subuniverse of V. Three-valued logics based on the strong Kleene tables are especially important and are deeply investigated in the literature; they are called *natural* in Avron (1991). We are thus led to the following definitions:

Definition 9 (the algebras $\mathcal{A}_{\mathbf{T}}^{3}$ and $\mathcal{A}_{\mathbf{E}}^{3}$). Let $\mathbf{S} \in \{\mathbf{T}, \mathbf{E}\}$, and let $V' = \{t, b, f\}$. We set $\mathcal{A}_{\mathbf{S}}^{3} = \langle V', \neg_{\mathbf{S}}^{3}, \wedge_{\mathbf{S}}^{3}, \vee_{\mathbf{S}}^{3} \rangle$, where each operation is the restriction to V' of the corresponding operation in $\mathcal{A}_{\mathbf{S}}^{4}$.

It turns out that there are several natural ways to endow $\mathcal{A}_{\mathbf{T}}^{3}$ and $\mathcal{A}_{\mathbf{E}}^{3}$ with binary relations, so as to build up FDE-matrices. Let us examine some of them.

Definition 10 (relations arising out of the tables). Let $\mathbf{S} \in \{\mathbf{T}, \mathbf{E}\}$. We define the following binary relations on V':

$\leq_{\mathbf{S}}^{3l} = \{\langle x, y \rangle : x \wedge_{\mathbf{S}}^{3} y = x\}$;
$\leq_{\mathbf{S}}^{3u} = \{\langle x, y \rangle : x \vee_{\mathbf{S}}^{3} y = y\}$;
$\leq_{\mathbf{S}}^{3} = \leq_{\mathbf{S}}^{3l} \cap \leq_{\mathbf{S}}^{3u}$.

Remark that in Definition 10 $\leq_{\mathbf{S}}^{3}$ is defined as the *intersection* of $\leq_{\mathbf{S}}^{3l}$ and $\leq_{\mathbf{S}}^{3u}$, while in Definition 7 we had used the union. However, this obviously makes no difference for $\leq^3_{\mathbf{T}}$, whereas if we had chosen union in defining $\leq^3_{\mathbf{E}}$ we would have obtained a useless FDE-matrix, satisfying exactly the classically valid fdes.

Definition 11 (more FDE-matrices). We define:

$\mathcal{M}_{\mathbf{M}} = \langle \mathcal{A}^3_{\mathbf{T}}, \leq^3_{\mathbf{T}} \rangle$;
$\mathcal{M}_{\mathbf{Kl}} = \langle \mathcal{A}^3_{\mathbf{T}}, \leq^3_{\mathbf{T}} \cup \leq^{3l}_{\mathbf{E}} \rangle$;
$\mathcal{M}_{\mathbf{P}} = \langle \mathcal{A}^3_{\mathbf{T}}, \leq^3_{\mathbf{T}} \cup \leq^{3u}_{\mathbf{E}} \rangle$;
$\mathcal{M}_{\mathbf{EM}} = \langle \mathcal{A}^3_{\mathbf{E}}, \leq^3_{\mathbf{T}} \rangle$;
$\mathcal{M}_{\mathbf{B}} = \langle \mathcal{A}^3_{\mathbf{E}}, \leq^3_{\mathbf{T}} \cup \leq^{3l}_{\mathbf{E}} \rangle$;
$\mathcal{M}_{\mathbf{H}} = \langle \mathcal{A}^3_{\mathbf{E}}, \leq^3_{\mathbf{T}} \cup \leq^{3u}_{\mathbf{E}} \rangle$;
$\mathcal{M}_{\mathbf{Eq}} = \langle \mathcal{A}^3_{\mathbf{E}}, \leq^3_{\mathbf{E}} \rangle$;
$\mathcal{M}_{\mathbf{D}} = \langle \mathcal{A}^3_{\mathbf{E}}, \leq^{3l}_{\mathbf{E}} \rangle$;
$\mathcal{M}_{\mathbf{DD}} = \langle \mathcal{A}^3_{\mathbf{E}}, \leq^{3u}_{\mathbf{E}} \rangle$.

Definition 12 (notions of validity). Let **S, S'** $\in \{$**E, T, M, Kl, P, EM, B, H, Eq, D, DD**$\}$. We say that a fde $\phi \rightarrow \psi$ is ***S**-valid* (in symbols, $\vDash_{\mathbf{S}} \phi \rightarrow \psi$) iff $\mathcal{M}_{\mathbf{S}} \vDash \phi \rightarrow \psi$. Occasionally, we shall refer to **S** as the logic determined by all **S**-valid fdes and write **S** $\subseteq$ **S'** just in case $\vDash_{\mathbf{S}} \phi \rightarrow \psi$ implies $\vDash_{\mathbf{S'}} \phi \rightarrow \psi$ for every ϕ, ψ in FOR(£).

It is noteworthy that $\vDash_{\mathbf{T}} \phi \rightarrow \psi$ iff $v(\psi) \in \{t, b\}$ whenever $v(\phi) \in \{t, b\}$. A proof of this fact can be found e.g. in Font (1997).

The logics introduced in Definition 12 are tightly related to several well-known logics:

1) **M** is related to Lukasiewicz's three-valued logic $\mathbf{L_3}$ in the following sense: for $\phi, \psi \in$ FOR(£), $\phi \rightarrow \psi$ is **M**-valid just in case it is a valid fde of $\mathbf{L_3}$. Remark, however, that the standard semantic deduction theorem does not hold in $\mathbf{L_3}$ and thus it is not the case that $\phi \rightarrow \psi$ is **M**-valid iff ψ is a logical consequence of ϕ in $\mathbf{L_3}$ (i.e. $v(\psi) = t$ whenever $v(\phi) = t$). Also, $\phi \rightarrow \psi$ is **M**-valid just in case it is a valid fde of **RM**[1]. Avron (1991) considers two possible ways of extracting from $\mathcal{M}_{\mathbf{M}}$ a consequence relation $\Gamma \vdash \Delta$ with multiple premisses and conclusions, which he respectively dubs $\vdash_{Luk}$ and $\vdash_{Sob}$ (the latter stands for *Sobocinski's* logic). They collapse onto each other when Γ, Δ are singletons.

2) **Kl** is related to strong Kleene's three-valued logic (Kleene, 1952) in the following sense: for $\phi, \psi \in$ FOR(£), $\phi \rightarrow \psi$ is **Kl**-valid just in case ψ is a logical consequence of ϕ in it (i.e. $v(\psi) = t$ whenever $v(\phi) = t$). Remark, however, that Kleene's logic has no tautologies and thus it is not the case that $\phi \rightarrow \psi$ is **Kl**-valid iff it is a valid fde of strong Kleene's logic. **Kl** is also known as Hao Wang's logic (Wang, 1961; Rose, 1963; Bolotov, 1986). Avron (1991) extracts from $\mathcal{M}_{\mathbf{Kl}}$ a consequence relation called $\vdash_{Kl}$.

[1] Actually, we have so named **M** after R.K. Meyer, who investigated this logic as the first-degree entailment fragment of **RM**.

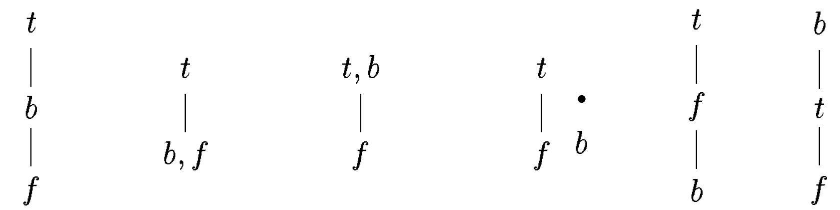
3) **P** is related to Priest's **LP** (Priest, 1979) and to D'Ottaviano's and Da Costa's $\mathbf{J_3}$ (D'Ottaviano and Da Costa, 1970) in the following sense: for $\phi, \psi \in$ FOR(£), $\phi \rightarrow \psi$ is **P**-valid iff it is a valid fde of $\mathbf{J_3}$ iff ψ is a logical consequence of ϕ in **LP** (i.e. $v(\psi) \in \{t, b\}$ whenever $v(\phi) \in \{t, b\}$). Avron (1991) extracts from $\mathcal{M}_{\mathbf{P}}$ a consequence relation called $\vdash_{Pac}$[2].

