Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 75, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

THREE NONTRIVIAL SOLUTIONS FOR NONLOCAL ANISOTROPIC INCLUSIONS UNDER NONRESONANCE

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Abstract. In this article, we study a pseudo-differential inclusion driven by a nonlocal anisotropic operator and a Clarke generalized subdifferential of a nonsmooth potential, which satisfies nonresonance conditions both at the origin and at infinity. We prove the existence of three nontrivial solutions: one positive, one negative and one of unknown sign, using variational methods based on nosmooth critical point theory, more precisely applying the second deformation theorem and spectral theory. Here, a nosmooth anisotropic version of the Hölder versus Sobolev minimizers relation play an important role.

1. Introduction

In this article, we consider a Dirichlet problem for a pseudo-differential inclusion, driven by a nonlocal integro-differential operator L_K with kernel K, of the form

$$\begin{gathered} L_K u \in \partial j(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \Omega^c, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a C^2 boundary $\partial\Omega$, $\Omega^c = \mathbb{R}^N \setminus \Omega$ and $\partial j(x,\cdot)$ denotes the Clarke generalized subdifferential of a potential $j : \Omega \times \mathbb{R} \to \mathbb{R}$.

Recently, nonlocal operators received big attention, because of their applications, in such fields as game theory, finance, image processing, and optimization; see [1, 9, 12, 22, 51] and the references therein. One reason is that such nonlocal operators are infinitesimal generators of Lévy-type stochastic processes. The common example is the fractional Laplacian. In this paper we consider the linear operator L_K, defined for any sufficiently smooth function $u : \mathbb{R}^N \to \mathbb{R}$ and all $x \in \mathbb{R}^N$, by

$$L_K u(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} (u(x) - u(y)) K(x-y)\, dy, \tag{1.2}$$

with the singular kernel $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ given by

$$K(y) = a\Big(\frac{y}{|y|}\Big)\frac{1}{|y|^{N+2s}} \text{ with } a \in L^1(S^{N-1}) \text{ even, } \inf_{S^{N-1}} a > 0,\ N > 2s,\ 0 < s < 1\,.$$

In the particular case $a \equiv 1$, we obtain the fractional Laplacian operator $(-\Delta)^s$. We point out that the kernel of L_K satisfies the following useful properties:

(i) $mK \in L^1(\mathbb{R}^N)$, where $m(y) = \min\{|y|^2, 1\}$;

2010 *Mathematics Subject Classification.* 47G20, 35R11, 34A60, 49J92, 58E05.
Key words and phrases. Integrodifferential operators; differential inclusions, nonsmooth analysis; critical point theory.

Submitted April 3, 2019. Published May 31, 2019.

(ii) there exists $\beta > 0$ such that $K(y) \geq \beta |y|^{-(N+2s)}$ for any $y \in \mathbb{R}^N \setminus \{0\}$;
(iii) $K(y) = K(-y)$ for any $y \in \mathbb{R}^N \setminus \{0\}$.

These operators have two typical features: *nonlocality* and *anisotropy*. The first means that the value of $L_K u(x)$ at any point $x \in \Omega$ depends not only on the values of u on a neighborhood of x, but actually on the whole $\mathbb{R}^N$, since $u(x)$ represents the expected value of a random variable tied to a process randomly jumping arbitrarily far from the point x. The second is due to the presence of the function a in the kernel, such function has the role to weight differently the different spacial directions.

Also notice that an operator to be an infinitesimal generator of a Lévy process, it should satisfy the L_K properties (see [51]) with the additional hypotheses that the process is symmetric, and the measure a is absolutely continuous on S^{N-1}.

Problem (1.1) can be referred to as a pseudo-differential inclusion in Ω, coupled with a Dirichlet-type condition in Ω^c (due to the nonlocal nature of the operator L_K).

Since Chang's pioneering work [13], variational methods based on nonsmooth critical point theory are used to study nonsmooth problems driven by nonlinear operators, such as the p-Laplacian. Such variational technique allows to establish several existence and multiplicity results for problems related to locally Lipschitz potentials, which can be equivalently formulated as either differential inclusions or hemivariational inequalities, see [3, 15, 27, 29, 32, 37, 42, 45, 46, 47] and the monographs [24, 43, 44].

Recently, nonlocal problems driven by fractional-type operators (both linear and nonlinear) have taken increasing relevance, because the nonlocal diffusion has important applications in the applied sciences (for instance, in mechanics, population dynamics, and probability). Another reason is the intrinsic mathematical interest: indeed, fractional operators induce a class of integral equations, exhibiting many common features with partial differential equations. Of the vast literature, we mention the results of [2, 5, 10, 11, 19, 26, 30, 33, 52, 54, 55] for the linear case, [4, 6, 17, 21, 28, 31, 34, 36, 38, 40, 49, 50, 53] for the p-case, as well as [12, 18, 39] for a general introduction to fractional operators.

Our work stands at the conjunction of these two branches of research. Inspired by [35], we will extend to the anisotropic case their result about the existence of at least two constant sign solutions, by applying nonsmooth critical point theory. Moreover, we shall prove the existence of three nontrivial weak solutions for problem (1.1) (one positive, one negative and one with unknown sign) under the assumptions that the nonsmooth potential satisfies nonresonance conditions both at the origin and at infinity. In particular the existence of the third solution will require a nonsmooth version of the Sobolev vs. Hölder minimizers result.

Our existence result is according to our knowledge the first one for nonlocal problems involving anisotropic operators and set-valued reactions in higher dimension, while we should mention [56, 57] for the ordinary case (the first based on fixed point methods, the second on nonsmooth variational methods). We also recall an application of nonsmooth analysis to a single-valued nonlocal equation in [16].

The paper has the following structure: in Section 2 we recall some basic notions from nonsmooth critical point theory, as well as some useful results on the operator L_K, in particular we show the nonsmooth anisotropic principle of equivalence of minimizers and in Section 3 we prove our main result.

2. Preliminary results

In this section, we collect some results that will be used in our arguments.

2.1. **Brief review of nonsmooth critical point theory.** We recall some basic definitions and results of nonsmooth critical point theory (see [14, 24, 43]). Let $(X, \|\cdot\|)$ be a real Banach space and $(X^*, \|\cdot\|_*)$ its topological dual. A functional $\varphi : X \to \mathbb{R}$ is said to be *locally Lipschitz continuous* if for every $u \in X$ there exist a neighborhood U of u and $L > 0$ such that

$$|\varphi(v) - \varphi(w)| \le L\|v - w\| \quad \text{for all } v, w \in U.$$

From now on, we assume φ to be locally Lipschitz continuous. The *generalized directional derivative* of φ at u along $v \in X$ is defined by

$$\varphi^\circ(u; v) = \limsup_{w \to u,\, t \to 0^+} \frac{\varphi(w + tv) - \varphi(w)}{t}.$$

The *Clarke generalized subdifferential* of φ at u is the set

$$\partial\varphi(u) = \big\{u^* \in X^* : \langle u^*, v\rangle \le \varphi^\circ(u; v) \text{ for all } v \in X\big\}.$$

A point u is said to be a *critical point* of φ if $0 \in \partial\varphi(u)$. In the following lemma we recall some useful properties of $\partial\varphi$ (see [24, Propositions 1.3.8-1.3.12]).

