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Containment Logics: Algebraic Completeness and Axiomatization

Abstract. The paper studies the containment companion (or, right variable inclusion companion) of a logic $\vdash$. This consists of the consequence relation $\vdash^r$ which satisfies all the inferences of $\vdash$, where the variables of the conclusion are *contained* into those of the set of premises, in case this is not inconsistent. In accordance with the work started in [10], we show that a different generalization of the Płonka sum construction, adapted from algebras to logical matrices, allows to provide a matrix-based semantics for containment logics. In particular, we provide an appropriate completeness theorem for a wide family of containment logics, and we show how to produce a complete Hilbert style axiomatization.

Keywords: Containment logic, Płonka sums, Abstract algebraic logic, Non-classical logics.

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1. Introduction

It is a recent discovery (see [9]) that the algebraic counterparts of weak Kleene logics are formed by a (subquasivarities of) regularized variety, whose members coincide with the *Płonka sum* of Boolean algebras, the algebraic semantics of propositional classical logic. Subsequently, the abstract construction of the Płonka sum of algebras has been generalized to logical matrices [10]. The main outcome is that the suggested notion provides an algebra-based semantics for a class of propositional logics, called *logics of left variable inclusion*, of which paraconsistent weak Kleene represents the most prominent example.

The logics in the weak Kleene "family"—essentially, Bochvar [8] and paraconsistent weak Kleene [29]—are syntactically characterized by imposing certain limitations on the inclusions of variables to classical propositional logic [14,50]. The extension of the construction of Płonka sums to logical matrices, introduced in [10], allows for an insightful investigation into the algebraic features of those logics, where the inclusion of variables runs from premises to conclusions. However, this is just one side of the coin of the *logics of variable inclusion*; the other side consisting of those logics verifying

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inferences in which the variables occurring in the conclusion are contained into the ones occurring in the premises. Consequence relations satisfying this feature are usually known as *containment logics*; the syntactic requirement which they share is a strengthened form of what Ferguson [24] understands as *Proscriptive Principle*, which also resembles the one defining logics of *analytic containment* [20,34].

The most famous example of containment logic is *Bochvar logic* B_3 [8], that is usually defined by a single matrix which features the presence of an infectious truth-value (a peculiarity shared by the twin-sister paraconsistent weak Kleene). B_3 has been successfully applied in different contexts: avoiding paradoxes in set-theory [8], modeling computer programs affected by errors [21] and non-sensical information databases [15], capturing the notion of truth in relation with on/off topic arguments [3].

The main condition defining containment logics mirrors the syntactic requirement defining logics of left variable inclusion. This work aims at answering the very natural question on whether it is possible to build a new generalization of the Płonka sum construction, suitable for obtaining a matrix semantics for containment logics. For this reason, the present paper may be understood as an ideal continuation of the path started in [10]. We also tried to closely match the structure and the theoretical framework of [10], in order to better underline how the intrinsic differences between the variable inclusion constraints at stake affect the algebraic treatment of these logics.

The paper is structured as follows. In Section 2, we recall all the preliminary notions needed to go through the reading of the whole paper. They basically consist of the basic notions of abstract algebraic logic and of the theory of Płonka sums. In Section 3, we formally introduce containment logics. By providing an adequate notion of Płonka sum for logical matrices, we obtain soundness and completeness for the *containment companion* $\vdash^r$ of an arbitrary (finitary) logic $\vdash$, with respect to the Płonka sum of the matrix models of $\vdash$. In Section 4, we focus on a specific (though very wide) class of logics, namely those possessing a binary term called partition function (a property shared by the vast majority of known logics). We provide a method for obtaining a Hilbert-style axiomatization for a logic $\vdash^r$ (Theorem 29) out of an axiomatization for (a finitary) logic $\vdash$. Finally, in Section 5, we put at work our machinery and characterize the axiomatization of containment companions of some well-known logics, namely of classical propositional logic, Belnap–Dunn and the Logic of Paradox.

2. Preliminaries

For standard background on universal algebra and abstract algebraic logic, we refer the reader, respectively, to [6], [11] and [27]. In this paper, algebraic languages are assumed not to contain constant symbols. Moreover, unless stated otherwise, we work within a fixed but arbitrary algebraic language. We denote algebras by $\mathbf{A}, \mathbf{B}, \mathbf{C} \ldots$ respectively with universes $A, B, C \ldots$. Let $\mathbf{Fm}$ be the algebra of formulas built up over a countably infinite set Var of variables (which we indicate by $x, y, z, \ldots$). Given a formula $\varphi \in Fm$, we denote by $Var(\varphi)$ the set of variables really occurring in φ. Similarly, given $\Gamma \subseteq Fm$, we set

$$Var(\Gamma) = \bigcup\{Var(\gamma)\colon \gamma \in \Gamma\}.$$

A *logic* is a substitution invariant consequence relation $\vdash \subseteq \mathcal{P}(Fm) \times Fm$ meaning that for every substitution $\sigma\colon \mathbf{Fm} \to \mathbf{Fm}$,

$$\text{if } \Gamma \vdash \varphi, \text{then } \sigma[\Gamma] \vdash \sigma(\varphi).$$

Given formulas φ, ψ, we write $\varphi \dashv\vdash \psi$ as a shorthand for $\varphi \vdash \psi$ and $\psi \vdash \varphi$. A logic $\vdash$ is *finitary* when for all $\Gamma \cup \{\varphi\} \subseteq Fm$:

$$\Gamma \vdash \varphi \iff \exists \Delta \subseteq \Gamma \text{ such that } \Delta \text{ is finite and } \Delta \vdash \varphi.$$

A *matrix* is a pair $\langle \mathbf{A}, F\rangle$ where $\mathbf{A}$ is an algebra and $F \subseteq A$. In this case, $\mathbf{A}$ is called the *algebraic reduct* of the matrix $\langle \mathbf{A}, F\rangle$.

Every class of matrices M defines a logic as follows:

$$\Gamma \vdash_{\mathsf{M}} \varphi \iff \text{for every } \langle \mathbf{A}, F\rangle \in \mathsf{M} \text{ and homomorphism } h\colon \mathbf{Fm} \to \mathbf{A}, \\ \text{if } h[\Gamma] \subseteq F, \text{then } h(\varphi) \in F.$$

We say that a logic $\vdash$ is *complete* with respect to a class of matrices M when $\vdash_{\mathsf{M}} = \vdash$. Sometimes, we will refer to such homomorphisms h as *evaluations*.

A matrix $\langle \mathbf{A}, F\rangle$ is a *model* of a logic $\vdash$ when

$$\text{if } \Gamma \vdash \varphi, \text{ then for every homomorphism } h\colon \mathbf{Fm} \to \mathbf{A}, \\ \text{if } h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F.$$

A set $F \subseteq A$ is a (deductive) *filter* of $\vdash$ on $\mathbf{A}$, or simply a $\vdash$-*filter*, when the matrix $\langle \mathbf{A}, F\rangle$ is a model of $\vdash$. We denote by $\mathcal{F}i_{\vdash}\mathbf{A}$ the set of all filters of $\vdash$ on $\mathbf{A}$.

Although the present paper does not address the study of reduced models (for containment logics), in order to make it self-contained, we recall those notions, concerning reduced models, that will be used. Let $\mathbf{A}$ be an algebra and $F \subseteq A$. A congruence θ of $\mathbf{A}$ is *compatible* with F when for every $a, b \in A$,

$$\text{if } a \in F \text{ and } \langle a, b\rangle \in \theta, \text{ then } b \in F.$$

The largest congruence of $\mathbf{A}$ which is compatible with F always exists, and is called the *Leibniz congruence* of F on $\mathbf{A}$. It is denoted by $\boldsymbol{\Omega}^{\mathbf{A}}F$. The *Suszko congruence* of F on $\mathbf{A}$, is defined as

$$\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}_{\vdash}F := \bigcap\{\boldsymbol{\Omega}^{\mathbf{A}}G : F \subseteq G \text{ and } G \in \mathcal{F}i_{\vdash}\mathbf{A}\}.$$

The Leibniz and Suszko congruences allow to single out a distinguished class of models of logics. More precisely, given a logic $\vdash$, we set

$$\begin{aligned}
\mathsf{Mod}(\vdash) &:= \{\langle \mathbf{A}, F\rangle : \langle \mathbf{A}, F\rangle \text{ is a model of } \vdash\};\\
\mathsf{Mod}^{*}(\vdash) &:= \{\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash) : \boldsymbol{\Omega}^{\mathbf{A}}F \text{ is the identity}\};\\
\mathsf{Mod}^{\mathrm{Su}}(\vdash) &:= \{\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash) : \widetilde{\boldsymbol{\Omega}}^{\mathbf{A}}_{\vdash}F \text{ is the identity}\}.
\end{aligned}$$

The above classes of matrices are called, respectively, the classes of models, *Leibniz reduced models* (or, simply reduced models), and *Suszko reduced models* of $\vdash$.

Given a logic $\vdash$, we set

$$\begin{aligned}
\mathsf{Alg}^{*}(\vdash) &= \{\mathbf{A} : \text{ there is } F \subseteq A \text{ s.t. } \langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{*}(\vdash)\}, \text{ and}\\
\mathsf{Alg}(\vdash) &= \{\mathbf{A} : \text{ there is } F \subseteq A \text{ s.t. } \langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)\}.
\end{aligned}$$

$\mathsf{Alg}(\vdash)$ is the class of algebraic reducts of matrices in $\mathsf{Mod}^{\mathrm{Su}}(\vdash)$. The class $\mathsf{Alg}(\vdash)$ is called the *algebraic counterpart* of $\vdash$ as, for the vast majority of logics $\vdash$, $\mathsf{Alg}(\vdash)$ is the class of algebras intuitively associated with $\vdash$.