4) **Eq**, **D**, **DD** respectively correspond to the fde fragments of Epstein's (1990) logics of *equality of contents*, *dependence* and *dual dependence*. Dependence logic is closely related, both in its motivation and in its technical developments, to Parry's (1933) calculus of *analytic implication* (for which see also Dunn, 1972). Its fdes are investigated e.g. in Deutsch (1981) or in Sinowjew and Wessel (1975). The fdes of dual dependence logic are motivated and studied in Paoli (1992). These approaches are extensively discussed and sharply criticized (under the name of "conceptivist" approaches to relevance) in Routley *et al.* (1982).

5) **B** bears the same relation to weak Kleene's three-valued logic, also known as Bochvar's logic, as **Kl** bears to strong Kleene's three-valued logic. On the other side, **H** bears the same relation to Halldén's (1949) *logic of nonsense* as **P** bears to **LP**.

Lemma 3. *a)* $\le^{3}_{\mathbf{T}}$, $\le^{3}_{\mathbf{T}} \cup \le^{3l}_{\mathbf{E}}$, $\le^{3}_{\mathbf{T}} \cup \le^{3u}_{\mathbf{E}}$, $\le^{3}_{\mathbf{E}}$, $\le^{3l}_{\mathbf{E}}$, $\le^{3u}_{\mathbf{E}}$ *are preorderings on* V'*; b) they are all partial orderings, except* $\le^{3}_{\mathbf{T}} \cup \le^{3l}_{\mathbf{E}}$ *and* $\le^{3}_{\mathbf{T}} \cup \le^{3u}_{\mathbf{E}}$*; c) they are all linear, except* $\le^{3}_{\mathbf{E}}$.

Proof. By inspection. Hasse diagrams for these preorderings are provided below.



$\le^{3}_{\mathbf{T}}$ $\quad \le^{3}_{\mathbf{T}} \cup \le^{3l}_{\mathbf{E}}$ $\quad \le^{3}_{\mathbf{T}} \cup \le^{3u}_{\mathbf{E}}$ $\quad \le^{3}_{\mathbf{E}}$ $\quad \le^{3l}_{\mathbf{E}}$ $\quad \le^{3u}_{\mathbf{E}}$ □

Epstein's logics of relatedness, dependence, dual dependence and equality of contents are called by Priest (2002) "filter logics", because they are the result of an attempt to single out a class of relevantly correct entailments (according to intuitive criteria of relevance which differ from one logic to another) by "filtering out" classically valid entailments through some kind of "sieve". On the level of fdes, such sieves are adequately characterized through appropriate variable sharing requirements, which also work for a wider range of logics. In fact, we have:

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[2] Reasons to prefer **M** to **Kl** or **P** are provided in Seymour Michael (2002).

Theorem 1 (characterization of filter logics)*. Let* ***S*** $\in \{$***E, EM, B, H, Eq, D, DD***$\}$*. Then* $\vDash_{S} \phi \to \psi$ *iff* $\phi \to \psi$ *is classically valid (*$\vDash_{K} \phi \to \psi$*) and:*

a) for ***E****,* $Var(\phi) \cap Var(\psi) \neq \emptyset$*;*

b) for ***D****,* $Var(\phi) \supseteq Var(\psi)$*;*

c) for ***DD****,* $Var(\phi) \subseteq Var(\psi)$*;*

d) for ***Eq****,* $Var(\phi) = Var(\psi)$*;*

e) for ***EM****, (either* $\vDash_{K} \neg\phi$ *or* $Var(\phi) \supseteq Var(\psi)$*) and (either* $\vDash_{K} \psi$ *or* $Var(\phi) \subseteq Var(\psi)$*)*

f) for ***B****, either* $\vDash_{K} \neg\phi$ *or* $Var(\phi) \supseteq Var(\psi)$*;*

g) for ***H****, either* $\vDash_{K} \psi$ *or* $Var(\phi) \subseteq Var(\psi)$*.*

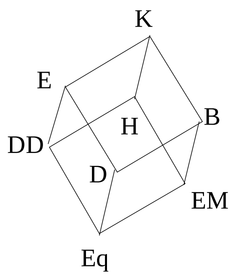
Proof. a) is proved in Paoli (1993); b) is proved e.g. in Deutsch (1981); c) is proved in Paoli (1992); a more general version of f) is proved in Urquhart (1986). We just prove e); the remaining items are taken care of similarly.

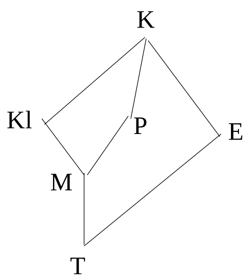
By Lemma 1, if $\vDash_{\mathbf{EM}} \phi \to \psi$, then $\phi \to \psi$ is classically valid. Now, suppose it is not the case that $\vDash_{\mathbf{K}} \neg\phi$ and it is not the case that Var(ϕ) $\supseteq$ Var(ψ). Then there exists a classical valuation v such that $v(\phi) = t$. Now, let w coincide with v as to the variables in ϕ, while $w(p) = b$ if p does not occur in ϕ. Clearly, $\langle \mathcal{M}_{\mathbf{EM}}, w \rangle$ is an interpretation for £ such that $w(\phi) = t, w(\psi) = b$. Since $\langle t, b \rangle \notin \leq^{3}_{\mathbf{T}}$, we conclude that it is not the case that $\vDash_{\mathbf{EM}} \phi \to \psi$. Likewise, if it is not the case that $\vDash_{\mathbf{K}} \psi$ and it is not the case that Var(ϕ) $\subseteq$ Var(ψ), a similar argument shows that it is not the case that $\vDash_{\mathbf{EM}} \phi \to \psi$ either.

Conversely, suppose that it is not the case that $\vDash_{\mathbf{EM}} \phi \to \psi$. Then there is an interpretation $\langle \mathcal{M}_{\mathbf{EM}}, v \rangle$ such that either a) $v(\phi) = t, v(\psi) = f$ or b) $v(\phi) = t, v(\psi) = b$, or else c) $v(\phi) = b, v(\psi) = f$. If a) holds, then by inspection of the tables one sees that $v(p) \in \{t, f\}$ for any variable $p \in$ Var(ϕ) $\cup$ Var(ψ). This immediately yields a falsifying classical valuation for $\phi \to \psi$. If b) holds, once again an inspection of the tables shows that $v(p) \in \{t, f\}$ for any $p \in$ Var(ϕ), while there must be q in Var(ψ) such that $v(q) = b$. Hence it cannot be Var(ψ) $\subseteq$ Var(ϕ). Moreover, let w coincide with v over the variables in ϕ, while $w(p) = f$ for any other variable. w is a classical valuation such that $w(\phi) = t$, which shows that $\neg\phi$ cannot be tautologous. Case c) is disposed of similarly. □

The next theorem provides an overview of the inclusion relationships among the logics defined above.

Theorem 2 (inclusion relationships among logics). *The inclusion relationships among the logics mentioned above are pictured below:*





A cube of logics in the filter family... ... and how **E** relates to logics in the Belnap family.

Moreover, all such inclusions are proper.

Proof. For a start, remark that as a consequence of Lemma 1, **Kl, P, E, B, H** $\subseteq$ **K**. For the same reason, we can also establish that **T** $\subseteq$ **M**, **D** $\subseteq$ **B**, **DD** $\subseteq$ **H**, **Eq** $\subseteq$ **EM**, **M** $\subseteq$ **Kl**, **M** $\subseteq$ **P**, **EM** $\subseteq$ **B**, **EM** $\subseteq$ **H**.