Lemma 2.1. *If $\varphi, \psi : X \to \mathbb{R}$ are locally Lipschitz continuous, then*

- (i) *$\partial\varphi(u)$ is convex, closed and weakly* compact for all $u \in X$;*
- (ii) *the multifunction $\partial\varphi : X \to 2^{X^*}$ is upper semicontinuous with respect to the weak* topology on X^*;*
- (iii) *if $\varphi \in C^1(X)$, then $\partial\varphi(u) = \{\varphi'(u)\}$ for all $u \in X$;*
- (iv) *$\partial(\lambda\varphi)(u) = \lambda\partial\varphi(u)$ for all $\lambda \in \mathbb{R}$, $u \in X$;*
- (v) *$\partial(\varphi + \psi)(u) \subseteq \partial\varphi(u) + \partial\psi(u)$ for all $u \in X$;*
- (vi) *if u is a local minimizer (or maximizer) of φ, then $0 \in \partial\varphi(u)$.*

We remark that in view of Lemma 2.1(i), for all $u \in X$,

$$m_\varphi(u) := \min_{u^* \in \partial\varphi(u)} \|u^*\|_*$$

is well defined and $u \in X$ is a critical point of φ if

$$m_\varphi(u) = 0.$$

The set of all critical points of φ is denoted by $K(\varphi)$. We shall use the level sets

$$K_c(\varphi) = \{u \in K(\varphi) : \varphi(u) = c\}, \quad \varphi^c = \{u \in X : \varphi(u) \le c\}, \text{ for } c \in \mathbb{R}.$$

We say that a locally Lipschitz function $\varphi : X \to \mathbb{R}$ satisfies the *Palais-Smale condition at level $c \in \mathbb{R}$* if every sequence $(u_n)_n \subset X$ such that

$$\varphi(u_n) \to c \quad \text{and} \quad m_\varphi(u_n) \to 0 \quad \text{as } n \to \infty$$

admits a strongly convergent subsequence. We say that φ satisfies the *Palais-Smale condition* if it satisfies the Palais-Smale condition for every $c \in \mathbb{R}$. Next, we recall the nonsmooth version of the mountain pass theorem (see [24, Theorem 2.1.1]).

Theorem 2.2. *Let X be a Banach space, $\varphi : X \to \mathbb{R}$ be a locally Lipschitz function satisfying the Palais-Smale condition, $u_0, \hat{u} \in X$, $r \in (0, \|\hat{u} - u_0\|)$ be such that*

$$\max\{\varphi(u_0), \varphi(\hat{u})\} < \eta_r = \inf_{\|u - u_0\| = r} \varphi(u),$$

moreover, let

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = \hat{u}\}, \quad c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)).$$

Then $c \geq \eta_r$, *and* $K_c(\varphi) \neq \emptyset$.

We will use the following nonsmooth second deformation theorem [24, Theorem 2.1.3].

Theorem 2.3. *Let* X *be a Banach space,* $\varphi : X \to \mathbb{R}$ *be a locally Lipschitz function satisfying the Palais-Smale condition, let* $a < b$ *be real numbers such that* $K_c(\varphi) = \emptyset$ *for all* $c \in (a, b)$ *and* $K_a(\varphi)$ *is a finite set. Then, there exists a continuous deformation*

$$h : [0,1] \times (\varphi^b \setminus K_b(\varphi)) \to (\varphi^b \setminus K_b(\varphi))$$

such that the following hold:

- (i) $h(0, u) = u$, $h(1, u) \in \varphi^a$ *for all* $u \in (\varphi^b \setminus K_b(\varphi))$,
- (ii) $h(t, u) = u$ *for all* $(t, u) \in [0, 1] \times \varphi^a$,
- (iii) $t \mapsto \varphi(h(t, u))$ *is decreasing in* $[0, 1]$ *for all* $u \in (\varphi^b \setminus K_b(\varphi))$.

In particular, by (i)-(ii) above we have that φ^a is a *strong deformation retract* of φ^b (see [41, Definition 5.33 (b)]). Moreover, we observe that, if a is the global minimum of φ and is attained at a unique point $u_0 \in X$, and there are no critical levels of φ in (a, b), then by Theorem 2.3 the set $\varphi^b \setminus K_b(\varphi)$ is contractible (see [41, Definition 6.22]).

Now we consider integral functionals defined on L^2-spaces by means of locally Lipschitz continuous potentials. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary and let j_0 be a potential satisfying the following:

- (H1) $j_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $j_0(\cdot, 0) = 0$, $j_0(\cdot, t)$ is measurable in Ω for all $t \in \mathbb{R}$, $j_0(x, \cdot)$ is locally Lipschitz continuous in $\mathbb{R}$ for a.e. $x \in \Omega$. Moreover, there exists $a_0 > 0$ such that for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, and all $\xi \in \partial j_0(x, t)$, we have $|\xi| \leq a_0|t|$.

For $u \in L^2(\Omega)$ we define the functional

$$J_0(u) = \int_\Omega j_0(x, u)\, dx, \tag{2.1}$$

and the set-valued Nemytzkij operator

$$N_0(u) = \{w \in L^2(\Omega) : w(x) \in \partial j_0(x, u(x)) \text{ for a.e. } x \in \Omega\}.$$

From [14, Theorem 2.7.5] we have the following lemma, which is a particular case of [35, Lemma 2.3].

Lemma 2.4. *If* $j_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfies* (H1), *then* $J_0 : L^2(\Omega) \to \mathbb{R}$, *defined by* (2.1), *is Lipschitz continuous on any bounded subset of* $L^2(\Omega)$. *Moreover, for all* $u \in L^2(\Omega)$, $w \in \partial J_0(u)$ *one has* $w \in N_0(u)$.