Trivial matrices have a central role in the whole paper. We say that a matrix $\langle \mathbf{A}, F\rangle$ is *trivial* if $F = A$. We denote by $\langle \mathbf{1}, \{1\}\rangle$ the trivial matrix, whose algebraic reduct $\mathbf{1}$ is the trivial algebra. Observe that the latter matrix is a model (resp. reduced, Suszko reduced model) of every logic. Moreover, if $\vdash$ is a logic and $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)$ is a trivial matrix, then $\langle \mathbf{A}, F\rangle = \langle \mathbf{1}, \{1\}\rangle$. A set of models of a logic $\vdash$ is said to be non trivial, if it does not contain trivial matrices. We indicate by $\mathsf{Mod}_{+}(\vdash)$ the set of non trivial models of a logic $\vdash$.

Płonka Sums

As standard references on Płonka sums we mention [38,39,41]. A *semilattice* is an algebra $\mathbf{A} = \langle A, \vee\rangle$, where $\vee$ is a binary associative, commutative and idempotent operation. Given a semilattice $\mathbf{A}$ and $a, b \in A$, we set

$$a \leq b \iff a \vee b = b.$$

It is easy to see that $\leq$ is a partial order on A.

Definition 1. A *direct system of algebras* consists of:

1. a semilattice $I = \langle I, \vee \rangle$;
2. a family of similar algebras $\{\mathbf{A}_i : i \in I\}$ with pairwise disjoint universes;
3. a homomorphism $f_{ij} \colon \mathbf{A}_i \to \mathbf{A}_j$, for every $i, j \in I$ such that $i \leq j$.

Moreover, f_{ii} is the identity map for every $i \in I$, and $f_{ik} = f_{jk} \circ f_{ij}$, for $i \leq j \leq k$.

Let X be a direct system of algebras as defined above. The *Płonka sum* of X, in symbols $\mathcal{P}_{\l}(X)$ or $\mathcal{P}_{\l}(\mathbf{A}_i)_{i \in I}$,[1] is the algebra in the same type defined as follows: the universe of $\mathcal{P}_{\l}(\mathbf{A}_i)_{i \in I}$ is the union $\bigcup_{i \in I} A_i$. Moreover, for every n-ary basic operation g and $a_1, \ldots, a_n \in \bigcup_{i \in I} A_i$, we set

$$g^{\mathcal{P}_{\l}(\mathbf{A}_i)_{i \in I}}(a_1, \ldots, a_n) \coloneqq g^{\mathbf{A}_j}(f_{i_1 j}(a_1), \ldots, f_{i_n j}(a_n)),$$

where $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$ and $j = i_1 \vee \cdots \vee i_n$.

Observe that if in the above display we replace g by any complex formula φ in n variables, we still have that

$$\varphi^{\mathcal{P}_{\l}(\mathbf{A}_i)_{i \in I}}(a_1, \ldots, a_n) = \varphi^{\mathbf{A}_j}(f_{i_1 j}(a_1), \ldots, f_{i_n j}(a_n)).$$

Notation: Given a formula φ, we will often write $\varphi^{\mathcal{P}_{\l}}$ instead of $\varphi^{\mathcal{P}_{\l}(\mathbf{A}_i)_{i \in I}}$ when no confusion shall occur.

The theory of Płonka sums is strictly related with a special kind of binary operation, called a partition function.

Definition 2. Let $\mathbf{A}$ be an algebra of type ν. A function $\cdot \colon A^2 \to A$ is a *partition function* in $\mathbf{A}$ if the following conditions are satisfied for all $a, b, c \in A$, $a_1, \ldots, a_n \in A$ and for any operation $g \in \nu$ of arity $n \geqslant 1$.

$\mathbf{P}_1$ $a \cdot a = a$;

$\mathbf{P}_2$ $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;

$\mathbf{P}_3$ $a \cdot (b \cdot c) = a \cdot (c \cdot b)$;

$\mathbf{P}_4$ $g(a_1, \ldots, a_n) \cdot b = g(a_1 \cdot b, \ldots, a_n \cdot b)$;

$\mathbf{P}_5$ $b \cdot g(a_1, \ldots, a_n) = b \cdot a_1 \cdot_{\ldots} \cdot a_n$.

[1]When no confusion shall occur, we will write $\mathcal{P}_{\l}(\mathbf{A}_i)$ instead of $\mathcal{P}_{\l}(\mathbf{A}_i)_{i \in I}$.

Different definitions of partition function appeared in the literature. We adopted the one which uses the minimal number of defining conditions (see [41]).

The next result underlines the connection between Płonka sums and partition functions:

THEOREM 3. [38, Thm. II] *Let* $\mathbf{A}$ *be an algebra of type* ν *with a partition function* $\cdot$. *The following conditions hold:*

(1) *A can be partitioned into $\{A_i : i \in I\}$ where any two elements $a, b \in A$ belong to the same component A_i exactly when*

$$a = a \cdot b \text{ and } b = b \cdot a.$$

Moreover, every A_i is the universe of a subalgebra $\mathbf{A}_i$ of $\mathbf{A}$.

(2) *The relation $\leq$ on I given by the rule*

$$i \leq j \iff \text{there exist } a \in A_i, b \in A_j \text{ s.t. } b \cdot a = b$$

is a semilattice order.

(3) *For all $i, j \in I$ such that $i \leq j$ and $b \in A_j$, the map $f_{ij} \colon A_i \to A_j$, defined by the rule $f_{ij}(x) = x \cdot b$ is a homomorphism. The definition of f_{ij} is independent from the choice of b, since $a \cdot b = a \cdot c$, for all $a \in A_i$ and $c \in A_j$.*

(4) *$Y = \langle\langle I, \leq\rangle, \{\mathbf{A}_i\}_{i\in I}, \{f_{ij} : i \leq j\}\rangle$ is a direct system of algebras such that $\mathcal{P}_{\l}(Y) = \mathbf{A}$.*

The statement of Theorem 3 displayed above relies on the assumption that the algebraic language contains no constant symbols.[2] It is worth remarking that the construction of Płonka sums preserves the validity of *regular identities*, i.e. identities of the form $\varphi \approx \psi$ such that $Var(\varphi) = Var(\psi)$.

3. Algebraic Completeness

The usual presentations of Kleene three-valued logics divide them into two families, depending on the meaning given to the connectives $\wedge, \vee$: *strong logics*—including Strong Kleene and the Logic of Paradox [43]—and *weak logics*—Bochvar logic (B_3) and paraconsistent weak Kleene (or Halldén logic).

[2]When considering types containing constants, then additional conditions should be added to the definition of partition function. This results into a decomposition over a semilattice having a least element: constants of the Płonka sum will belong to the algebra whose index is the least element. For further details, see [40].

$\wedge$	0	1/2	1
0	0	1/2	0
1/2	1/2	1/2	1/2
1	0	1/2	1

$\vee$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1/2
1	1	1/2	1

$\neg$	
1	0
1/2	1/2
0	1

Figure 1. The algebra **WK** of weak Kleene tables

Logics in each family differentiate upon the choice of the truth-set: $\{1\}$ in Strong Kleene and Bochvar, $\{1, \frac{1}{2}\}$ in the logic of Paradox and paraconsistent weak Kleene. Bochvar [8] is the logic induced by the matrix $\langle \mathbf{WK}, \{1\}\rangle$ of the so-called weak Kleene tables[3] displayed in Figure 1.

It is not difficult to check that the algebra **WK** is the Płonka sum of the two-element Boolean algebra $\mathbf{B}_2$ and the trivial (Boolean) algebra $\mathbf{\frac{1}{2}}$ (over the index set given by the two-element semilattice).[4]

Bochvar logic can be equivalently presented as follows.

THEOREM 4. [50, Theorem 2.3.1] *The following are equivalent:*

(1) $\Gamma \vdash_{\mathrm{B}_3} \varphi$*;*

(2) $\Gamma \vdash_{\mathrm{CL}} \varphi$ *with* $Var(\varphi) \subseteq Var(\Gamma)$ *or* Γ *is an inconsistent set of formulas.*

In other words, Bochvar logic is the consequence relation obtained out of classical logic, imposing the constraint that variables of the conclusion (formula) shall be included into those of the set of premises, when the latter is not inconsistent. For this reason, B_3 is often referred to as a *containment logic*, see for e.g. [24,36].

The following definition originates in [33] (but see also [13,47]) and generalizes the notion of inconsistent set to an arbitrary logic $\vdash$.

DEFINITION 5. A set of formulas Σ is an antitheorem of a logic $\vdash$ if $\sigma[\Sigma] \vdash \varphi$, for every substitution $\sigma\colon \mathbf{Fm} \to \mathbf{Fm}$ and formula φ.