Theorem 1, given the fact that each formula in FOR(£) contains at least a propositional variable, implies that **Eq** $\subseteq$ **D**, **Eq** $\subseteq$ **DD**, **D** $\subseteq$ **E**, **DD** $\subseteq$ **E**.

Ir remains to prove that **T** $\subseteq$ **E**. By Theorem 1, it suffices to show that if $\vDash_{\mathbf{T}} \phi \to \psi$, then $\phi \to \psi$ is classically valid and $\mathrm{Var}(\phi) \cap \mathrm{Var}(\psi) \neq \emptyset$. The former property follows from Lemma 1. As to the latter, suppose that $\mathrm{Var}(\phi) \cap \mathrm{Var}(\psi) = \emptyset$. Then let $v(p) = b$ for $p \in \mathrm{Var}(\phi)$ and $v(q) = n$ for $q \in \mathrm{Var}(\psi)$. Hence $v(\phi) = b$, $v(\psi) = n$ and, since $\langle b, n \rangle \notin \leq^{4}_{\mathbf{T}}$, the interpretation $\langle \mathcal{M}_{\mathbf{T}}, v \rangle$ falsifies $\phi \to \psi$.

The next table shows that all the preceding inclusions are proper (Y means that the given formula is valid, N means that it is not):

	K	**Kl**	**P**	**M**	**T**	**Eq**	**D**	**DD**	**E**	**H**	**EM**	**B**
$\phi \wedge \neg\phi \to \psi$	Y	Y	N	N	N	N	N	N	N	N	N	Y
$\phi \to \psi \vee \neg\psi$	Y	N	Y	N	N	N	N	N	N	Y	N	N
$\phi \wedge (\neg\phi \vee \psi) \to \psi$	Y	Y	N	N	N	N	Y	N	Y	N	N	Y
$\phi \wedge \psi \to \phi$	Y	Y	Y	Y	Y	N	Y	N	Y	N	N	Y
$\phi \to \phi \vee \psi$	Y	Y	Y	Y	Y	N	N	Y	Y	Y	N	N
$\phi \wedge \neg\phi \to \psi \vee \neg\psi$	Y	Y	Y	Y	N	N	N	N	N	Y	Y	Y
$(\phi \wedge \neg\phi) \vee (\psi \wedge \neg\psi) \to \phi \wedge \psi$	Y	Y	N	N	N	Y	Y	Y	Y	Y	Y	Y
$\phi \vee \psi \to (\phi \vee \neg\phi) \wedge (\psi \vee \neg\psi)$	Y	N	Y	N	N	Y	Y	Y	Y	Y	Y	Y

□

Such a table also yields additional nontrivial information: we can infer e.g. that **Eq** is not included in **T**, **H** is not included in **P**, **EM** is not included in **M**, and **B** is not included in **Kl**.

Of course, many more rivals to **T** than we mentioned could be examined within this framework. It could be worthwhile, for example, to find out what first-degree entailment systems could be extracted from the logics considered in Carnielli and Marcos (2002). We leave this as a task to the interested reader.

4. Proof theory

It is usually believed that logics in the filter family are difficult to compare with logics in the Belnap family, especially on the proof-theoretical side. In this section, we shall discuss several well-known proof-theoretic presentations of logics in the latter group, which however - interestingly enough - can be also adapted for at least one filter logic, **E**. We hope that such a perspective can facilitate further comparisons across both groups.

Remark, in any case, that the formalisms hereafter considered are by no means the sole employable tools: as regards **T** and its generalizations (such as Belnap's four-valued logic), for instance, several sequent calculi (e.g. Anderson and Belnap, 1975; Font, 1997; Arieli and Avron, 1998) as well as natural deduction calculi (Tamminga and Tanaka, 1999) have been suggested.

4.1 Hilbert-style systems

T admits of a simple Hilbert-style axiomatization (Anderson and Belnap, 1975), whence appropriate postulates can be extracted also for **Kl** (Rose, 1963), **P**, **M** (see e.g. Dunn, 1986) and **E** (Paoli, 1993).

Definition 13 (Hilbert-type systems). The calculus **HT**, based on the language £, has the following axioms and rules:

A1.1 $\phi \wedge \psi \to \phi$ A1.2 $\phi \wedge \psi \to \psi$
A2.1 $\phi \to \phi \vee \psi$ A2.2 $\psi \to \phi \vee \psi$
A3.1 $\phi \to \neg\neg\phi$ A3.2 $\neg\neg\phi \to \phi$
A4 $\phi \wedge (\psi \vee \chi) \to (\phi \wedge \psi) \vee \chi$
R1 $\phi \to \psi, \psi \to \chi \vdash \phi \to \chi$ R2 $\phi \to \psi, \phi \to \chi \vdash \phi \to \psi \wedge \chi$
R3 $\phi \to \chi, \psi \to \chi \vdash \phi \vee \psi \to \chi$ R4 $\phi \to \psi \vdash \neg\psi \to \neg\phi$.

Now, consider the following additional postulates:

A5 $\phi \wedge \neg\phi \to \psi \vee \neg\psi$ A6 $\phi \to \phi \wedge (\psi \vee \neg\psi)$
A7.1 $\neg(\phi \wedge \psi) \to \neg\phi \vee \neg\psi$ A7.2 $\neg\phi \vee \neg\psi \to \neg(\phi \wedge \psi)$
A7.3 $\neg(\phi \vee \psi) \to \neg\phi \wedge \neg\psi$ A7.4 $\neg\phi \wedge \neg\psi \to \neg(\phi \vee \psi)$
A8 $\phi \wedge \neg\phi \to \psi$ A9 $\phi \to \psi \vee \neg\psi$
R1' $\phi \to \psi, \psi \to \chi \vdash \phi \to \chi$, if ϕ and χ share a variable.

The calculus **HKl** can be obtained from **HT** by adding A7 and A8 and dropping R4. From **HKl** we get **HP** replacing A8 by A9. **HM** is obtained from **HT** by adjoining A5.

Finally, **HE** differs from **HT** in that it has A6 as an extra axiom and the weaker rule R1' in place of R1.

The notions of proof and provability are defined as usual. We write $\vdash_{\mathbf{HS}} \phi \rightarrow \psi$ to mean that the fde $\phi \rightarrow \psi$ is provable in the calculus **HS** (where $\mathbf{S} \in \{\mathbf{T}, \mathbf{Kl}, \mathbf{P}, \mathbf{M}, \mathbf{E}\}$). We shall also use the abbreviation $\vdash_{\mathbf{HS}} \phi \leftrightarrow \psi$ to mean that both $\vdash_{\mathbf{HS}} \phi \rightarrow \psi$ and $\vdash_{\mathbf{HS}} \psi \rightarrow \phi$.