2.2. **Variational formulation of the problem.** In this section we gather some useful results related to the nonlocal anisotropic operator L_K defined in (1.2). We begin to fix a functional-analytical framework, inspired by the fractional Sobolev spaces $H^s_0(\Omega)$ [18] in order to correctly encode the Dirichlet boundary datum in the variational formulation. We introduce the Hilbert space (see [54])

$$X(\Omega) = \big\{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} |u(x) - u(y)|^2 K(x - y)\, dx\, dy < \infty,\ u = 0 \text{ a.e. in } \Omega^c\big\},$$

endowed with the scalar product

$$\langle u, v\rangle = \int_{\mathbb{R}^{2N}} (u(x)-u(y))(v(x)-v(y))K(x-y)\,dx\,dy,$$

which induces the norm

$$\|u\|_{X(\Omega)} = (\langle u, u\rangle)^{1/2} = \Big(\int_{\mathbb{R}^{2N}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy\Big)^{1/2}.$$

For simplicity we use $\|u\|$ instead of $\|u\|_{X(\Omega)}$ to denote the norm of $X(\Omega)$. For all $q\in[1,\infty]$, $\|\cdot\|_q$ will denote the standard norm of $L^q(\Omega)$ (or $L^q(\mathbb{R}^N)$, which will be clear from the context).

Moreover, we denote by $(X(\Omega)^*, \|\cdot\|_*)$ the topological dual of $(X(\Omega), \|\cdot\|)$ and by $\langle\cdot,\cdot\rangle$ the scalar product of $X(\Omega)$ (or the duality pairing between $X(\Omega)^*$ and $X(\Omega)$).

Applying the fractional Sobolev inequality and the continuous embedding of $X(\Omega)$ in $H^s_0(\Omega)$ (see [54, Subsection 2.2]), we obtain that the embedding $X(\Omega)\hookrightarrow L^q(\Omega)$ is continuous for all $q\in[1,2^*_s]$ and compact if $q\in[1,2^*_s)$ (see [18, Theorem 6.7, Corollary 7.2]), here $2^*_s = 2N/(N-2s)$ is the fractional critical exponent.

Let $A : X(\Omega)\to X(\Omega)^*$ be the linear map defined by

$$\langle A(u), v\rangle = \int_{\mathbb{R}^{2N}} (u(x)-u(y))(v(x)-v(y))K(x-y)\,dx\,dy$$

for all $u, v\in X(\Omega)$. Now we consider the problem

$$\begin{gathered} L_K u \in \partial j_0(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \Omega^c, \end{gathered} \tag{2.2}$$

where j_0 satisfies (H1).

Definition 2.5. A function $u\in X(\Omega)$ is said to be a (weak) solution of (2.2) if there exists $w\in N_0(u)$ such that for all $v\in X(\Omega)$

$$\langle A(u), v\rangle = \int_\Omega wv\,dx. \tag{2.3}$$

By the embedding of $X(\Omega)$ in $L^2(\Omega)$, we have that $L^2(\Omega)$ is embedded in $X(\Omega)^*$, so (2.3) can be rephrased by

$$A(u) = w \quad \text{in } X(\Omega)^*. \tag{2.4}$$

By means of (2.4), problem (1.1) may be seen as a pseudodifferential equation (with single-valued right hand side), to which we can apply most recent results from fractional calculus of variations. In [35, Lemma 2.5] the authors proved an uniform L^∞-bounds for the fractional p-Laplacian $(-\Delta)^s_p$, in particular this holds in the case $p=2$, namely for the fractional Laplacian $(-\Delta)^s$. Using the previous fact and the embedding of $X(\Omega)$ in $H^s_0(\Omega)$, we obtain that

$$\|u\|_\infty \le C_0(1+\|u\|_{H^s_0(\Omega)}) \le C(1+\|u\|).$$

Hence, we have the following lemma.

Lemma 2.6. *If j_0 satisfies* (H1)*, then there exists $C>0$ such that for all solutions $u\in X(\Omega)$ of* (2.2) *one has $u\in L^\infty(\Omega)$ and*

$$\|u\|_\infty \le C(1+\|u\|).$$

From the literature about fractional equations, we know that solutions of such problems exhibit good interior regularity properties, but they may have a singular behaviour on the boundary. This is the reason why we consider the following weighted Hölder-type spaces $C^0_\delta(\overline{\Omega})$ and $C^\alpha_\delta(\overline{\Omega})$, instead of the usual space $C^1(\overline{\Omega})$. We define the spaces

$$C^0_\delta(\overline{\Omega}) = \{u \in C^0(\overline{\Omega}) : u/\delta^s \in C^0(\overline{\Omega})\},$$
$$C^\alpha_\delta(\overline{\Omega}) = \{u \in C^0(\overline{\Omega}) : u/\delta^s \in C^\alpha(\overline{\Omega})\} \quad (\alpha \in (0,1)),$$

where $\delta(x) = \operatorname{dist}(x, \Omega^c)$ with $x \in \overline{\Omega}$, endowed with the norms

$$\|u\|_{0,\delta} = \|\frac{u}{\delta^s}\|_\infty, \quad \|u\|_{\alpha,\delta} = \|u\|_{0,\delta} + \sup_{x\neq y} \frac{|u(x)/\delta^s(x) - u(y)/\delta^s(y)|}{|x-y|^\alpha},$$

respectively. For all $0 \le \alpha < \beta < 1$, the embedding $C^\beta_\delta(\overline{\Omega}) \hookrightarrow C^\alpha_\delta(\overline{\Omega})$ is continuous and compact. In this case, the positive cone $C^0_\delta(\overline{\Omega})_+$ has a nonempty interior given by

$$\operatorname{int}(C^0_\delta(\overline{\Omega})_+) = \big\{u \in C^0_\delta(\overline{\Omega}) : \frac{u(x)}{\delta^s(x)} > 0 \text{ for all } x \in \overline{\Omega}\big\}.$$

Lemma 2.7. *If j_0 satisfies* (H1)*, then there exist $\alpha \in (0,s)$ and $C > 0$ such that for all solutions $u \in X(\Omega)$ of* (2.2) *one has $u \in C^\alpha_\delta(\overline{\Omega})$ and*

$$\|u\|_{C^\alpha_\delta(\overline{\Omega})} \le C_1\big(1 + \|u\|\big).$$

Proof. From Lemma 2.6, we obtain $u \in L^\infty(\Omega)$ such that $\|u\|_\infty \le C(1+\|u\|)$, with $C > 0$ independent of u. Let $w \in N_0(u)$ be as in Definition 2.5. Then by (H1), we have

$$\|w\|_\infty \le a_0\|u\|_\infty.$$

Now [51, Proposition 7.2 - Theorem 7.4] imply $u \in C^\alpha_\delta(\overline{\Omega})$ and

$$\|u\|_{C^\alpha_\delta(\overline{\Omega})} \le (c_0 + c\|w\|_\infty) \le C_1\big(1 + \|u\|\big),$$

with $c_0, c, C_1 > 0$ independent of u. □

The regularity C^s is the best result that we can obtain in the fractional framework, as was pointed out in [52] even for the fractional Laplacian. In particular, solutions do not, in general, admit an outward normal derivative at the points of $\partial\Omega$ and, for this reason, the Hopf property is stated in terms of a Hölder-type quotient (see [17] and Lemma 3.2 below).