Observe that, if the set $\Sigma(y_1, \ldots, y_n)$, where the variables $y_1, \ldots, y_n$ really occur, is an antitheorem for $\vdash$, then, by substitution, also $\Sigma(x)$ (where only x occurs) is an antitheorem for $\vdash$. In other words, if a logic $\vdash$ possesses an antitheorem Σ, then it possesses an antitheorem in one variable only. When

[3] In accordance with [50], here we are actually considering Bochvar "internal calculus", which is only one of the two logics introduced in [8]. The "external calculus" consists of a linguistic extension of the weak Kleene tables, with a connective t, interpreted (for every evaluation h) as $h(\mathrm{t}\varphi) = 1$ if and only if $h(\varphi) = 1$ (for further details, see [8,25,32]).

[4] We refer the reader interested in further details directly to [9].

referring to this fact, we will write $\Sigma(x)$. The most intuitive example one can keep in mind is the following: for any formula φ, the set $\{\varphi, \neg\varphi\}$ is an antitheorem of intuitionistic, classical and both local and global modal logics.

The above presented characterization of Bochvar logic suggests that a logic $\vdash^r$ satisfying an analogous criterion on the inclusion of variables as that of Theorem 4 can be associated to any arbitrary logic $\vdash$.

Definition 6. Let $\vdash$ be a logic. $\vdash^r$ is the logic defined as

$$\Gamma \vdash^r \varphi \iff \begin{cases} \Gamma \vdash \varphi \text{ and } Var(\varphi) \subseteq Var(\Gamma) \text{ or} \\ \Sigma \subseteq \Gamma, \end{cases}$$

where Σ is an antitheorem of $\vdash$.

We will refer to $\vdash^r$ as the *containment companion* (or, *right variable inclusion companion*) of the logic $\vdash$. A comment on Definition 6 is in order.[5] Observe, at first, that it follows from the definition that $\vdash^r$ and $\vdash$ have the same antitheorems. This motivates our choice of inserting the condition on antitheorems for defining the containment companion ($\vdash^r$) of a logic $\vdash$. Indeed, the "dual" notion of logic of left variable inclusion ($\vdash^l$), considered in [10]—where the inclusion of variables works from premises into conclusion (so, from left to right)—has the same theorems of $\vdash$. This feature is dually restored for inclusion of variables from conclusion to premises (right to left) by requiring $\vdash^r$ to have the same antitheorems of $\vdash$. It is useful to remark that, for logics possessing no antitheorems (as, for instance, Belnap–Dunn or the Logic of Paradox), the containment companion of a logic coincides with the "analytic fragment" (see, for instance, [23]).

Bochvar logic is not the unique example of containment logic that can be found in literature. Indeed, the logic $\mathbf{S}_{\text{fde}}$, introduced by Deutsch [18] (see also [4,16,48]), can be counted as the containment companion of the Logic of Paradox: a result that has firstly been shown in [23, Observation 9] (and that, also, follows from our analysis, see Subsection 5.3). Moreover, the logic $\mathbf{FDE}_{\varphi}$, introduced by Priest [44], and, independently by Daniels [17], is actually the "containment companion" of Belnap–Dunn logic (on which we will come back in Section 5.2). This fact is actually proved in [22, Theorem 28].[6] Also one of the four-valued logics introduced by Tomova (see Example 14) is a containment logic, more precisely the containment companion of PWK.

[5] We thank an anonymous reviewer for suggesting a clarification on this matter.

[6] We thank an anonymous reviewer for suggesting this literature, of whose existence we were not aware.

LEMMA 7. *Let $\vdash$ be a finitary logic. Then $\vdash^r$ is finitary.*

PROOF. Suppose that $\Gamma \vdash^r \varphi$. By definition of $\vdash^r$, two cases are to be considered:

i) $\Gamma \vdash \varphi$ with $Var(\varphi) \subseteq Var(\Gamma)$;

ii) Γ contains an antitheorem Σ of $\vdash$.

i) Clearly, $\mid Var(\varphi) \mid < \infty$, hence there exist formulas $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $Var(\varphi) \subseteq Var(\gamma_1) \cup \cdots \cup Var(\gamma_n)$. Since $\vdash$ is finitary, then there exists a finite set $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \varphi$. If $Var(\varphi) \subseteq Var(\Gamma')$, then also $\Gamma' \vdash^r \varphi$, i.e. $\vdash^r$ is finitary. So, suppose that $Var(\varphi) \nsubseteq Var(\Gamma')$. Consider $\Delta = \Gamma' \cup \{\gamma_1, \ldots, \gamma_n\}$. Obviously, Δ is finite, $Var(\varphi) \subseteq Var(\Delta)$ and $\Delta \subseteq \Gamma$. By monotonicity of $\vdash$, $\Delta \vdash \varphi$, hence $\Delta \vdash^r \varphi$, i.e. $\vdash^r$ is finitary.

ii) Since Σ is an antitheorem for $\vdash$, then $\Sigma \vdash \varphi$. Hence $\Sigma \vdash_r \varphi$ and Σ is finite (as $\vdash$ is finitary). ■

Since the algebra **WK** is a Płonka sum (of Boolean algebras), it makes sense to ask whether the matrix $\langle \mathbf{WK}, \{1\} \rangle$ can be constructed as Płonka sum of (two) matrices. To the best of our knowledge, the construction of Płonka sums between matrices has been developed exclusively in [10]. However, it is not difficult to check that the above-mentioned construction, when applied to the matrices $\langle \mathbf{B}_2, \{1\} \rangle$ and $\langle \mathbf{\frac{1}{2}}, \emptyset \rangle$ (where $\mathbf{B}_2$ and $\mathbf{\frac{1}{2}}$ stand for the two-element Boolean algebra and the trivial algebra, respectively), does not result in $\langle \mathbf{WK}, \{1\} \rangle$. This suggests that a different notion of direct system of logical matrices shall be introduced.

DEFINITION 8. An *r-direct system* of matrices consists of:

(i) A semilattice $I = \langle I, \vee \rangle$.

(ii) A family of matrices $\{\langle \mathbf{A}_i, F_i \rangle : i \in I\}$ such that $I^+ \coloneqq \{i \in I : F_i \neq \emptyset\}$ is a sub-semilattice of I.

(iii) a homomorphism $f_{ij} \colon \mathbf{A}_i \to \mathbf{A}_j$, for every $i, j \in I$ such that $i \leq j$, satisfying also that:

- f_{ii} is the identity map, for every $i \in I$;
- if $i \leq j \leq k$, then $f_{ik} = f_{jk} \circ f_{ij}$;
- if $F_j \neq \emptyset$ then $f_{ij}^{-1}[F_j] = F_i$, for any $i \leq j$.

As the nomenclature highlights, the above introduced notion of direct system of matrices is essentially different from the one in [10]. The main difference concerns the interplay between homomorphisms of the system and filters of matrices.

Given a r-direct system of matrices X, we define a new matrix as

$$\mathcal{P}_{\ell}(X) \coloneqq \langle \mathcal{P}_{\ell}(\mathbf{A}_i)_{i\in I}, \bigcup_{i\in I} F_i\rangle.$$

We will refer to the matrix $\mathcal{P}_{\ell}(X)$ as the *Płonka sum* over the r-direct system of matrices X. Given a class M of matrices, $\mathcal{P}_{\ell}(\mathsf{M})$ will denote the class of all Płonka sums of r-direct systems of matrices in M.

Let $h\colon \mathbf{Fm} \to \mathcal{P}_{\ell}(\mathbf{A}_i)$ be a homomorphism from the formula algebra into a generic Płonka sum of algebras. Then, for any formula $\varphi \in Fm$, we set

$$i_h(\varphi) \coloneqq \bigvee\{i \in I : h(x) \in A_i, x \in Var(\varphi)\}.$$

In words, $i_h(\varphi)$ indicates the index where the formula φ is interpreted by the homomorphism h, into a Płonka sum. Moreover, for any $\Gamma \subseteq Fm$, we set $i_h(\Gamma) \coloneqq \bigvee\{i_h(x)\colon x \in Var(\Gamma)\}$.

Remark 9. Notice that the index $i_h(\Gamma)$ is defined provided that the set $Var(\Gamma)$ is finite. For several results, we will assume that the logic $\vdash$ is finitary. Hence, by Lemma 7, also $\vdash^r$ is finitary, and this allows us to consider finite sets $\Gamma \subseteq Fm$, for which the existence of $i_h(\Gamma)$ is assured. Moreover, observe that, for every homomorphism $h\colon \mathbf{Fm} \to \mathcal{P}_{\ell}(X)$ from the formula algebra into a generic Płonka sum over an r-direct system of matrices X, and every $\Gamma \cup \{\varphi\} \subseteq Fm$, it is immediate to check that $Var(\varphi) \subseteq Var(\Gamma)$ implies $i_h(\varphi) \leq i_h(\Gamma)$.

Lemma 10. *Let X be an r-direct system of non trivial models of a finitary logic $\vdash$. Then $\mathcal{P}_{\ell}(X) = \langle \mathcal{P}_{\ell}(\mathbf{A}_i), \bigcup_{i\in I} F_i\rangle$ is a model of $\vdash^r$.*

Proof. Let X be an r-direct system of non trivial models of $\vdash$. Assume $\Gamma \vdash^r \varphi$. Since $\vdash$ is finitary, so it is also $\vdash^r$ (by Lemma 7), there exists a finite subset $\Delta \subseteq \Gamma$, such that $\Delta \vdash^r \varphi$. We distinguish the following cases:

(a) $\Sigma \subseteq \Delta$, where Σ is an antitheorem of $\vdash$;

(b) $\Delta \vdash \varphi$ with $Var(\varphi) \subseteq Var(\Delta)$.