Lemma 4 (theorems of HT, HE). *a) The following theorems are provable in* ***HT****: (T1)* $\phi \rightarrow \phi$*; (T2)* $\phi \leftrightarrow \phi \wedge \phi$*; (T3)* $\phi \leftrightarrow \phi \vee \phi$*; (T4)* $\phi \wedge \psi \leftrightarrow \psi \wedge \phi$*; (T5)* $\phi \vee \psi \leftrightarrow \psi \vee \phi$*; (T6)* $\phi \wedge (\psi \wedge \chi) \leftrightarrow (\phi \wedge \psi) \wedge \chi$*; (T7)* $\phi \vee (\psi \vee \chi) \leftrightarrow (\phi \vee \psi) \vee \chi$*; (T8)* $\phi \vee (\psi \wedge \chi) \leftrightarrow (\phi \vee \psi) \wedge (\phi \vee \chi)$*; (T9)* $\phi \wedge (\psi \vee \chi) \leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$*; (T10)* $\neg(\phi \wedge \psi) \leftrightarrow \neg\phi \vee \neg\psi$*; (T11)* $\neg(\phi \vee \psi) \leftrightarrow \neg\phi \wedge \neg\psi$*.*

b) Besides T1-T11, the following theorems are also provable in ***HE****: (T12)* $\phi \wedge (\neg\phi \vee \psi) \rightarrow \psi$*; (T13)* $\phi \vee (\psi \wedge \neg\psi) \rightarrow \phi$*; (T14)* $\phi \wedge \neg\phi \rightarrow \psi$*, if* ϕ *and* ψ *share a variable; (T15)* $\phi \rightarrow \psi \vee \neg\psi$*, if* ϕ *and* ψ *share a variable.*

Proof. Proof sketches for some of the T1-T15 are provided e.g. in Anderson and Belnap (1975) or Paoli (1993). □

We also state without a proof the following results (Lemma 5 for **HT** can be found in Anderson and Belnap, 1975 and can be straightforwardly extended to **HE** using restricted transitivity, while Lemma 6 is proved in Paoli, 1993):

Lemma 5 (derivable rules). *Let* $\boldsymbol{S} \in \{\boldsymbol{T}, \boldsymbol{M}, \boldsymbol{E}\}$*. The following rules are derivable in* ***HS****: (R5)* $\phi \leftrightarrow \psi \vdash \neg\phi \leftrightarrow \neg\psi$*; (R6)* $\phi \leftrightarrow \psi, \chi \leftrightarrow \sigma \vdash \phi \wedge \chi \leftrightarrow \psi \wedge \sigma$*; (R7)* $\phi \leftrightarrow \psi, \chi \leftrightarrow \sigma \vdash \phi \vee \chi \leftrightarrow \psi \vee \sigma$*.* □

Lemma 6 (derivable rules in HE). *a) If* $(Var(\phi_1) \cup ... \cup Var(\phi_n)) \cap Var(\psi) \neq \emptyset$ *and for some* $i, j \leq n$ *it is* $\phi_i = \neg\phi_j$*, then* $\vdash_{\boldsymbol{HE}} \phi_1 \wedge ... \wedge \phi_n \rightarrow \psi$*; b) If* $Var(\phi) \cap (Var(\psi_1) \cup ... \cup Var(\psi_n)) \neq \emptyset$ *and for some* $i, j \leq n$ *it is* $\psi_i = \neg\psi_j$*, then* $\vdash_{\boldsymbol{HE}} \phi \rightarrow \psi_1 \vee ... \vee \psi_n$*; c) If* $(Var(\phi_1) \cup ... \cup Var(\phi_n)) \cap Var(\psi) \neq \emptyset$ *and each* ϕ_i *is such that either* $\vdash_{\boldsymbol{HE}} \phi_i \rightarrow \psi$ *or it is a generalized conjunction* $\chi_1 \wedge ... \wedge \chi_m$ *where for some* $i, j \leq m$ *it is* $\chi_i = \neg\chi_j$*, then* $\vdash_{\boldsymbol{HE}} \phi_1 \vee ... \vee \phi_n \rightarrow \psi$*; d) If* $Var(\phi) \cap (Var(\psi_1) \cup ... \cup Var(\psi_n)) \neq \emptyset$ *and each* ψ_i *is such that either* $\vdash_{\boldsymbol{HE}} \phi \rightarrow \psi_i$ *or it is a generalized disjunction* $\chi_1 \vee ... \vee \chi_m$ *where for some* $i, j \leq m$ *it is* $\chi_i = \neg\chi_j$*, then* $\vdash_{\boldsymbol{HE}} \phi \rightarrow \psi_1 \wedge ... \wedge \psi_n$*.* □

The following version of the replacement theorem is available in **HT**, **HM**, **HE**:

Lemma 7 (replacement). *Let* $\boldsymbol{S} \in \{\boldsymbol{T}, \boldsymbol{M}, \boldsymbol{E}\}$*.* $\vdash_{\boldsymbol{HS}} \phi \leftrightarrow \psi$ *implies* $\vdash_{\boldsymbol{HS}} \chi \leftrightarrow \chi[\phi/\psi]$*, where* $\chi[\phi/\psi]$ *is obtained by replacing zero or more occurrences of* ϕ *in* χ *by* ψ*.*

Proof. Induction on the complexity of χ, using Lemma 5. □

Remark, however, that stronger replacement results do not hold unrestrictedly for **HE**: for example, $\vdash_{\mathbf{HS}}\phi \leftrightarrow \psi$ and $\vdash_{\mathbf{HS}}\phi \rightarrow \chi$ do not necessarily imply $\vdash_{\mathbf{HS}}\psi \rightarrow \chi$ (let e.g. ϕ be $p \wedge (q \vee \neg q)$, ψ be p, χ be $q \vee \neg q$). Anyway, this holds in case ϕ and ψ contain the same variables; this is just a special case of a more general result proved in Paoli (1996).

Lemma 8 (restricted strong replacement). *Let* $\boldsymbol{S} \in \{\boldsymbol{T}, \boldsymbol{M}, \boldsymbol{E}\}$. *Let* $\phi, \psi \in$ *FOR(£) and Var(*ϕ*) = Var(*ψ*). a)* $\vdash_{\boldsymbol{HS}}\phi \leftrightarrow \psi$ *and* $\vdash_{\boldsymbol{HS}}\chi \rightarrow \sigma$ *imply* $\vdash_{\boldsymbol{HS}}\chi[\phi/\psi] \rightarrow \sigma$*; b)* $\vdash_{\boldsymbol{HS}}\phi \leftrightarrow \psi$ *and* $\vdash_{\boldsymbol{HS}}\sigma \rightarrow \chi$ *imply* $\vdash_{\boldsymbol{HS}}\sigma \rightarrow \chi[\phi/\psi]$. ☐

Appropriate versions of Lemma 7 and Lemma 8 also hold for **HKl** and **HP**, although here there is a slight complication due to the fact that R5 fails in general. However, if **S** $\in \{$**Kl, P**$\}$, it is possible to prove that $\vdash_{\mathbf{HS}}\phi \leftrightarrow \psi$ *and* $\vdash_{\mathbf{HS}}\neg\phi \leftrightarrow \neg\psi$ together imply $\vdash_{\mathbf{HS}}\chi \leftrightarrow \chi[\phi/\psi]$; also Lemma 8 can be modified accordingly (cf. Rose, 1963).

4.2 S-entailments

In the next definition we recall some basic syntactical notions. Apart from minor modifications, the terminology is drawn from Anderson and Belnap (1975).

Definition 14 (some syntactic notions). A *literal* is either a variable in VAR(£) or the negation of such. L, L'... are used as metavariables for literals. By the *complementary* L^c of a literal L we mean $\neg p$ if L is the variable p, q if L is the negated variable $\neg q$. A *primitive conjunction (disjunction)* is a generalized conjunction (disjunction) of literals. A *conjunctive (disjunctive) normal form* is a generalized conjunction (disjunction) of primitive disjunctions (conjunctions). We shall often use the abbreviations "cnf" for "conjunctive normal form" and "dnf" for "disjunctive normal form".

In all of the systems considered above, given a formula ϕ, it is always possible to find a dnf ϕ^* and a cnf ϕ^{**} which are equivalent to ϕ and contain the same variables as ϕ. In fact, we have:

Theorem 3 (normal form). *Let* $\boldsymbol{S} \in \{\boldsymbol{T}, \boldsymbol{M}, \boldsymbol{Kl}, \boldsymbol{P}, \boldsymbol{E}\}$ *and* $\phi \in$ *FOR(£). Then there exist a dnf* ϕ^* *and a cnf* ϕ^{**}*, both containing the same variables as* ϕ*, such that* $\vdash_{\boldsymbol{HS}}\phi \leftrightarrow \phi^*$, $\vdash_{\boldsymbol{HS}}\phi \leftrightarrow \phi^{**}$.