2.3. **Equivalence of minimizers in the two topologies.** In the next theorem we prove an useful topological result, regarding the minimizers in the $X(\Omega)$-topology and in the $C^0_\delta(\overline{\Omega})$-topology, respectively. This is a nonsmooth anisotropic version of the result of [22], previously proved in [30, Theorem 1.1] and [5, Proposition 2.5], which in turn is inspired by [8].

Theorem 2.8 (Hölder vs Sobolev minimizers)**.** *If j_0 satisfies* $(\mathbf{H}_0)$*, then for all $u_0 \in X(\Omega)$ the following statements are equivalent:*

- (i) *there exists $\rho > 0$ such that $\varphi(u_0 + v) \ge \varphi(u_0)$ for all $v \in X(\Omega) \cap C^0_\delta(\overline{\Omega})$, $\|v\|_{0,\delta} \le \rho$;*
- (ii) *there exists $\epsilon > 0$ such that $\varphi(u_0 + v) \ge \varphi(u_0)$ for all $v \in X(\Omega)$, $\|v\| \le \epsilon$.*

We remark that, contrary to the result in [8] for the local case $s = 1$, there is no known relationship between the topologies of $X(\Omega)$ and $C^0_\delta(\overline{\Omega})$.

Proof. Let φ be the locally Lipschitz energy functional

$$\varphi(u) = \frac{\|u\|^2}{2} - \int_\Omega j_0(x, u(x))\, dx.$$

(i) $\Rightarrow$ (ii) **Case $u_0 = 0$.** We point out that $\varphi(0) = 0$, hence we can rewrite the hypothesis as

$$\inf_{u \in X(\Omega) \cap \overline{B}^\delta_\rho} \varphi(u) = 0,$$

where $\overline{B}^\delta_\rho$ denotes the closed ball in $C^0_\delta(\overline{\Omega})$ centered at 0 with radius ρ. We suppose by contradiction that (i) holds and that there exist a sequence $(\epsilon_n)_n \in (0, \infty)$ such that $\epsilon_n \to 0$ and for all $n \in N$ we have

$$\inf_{u \in \overline{B}^X_{\epsilon_n}} \varphi(u) = m_n < 0,$$

where $\overline{B}^X_{\epsilon_n}$ denotes the closed ball in $X(\Omega)$ centered at 0 with radius ϵ_n. Furthermore, the functional $u \mapsto \|u\|^2/2$ is convex, hence weakly l.s.c. in $X(\Omega)$, while J_0 is continuous in $L^2(\Omega)$, which, by the compact embedding $X(\Omega) \hookrightarrow L^2(\Omega)$ and the Eberlein-Smulyan theorem, implies that J is sequentially weakly continuous in $X(\Omega)$. Hence, φ is sequentially weakly l.s.c. in $X(\Omega)$. As a consequence, m_n is attained at some $u_n \in \overline{B}^X_{\epsilon_n}$ for all $n \in N$.
We state that, for all $n \in \mathbb{N}$, there exist $\mu_n \le 0$, $w_n \in N(u_n)$ such that for all $v \in X(\Omega)$,

$$\langle A(u_n), v\rangle - \int_\Omega w_n v\, dx = \mu_n \langle A(u_n), v\rangle . \tag{2.5}$$

Indeed, if $u_n \in B^X_{\epsilon_n}$, then u_n is a local minimizer of φ in $X(\Omega)$, hence a critical point, so (2.5) holds with $\mu_n = 0$. If $u_n \in \partial B^X_{\epsilon_n}$, then u_n minimizes φ restricted to the C^1-Banach manifold

$$\big\{u \in X(\Omega) : \|u\|^2 = \epsilon_n^2\big\},$$

so we can find a Lagrange multiplier $\mu_n \in \mathbb{R}$ such that (2.5) holds. More precisely, testing (2.5) with $-u_n$, we obtain

$$\langle B(u_n), -u_n\rangle := \langle A(u_n), -u_n\rangle - \int_\Omega w_n(-u_n)\, dx = -\mu_n \|u_n\|^2,$$

where $B(u_n) \in X(\Omega)^*$, so recalling that $\varphi(u) \ge \varphi(u_n)$ for all $u \in B^X_{\epsilon_n}$, applying the definition of generalized subdifferential, the properties of the generalized directional derivative (see [24, Proposition 1.3.7], and Lemma 2.1(vi), we obtain

$$\langle B(u_n), -u_n\rangle \ge \varphi^0(u_n, -u_n) \ge 0,$$

hence $\mu_n \le 0$.

Putting $C_n = (1 - \mu_n)^{-1} \in (0, 1]$, we obtain that for all $n \in \mathbb{N}$, $u_n \in X(\Omega)$ is a weak solution of the auxiliary boundary value problem

$$\begin{gathered} L_K u_n = C_n w_n \quad \text{in } \Omega \\ u_n = 0 \quad \text{in } \Omega^c, \end{gathered}$$

where $C_n w_n \in N(u_n)$ for all $n \in N$. By Lemma 2.6, $u_n \in L^\infty(\Omega)$, so by Lemma 2.7 we have $u_n \in C^\alpha_\delta(\overline{\Omega})$. Hence $(u_n)_n$ is bounded in $C^\alpha_\delta(\overline{\Omega})$, by the compact embedding $C^\alpha_\delta(\overline{\Omega}) \hookrightarrow C^0_\delta(\overline{\Omega})$, up to a subsequence, we have that $(u_n)_n$ is strongly convergent in $C^0_\delta(\overline{\Omega})$, hence $(u_n)_n$ is uniformly convergent in $\overline{\Omega}$. Since $u_n \to 0$ in $X(\Omega)$, passing to a subsequence, we may assume $u_n(x) \to 0$ a.e. in Ω, so this implies $u_n \to 0$ in $C^0_\delta(\overline{\Omega})$. Consequently for $n \in \mathbb{N}$ big enough we have $\|u_n\|_{0,\delta} \le \rho$ together with $\varphi(u_n) = m_n < 0$, a contradiction.