Since X contains non-trivial models only, the case (a) easily follows by noticing that, for any homomorphism $h\colon \mathbf{Fm} \to \mathcal{P}_{\ell}(\mathbf{A}_i)$, $h[\Sigma] \nsubseteq F = \bigcup_{i\in I} F_i$. Therefore $\Sigma \vdash_{\mathcal{P}_{\ell}(X)} \varphi$, hence also $\Delta \vdash_{\mathcal{P}_{\ell}(X)} \varphi$.
Suppose (b) is the case, i.e. $\Delta \vdash \varphi$ with $Var(\varphi) \subseteq Var(\Delta)$. Let $h\colon \mathbf{Fm} \to \mathcal{P}_{\ell}(\mathbf{A}_i)$ be a homomorphism such that $h[\Delta] \subseteq F$. Since Δ is a finite set, then we can fix $j \coloneqq i_h(\Delta)$ and, for any formula $\delta \in \Delta$, we have $h(\delta) \in F_{i_h(\delta)}$. This implies that each $i_h(\delta) \in I^+$ and, as I^+ forms a sub-semilattice of I, we have that $j \in I^+$.

Now, define $g\colon \mathbf{Fm} \to \mathbf{A}_j$ as

$$g(x) \coloneqq f_{i_h(x)j} \circ h(x),$$

for every $x \in Var(\Delta)$. For any $\delta \in \Delta$, we have $g(\delta) = f_{i_h(\delta)j} \circ h(\delta)$, hence $g[\Delta] \subseteq F_j$. From the fact that $\Delta \vdash \varphi$ and $\langle \mathbf{A}_j, F_j \rangle \in \mathsf{Mod}(\vdash)$, it follows that $g(\varphi) \in F_j$. Setting $k \coloneqq i_h(\varphi)$, by Remark 9, we have $k \leq j$ and this, together with the observation that $F_j \neq \emptyset$, implies $f_{kj}^{-1}[F_j] = F_k$. Moreover, we claim that $F_k \neq \emptyset$. Suppose, by contradiction, that $F_k = \emptyset$. Then, by definition of r-direct system of matrices, we have that $f_{kj}^{-1}[F_j] = \emptyset$, that is: there exists no $a \in A_k$ such that $f_{kj}(a) \in F_j$. On the other hand, since $Var(\varphi) \subseteq Var(\Delta)$, then $g(\varphi) = f_{kj} \circ h(\varphi) \in F_j$, a contradiction.

From the fact that $g(\varphi) \in F_j$ together with $f_{kj}^{-1}[F_j] = F_k$, we conclude $h(\varphi) \in F_k$. This proves that $h(\varphi) \in F_k \subseteq \bigcup_{i \in I} F_i$. ■

Remark 11. Observe that the assumption on the non-triviality of models of the logic $\vdash$ is crucial in Lemma 10, as witnessed by the following example. Let $\vdash$ be a theoremless logic possessing an antitheorem Σ (an example is the almost inconsistent logic). Set $X = \langle \mathbf{A} \oplus \mathbf{1}, A \rangle$ to be the r-direct system of models of $\vdash$, consisting of the two algebras $\mathbf{A}$ and $\mathbf{1}$ with the unique homomorphism $f\colon \mathbf{A} \to \mathbf{1}$ (plus the identity homomorphisms). Recall that an antitheorem can be expressed in one varible only, thus $\Sigma(x) \vdash y$, for an arbitrary variable y, and therefore $\Sigma(x) \vdash^r y$. However, $\mathcal{P}_{\l}(X)$ is not a model of the latter inference (consider, for instance, an evaluation $v\colon \mathbf{Fm} \to \mathcal{P}_{\l}(\mathbf{A} \oplus \mathbf{1})$ such that $v(x) = a \in A$ and $v(y) = 1$).

Observe that, if the logic $\vdash$ does not possess an antitheorem, then the following holds:

Corollary 12. *Let X be an r-direct system of models of a finitary logic $\vdash$ possessing no antitheorems. Then $\mathcal{P}_{\l}(X)$ is a model of $\vdash^r$.*

Given a logic $\vdash$ which is complete with respect to a class M of matrices, we set $\mathsf{M}^{\emptyset} \coloneqq \mathsf{M} \cup \langle \mathbf{A}, \emptyset \rangle$, for any arbitrary $\mathbf{A} \in \mathsf{Alg}(\vdash)$.

Theorem 13. *Let $\vdash$ be a finitary logic which is complete with respect to a class of non trivial matrices M. Then $\vdash^r$ is complete with respect to $\mathcal{P}_{\l}(\mathsf{M}^{\emptyset})$.*

Proof. We aim at showing that $\vdash^r = \vdash_{\mathcal{P}_{\l}(\mathsf{M}^{\emptyset})}$.
($\vdash^r \subseteq \vdash_{\mathcal{P}_{\l}(\mathsf{M}^{\emptyset})}$). Consider $\Gamma \vdash^r \varphi$ and a Płonka sum $\langle \mathcal{P}_{\l}(\mathbf{A}_i), \bigcup_{i \in I} F_i \rangle$ of matrices in $\mathsf{M}^{\emptyset}$. Set $\mathbf{A} = \mathcal{P}_{\l}(\mathbf{A}_i)$ The cases in which Γ contains an antitheorem of $\vdash$ or $\langle \mathbf{A}, \emptyset \rangle$ is a model of $\vdash$ follow by Lemma 10.

So, assume $\langle \mathbf{A}, \emptyset \rangle$ is not a model of $\vdash$ and that Γ does not contain an antitheorem of $\vdash$. Let $h\colon \mathbf{Fm} \to \mathbf{A}$ be a homomorphism such that

$h[\Gamma] \subseteq \bigcup_{i\in I} F_i$. Suppose, in view of a contradiction, that $h(\varphi) \notin \bigcup_{i\in I} F_i$. Set $i_h(\varphi) = j$ and $i_h(\Gamma) = k$; since $Var(\varphi) \subseteq Var(\Gamma)$ then $j \leq k$, by Remark 9. We define a homomorphism $v\colon \mathbf{Fm} \to \mathbf{A}_k$, as follows

$$v(x) := f_{lk} \circ h(x),$$

where $l = i_h(x)$. Clearly, $v[\Gamma] = f_{kk} \circ h[\Gamma] = h[\Gamma] \subseteq F_k$ and $v(\varphi) = f_{jk} \circ h(\varphi) \in A_k \smallsetminus F_k$, since $h(\varphi) \in A_j \smallsetminus F_j$ and $F_j = f_{jk}^{-1}[F_k]$ (as we know that $F_k \neq \emptyset$). Therefore, we have $\Gamma \not\vdash \varphi$, which is a contradiction.
($\vdash_{\mathcal{P}_l(\mathsf{M}^{\emptyset})} \subseteq \vdash^r$). By contraposition, we prove that $\Gamma \nvdash^r \varphi$ implies $\Gamma \nvdash_{\mathcal{P}_l(\mathsf{M}^{\emptyset})} \varphi$. To this end, assume $\Gamma \nvdash^r \varphi$. If $Var(\varphi) \subseteq Var(\Gamma)$, clearly $\Gamma \nvdash \varphi$. Therefore there exists a matrix $\langle \mathbf{A}_i, F_i\rangle \in \mathsf{M}$ and a homomorphism $h\colon \mathbf{Fm} \to \mathbf{A}_i$ such that $h[\Gamma] \subseteq F_i$ and $h(\varphi) \notin F_i$. Upon considering the r-direct system $X = \langle\langle \mathbf{A}_i, F_i\rangle, \{i\}, id\rangle$ and the homomorphism h, we immediately obtain $\Gamma \nvdash_{\mathcal{P}_l(\mathsf{M}^{\emptyset})} \varphi$.

The only other case to consider is $Var(\varphi) \nsubseteq Var(\Gamma)$. Preliminarily, observe that the assumption $\Gamma \nvdash^r \varphi$ implies that Γ contains no antitheorem Σ for $\vdash$. Therefore, since M is a class of models complete with respect to $\vdash$, there exists a matrix $\langle \mathbf{B}, G\rangle \in \mathsf{M}$ and a homomorphism $v\colon \mathbf{Fm} \to \mathbf{B}$ such that $v[\Gamma] \subseteq G$ and $v(\varphi) \notin G$. Consider any r-direct system formed by the matrices $\langle \mathbf{B}, G\rangle$ and $\langle \mathbf{A}, \emptyset\rangle$ for an appropriate $\mathbf{A} \in \mathsf{Alg}(\vdash)$ (the choice $\mathbf{A} = \mathbf{1}$ is always appropriate), with $\langle \mathbf{A}, \emptyset\rangle$ indexed as top element. Denote by $\mathbf{B} \oplus \mathbf{A}^{\emptyset}$ a Płonka sum over the r-direct system just described.