*Proof.*We proceed in the standard way using A3, A4, T4-T11 and the appropriate versions of the replacement theorems. Remark that transitivity is not impaired while performing normal form moves in **HE**, because in all of the above-mentioned axioms and theorems both formulae in the (co-)entailment contain exactly the same variables. ☐

Applying canonical methods, we can select for each formula $\phi \in$ FOR(£), among the dnfs and cnfs whose existence is guaranteed by the previous theorem, a designated one. Thus, we shall henceforth speak of *the* dnf (cnf) of the formula ϕ. Now we have all we need to introduce the important notion of ***S****-entailment*.

Definition 15 (explicit S-entailment). Let ϕ be a primitive conjunction and ψ a primitive disjunction, and consider the following conditions:

a) ϕ and ψ share a literal;

b) ϕ contains complementary literals;

c) ψ contains complementary literals;

d) ϕ and ψ share a variable.

A fde $\phi \rightarrow \psi$ is said to be a *primitive **T**-entailment* iff a) holds; a *primitive **Kl**-entailment* iff either a) or b) holds; a *primitive **P**-entailment* iff either a) or c) holds; a *primitive **M**-entailment* iff either a) holds or else both b) and c) hold.

Now, let $\phi = \phi_1 \vee ... \vee \phi_n$ be a dnf, $\psi = \psi_1 \wedge ... \wedge \psi_m$ be a cnf, and let **S** $\in \{$**T, M, Kl**, **P**$\}$. The fde $\phi \rightarrow \psi$ is an *explicit **S**-entailment* iff for every $i \leq n, j \leq m$, $\phi_i \rightarrow \psi_j$ is a primitive **S**-entailment. On the other hand, $\phi \rightarrow \psi$ is called an *explicit **E**-entailment* iff: 1) for every $i \leq n, j \leq m$, either a) or b) or c) hold of $\phi_i \rightarrow \psi_j$; 2) for some $k \leq n, l \leq m$, d) holds of $\phi_k \rightarrow \psi_l$.

Definition 16 (S-entailment). Let **S** $\in \{$**T, M, E**, **Kl**, **P**$\}$. Let $\phi \rightarrow \psi$ be a fde, and let ϕ^* (ψ^*) be the dnf (cnf) of ϕ (of ψ). We say that $\phi \rightarrow \psi$ is an ***S**-entailment* iff $\phi^* \rightarrow \psi^*$ is an explicit **S**-entailment.

4.3 Tableaux

Several tableaux systems for **T** have been suggested in the literature, and these are conveniently classified into two main approaches. Dunn (1976) and Fitting (1994) employ a variant of Jeffrey's method of *coupled trees*; tableaux of this kind will not be considered here. On the other hand, D'Agostino (1990) and Priest (2001) investigate tableaux calculi of a more traditional sort, which - as we shall show - can be extended also to **Kl, P, M** and **E** (cf. also Bloesch, 1993, where something similar is done for **LP**)[3]. Throughout this subsection we shall mainly stick to Priest's notation, but we shall occasionally resort to D'Agostino's terminology as well, providing also a quick "translation method" from the former to the latter formalism[4].

Definition 17 (signed formulae). By a *signed formula* we mean an ordered pair $\langle \phi, s \rangle$, where $\phi \in$ FOR(£) and $s \in \{+, -\}$. A member of this last set is called a *sign*. The *complementary* $s^c$of a sign s is $+$ if s is $-$, and $-$ otherwise. A signed formula is said to be *positive* iff it has the form $\langle \phi, + \rangle$, *negative* otherwise. We also stipulate that:

- the *conjugate* of the signed literal $\langle L, s \rangle$ is $\langle L, s^c \rangle$;
- the *converse* of the signed literal $\langle L, s \rangle$ is $\langle L^c, s^c \rangle$;
- the *weak conjugate* of the signed literal $\langle L, s \rangle$ is $\langle L^c, s \rangle$.

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3 Tableaux for full propositional symmetric relatedness logic are discussed in Carnielli (1987).

4 Remark that these tableaux calculi are basically grounded on the same ideas as the sequent calculi mentioned at the beginning of § 4.

When referring to signed formulae, hereafter, angles are omitted whenever this is possible. As the reader will notice, the tableaux systems for **T, M**, **Kl**, **P** and **E** contain exactly the same rules for the logical connectives; they only differ in their *closure rules*.

Definition 18 (fde-tableau). A *fde-tableau* is a labelled tree $\mathcal{T}$ whose vertices are labelled by signed formulae. The signed formulae labelling the root of $\mathcal{T}$ and its immediate successor(s) are called *initial formulae* of $\mathcal{T}$. The other formulae occurring in $\mathcal{T}$ are called *non-initial*. Non-initial formulae in $\mathcal{T}$ are obtained from predecessor vertices by one of the rules below (where the usual typographical conventions are adopted):

$$\frac{\neg\neg\phi,\ +}{\phi,\ +} \qquad \frac{\neg\neg\phi,\ -}{\phi,\ -} \qquad \frac{\phi\wedge\psi,\ +}{\begin{array}{c}\phi,\ +\\ \psi,\ +\end{array}} \qquad \frac{\phi\wedge\psi,\ -}{\phi,\ -\,|\,\psi,\ -} \qquad \frac{\phi\vee\psi,\ +}{\phi,\ +\,|\,\psi,\ +}$$

$$\frac{\phi\vee\psi,\ -}{\begin{array}{c}\phi,\ -\\ \psi,\ -\end{array}} \qquad \frac{\neg(\phi\wedge\psi),\ +}{\neg\phi,\ +\,|\,\neg\psi,\ +} \qquad \frac{\neg(\phi\wedge\psi),\ -}{\begin{array}{c}\neg\phi,\ -\\ \neg\psi,\ -\end{array}} \qquad \frac{\neg(\phi\vee\psi),\ +}{\begin{array}{c}\neg\phi,\ +\\ \neg\psi,\ +\end{array}} \qquad \frac{\neg(\phi\vee\psi),\ -}{\neg\phi,\ -\,|\,\neg\psi,\ -}$$

If $\phi\rightarrow\psi$ is a fde, we say that the fde-tableau $\mathcal{T}$ is *for* $\phi\rightarrow\psi$ iff its root is labelled by $\langle\phi,\ +\rangle$ and its sole immediate successor is labelled by $\langle\psi,\ -\rangle$.

A branch of a fde-tableau $\mathcal{T}$ is called:

- *closed*, iff it contains conjugate vertices[5];
- $+$ -*semiclosed*, iff it contains positive weak conjugate vertices;
- $-$ -*semiclosed*, iff it contains negative weak conjugate vertices;
- *linked*, iff it contains either conjugate vertices or converse vertices.

Definition 19 (completed fde-tableau). A branch r of a fde-tableau $\mathcal{T}$ is called *complete* iff it contains both conclusions of a non-branching rule (α-rule, in Smullyan's standard jargon) and at least one conclusion of a branching rule (β-rule) provided it contains their premisses. A tableau is said to be *completed* iff all of its branches are complete.