(i) $\Rightarrow$ (ii), **Case** $u_0 \neq 0$**.** For all $v \in C^\infty_0(\Omega)$, we stress that in particular $v \in X(\Omega) \cap C^0_\delta(\overline{\Omega})$, so the minimality assures

$$\langle A(u_0), v\rangle = \int_\Omega w_0 v\,dx \quad \text{for some } w \in N_0(u) \text{ and all } v \in C^\infty_c(\Omega). \tag{2.6}$$

Since $C^\infty_0(\Omega)$ is dense in $X(\Omega)$ (see [20, Theorem 6], [39, Theorem 2.6]), and $A(u_0) \in X(\Omega)^*$, equality (2.6) holds for all $v \in X(\Omega)$, namely u_0 is a weak solution of (2.2). From Lemma 2.6, we obtain $u_0 \in L^\infty(\Omega)$, hence $w_0 \in L^\infty(\Omega)$. Applying Lemma 2.7, we have that $u_0 \in C^0_\delta(\overline{\Omega})$. For $(x,t) \in \Omega \times \mathbb{R}$ we define

$$\tilde{j}(x,t) = j(x, u_0(x) + t) - j(x, u_0(x)) - w_0(x)t,$$

and for $v \in X(\Omega)$, we define

$$\tilde{\varphi}(v) = \frac{\|v\|^2}{2} - \int_\Omega \tilde{j}(x, v(x))\,dx,$$

where $\tilde{\varphi}$ is locally Lipschitz, $\tilde{j}$ satisfies (H1) and $\tilde{w} \in \tilde{N}(v)$. Moreover, by (2.6), for $v \in X(\Omega)$ we obtain

$$\tilde{\varphi}(v) = \frac{1}{2}(\|u_0 + v\|^2 - \|u_0\|^2) - \int_\Omega (j(x, u_0 + v) - j(x, u_0))\,dx = \varphi(u_0 + v) - \varphi(u_0),$$

in particular $\tilde{\varphi}(0) = 0$. Hence, we can rephrase hypothesis (i) as

$$\inf_{v \in X(\Omega) \cap \overline{B}^\delta_\rho} \tilde{\varphi}(v) = 0.$$

Recalling the previous case, we can find $\epsilon > 0$ such that for all $v \in X(\Omega)$, $\|v\| \le \epsilon$, we obtain $\tilde{\varphi}(v) \ge 0$, that is to say $\varphi(u_0 + v) \ge \varphi(u_0)$.

(ii) $\Rightarrow$ (i) We argue by contradiction. We suppose that there exists a sequence $(u_n)_n$ in $X(\Omega) \cap C^0_\delta(\overline{\Omega})$ such that $u_n \to u_0$ in $C^0_\delta(\overline{\Omega})$ and $\varphi(u_n) < \varphi(u_0)$. We note that

$$\int_\Omega j(x, u_n)\,dx \to \int_\Omega j(x, u_0)\,dx \quad \text{as } n \to \infty,$$

and this, together with $\varphi(u_n) < \varphi(u_0)$, means that

$$\limsup_n \|u_n\|^2 \le \|u_0\|^2.$$

Furthermore $(u_n)_n$ is bounded in $X(\Omega)$, so (up to a subsequence) $(u_n)_n$ converges weakly in $X(\Omega)$ to u_0, hence, by [7, Proposition 3.32], $u_n \to u_0$ in $X(\Omega)$. For $n \in \mathbb{N}$ big enough, we have $\|u_n - u_0\| \le \epsilon$ and recalling that $\varphi(u_n) < \varphi(u_0)$, we obtain a contradiction. $\square$

Remark 2.9. We stress that the proof of the case $u_0 \neq 0$, (i) $\Rightarrow$ (ii) requires $p = 2$. This is the main difference with the nonlinear case (see [35]) and this explains why we have one more solution only in the linear case, as we will see in the sequel.

In analogy to the case of the Laplacian, the spectrum of L_K is defined by a sequence $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$ of variational eigenvalues with min-max characterizations (see [22, 23, 25, 33, 39, 55] for a detailed description of such eigenvalues). Here we shall only use some properties of λ_1 and λ_2.

Lemma 2.10. *The principal eigenvalue λ_1 of operator L_K in $X(\Omega)$ is simple and isolated (as an element of the spectrum), with the following variational characterization*

$$\lambda_1 = \inf_{u \in X(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_2^2}.$$

The corresponding positive and $L^2(\Omega)$-normalized eigenfunction $u_1 \in \operatorname{int}(C^0_\delta(\overline{\Omega})_+)$.

The second eigenvalue λ_2 has the variational characterization

$$\lambda_2 = \inf_{\gamma \in \Gamma_1} \sup_{t \in [0,1]} \|\gamma(t)\|^2,$$

where Γ_1 is the family of paths $\gamma \in C([0,1], X(\Omega))$ such that $\gamma(0) = u_1$, $\gamma(1) = -u_1$, and $\|\gamma(t)\|_2 = 1$ for all $t \in [0,1]$ (see [25]).

Using the hypothesis of nonresonance at infinity, we can show the coercivity of φ, and this is fundamental to obtain the constant sign solutions of (1.1).

Lemma 2.11. *Let $\theta \in L^\infty(\Omega)_+$ be such that $\theta \leq \lambda_1$, $\theta \not\equiv \lambda_1$, and $\psi \in C^1(X(\Omega))$ be defined by*

$$\psi(u) = \|u\|^2 - \int_\Omega \theta(x)|u|^2\,dx.$$

Then there exists $\theta_0 \in (0, \infty)$ such that for all $u \in X(\Omega)$,

$$\psi(u) \geq \theta_0 \|u\|^2.$$

Proof. The claim follows from [35, Proposition 2.9] and recalling that $X(\Omega)$ is embedded in $H^s_0(\Omega)$. $\square$

3. A multiplicity result

In this section, we prove the existence of three nontrivial solutions of problem (1.1) (one positive, one negative and one of unknown sign), by means of the (nonsmooth) second deformation theorem and spectral theory. Precisely, on the nonsmooth potential j we will assume the following:

(H2) $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(\cdot, 0) = 0$, $j(\cdot, t)$ is measurable in Ω for all $t \in \mathbb{R}$, $j(x, \cdot)$ is locally Lipschitz continuous in $\mathbb{R}$ for a.e. $x \in \Omega$. Moreover,

- (i) for all $\rho > 0$ there exists $a_\rho \in L^\infty(\Omega)_+$ such that for a.e. $x \in \Omega$, all $|t| \leq \rho$, and all $\xi \in \partial j(x,t)$, we have $|\xi| \leq a_\rho(x)$;
- (ii) there exists $\theta \in L^\infty(\Omega)_+$ such that $\theta \leq \lambda_1$, $\theta \not\equiv \lambda_1$, and uniformly for a.e. $x \in \Omega$
$$\limsup_{|t|\to\infty} \max_{\xi \in \partial j(x,t)} \frac{\xi}{t} \leq \theta(x);$$
- (iii) there exist $\eta_1, \eta_2 \in L^\infty(\Omega)_+$, $\inf_\Omega \eta_1 > \lambda_2$ such that uniformly for a.e. $x \in \Omega$
$$\eta_1(x) \leq \liminf_{t\to 0} \min_{\xi \in \partial j(x,t)} \frac{\xi}{t} \leq \limsup_{t\to 0} \max_{\xi \in \partial j(x,t)} \frac{\xi}{t} \leq \eta_2(x);$$
- (iv) for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, and all $\xi \in \partial j(x,t)$, we have $\xi t \geq 0$.