The homomorphism $g\colon \mathbf{Fm} \to \mathbf{B} \oplus \mathbf{A}^{\emptyset}$ defined as

$$g(x) := \begin{cases} v(x) & \text{if } x \in Var(\Gamma), \\ a & \text{otherwise.} \end{cases}$$

for arbitrary $a \in A$ easily witnesses $\Gamma \nvdash_{\mathcal{P}_l(\mathsf{M}^{\emptyset})} \varphi$, as desired. ■

EXAMPLE 14. Let $\mathbf{K_4} = \langle \{0, 1, \frac{1}{2}, n\}, \neg, \wedge, \vee\rangle$ be the algebra given by the following tables

$\neg$	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
n	n
0	1

$\wedge$	0	$\frac{1}{2}$	n	1
0	0	$\frac{1}{2}$	n	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	n	$\frac{1}{2}$
n	n	n	n	n
1	0	$\frac{1}{2}$	n	1

$\vee$	0	$\frac{1}{2}$	n	1
0	0	$\frac{1}{2}$	n	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	n	$\frac{1}{2}$
n	n	n	n	n
1	1	$\frac{1}{2}$	n	1

The logic $\mathbf{K^w_{4n}} = \langle \mathbf{K_4}, \{1, \frac{1}{2}\}\rangle$ is included among the four-valued regular logics counted by Tomova (see [37,49]). Observe that $\langle \mathbf{K_4}, \{1, \frac{1}{2}\}\rangle$ is the Płonka sum (over the r-direct system) of the matrices $\langle \mathbf{WK}, \{1, \frac{1}{2}\}\rangle$ and

$\langle \mathbf{n}, \emptyset \rangle$. Since PWK is complete with respect to $\langle \mathbf{WK}, \{1, \frac{1}{2}\} \rangle$, then, it follows by Theorem 13, that $\mathbf{K^w_{4n}} = \vdash^r_{\mathsf{PWK}}$, i.e. $\mathbf{K^w_{4n}}$ is the containment companion of PWK. ■

As PWK is the *left variable inclusion* companion of classical logic (see [10]), the above example shows that the constructions yielding the (two) companions (left variable inclusion and containment) can actually be iterated, in alternation, starting from an arbitrary logic $\vdash$ (for further details see [42]).

REMARK 15. Observe that if $\vdash$ is a logic which is complete with respect to a finite set of finite matrices, then so is $\vdash^r$ (by Theorem 13). This means that the containment companion of a logic $\vdash$ preserves "finite valuedness", a notion introduced and studied in [12].

Theorem 13 provides a complete class of matrices for an arbitrary logic of right variable inclusion. This class is obtained performing Płonka sums over r-direct systems of models of $\vdash$ together with the matrices $\langle \mathbf{A}, \emptyset \rangle$ for any $\mathbf{A} \in \mathsf{Alg}(\vdash)$. Obviously, it is not generally the case that the matrix $\langle \mathbf{A}, \emptyset \rangle$ is a model of a logic $\vdash$. For this reason, it is not always true that Płonka sums over an r-direct systems of models of $\vdash$ provide a complete matrix semantics for $\vdash^r$. In this sense, the right variable inclusion companion of a logic is a logic of "Płonka sums" (of matrices) in a weaker sense compared to the case of the left variable inclusion companion, fully described in [10]. Nonetheless, if $\langle \mathbf{1}, \emptyset \rangle \in \mathsf{Mod}(\vdash)$, the correspondence between $\vdash^r$ and Płonka sums is fully recovered. This is actually the case of every theoremless logic, such as Strong Kleene Logic, or $\vdash^{\wedge,\vee}_{\mathrm{CL}}$, the conjunction and disjunction fragment of classical logic.

COROLLARY 16. *A finitary containment logic $\vdash^r$ is complete w.r.t. any of the following classes of matrices:*

$$\mathcal{P}_{\ell}(\mathsf{Mod}_{+}(\vdash) \cup \langle \mathbf{A}, \emptyset \rangle),\ \mathcal{P}_{\ell}(\mathsf{Mod}^{*}_{+}(\vdash) \cup \langle \mathbf{A}, \emptyset \rangle),\ \mathcal{P}_{\ell}(\mathsf{Mod}^{\mathrm{Su}}_{+}(\vdash) \cup \langle \mathbf{A}, \emptyset \rangle),$$

for $\mathbf{A} \in \mathsf{Alg}(\vdash)$*.*

Moreover, observing that if $\langle \mathbf{1}, \emptyset \rangle \in \mathsf{Mod}(\vdash)$ then $\langle \mathbf{1}, \emptyset \rangle \in \mathsf{Mod}^{*}(\vdash)$, the following holds

COROLLARY 17. *Let $\vdash$ be a finitary logic such that $\langle \mathbf{1}, \emptyset \rangle \in \mathsf{Mod}(\vdash)$. Then $\vdash^r$ is complete w.r.t. any of the following classes of matrices:*

$$\mathcal{P}_{\ell}(\mathsf{Mod}_{+}(\vdash)),\ \mathcal{P}_{\ell}(\mathsf{Mod}^{*}_{+}(\vdash)),\ \mathcal{P}_{\ell}(\mathsf{Mod}^{\mathrm{Su}}_{+}(\vdash)).$$

In case $\vdash$ does not possess antitheorems, then the above corollaries can be restated as follows

Corollary 18. *Let $\vdash$ be a finitary logic without antitheorems. Then $\vdash^r$ is complete w.r.t. any of the following classes of matrices:*

$$\mathcal{P}_{\ell}(\mathsf{Mod}(\vdash) \cup \langle \mathbf{A}, \emptyset \rangle),\ \mathcal{P}_{\ell}(\mathsf{Mod}^{*}(\vdash) \cup \langle \mathbf{A}, \emptyset \rangle),\ \mathcal{P}_{\ell}(\mathsf{Mod}^{\mathrm{Su}}(\vdash) \cup \langle \mathbf{A}, \emptyset \rangle),$$

for any $\mathbf{A} \in \mathsf{Alg}(\vdash)$.

Corollary 19. *Let $\vdash$ a finitary logic without antitheorems such that $\langle \mathbf{1}, \emptyset \rangle \in \mathsf{Mod}(\vdash)$, then $\vdash^r$ is complete w.r.t. any of the following classes of matrices:*

$$\mathcal{P}_{\ell}(\mathsf{Mod}(\vdash)),\ \mathcal{P}_{\ell}(\mathsf{Mod}^{*}(\vdash)),\ \mathcal{P}_{\ell}(\mathsf{Mod}^{\mathrm{Su}}(\vdash)).$$

4. Hilbert Style Calculi (For Logics with r-Partition Function)

Partitions functions, which have been defined for algebras (see Definition 2), can be defined also for logics.[7]

Definition 20. A logic $\vdash$ has an *r-partition function* if there is a formula $x * y$, in which the variables x and y really occur, such that:

(i) $x, y \vdash x * y$,

(ii) $x * y \vdash x$,

(iii) $\varphi(\varepsilon, \vec{z}\,) \dashv\vdash \varphi(\delta, \vec{z}\,)$,

for every formula $\varphi(v, \vec{z}\,)$ and every identity of the form $\varepsilon \approx \delta$ in Definition 2.

Condition (iii) in the Definition of r-partition function is actually equivalent to say that the term operation $*$ is a partition function in every algebra $\mathbf{A} \in \mathsf{Alg}(\vdash)$. This is the consequence of the following (known) fact in abstract algebraic logic.

Lemma 21. *Let $\vdash$ be a logic and $\varepsilon, \delta \in Fm$. The following are equivalent:*

(i) $\mathsf{Alg}(\vdash) \vDash \varepsilon \approx \delta$*;*

(ii) $\varphi(\varepsilon, \vec{z}\,) \dashv\vdash \varphi(\delta, \vec{z}\,)$*, for every formula* $\varphi(v, \vec{z}\,)$*.*

Proof. See [27, Lemma 5.74(1)] and [27, Theorem 5.76]. ■

[7] Partition functions have actually been defined for logics of left variable inclusions in [10]. The definition given here is obviously different (as highlighted by the nomenclature), as it takes into account the instrinsic difference between right and left variable inclusion constraints.

From now on, we will denote both the formula $x * y$ and the term operation $*$ as r-partition functions with respect to a logic $\vdash$.

The definition of partition functions for an arbitrary logic is introduced also in [10, Definition 16]. It shall be noticed that Definition 20 is essentially different (this also motivates the choice of the terminology *r*-partition function). Nevertheless, in most cases (for instance, all substructural logics, classical and modal logics) the very same formula plays both the role of a r-partition function and of a partition function in the sense of [10, Definition 16].

EXAMPLE 22. Logics with an r-partition function abound in the literature. Indeed, the term $x * y \coloneqq x \wedge (x \vee y)$ is a partition function for every logic $\vdash$ such that $\mathsf{Alg}(\vdash)$ has a lattice reduct. Such examples include all modal and substructural logics [28]. On the other hand, the term $x * y \coloneqq (y \to y) \to x$ as an r-partition function for all the logics $\vdash$ whose class $\mathsf{Alg}(\vdash)$ possesses a Hilbert algebra (see [19]) or a BCK algebra (see [30]) reduct.

REMARK 23. It is easily checked that a logic $\vdash$ has r-partition function $*$ if and only if $\vdash^r$ has r-partition function $*$.

In the following, we extend Płonka representation theorem to r-direct systems of logical matrices.

THEOREM 24. *Let $\vdash$ be a logic with r-partition function $*$, and $\langle \mathbf{A}, F\rangle$ be a model of $\vdash$ such that $\mathbf{A} \in \mathsf{Alg}(\vdash)$. Then Theorem 3 holds for $\mathbf{A}$. Moreover, by setting $F_i \coloneqq F \cap A_i$ for every $i \in I$, the triple*

$$X = \langle \langle I, \leq \rangle, \{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}, \{f_{ij} : i \leq j\}\rangle$$

is an r-direct system of matrices such that $\mathcal{P}_{\nmid}(X) = \langle \mathbf{A}, F\rangle$.