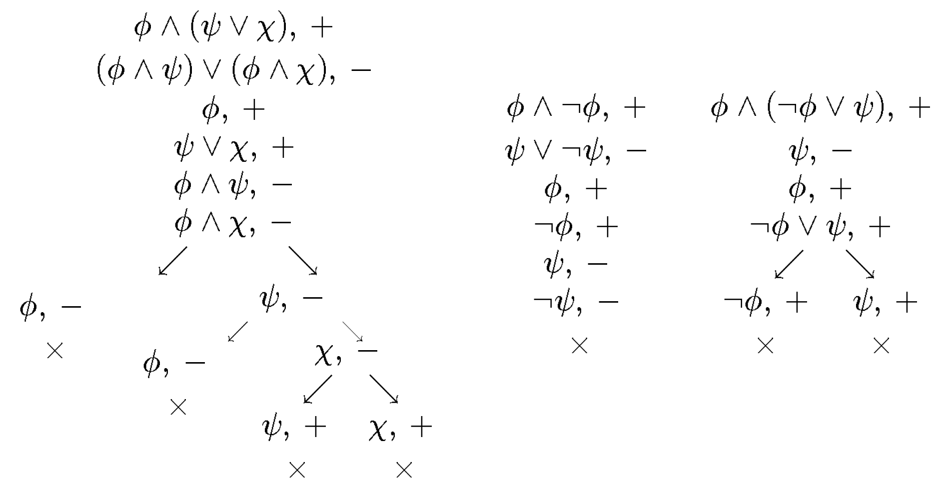
Definition 20 (TS-provability). We say that the fde $\phi\rightarrow\psi$ is:

- *provable in **TT*** (in symbols, $\vdash_{\mathbf{TT}}\phi\rightarrow\psi$) iff there is a completed fde-tableau $\mathcal{T}$ for $\phi\rightarrow\psi$ whose branches are all closed;
- *provable in **TKl*** (in symbols, $\vdash_{\mathbf{TKl}}\phi\rightarrow\psi$) iff there is a completed fde-tableau $\mathcal{T}$ for $\phi\rightarrow\psi$ whose branches are either closed or $+$ -semiclosed;
- *provable in **TP*** (in symbols, $\vdash_{\mathbf{TP}}\phi\rightarrow\psi$) iff there is a completed fde-tableau $\mathcal{T}$ for $\phi\rightarrow\psi$ whose branches are either closed or $-$ -semiclosed;
- *provable in **TM*** (in symbols, $\vdash_{\mathbf{TM}}\phi\rightarrow\psi$) iff there is a completed fde-tableau $\mathcal{T}$ for $\phi\rightarrow\psi$ whose branches are either closed or else both $+$ -semiclosed and $-$ -semiclosed;

[5] Throughout the rest of the paper, we shall allow ourselves a notational abuse and identify vertices in fde-tableaux with the signed formulae which label them.

- *provable in **TE*** (in symbols, $\vdash_{\mathbf{TE}}\phi \to \psi$) iff there is a completed fde-tableau $\mathcal{T}$ for $\phi \to \psi$ whose branches are either closed or $+$-semiclosed or $-$-semiclosed, and such that at least one branch is linked.

Example 1. Here are some examples of fde-tableaux for fdes which are provable, respectively, in **TT, TM, TE**:



Remark that, in order to obtain tableaux calculi in the style of D'Agostino (1990) for the logics at issue, it is sufficient to abide by the following translation rules:

- replace each label of the form $\langle \phi,\ + \rangle$ by a label of the form $t\phi$;

- replace each label of the form $\langle \phi,\ - \rangle$ by a label of the form $f\phi$;

- replace each label of the form $\langle \neg\phi,\ + \rangle$ by a label of the form $f{*}\phi$;

- replace each label of the form $\langle \neg\phi,\ - \rangle$ by a label of the form $t{*}\phi$.

The following definitions will be useful in what follows.

Definition 21 (faithful interpretation). Let $\mathcal{I} = \langle \mathcal{M}_{\mathbf{T}}, v \rangle$ be an interpretation for £. We say that $\mathcal{I}$ is ***T**-faithful* to the signed formula $\langle \phi, s \rangle$ iff one of the following conditions holds:

- $\langle \phi, s \rangle$ has the form $\langle \psi,\ + \rangle$ and $v(\psi) \in \{t, b\}$;
- $\langle \phi, s \rangle$ has the form $\langle \psi,\ - \rangle$ and $v(\psi) \in \{f, n\}$;
- $\langle \phi, s \rangle$ has the form $\langle \neg\psi,\ + \rangle$ and $v(\psi) \in \{f, b\}$;
- $\langle \phi, s \rangle$ has the form $\langle \neg\psi,\ - \rangle$ and $v(\psi) \in \{t, n\}$.

An interpretation for £ $\mathcal{J} = \langle \mathcal{M}_{\mathbf{E}}, v \rangle$ is called ***Kl**-faithful* (***P**-faithful*) to the signed formula $\langle \phi, s \rangle$ iff one of the following conditions holds:

- $\langle \phi, s \rangle$ has the form $\langle \psi,\ + \rangle$ and $v(\psi) = t$ (and $v(\psi) \in \{t, b\}$);
- $\langle \phi, s \rangle$ has the form $\langle \psi,\ - \rangle$ and $v(\psi) \in \{f, b\}$ (and $v(\psi) = f$);

- $\langle \phi, s \rangle$ has the form $\langle \neg\psi, + \rangle$ and $v(\psi) = f$ (and $v(\psi) \in \{f, b\}$);
- $\langle \phi, s \rangle$ has the form $\langle \neg\psi, - \rangle$ and $v(\psi) \in \{t, b\}$ (and $v(\psi) = t$).

A set S of signed formulae is called ***T**- (**Kl**-, **P**-) satisfiable* iff there is an interpretation $\mathcal{I}$ for £ which is **T**- (**Kl**-, **P**-) faithful to every signed formula in S; it is called ***M**-satisfiable* iff there is an interpretation $\mathcal{I}$ for £ which is either **Kl**-faithful to every signed formula in S or **P**-faithful to every signed formula in S.

Definition 22 (Hintikka set). A set S of signed formulae is a ***T**-Hintikka set* iff the following conditions hold:

- for every literal L, it is not the case that both $\langle L, s \rangle$ and its conjugate $\langle L, s^c \rangle$ occur in S;
- if ϕ is a possible premiss for an α-rule of a fde-tableau, the conclusion(s) that would result from the application of the rule to ϕ occur in S;
- if ϕ is a possible premiss for a β-rule of a fde-tableau, at least one of the conclusions that would result from the application of the rule to ϕ occurs in S.

The definition of ***Kl**-Hintikka set* is identical, except for the fact that a further condition is added: for any literal L, it is not the case that both $\langle L, + \rangle$ and its weak conjugate $\langle L^c, + \rangle$ occur in S. **P**-Hintikka sets and **M**-Hintikka sets are defined with the obvious modifications.

4.4 Equivalence results

We are now in a position to prove the equivalence of the previous formulations. As regards **T**, the equivalence of (A), (B), (C) below can be found already in Anderson and Belnap (1975); the equivalence of (D) and the above was first shown by D'Agostino (1990). Some of the corresponding results for **Kl, P** and **M** can be found e.g. in Rose (1963), Muravitsky (1995), Priest (2001). As regards **E**, the equivalence of (A), (B), (C) has been established in Paoli (1993).

***Theorem 4 (equivalence of the previous formulations)**. Let **S** $\in$ {**T, M, Kl, P, E**} and let $\phi \to \psi$ be a fde. The following are equivalent:*

(A) $\vdash_{HS} \phi \to \psi$;
(B) $\models_{S} \phi \to \psi$;
*(C) $\phi \to \psi$ is an **S**-entailment;*
(D) $\vdash_{TS} \phi \to \psi$.

Proof. (A) $\Rightarrow$ (B). The proof proceeds by a standard induction on the length of the proof of $\phi \to \psi$ in **HS**.

(B) $\Rightarrow$ (C) Let $\phi' = \phi_1 \vee ... \vee \phi_n$ ($\psi' = \psi_1 \wedge ... \wedge \psi_m$) be the dnf (cnf) of ϕ (of ψ). Now, suppose that $\phi \to \psi$ is not an **S**-entailment, whence $\phi' \to \psi'$ is not an explicit **S**-entailment. We shall show that $\phi' \to \psi'$ is not **S**-valid, whence $\phi \to \psi$ is not **S**-valid either, since conversion into normal form does not affect the value of a formula.