Clearly, by hypothesis (H2), problem (1.1) always has the zero solution. The hypothesis (H2) (ii)-(iii) produce a nonresonance phenomenon both at infinity and at the origin, where we indicate with λ_1 and λ_2 the principal and the second eigenvalue of L_K with Dirichlet conditions in Ω. Here we give an example of a potential satisfying (H2).

Example 3.1. Let $\theta, \eta \in L^\infty(\Omega)_+$ be such that $\theta < \lambda_1 < \lambda_2 < \eta$, and $j : \Omega\times\mathbb{R} \to \mathbb{R}$ be defined for all $(x,t) \in \Omega \times \mathbb{R}$ by

$$j(x,t) = \begin{cases} \frac{\eta(x)}{2}|t|^2 & \text{if } |t| \le 1 \\ \frac{\theta(x)}{2}|t|^2 + \ln(|t|^2) + \frac{\eta(x)-\theta(x)}{2} & \text{if } |t| > 1. \end{cases}$$

As a first step we define two truncated, nonsmooth energy functionals, setting for all $u \in X(\Omega)$,

$$\varphi_\pm(u) = \frac{\|u\|^2}{2} - \int_\Omega j_\pm(x,u)\,dx,$$

where for all $(x,t) \in \Omega \times \mathbb{R}$,

$$j_\pm(x,t) = j(x, \pm t^\pm), \quad \text{with } t^\pm = \max\{\pm t, 0\}.$$

Such functionals $\varphi_\pm$ allow us to find constant sign solutions of (1.1), as explained by the following lemma.

Lemma 3.2. *The functional $\varphi_+ : X(\Omega) \to \mathbb{R}$ is locally Lipschitz continuous. Moreover, if $u \in X(\Omega) \setminus \{0\}$ is a critical point of φ_+, then $u \in C^\alpha_\delta(\overline{\Omega})$ is a solution of* (1.1) *such that*

(i) *$u(x) > 0$ for all $x \in \Omega$;*
(ii) *for all $y \in \partial\Omega$,*

$$\liminf_{x\to y,\, x\in\Omega} \frac{u(x)}{\operatorname{dist}(x,\Omega^c)^s} > 0.$$

Analogously, the functional $\varphi_- : X(\Omega) \to \mathbb{R}$ is locally Lipschitz continuous. Furthermore, if $u \in X(\Omega) \setminus \{0\}$ is a critical point of φ_-, then $u \in C^\alpha_\delta(\overline{\Omega})$ is a solution of (1.1) *such that*

(i) *$u(x) < 0$ for all $x \in \Omega$;*
(ii) *for all $y \in \partial\Omega$,*

$$\limsup_{x\to y,\, x\in\Omega} \frac{u(x)}{\operatorname{dist}(x,\Omega^c)^s} < 0.$$

Proof. By [35, Lemma 3.1, Lemma 3.2] this result holds in the case $p = 2$, namely for $(-\Delta)^s$. Exploiting the embedding of $X(\Omega)$ in $H^s_0(\Omega)$ and recalling the strong maximum principle (consequence of [51, Lemma 7.3]) and the Hopf lemma (see [51, Lemma 7.3]) for L_K we obtain the thesis. □

Now we can prove our main result, where Theorem 2.8 plays an essential part to relate critical points of $\varphi_\pm$ with critical points of φ.

Theorem 3.3. *If* (H2) *holds, then problem* (1.1) *admits at least three nontrivial solutions $u_\pm \in \pm\operatorname{int}(C^0_\delta(\overline{\Omega})_+)$, and $\tilde{u} \in C^0_\delta(\overline{\Omega}) \setminus \{0\}$.*

Proof. We focus on the truncated functional φ_+ and we show the existence of the positive solution, that will be a global minimizer of such functional. First of all, the generalized subdifferential $\partial j_+(x,\cdot)$ for all $t\in\mathbb{R}$ is given by

$$\begin{gathered}\partial j_+(x,t)=\{0\}\quad\text{if } t<0,\\ \partial j_+(x,t)\subseteq\{\mu\xi:\mu\in[0,1],\,\xi\in\partial j(x,0)\}\quad\text{if } t=0,\\ \partial j_+(x,t)=\partial j(x,t)\quad\text{if } t>0.\end{gathered}\tag{3.1}$$

Using (H2)(ii), for any $\varepsilon>0$ we can find $\rho>0$ such that for a.e. $x\in\Omega$, all $t>\rho$ and all $\xi\in\partial j_+(x,t)$ we have

$$|\xi|\le(\theta(x)+\varepsilon)t$$

(we note that $\partial j_+(x,t)=\partial j(x,t)$ for $t>0$). From (H2)(i) and using (3.1), there exists $a_\rho\in L^\infty(\Omega)_+$ such that for a.e. $x\in\Omega$, all $t\le\rho$ and all $\xi\in\partial j_+(x,t)$

$$|\xi|\le a_\rho(x).$$

Hence, for a.e. $x\in\Omega$, all $t\in\mathbb{R}$ and all $\xi\in\partial j_+(x,t)$ we obtain

$$|\xi|\le a_\rho(x)+(\theta(x)+\varepsilon)|t|.\tag{3.2}$$

From the Rademacher theorem and [14, Proposition 2.2.2], we know that for a.e. $x\in\Omega$ the mapping $j_+(x,\cdot)$ is differentiable for a.e. $t\in\mathbb{R}$ with

$$\frac{d}{dt}j_+(x,t)\in\partial j_+(x,t).$$

Hence, integrating and applying (3.2), we obtain for a.e. $x\in\Omega$ and all $t\in\mathbb{R}$

$$j_+(x,t)\le a_\rho(x)|t|+(\theta(x)+\varepsilon)\frac{|t|^2}{2}.\tag{3.3}$$

Applying (3.3), Lemmas 2.10, 2.11, and the continuous embedding $X(\Omega)\hookrightarrow L^1(\Omega)$, for all $u\in X(\Omega)$ we have

$$\begin{aligned}\varphi_+(u)&\ge\frac{\|u\|^2}{2}-\int_\Omega\Big(a_\rho(x)|u|+(\theta(x)+\varepsilon)\frac{|u|^2}{2}\Big)\,dx\\ &\ge\frac12\Big(\|u\|^2-\int_\Omega\theta(x)|u|^2\,dx\Big)-\|a_\rho\|_\infty\|u\|_1-\frac{\varepsilon}{2}\|u\|_2^2\\ &\ge\frac12\Big(\theta_0-\frac{\varepsilon}{\lambda_1}\Big)\|u\|^2-c\|u\|\quad\text{for some }\ c>0.\end{aligned}$$