PROOF. Theorem 3 holds for $\mathbf{A}$, by simply observing that $*$ is a partition function for $\mathbf{A}$. For the remaining part, it will be enough to show:

(a) for every $i, j \in I$ such that $i \leq j$, if $F_j \neq \emptyset$ then $f_{ij}^{-1}[F_j] = F_i$;

(b) I^+ is a sub-semilattice of I.

In order to prove (a), consider $i, j \in I$ such that $i \leq j$ and let F_j be non-empty. Assume, in view of a contradiction, that $f_{ij}^{-1}[F_j] \neq F_i$. This implies that $F_i \nsubseteq f_{ij}^{-1}[F_j]$ or that $f_{ij}^{-1}[F_j] \nsubseteq F_i$. The first case immediately leads to the contradiction that $x, y \nvdash x * y$, while the second case contradicts $x * y \vdash x$. This proves (a).

In order to prove (b), consider $i, j \in I^+$ and let $k = i \vee j$, with $i, j, k \in I$. As $*$ is a *r*-partition function for $\vdash$, $x, y \vdash x * y$. Since $i, j \in I^+$, then F_i

and F_j are non-empty, therefore there exist two elements $a \in F_i$, $b \in F_j$. We have $a *^{\mathbf{A}} b = f_{ik}(a) *^{\mathbf{A}_k} f_{jk}(b) \in A_k$. This, together with the fact that $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}(\vdash)$ implies $a * b \in F_k$, i.e. $F_k \neq \emptyset$. So $k \in I^+$ and this proves (b). ∎

Given a logic $\vdash$ with an r-partition function $*$ and a model $\langle \mathbf{A}, F \rangle$ of $\vdash$ such that $\mathbf{A} \in \mathsf{Alg}(\vdash)$, we call *Płonka fibers* of $\langle \mathbf{A}, F \rangle$ the members of the class of matrices $\{\langle \mathbf{A}_i, F_i \rangle\}_{i \in I}$ given by the decomposition in Theorem 24. From now on, when considering a model $\langle \mathbf{A}, F \rangle$ of a logic $\vdash$ with r-partition function, we will assume that $\langle \mathbf{A}, F \rangle = \mathcal{P}_{\mathit{l}}(X)$, for a given direct system $X = \langle \langle I, \leq \rangle, \{\langle \mathbf{A}_i, F_i \rangle\}_{i \in I}, \{f_{ij} : i \leq j\} \rangle$, without explicitly mentioning the r-direct system X.

Lemma 25. *Let $\vdash^r$ be a logic with r-partition function $*$, and $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}(\vdash^r)$, with $\mathbf{A} \in \mathsf{Alg}(\vdash^r)$. Then, the Płonka fibers $\langle \mathbf{A}_i, F_i \rangle$, such that $i \in I^+$, are models of $\vdash$.*

Proof. Let $\Gamma \vdash \varphi$ and suppose, by contradiction, that there exists a matrix $\langle \mathbf{A}_j, F_j \rangle$, with $j \in I^+$, and a homomorphism $h \colon \mathbf{Fm} \to \mathbf{A}_j$ such that $h[\Gamma] \subseteq F_j$ and $h(\varphi) \notin F_j$. Preliminarily, observe that $Var(\varphi) \nsubseteq Var(\Gamma)$ and, moreover, if $\vdash$ has an antitheorem Σ, then $\Sigma \nsubseteq \Gamma$, for otherwise $\Gamma \vdash^r \varphi$, which is in contradiction with our assumption that $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}(\vdash^r)$. Denote by X the (non-empty) set of variables occurring in φ but not in Γ and, for $\gamma \in \Gamma$, let $X_\gamma \coloneqq \{\gamma * x : x \in X\}$ and $\Gamma^-_\gamma \coloneqq \Gamma \smallsetminus \{\gamma\}$. Since $*$ is an r-partition function for $\vdash^r$, we have $\gamma * x \vdash^r \gamma$. Therefore $\gamma * x \vdash \gamma$ and $X_\gamma \vdash \gamma$, which implies $X_\gamma, \Gamma^-_\gamma \vdash \varphi$, for any $\gamma \in \Gamma$. Observe that $Var(\varphi) \subseteq Var(X_\gamma) \cup Var(\Gamma^-_\gamma)$, hence $X_\gamma, \Gamma^-_\gamma \vdash^r \varphi$.

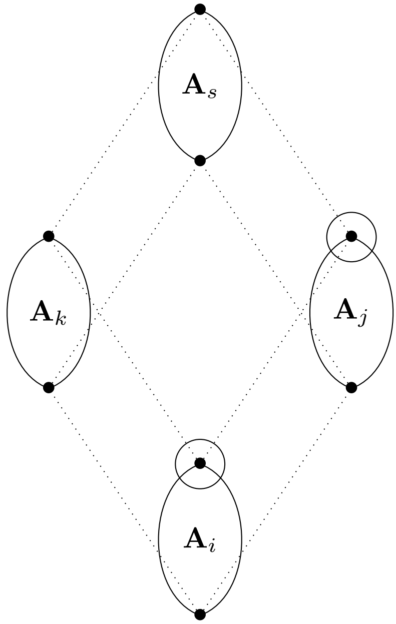
Since $h(\gamma), h(\varphi) \in A_j$ and $x \in Var(\varphi)$, for every $x \in X$, we have that $h(\gamma * x) = h(\gamma)$, whence $h[X_\gamma] = h(\gamma)$. Now, for an arbitrary $a \in A$, we define a homomorphism $g \colon \mathbf{Fm} \to \mathbf{A}$, as follows

$$g(x) \coloneqq \begin{cases} h(x) \text{ if } \ x \in Var(\Gamma) \cup Var(\varphi) \\ a \text{ otherwise.} \end{cases}$$

We have $g[X_\gamma] = h[X_\gamma] = h(\gamma) \in F_j$, $g[\Gamma^-_\gamma] = h[\Gamma^-_\gamma] \in F_j$ and $g(\varphi) = h(\varphi) \notin F_j$. A contradiction. ∎

The following example provides a simple instance of the above Lemma 25.

Example 26. Consider the matrix $\langle \mathbf{B}, F \rangle$ constructed as Płonka sum of the (non-trivial) Boolean algebras $\mathbf{A}_i, \mathbf{A}_j, \mathbf{A}_k, \mathbf{A}_s$ over the r-direct system depicted below (the index set is the four-element Boolean algebra): circles indicate filters, consisting of the top elements $1_i, 1_j$ of $\mathbf{A}_i, \mathbf{A}_j$, respectively. Thus, $F = \{1_i, 1_j\}$. Dotted lines represent arbitrary Płonka homomorphisms.



By Theorem 13, $\langle \mathbf{B}, F \rangle$ is a model of Bochvar logic B_3. It can be easily checked that $\mathbf{B} \in \mathsf{Alg}(\mathsf{B}_3)$. Moreover, $I^+ = \{i, j\}$ and clearly $\langle \mathbf{A}_i, 1_i \rangle, \langle \mathbf{A}_j, 1_j \rangle \in \mathsf{Mod}(\mathrm{CL})$. ∎

The presence of an r-partition function yields an important syntactic consequence: it allows to adapt a Hilbert style calculus of a logic $\vdash$ into a calculus, for its containment companion $\vdash^r$. Despite $\vdash^r$ is defined via a linguistic restriction constraint (on the inclusion of variables), the axiomatization that we obtain is free of any (linguistic) restriction. Throughout the remaining part of this section, we implicitly assume that the logic $\vdash$ possesses an antitheorem. Our analysis can be easily adapted to the case where $\vdash$ does not have antitheorems (see Remark 30).

In what follows, by a *Hilbert-style calculus with finite rules*, we understand a (possibly infinite) set of Hilbert-style rules, each of which has finitely many premises.

Definition 27. Let $\mathcal{H}$ be a Hilbert-style calculus with finite rules, which determines a logic $\vdash$ with a r-partition function $*$ and an antitheorem Σ. Let $\mathcal{H}^r$ be the Hilbert-style calculus given by the following rules:

$$\alpha * \varphi \rhd \varphi \tag{H0}$$

$$\alpha, \beta \rhd \alpha * \beta \tag{H1}$$

$$\alpha * \beta \rhd \alpha \tag{H2}$$

$$\{\gamma_1, \ldots, \gamma_n\} \smallsetminus \{\gamma_i\}, \gamma_i * \psi \rhd \psi \tag{H3}$$

$$\Sigma \rhd \lambda \tag{H4}$$

$$\chi(\delta, \vec{z}\,) \lhd \rhd \chi(\varepsilon, \vec{z}\,) \tag{H5}$$

for every

(i) $\rhd\varphi$ axiom in $\mathcal{H}$;

(ii) $\gamma_1, \ldots, \gamma_n \rhd \psi$ rule in $\mathcal{H}$ (and γ_i such that $i \in \{i, \ldots, n\}$);

(iii) $\delta \approx \varepsilon$ equation in the definition of partition function, and formula $\chi(v, \vec{z}\,)$.