Let us consider the case of **T** first. If $\phi' \rightarrow \psi'$ is not an explicit **T**-entailment, there exist $i \leq n, j \leq m$ such that ϕ_i and ψ_j share no literal. Now, let $v : \mathcal{F}(£) \rightarrow \mathcal{A}^4_{\mathbf{T}}$ be such that, for any variable p:

- if p occurs, $\neg p$ does not occur in ϕ_i, $v(p) = t$;
- if p occurs, $\neg p$ occurs in ϕ_i, $v(p) = b$;
- if p does not occur, $\neg p$ occurs in ϕ_i, $v(p) = f$;
- if p does not occur, $\neg p$ does not occur in ϕ_i, $v(p) = n$.

Since ϕ_i and ψ_j share no literal, one can see from the tables that $v(\phi_i) \in \{t, b\}$ and $v(\psi_j) \in \{f, n\}$. But then also $v(\phi') \in \{t, b\}$ and $v(\psi') \in \{f, n\}$. Since $\langle t, f\rangle, \langle t, n\rangle, \langle b, n\rangle, \langle b, f\rangle \notin \leq^4_{\mathbf{T}}$, the interpretation $\langle \mathcal{M}_{\mathbf{T}}, v\rangle$ falsifies $\phi' \rightarrow \psi'$.

If $\phi' \rightarrow \psi'$ is not an explicit **Kl**-entailment, there exist $i \leq n, j \leq m$ such that ϕ_i and ψ_j share no literal, and moreover it is not the case that ϕ_i contains complementary literals. Now, let $v : \mathcal{F}(£) \rightarrow \mathcal{A}^3_{\mathbf{T}}$ be such that, for any variable p:

- if p occurs, $\neg p$ does not occur in ϕ_i, $v(p) = t$;
- if p does not occur, $\neg p$ occurs in ϕ_i, $v(p) = f$;
- if p does not occur, $\neg p$ does not occur in ϕ_i, $v(p) = b$.

Since ϕ_i and ψ_j share no literal, one can see from the tables that $v(\phi_i) = t$ and $v(\psi_j) \in \{f, b\}$. But then also $v(\phi') = t$ and $v(\psi') \in \{f, b\}$. Since $\langle t, f\rangle, \langle t, b\rangle \notin \leq^3_{\mathbf{T}}$, the interpretation $\langle \mathcal{M}_{\mathbf{Kl}}, v\rangle$ falsifies $\phi' \rightarrow \psi'$.

The cases of **P** and **M** are handled analogously, while the result for **E**-entailments follows easily from Theorem 1 and well-known properties of classically valid fdes.

(C) $\Rightarrow$ (A) First remark that, by A1-A2, T1, R1, $\vdash_{\mathbf{HT}} \phi \rightarrow \psi$ whenever it is a primitive **T**-entailment. Next, observe that by R2-R3 all explicit **T**-entailments are provable in **HT**. Finally, by R1 all **T**-entailments are provable in **HT**. The same proof also goes through for **M**, **Kl**, and **P**, with minor differences. As for **E**, let $\phi \rightarrow \psi$ be an explicit **E**-entailment; then there are ϕ_i, ψ_j sharing a variable. For such formulae we also have that either ϕ_i and ψ_j share a literal, or at least one of ϕ_i, ψ_j contains complementary literals. Then, by Lemma 6, A1-A2, T1, R1', $\vdash_{\mathbf{HE}} \phi_i \rightarrow \psi_j$ and $\vdash_{\mathbf{HE}} \phi \rightarrow \psi$.

(B) $\Rightarrow$ (D). For **TT**, we prove that every complete open branch of a fde-tableau is **T**-satisfiable, whence the implication follows by the remark made after Definition 12. Since the set of the signed formulae occurring in a complete open branch of a fde-tableau is a **T**-Hintikka set, it will be enough to show that any **T**-Hintikka set S is **T**-satisfiable. Now, let $\mathcal{I} = \langle \mathcal{M}_{\mathbf{T}}, v\rangle$ be constructed as follows:

- if $\langle p, +\rangle \in S$ and $\langle \neg p, +\rangle \notin S$, then $v(p) = t$;
- if $\langle p, +\rangle \in S$ and $\langle \neg p, +\rangle \in S$, then $v(p) = b$;
- if $\langle p, -\rangle \in S$ and $\langle \neg p, -\rangle \notin S$, then $v(p) = f$;
- if $\langle p, -\rangle \in S$ and $\langle \neg p, -\rangle \in S$, then $v(p) = n$;
- if $\langle \neg p, -\rangle \in S$ and $\langle p, -\rangle \notin S$, then $v(p) = t$;
- if $\langle \neg p, +\rangle \in S$ and $\langle p, +\rangle \notin S$, then $v(p) = f$.

Given Definition 22, $\mathcal{I}$ is well-defined because the cases envisaged above do not clash and exhaust all the alternatives. Moreover, $\mathcal{I}$ is **T**-faithful to all the signed literals in S. Upon remarking that a valuation which is **T**-faithful to the possible conclusion(s) of an α-rule is also **T**-faithful to the respective premiss, while a valuation which is **T**-faithful to at least one of the possible conclusions of a β-rule is also **T**-faithful to the respective premiss, we conclude by induction that $\mathcal{I}$ is **T**-faithful to all the signed formulae in S.

A similar argument works for **TKl**. By hypothesis, a **Kl**-Hintikka set S does not contain positive weak conjugate literals. The crucial interpretation is constructed as follows:

- if $\langle p, + \rangle \in S$ and $\langle \neg p, + \rangle \notin S$, then $v(p) = t$;
- if $\langle p, - \rangle \in S$ and $\langle \neg p, - \rangle \notin S$, then $v(p) = f$;
- if $\langle p, - \rangle \in S$ and $\langle \neg p, - \rangle \in S$, then $v(p) = b$;
- if $\langle \neg p, - \rangle \in S$ and $\langle p, - \rangle \notin S$, then $v(p) = t$;
- if $\langle \neg p, + \rangle \in S$ and $\langle p, + \rangle \notin S$, then $v(p) = f$.

Such an interpretation can be shown to be **Kl**-faithful to all the signed formulae in S. With **TP** and **TM**, we proceed similarly.

As to **TE**, suppose $\vDash_{\mathbf{E}} \phi \to \psi$. Then by Theorem 1 $\vDash_{\mathbf{K}} \phi \to \psi$, whence we can construct a closed completed classical tableau refuting $\neg(\phi \to \psi)$, which we can easily turn into a completed fde-tableau for $\phi \to \psi$ whose branches are either closed or $+$-semiclosed or $-$-semiclosed. Once more by Theorem 1, however, Var(ϕ) $\cap$ Var(ψ) $\neq \emptyset$. Hence at least one of such branches is linked; consequently, $\phi \to \psi$ is provable in **TE**.