If we choose $\varepsilon\in(0,\theta_0\lambda_1)$ in the last term of the inequality, then $\varphi_+(u)$ tends to $+\infty$ as $\|u\|\to\infty$, hence φ_+ is coercive in $X(\Omega)$. Furthermore, the functional $u\mapsto\|u\|^2/2$ is convex, so weakly l.s.c. in $X(\Omega)$, while J_+ is continuous in $L^2(\Omega)$, which, by the compact embedding $X(\Omega)\hookrightarrow L^2(\Omega)$ and the Eberlein-Smulyan theorem, implies that J_+ is sequentially weakly continuous in $X(\Omega)$. Hence, φ_+ is sequentially weakly l.s.c. in $X(\Omega)$. Consequently, there exists $u_+\in X(\Omega)$ such that

$$\varphi_+(u_+)=\inf_{u\in X(\Omega)}\varphi_+(u)=:m_+.\tag{3.4}$$

From Lemma 2.1(vi), u_+ is a critical point of φ_+. We state that

$$m_+<0.\tag{3.5}$$

Indeed, by (H2)(iii), for any $\varepsilon>0$, we can find $\delta>0$ such that for a.e. $x\in\Omega$, all $t\in[0,\delta)$, and all $\xi\in\partial j_+(x,t)$

$$\xi\ge(\eta_1(x)-\varepsilon)t.$$

Arguing as before, integrating we have

$$j_+(x,t) \geq \frac{\eta_1(x) - \varepsilon}{2} t^2. \tag{3.6}$$

Let $u_1 \in X(\Omega) \cap C^\alpha_\delta(\overline{\Omega})$ be the first eigenfunction. We can find $\mu > 0$ such that $0 < \mu u_1(x) \leq \delta$ for all $x \in \Omega$. Then, applying (3.6) and Lemma 2.10, we obtain

$$\begin{aligned} \varphi_+(\mu u_1) &\leq \frac{\mu^2}{2}\|u_1\|^2 - \frac{\mu^2}{2}\int_\Omega (\eta_1(x) - \varepsilon)u_1^2 \, dx \\ &= \frac{\mu^2}{2}\Big(\int_\Omega (\lambda_1 - \eta_1(x))u_1^2 \, dx + \varepsilon\Big). \end{aligned}$$

Using the fact that $\inf_\Omega \eta_1 > \lambda_2$ with $\lambda_2 > \lambda_1$, and that $u_1(x) > 0$ for all $x \in \Omega$, we obtain

$$\int_\Omega (\lambda_1 - \eta_1(x))u_1^2 \, dx < 0.$$

Hence, for $\varepsilon > 0$ small enough, the estimates above imply $\varphi_+(\mu u_1) < 0$. Therefore, (3.5) is true.

Moreover, from (3.4) we obtain $u_+ \neq 0$. From Lemma 3.2 we have that $u_+ \in C^\alpha_\delta(\overline{\Omega})$, $u_+(x) > 0$ for all $x \in \Omega$, and

$$\liminf_{x \to y.\, x\in\Omega} \frac{u_+(x)}{\operatorname{dist}(x, \Omega^c)} > 0$$

for all $y \in \partial\Omega$, so we deduce $u_+ \in \operatorname{int}(C^0_\delta(\overline{\Omega})_+)$. Noting that $\varphi \equiv \varphi_+$ on $C^0_\delta(\overline{\Omega})_+$, we see that u_+ is a Hölder local minimizer of φ, hence by Theorem 2.8, u_+ is as well a Sobolev local minimizer of φ. In particular, $u_+ \in K(\varphi)$ is a positive solution of (1.1).

Working on φ_- and recalling Lemma 3.2, we can find another solution $u_- \in C^\alpha_\delta(\overline{\Omega})$ such that $u_-(x) < 0$ for all $x \in \Omega$, and

$$\limsup_{x \to y,\, x\in\Omega} \frac{u_-(x)}{\operatorname{dist}(x, \Omega^c)} < 0$$

for all $y \in \partial\Omega$. Therefore $u_- \in -\operatorname{int}(C^0_\delta(\overline{\Omega})_+)$ and similarly u_- is a local minimizer of φ.

We want to show the existence of another nontrivial solution, and in order to do it, first we observe that φ is coercive. Now we show that φ and $\varphi_\pm$ satisfy the Palais-Smale condition.

Let $(u_n)_n$ be a bounded sequence in $X(\Omega)$ such that $(\varphi(u_n))$ is bounded and $m_\varphi(u_n) \to 0$. By Lemma 2.1(i), the definition of $m_\varphi(u_n)$, and recalling that $\partial\varphi(u_n) \subset A(u_n) - N(u_n)$ for all $n \in \mathbb{N}$, there exists $w_n \in N(u_n)$ such that $m_\varphi(u_n) = \|A(u_n) - w_n\|_*$. From the reflexivity of $X(\Omega)$ and the compact embedding $X(\Omega) \to L^2(\Omega)$, passing if necessary to a subsequence, we have $u_n \rightharpoonup u$ in $X(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$ for some $u \in X(\Omega)$. Besides, by (H1) we see that $(w_n)_n$ is bounded in $L^2(\Omega)$. By what was stated above, we have

$$\begin{aligned} \|u_n - u\|^2 &= \langle A(u_n), u_n - u\rangle - \langle A(u), u_n - u\rangle \\ &= \langle A(u_n) - w_n, u_n - u\rangle + \int_\Omega w_n(u_n - u)\, dx - \langle A(u), u_n - u\rangle \\ &\leq m_\varphi(u_n)\|u_n - u\| + \|w_n\|_2\|u_n - u\|_2 - \langle A(u), u_n - u\rangle \end{aligned}$$

for all $n \in \mathbb{N}$ and the latter tends to 0 as $n \to \infty$. Thus, $u_n \to u$ in $X(\Omega)$.