LEMMA 28. *Let $\vdash$ be a logic with a r-partition function $*$, an antitheorem Σ and let $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash_{\mathcal{H}^r})$. Then:*

(i) $\langle \mathbf{A}, F\rangle = \mathcal{P}_{\!l}(X)$, where $X = \langle\langle I, \leq\rangle, \{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}, \{f_{ij} \colon i \leq j\}\rangle$ is an r-direct system of matrices;

(ii) if X contains a trivial matrix then $\mathbf{A} = \mathbf{1}$.

PROOF. (i) Since $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash_{\mathcal{H}^r})$, $\mathbf{A} \in \mathsf{Alg}(\vdash_{\mathcal{H}^r})$. Moreover, observe that $*$ is a r-partition function for $\vdash_{\mathcal{H}^r}$ (thanks to conditions (H1), (H2), (H5)). These facts, together with Theorem 24, imply that $\langle \mathbf{A}, F\rangle = \mathcal{P}_{\!l}(X)$, where $X = \langle\{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}, \{f_{ij} : i \leq j\}, \langle I, \leq\rangle\rangle$ is an r-direct system of matrices.

(ii) Suppose that, for some $j \in I$, $\langle \mathbf{A}_j, F_j\rangle$ is a trivial fiber of $\langle \mathbf{A}, F\rangle$, i.e. $F_j = A_j$. Since $\Sigma(x)$ is an antitheorem (for $\vdash$) and (H4) is a rule of $\mathcal{H}^r$, then, for every $i \in I$, we have $A_i = F_i$, i.e. each fiber is trivial. Indeed, if there exists a non trivial fiber $\langle \mathbf{A}_k, F_k\rangle$ and an element $c \in A_k \smallsetminus F_k$, then the evaluation $h\colon \mathbf{Fm} \to \mathbf{A}$, defined as $h(x) = a$, $h(y) = c$ (for an arbitrary $a \in \mathbf{A}_j$) is such that $h[\Sigma(x)] \subseteq F$ while $h(y) \notin F$, against the fact that $\Sigma(x) \vdash_{\mathcal{H}^r} y$. Moreover, the facts that each fiber is trivial and that $\widetilde{\boldsymbol{\Omega}}^{\mathbf{A}} F = id$ immediately imply $\mathbf{A} = \mathbf{1}$. ∎

THEOREM 29. *Let $\vdash$ be a finitary logic with an r-partition function $*$ and an antitheorem Σ. Let, moreover, $\mathcal{H}$ be a Hilbert style calculus with finite rules. If $\mathcal{H}$ is complete for $\vdash$, then $\mathcal{H}^r$ is complete for $\vdash^r$.*

PROOF. Let us denote with $\vdash_{\mathcal{H}^r}$ the logic defined by $\mathcal{H}^r$. We show that $\vdash_{\mathcal{H}^r} = \vdash^r$.

($\subseteq$). It is immediate to check that every rule of $\mathcal{H}^r$ holds in $\vdash^r$.

($\supseteq$). We now show that $\mathsf{Mod}^{\mathrm{Su}}(\vdash_{\mathcal{H}^r}) \subseteq \mathsf{Mod}(\vdash^r)$. So let $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}$ $(\vdash_{\mathcal{H}^r})$. By Lemma 28-(i), we know that $\langle \mathbf{A}, F\rangle \cong \mathcal{P}_{\mathfrak{l}}(X)$, where $X = \langle\{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}, \{f_{ij} : i \leq j\}, \langle I, \leq\rangle\rangle$ is an r-direct system of matrices. The fact that the matrix $\langle \mathbf{A}_i, F_i\rangle \in \mathsf{Mod}(\vdash_{\mathcal{H}})$ for each $i \in I^+$ can be proved on the ground of (H0) and (H3) by adapting the proof of Lemma 25 to the calculus $\mathcal{H}^r$. Recalling that $\mathcal{H}$ is complete for $\vdash$ we obtain that $\langle \mathbf{A}_i, F_i\rangle \in \mathsf{Mod}(\vdash)$, for each $i \in I^+$. Moreover, by Lemma 28-(ii), we know that if X contains a trivial matrix $\langle \mathbf{A}_j, F_j\rangle$, then $\mathbf{A} = \mathbf{1}$. Therefore, two cases may arise: (1) $\mathbf{A} = \mathbf{1}$, (2) X contains no trivial fibers. If (1), then clearly $\langle \mathbf{A}, F\rangle \in \{\langle \mathbf{1}, \emptyset\rangle, \langle \mathbf{1}, \{1\}\rangle\}$. As $\vdash^r$ is a theoremless logic $\{\langle \mathbf{1}, \emptyset\rangle, \langle \mathbf{1}, 1\rangle\} \subseteq \mathsf{Mod}(\vdash^r)$. If (2), then we can apply Lemma 10, so $\langle \mathbf{A}, F\rangle = \mathcal{P}_{\mathfrak{l}}(X) \in \mathsf{Mod}(\vdash^r)$. ■

Remark 30. It is easy to check that if the logic $\vdash$ does not possess antitheorems, then a Hilbert-style calculus for $\vdash^r$ can be defined by simply dropping condition (H4) from Definition 27. The completeness of $\vdash^r$ with respect to such a calculus can be proven by adapting the strategy in the proof of Theorem 29.

5. Examples of Axiomatizations

In this last section, we show how to obtain Hilbert-style axiomatizations of some containment logics.

5.1. Bochvar Logic

Bochvar logic is the containment companion of classical logic. Consider the following Hilbert-style axiomatization of classical propositional logic:

($\mathbf{CL}_1$) $\rhd \varphi \to \varphi$

($\mathbf{CL}_2$) $\rhd \varphi \to (\psi \to \varphi)$

($\mathbf{CL}_3$) $\rhd \varphi \to (\psi \to \chi) \to (\varphi \to \psi) \to (\varphi \to \chi)$

($\mathbf{CL}_4$) $\rhd (\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

($\mathbf{CL}_5$) $\varphi, \varphi \to \psi \rhd \varphi$

Theorem 29 allows to provide the following complete Hilbert style calculus for Bochvar logic B_3.

($\mathbf{CL}_1^r$) $\alpha * (\varphi \to \varphi) \rhd \varphi \to \varphi$

($\mathbf{CL}_2^r$) $\alpha * (\varphi \to (\psi \to \varphi) \rhd \varphi \to (\psi \to \varphi)$

($\mathbf{CL}_3^r$) $\alpha * (\varphi \to (\psi \to \chi) \to (\varphi \to \psi) \to (\varphi \to \chi)) \rhd \varphi \to (\psi \to \chi) \to (\varphi \to \psi) \to (\varphi \to \chi)$

($\mathbf{CL}_4^r$) $\alpha * ((\neg\varphi \to \neg\psi) \to (\psi \to \varphi)) \rhd (\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$

($\mathbf{CL}_5^r$) $\varphi * \psi, \varphi \to \psi \rhd \psi$

($\mathbf{CL}_6^r$) $\varphi, \psi \rhd \varphi * \psi$

($\mathbf{CL}_7^r$) $\varphi * \psi \rhd \varphi$

($\mathbf{CL}_8^r$) $\alpha, \neg\alpha \rhd \varphi$

($\mathbf{CL}_9^r$) $\chi(\delta, \vec{z}\,) \lhd \rhd \chi(\varepsilon, \vec{z}\,)$ for every formula $\chi(x, \vec{z})$ and equation $\delta \approx \varepsilon$ in Definition 2,

where $\varphi * \psi$ is an abbreviation for $\varphi \wedge (\varphi \vee \psi)$.

5.2. The Containment Companion of Belnap–Dunn

Belnap–Dunn B is a logic originally introduced as *First Degree Entailment* within the research enterprise on relevance and entailment logic [2,5].

Consider the algebraic language of type 1, 2, 2, containing $\neg, \vee, \wedge$. Recall that a *De Morgan lattice* is an algebra $\mathbf{A} = \langle A, \neg, \vee, \wedge\rangle$ of type 1, 2, 2 such that:

(i) $\langle A, \wedge, \vee\rangle$ is a distributive lattice;

(ii) $\neg$ satisfies the following equations:

$$x \approx \neg\neg x, \qquad \neg(x \wedge y) \approx \neg x \vee \neg y, \qquad \neg(x \vee y) \approx \neg x \wedge \neg y.$$

De Morgan lattices, originally introduced by Moisil [35] and, independently, by Kalman [31] (under the name of *distributive i-lattices*) form a variety, which is generated by the four element algebra $\mathbf{M}_4=\langle\{0, b, n, 1\}, \neg, \vee, \wedge\rangle$, whose lattice reduct is displayed in Figure 2 and negation in the following table:

$\neg$	
1	0
b	b
n	n
0	1

B is the logic induced by the matrix $\langle \mathbf{M}_4, \{1, b\}\rangle$ (or, equivalently, by $\langle \mathbf{M}_4, \{1, n\}\rangle$, see [26, Proposition 2.3]). B is finitary and theoremless (purely inferential). Moreover, the class $\mathsf{Alg}(\mathsf{B})$ coincides with the variety of De

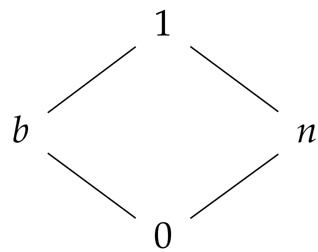


Figure 2. Hasse diagram of the De Morgan lattice $\mathbf{M}_4$.