(D) $\Rightarrow$ (C). For **TT**, just remark that any interpretation $\mathcal{I}$ which is **T**-faithful to the premiss of an α-rule is **T**-faithful to its conclusion(s) as well, while any interpretation $\mathcal{I}$ which is **T**-faithful to the premiss of a β-rule is **T**-faithful to at least one of its conclusions as well. Reasoning inductively, it follows that if $\mathcal{I}$ is **T**-faithful to $\langle \phi, + \rangle$, $\langle \psi, - \rangle$, then in any $\mathcal{T}$ for $\phi \to \psi$ there is at least a branch r such that $\mathcal{I}$ is **T**-faithful to all the signed formulae in r. However, no interpretation can be **T**-faithful to both $\langle \chi, + \rangle$ and $\langle \chi, - \rangle$, for any χ. Hence, if $\mathcal{T}$ is a fde-tableau for $\phi \to \psi$ all of whose branches are closed, then for any interpretation $\mathcal{I} = \langle \mathcal{M}_{\mathbf{T}}, v \rangle$, if $v(\phi) \in \{t, b\}$, then $v(\psi) \in \{t, b\}$. By the remark following Definition 12, then, $\vDash_{\mathbf{S}} \phi \to \psi$. Similar arguments apply to **TM**, **TKl**, **TP**. Finally, if $\phi \to \psi$ is provable in **TE**, it is easy to build a closed completed classical tableau refuting $\neg(\phi \to \psi)$. The fact that at least one branch must be linked yields variable sharing, whence by Theorem 1 $\vDash_{\mathbf{E}} \phi \to \psi$. ☐

4.5 Interpolation

Anderson and Belnap (1975) notice that **HT** admits of a "perfect" interpolation theorem: if $\phi \to \psi$ is provable in it, there is an interpolation formula χ such that both $\phi \to \chi$ and $\chi \to \psi$ are provable as well, and χ contains no variable which is not in both ϕ and ψ. The qualification "perfect" applies to it in contrast to the analogous result for classical logic - Craig's theorem - which establishes a weaker property: if either ϕ is contradictory or ψ is valid, we have no guarantee of the existence of the interpolation formula.

What about the other systems at issue? Clearly, **HKl** and **HP** cannot enjoy perfect interpolation, because they lack the variable sharing property. Adapting an argument by R.K. Meyer, we shall see that **HM** does not allow even a *weak* Craig-style interpolation theorem. On the other hand, **HE** enjoys perfect interpolation; this shows that, among the subclassical logics, **E** is a "maximal perfect interpolation logic": in fact, if **E** $\subset$ **S** $\subset$ **K**, then **S** contains a fde $\phi \to \psi$ where ϕ and ψ share no variable, so that there can be no interpolation formula.

Theorem 5 (interpolation). *a) Let* $\mathbf{S} \in \{\mathbf{T}, \mathbf{E}\}$. *If* $\vdash_{\mathbf{HS}} \phi \to \psi$, *then there is* $\chi \in$ *FOR(£) such that* $\vdash_{\mathbf{HS}} \phi \to \chi$, $\vdash_{\mathbf{HS}} \chi \to \psi$ *and Var*(χ) $\subseteq$ *Var*(ϕ) $\cap$ *Var*(ψ); *b) There are* $\phi, \psi \in$ *FOR(£) such that* $\vdash_{\mathbf{HM}} \phi \to \psi$, ϕ *is not classically inconsistent,* ψ *is not classically valid and there is no* $\chi \in$ *FOR(£) such that* $\vdash_{\mathbf{HM}} \phi \to \chi$, $\vdash_{\mathbf{HM}} \chi \to \psi$ *and Var*(χ) $\subseteq$ *Var*(ϕ) $\cap$ *Var*(ψ).

Proof. a) Suppose $\vdash_{\mathbf{HT}} \phi \to \psi$. By Theorem 4 $\phi' \to \psi'$ (where $\phi' = \phi_1 \vee ... \vee \phi_n$ is the dnf of ϕ and $\psi' = \psi_1 \wedge ... \wedge \psi_m$ is the cnf of ψ) is an explicit **T**-entailment. For $i \leq n$, let ϕ_i^* be a subconjunction of ϕ_i which contains just the literals shared by ϕ_i with each conjunct of ψ', and let χ be $\phi_1^* \vee ... \vee \phi_n^*$. By A1-A2, R1, R3, we obtain $\vdash_{\mathbf{HT}} \phi' \to \chi$, whence by R1 $\vdash_{\mathbf{HT}} \phi \to \chi$. Furthermore, by A1-A2, R1-R3, $\vdash_{\mathbf{HT}} \chi \to \psi'$, whence by R1 $\vdash_{\mathbf{HT}} \chi \to \psi$. Finally, given the fact that ϕ and ϕ' (respectively ψ and ψ') have the same variables, Var(χ) $\subseteq$ Var(ϕ) $\cap$ Var(ψ).

A similar proof works also for **HE**. Thus, suppose $\vdash_{\mathbf{HE}} \phi \to \psi$, and consider the fde $\phi' \to \psi'$ rewritten in normal form. In general, some ϕ_i's and some ψ_j's may contain complementary literals, other ones may not. For the sake of simplicity, suppose that for $p \leq i$ (for $q \leq j$) ϕ_p (ψ_q) contains no complementary literals, whereas for $p > i$ (for $q > j$) it does. In the following schema, primitive conjunctions and disjunctions which are free of complementary literals are printed in bold:

$$\boldsymbol{\phi_1} \vee ... \vee \boldsymbol{\phi_i} \vee \phi_{i+1} \vee ... \vee \phi_n$$
$$\boldsymbol{\psi_1} \wedge ... \wedge \boldsymbol{\psi_j} \wedge \psi_{j+1} \wedge ... \wedge \psi_m$$

We distinguish four cases: a1) $i, j > 0$; a2) $i = 0$, $j > 0$; a3) $i > 0, j = 0$; a4) $i, j = 0$.

a1) Consider $\phi_1 \vee ... \vee \phi_i \to \psi_1 \wedge ... \wedge \psi_j$. By our definition of explicit **E**-entailment, each ϕ_p shares a literal with each ψ_q; hence the previous fde is an explicit **T**-entailment. Proceeding as above, thus, we get a formula χ interpolating between $\phi_1 \vee ... \vee \phi_i$ and $\psi_1 \wedge ... \wedge \psi_j$ (transitivity moves are countenanced by our variable sharing hypothesis). Should n be greater than i, by Lemma 6 we get $\vdash_{\mathbf{HE}} \phi' \to \chi$, whence by R1' $\vdash_{\mathbf{HE}} \phi \to \chi$; should m be greater than j, once again by Lemma 6 we get $\vdash_{\mathbf{HE}} \chi \to \psi'$, whence by R1' $\vdash_{\mathbf{HE}} \chi \to \psi$. Our construction also yields Var(χ) $\subseteq$ Var(ϕ) $\cap$ Var(ψ).

a2) Consider ϕ_k, ψ_l with a variable in common, say p. The formula $\chi = p \wedge \neg p$ has no variables not in both ϕ' and ψ'; moreover, by Lemma 6 we have $\vdash_{\mathbf{HE}} \phi_k \to \chi$ and then $\vdash_{\mathbf{HE}} \phi' \to \chi$ (whence as usual $\vdash_{\mathbf{HE}} \phi \to \chi$). Applying Lemma 6 again, we get $\vdash_{\mathbf{HE}} \chi \to \psi'$ and thus $\vdash_{\mathbf{HE}} \chi \to \psi$.

a3) Dual of the above, with $\chi = p \vee \neg p$.

a4) We may proceed as in case a2) or as in case a3).

b) Let $\phi = s \vee (p \wedge q \wedge \neg q), \psi = (s \vee p) \wedge (s \vee r \vee \neg r)$. $\phi \to \psi$ is an explicit **M**-entailment, hence it is provable in **HM** by Theorem 4. It can be readily checked, too, that neither ϕ is classically contradictory nor is ψ classically valid. Now, suppose that there is $\chi \in$ FOR(£) such that Var(χ) $\subseteq \{p, s\}$ and $\vdash_{\mathbf{HM}} \phi \to \chi$, $\vdash_{\mathbf{HM}} \chi \to \psi$. By Theorem 4, it would be $\vDash_{\mathbf{M}} \phi \to \chi$, $\vDash_{\mathbf{M}} \chi \to \psi$. However, let v be such that

$v(p)=t, v(q)=v(r)=b, v(s)=f$. Then $v(\phi)=v(\psi)=b$, $v(\chi)\in\{t,f\}$. Hence either $\langle v(\phi),v(\chi)\rangle \notin \ \leq^{3}_{\mathbf{T}}$ or $\langle v(\chi),v(\psi)\rangle \notin \ \leq^{3}_{\mathbf{T}}$, contradicting our hypothesis. □

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