From (H2), we have $0 \in K(\varphi)$, while from the first part of the proof we already know that $u_\pm \in K(\varphi) \setminus \{0\}$. By contradiction, we suppose there is no more critical point $\tilde{u} \in X(\Omega)$, which means

$$K(\varphi) = \{0, u_+, u_-\}. \tag{3.7}$$

Without loss of generality, we assume that $\varphi(u_+) \geq \varphi(u_-)$ and that u_+ is a strict local minimizer of φ, so we can find $r \in (0, \|u_+ - u_-\|)$ such that $\varphi(u) > \varphi(u_+)$ for all $u \in X(\Omega)$ and $0 < \|u - u_+\| \leq r$. Furthermore, we have

$$\eta_r = \inf_{\|u-u_+\|=r} \varphi(u) > \varphi(u_+). \tag{3.8}$$

We could also find a sequence $(u_n)_n$ in $X(\Omega)$ such that $\|u_n - u_+\| = r$ for all $n \in \mathbb{N}$, $\varphi(u_n) \to \varphi(u_+)$ and $m_\varphi(u_n) \to 0$. Then by Palais - Smale condition, we would have $u_n \to \overline{u}$ in $X(\Omega)$ for some $\overline{u} \in X(\Omega)$ and $\|\overline{u} - u_+\| = r$, hence in turn $\varphi(\overline{u}) = \varphi(u_+)$, which is a contradiction. Now we introduce

$$\Gamma = \{\gamma \in C([0,1], X(\Omega)) : \gamma(0) = u_+, \gamma(1) = u_-\} \text{ and } c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)).$$

From Theorem 2.2, we have $c \geq \eta_r$ and there exists $\tilde{u} \in K_c(\varphi)$. By (3.8), $\tilde{u} \neq u_\pm$. Hence, from (3.7) we deduce that $\tilde{u} = 0$, so $c = 0$. In order to achieve a contradiction, we will construct a path $\gamma \in \Gamma$ such that

$$\max_{t\in[0,1]} \varphi(\gamma(t)) < 0, \tag{3.9}$$

so that $c < 0$. Let $0 < \eta_1' < \eta_1(x)$ and $\tau > 0$ be such that

$$\eta_1' > \lambda_2 + \tau. \tag{3.10}$$

By (H2)(iii), there exists $\sigma > 0$ such that $j(x,t) > \eta_1' \frac{|t|^2}{2}$ for a.e. in Ω and all $|t| \leq \sigma$. Moreover, by Lemma 2.10, there exists $\gamma_1 \in \Gamma_1$ such that

$$\max_{t\in[0,1]} \|\gamma_1(t)\|^2 < \lambda_2 + \tau. \tag{3.11}$$

Since $C_0^\infty(\Omega)$ is dense in $X(\Omega)$ (see [20, Theorem 6], [39, Theorem 2.6]), we can picking out $\gamma_1(t) \in L^\infty(\Omega)$ for all $t \in [0,1]$ and γ_1 continuous with respect to the L^∞-topology. Hence, by choosing $\tilde{\mu} > 0$ small enough, we have $\|\tilde{\mu}\gamma_1(t)\|_\infty \leq \sigma$ for all $t \in [0,1]$. We define $\tilde{\gamma}(t) = \tilde{\mu}\gamma_1(t)$. Therefore, by (3.11) and recalling that $\|\gamma_1(t)\|_2 = 1$ (Lemma 2.10), we obtain for all $t \in [0,1]$ that

$$\varphi(\tilde{\gamma}(t)) \leq \frac{\tilde{\mu}^2}{2}\|\gamma_1(t)\|^2 - \int_\Omega \eta_1' \frac{\tilde{\mu}^2}{2}|\gamma_1(t)|^2\,dx \leq \frac{\tilde{\mu}^2}{2}(\lambda_2 + \tau - \eta_1') < 0,$$

and the latter is negative by (3.10). Then $\tilde{\gamma}$ is a continuous path joining $\tilde{\mu}u_1$ and $-\tilde{\mu}u_1$ such that

$$\max_{t\in[0,1]} \varphi(\tilde{\gamma}(t)) < 0. \tag{3.12}$$

By (H2)(iv) and Lemma 3.2, we see that $K(\varphi_+) \subset K(\varphi)$, actually, by (3.7), we obtain $K(\varphi_+) = \{0, u_+\}$. We fix $a = \varphi_+(u_+)$ and $b = 0$, in this way $\varphi_+^a = \{u_+\}$ and φ_+ fulfill all the hypothesis of Theorem 2.3, so there exists a continuous deformation $h_+ : [0,1] \times (\varphi_+^0 \setminus \{0\}) \to (\varphi_+^0 \setminus \{0\})$ such that

$$\begin{gathered}
h_+(0,u) = u, \quad h_+(1,u) = u_+ \quad \text{for all } u \in (\varphi_+^0 \setminus \{0\}), \\
h_+(t,u_+) = u_+ \quad \text{for all } t \in [0,1], \\
t \mapsto \varphi_+(h_+(t,u)) \quad \text{is decreasing for all } u \in (\varphi_+^0 \setminus \{0\}).
\end{gathered}$$

Moreover, the set $\varphi_+^0 \setminus \{0\}$ is contractible. We define

$$\gamma_+(t) = h_+(t, \tilde{\mu} u_1)$$

for all $t \in [0,1]$. Then $\gamma_+ \in C([0,1], X(\Omega))$ is a path joining $\tilde{\mu} u_1$ and u_+, such that $\varphi_+(\gamma_+(t)) < 0$ for all $t \in [0,1]$. Observing that $\varphi(u) \le \varphi_+(u)$ for all $u \in X(\Omega)$, we obtain

$$\varphi_+(u) - \varphi(u) = \int_\Omega (j(x,u) - j_+(x,u))\,dx = \int_{\{u<0\}} j(x,u)\,dx,$$

and the latter is non negative by (H2)(iv). Hence we obtain

$$\max_{t \in [0,1]} \varphi(\gamma_+(t)) < 0. \tag{3.13}$$

In the same way, we construct a path $\gamma_- \in C([0,1], X(\Omega))$ joining $-\tilde{\mu} u_1$ and u_-, such that

$$\max_{t \in [0,1]} \varphi(\gamma_-(t)) < 0. \tag{3.14}$$

Concatenating $\gamma_+, \tilde{\gamma}$ and γ_- (with a convenient changes of parameter) and using (3.12)-(3.14), we construct a path $\gamma \in \Gamma$ satisfying (3.9), against (3.7) and the definition of the mountain pass level c. Hence, we deduce that there exists a fourth critical point $\tilde{u} \in K(\varphi) \setminus \{0, u_+, u_-\}$, that is a nontrivial solution of (1.1). $\square$

Acknowledgements. S. Frassu was supported by the University of Cagliari within the framework of PlaceDoc Programme and the hospitality of the University of Aveiro during the research internship she spent within the Functional Analysis and Applications Group of CIDMA - the Center for Research and Development in Mathematics and Applications. S. Frassu is member of GNAMPA (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni) of INdAM (Istituto Nazionale di Alta Matematica "Francesco Severi").

E. M. Rocha, V. Staicu were supported by the Portuguese Foundation for Science and Technology (FCT), through CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2013.

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