Morgan lattices [26, Theorem 4.1]. Observe that the set $\{\varphi, \neg\varphi\}$ is not an antitheorem of B (indeed $\varphi, \neg\varphi \nvdash_{\mathsf{B}} \psi$). It is not difficult to check that B does not possess antitheorems.

Recall that a *lattice filter* of a De Morgan lattice $\mathbf{A}$ is a subset $F \subseteq A$ such that $x \wedge y \in F$ if and only if $x \in F$ and $y \in F$. The class of matrices

$$\mathsf{M} = \{\langle \mathbf{A}, F\rangle : \ \mathbf{A} \text{ De Morgan algebra, } F \subseteq A \text{ lattice filter}\}$$

is complete for B [26, Corollary 2.6]. Observe that $\langle \mathbf{A}, \emptyset\rangle \in \mathsf{M}$, for any De Morgan lattice $\mathbf{A}$. As already mentioned, the containment companion of Belnap–Dunn is the logic $\mathbf{FDE}_{\varphi}$ introduced, independently, in [44] and [17] (the fact that $\vdash_{\mathbf{FDE}_{\varphi}} = \vdash^{r}_{\mathsf{B}}$ is proven in [22]). $\mathbf{FDE}_{\varphi}$ is introduced in [44] as the logic induced by the matrix $\langle \mathbf{M}_4 \oplus \mathbf{e}, \{1, b\}\rangle$, where $\mathbf{M}_4 \oplus \mathbf{e}$ is the Płonka sum of $\mathbf{M}_4$ with the trivial algebra $\mathbf{e}$. From our analysis (see Theorem 13), it follows that $\vdash^{r}_{\mathsf{B}}$ is complete with respect to $\mathcal{P}_{\ell}(\mathsf{M})$, i.e. the class of all Płonka sums over r-direct systems of matrices in M.

We present the Hilbert-style axiomatization for B which is introduced in [26] (and, independently in [45]). Since B is theoremless, the calculus has no axioms and the following rules:

(B_1) $\varphi \wedge \psi \rhd \varphi$;

(B_2) $\varphi \wedge \psi \rhd \psi$;

(B_3) $\varphi, \psi \rhd \varphi \wedge \psi$;

(B_4) $\varphi \rhd \varphi \vee \psi$;

(B_5) $\varphi \vee \psi \rhd \psi \vee \varphi$;

(B_6) $\varphi \vee \varphi \rhd \varphi$;

(B_7) $\varphi \vee (\psi \vee \chi) \rhd (\varphi \vee \psi) \vee \chi$;

(B_8) $\varphi \vee (\psi \wedge \chi) \rhd (\varphi \vee \psi) \wedge (\varphi \vee \chi)$;

(B_9) $(\varphi \vee \psi) \wedge (\varphi \vee \chi) \rhd \varphi \vee (\psi \wedge \chi)$;

(B_{10}) $\varphi \vee \psi \rhd \neg\neg\varphi \vee \psi$;

(B_{11}) $\neg\neg\varphi \vee \psi \rhd \varphi \vee \psi$

(B_{12}) $\neg(\varphi \vee \psi) \vee \chi \rhd (\neg\varphi \wedge \neg\psi) \vee \chi$;

(B_{13}) $(\neg\varphi \wedge \neg\psi) \vee \chi \rhd \neg(\varphi \vee \psi) \vee \chi$;

(B_{14}) $\neg(\varphi \wedge \psi) \vee \chi \rhd (\neg\varphi \vee \neg\psi) \vee \chi$;

(B_{15}) $(\neg\varphi \vee \neg\psi) \vee \chi \rhd \neg(\varphi \wedge \psi) \vee \chi$.

A Hilbert-style axiomatization of $\vdash^r_{\mathsf{B}}$, (see Definition 27 and Theorem 29) is given by the following ($\varphi * \psi$ is an abbreviation for $\varphi \wedge (\varphi \vee \psi)$):

(B^r_1) $\varphi, \psi \rhd \varphi * \psi$;

(B^r_2) $\varphi * \psi \rhd \varphi$;

(B^r_3) $(\varphi \wedge \psi) * \varphi \rhd \varphi$;

(B^r_4) $(\varphi \wedge \psi) * \psi \rhd \psi$;

(B^r_5) $\varphi * (\varphi \wedge \psi), \psi \rhd \varphi \wedge \psi$;

(B^r_6) $\varphi, \psi * (\varphi \wedge \psi) \rhd \varphi \wedge \psi$;

(B^r_7) $\varphi * (\varphi \vee \psi) \rhd \varphi \vee \psi$;

(B^r_8) $(\varphi \vee \psi) * (\psi \vee \varphi) \rhd \psi \vee \varphi$;

(B^r_9) $(\varphi \vee \varphi) * \varphi \rhd \varphi$;

(B^r_{10}) $(\varphi \vee (\psi \vee \chi)) * ((\varphi \vee \psi) \vee \chi) \rhd (\varphi \vee \psi) \vee \chi$;

(B^r_{11}) $\varphi \vee (\psi \wedge \chi) * ((\varphi \vee \psi) \wedge (\varphi \vee \chi)) \rhd (\varphi \vee \psi) \wedge (\varphi \vee \chi)$;

(B^r_{12}) $((\varphi \vee \psi) \wedge (\varphi \vee \chi)) * (\varphi \vee (\psi \wedge \chi)) \rhd \varphi \vee (\psi \wedge \chi)$;

(B^r_{13}) $(\varphi \vee \psi) * (\neg\neg\varphi \vee \psi) \rhd \neg\neg\varphi \vee \psi$;

(B^r_{14}) $(\neg\neg\varphi \vee \psi) * (\varphi \vee \psi) \rhd \varphi \vee \psi$;

(B^r_{15}) $(\neg(\varphi \vee \psi) \vee \chi) * ((\neg\varphi \wedge \neg\psi) \vee \chi) \rhd (\neg\varphi \wedge \neg\psi) \vee \chi$;

(B^r_{16}) $((\neg\varphi \wedge \neg\psi) \vee \chi) * (\neg(\varphi \vee \psi) \vee \chi) \rhd \neg(\varphi \vee \psi) \vee \chi$;

(B^r_{17}) $(\neg(\varphi \wedge \psi) \vee \chi) * ((\neg\varphi \vee \neg\psi) \vee \chi) \rhd (\neg\varphi \vee \neg\psi) \vee \chi$;

(B^r_{18}) $((\neg\varphi \vee \neg\psi) \vee \chi) * (\neg(\varphi \wedge \psi) \vee \chi) \rhd \neg(\varphi \wedge \psi) \vee \chi$;

(B^r_{19}) $\chi(\delta, \vec{z}\,) \lhd \rhd \chi(\varepsilon, \vec{z}\,)$ for every formula $\chi(x, \vec{z})$ and equation $\delta \approx \varepsilon$ in Definition 2.

5.3. The Relevance Logic $\mathbf{S}_{\mathbf{fde}}$

The logic $\mathbf{S}_{\mathrm{fde}}$ has been introduced by Deutsch [18]: it is induced the matrix $\langle \mathbf{S}_4, \{1, \frac{1}{2}\}\rangle$, whose algebraic reduct $\mathbf{S}_4 = \langle\{0, \frac{1}{2}, m, 1\}, \neg, \wedge, \vee\rangle$ is given in the following tables.

$\neg$	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
m	m
0	1

$\wedge$	0	$\frac{1}{2}$	m	1
0	0	0	m	0
$\frac{1}{2}$	0	$\frac{1}{2}$	m	$\frac{1}{2}$
m	m	m	m	m
1	0	$\frac{1}{2}$	m	1

$\vee$	0	$\frac{1}{2}$	m	1
0	0	$\frac{1}{2}$	m	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	m	1
m	m	m	m	m
1	1	1	m	1

Recall that a *Kleene lattice* is a De Morgan lattice satisfying $x \wedge \neg x \leq y \vee \neg y$. Kleene lattices form a variety (KL), generated by the 3-element algebra $\mathbf{SK} = \langle\{0, 1, \frac{1}{2}\}, \neg, \vee, \wedge\rangle$, which is a subalgebra of $\mathbf{S}_4$ (and also isomorphic to the two three-element subalgebras of $\mathbf{M}_4$).

The logic of Paradox LP (see [7,43,46]) is defined by the matrix $\langle \mathbf{SK}, \{1, \frac{1}{2}\}\rangle$. The algebraic counterpart of LP is exactly the variety of Kleene lattice, i.e. $\mathsf{KL} = \mathsf{Alg}(\mathsf{LP})$. Ferguson [23] showed that $\vdash_{\mathsf{S}_{\mathrm{fde}}} = \vdash^r_{\mathsf{LP}}$. A fact that also follows from Theorem 13, by observing that the matrix $\langle \mathbf{S}_4, \{1, \frac{1}{2}\}\rangle$ is the Płonka sum over the r-direct system of the two matrices $\langle \mathbf{SK}, \{1, \frac{1}{2}\}\rangle$ and $\langle \mathbf{m}, \emptyset\rangle$.

A finite Hilbert style calculus for LP (see for instance [1]) can be obtained by adding the axiom

(LP_1) $\rhd \varphi \vee \neg\varphi$

to the calculus for the logic B described above. Therefore, by Theorem 29, the calculus consisting of (B^r_1) – (B^r_{19}) and

(LP^r_1) $\alpha * \varphi \vee \neg\varphi \rhd \varphi \vee \neg\varphi$

is complete for $\mathbf{S}_{\mathrm{fde}}$.

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