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Keywords: Variable structure systems; Sliding mode control; Consensus control; Multi-agent systems; Distributed control; Large scale systems; Robust control; Finite-time control; Discontinuous systems; Nonsmooth analysis; Filippov theory

On the Variable Structure Control Approach with Sliding Modes to Robust Finite-Time Consensus Problems: a Methodological Overview Based on Nonsmooth Analysis

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Abstract

When controlling cyber-physical systems via consensus algorithms, the robustness issue is of paramount importance because of model mismatching and/or disturbances that generally modify the Laplacian flow dynamics associated to the overall network, compromising the possibility to achieve any expected, orchestrated emergent behavior. To face this issue, the Variable Structure Control (VSC) approach has recently been shown to be an effective tool in designing control protocols over networks, providing robustness and often yielding finite-time convergence. Moreover, VSC further enables the decoupling of simultaneous control objectives where reaching a consensus state is only part of a more complex task, for instance as it happens in distributed optimization. Thus motivated, this paper overviews, from a tutorial perspective, some selected recent advances in the application of VSC with Sliding Modes to design robust consensus controllers in both the leader-less and the leader-following settings. Efforts are also made to unify the notation and to discuss the theoretical foundations of the nonsmooth analysis tools at the basis of their design for a wide readership unfamiliar with these problems and formal tools. To this aim, examples supporting the treatment, and numerical simulations, are given and discussed in detail. Finally, hints for future investigations along with some current open problems are provided.

Keywords: Variable structure systems, sliding mode control, consensus control, multi-agent systems, distributed control, large scale systems, robust control, finite-time control, discontinuous systems, nonsmooth analysis, Filippov theory
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1. Introduction

The innovation and progress in communication technologies and the extensive utilization of pervasive communication systems changed the way how we connect to the real world (Huang et al. (2021)). This trend towards more connectivity also laid the foundations to significantly modify the *paradigm* of analysis and control of dynamical systems so that nowadays it is common to refer to a *multi-agent system* (MAS) whenever multi-variable coupled dynamical systems are dealt with (Chen et al. (2019)).

At the origin, the MAS paradigm was referred to *software decision-making entities* (or simply *agents*) tasked to decrease the computational overhead of *large-scale algorithms* and, using certain *individual* computational and sensing capabilities to perform, in a discrete-time setting, and over an interaction *graph* G, parallel calculations until they reach a common *agreement*. Such a decision-making process is commonly referred to *consensus problem* (Wooldridge (2009)).

The consensus problem is widely recognized as a challenging task due to the incomplete or inconsistent information that agents have about one another and the interaction network, both at the communication and physical levels. Furthermore, the interaction networks may be subject to unmodeled phenomena such as disturbances, failures, and delays, which can further complicate the coordination of actions among agents (Chen et al. (2019)). It is worth noting that the different types of consensus problems are classified based on the operating assumptions and requirements of the system. Among these, two main classes of problems stand out: the *leaderless* and *leader-following* consensus problems.

Leaderless problems refer to tasks in which a set of peer agents interacts with neighbors to reach a common agreement on a certain quantity, without relying on any higher-level or central authority. This quantity may be unknown a-priori and distributedly computed through their interaction. Some early applications of the leaderless paradigm to networked control and identification problems include measurement average estimation in wireless sensor networks (Olfati-Saber et al. (2007)),

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distributed Kalman filtering (Carli et al. (2008)), optimizations (Nedic & Ozdaglar (2009)), and the Google PageRank computation (Ishii & Tempo (2010)).

To improve the reliability and flexibility of geographically distributed control architectures, consensus problems have been extended to the continuous-time setting, primarily for the real-time control of physically coupled actuated agents operating over a communication network (Dörfler et al. (2015)).

Contemporary, and remaining within this *cyber-physical* setting, the *leader-following consensus* originated as an extension of *Vicsek's birth flocking model* (Jadbabaie et al. (2003)). This involves introducing a designated leader agent to coordinate a set of peer mobile robots to track the leader state while maintaining the consensus among themselves, and by utilizing only local sensing and interaction rules.

Thus the three main characteristics of consensus control schemes can easily be recognized:

- *autonomy*, which refers to each agent being partially independent, self-aware, or autonomous;
- *local views*, where no agent has a global view of the network due to its complexity or local privacy and communication constraints. Therefore, only relative standpoints concerning neighboring agents and the environment are available;
- *decentralization*, the lack of a monolithic decision-making scheme removes problems, such as time-scale separation latency, and enhances the flexibility, responsiveness and reconfigurability of the control system;
- *self-organization*, because by exploiting ad-hoc designed individual strategies, even simple for each agent, and local interactions over a communication network, the MAS is enabled to give rise to a desired *emergent behavior*.

These advancements gave rise to various control applications where agents are not limited to being only abstract software entities (Wooldridge (2009)) or mobile robots (Oh et al. (2015)). Instead, they are effective controlled systems that interact or compete over an interaction network, where couplings can be either physical or virtualized, and the distinction no longer matters (Chen et al. (2019)). Examples of these applications include inverter-based control of power generators in DC (Cucuzzella et al. (2018)) and AC microgrids (Simpson-Porco et al. (2013); Pilloni et al. (2018)), synchronization of mechanical oscillators (Pavlov et al. (2022)) and infinite-dimensional thermal systems (Pilloni et al. (2015)), modeling and control of hydraulic networks (Lee et al. (2017)), and more recently, applications to chemical reactors (Dorneanu et al. (2018, 2020)). This rapid development of applications, coupled with the implementation of contemporary control standards such as the Internet of Things (IoT) and Industry 4.0 (Lin et al. (2022); Pilloni et al. (2017b); Dorneanu et al. (2020)), has brought to light issues of resilience and robustness for the overall cyber-physical networked system (Bodkhe et al. (2020)).

Ensuring *robustness* to consensus controllers is a crucial issue in real-world problems (Zambelli & Ferrara (2021)).

On the one hand, the robustness against agents' misfunctioning or deliberately uncooperative behaviors has to be considered. The latter includes both the case in which the misfunctioning agent is oblivious and the case it is acting intentionally in a harmful way (refer to Ishii et al. (2022); Franceschelli et al. (2016), as well as Section 4).

On the other hand, ensuring the network's robustness to perturbations and uncertainties is equally crucial. Indeed, as also discussed in later sections, consensus protocols require a well-defined eigenstructure for the networked dynamics. If this is absent or corrupted, no protocol can function as intended anymore (Pilloni et al. (2020)).

To achieve *robust consensus* in MAS, various robust control techniques have been investigated. These include stochastic control to tolerate either packet dropping, switching communications (Boyd et al. (2006)), or measurement noise (Li & Zhang (2009)), as well as H_∞, H_2 and Markovian jump approaches to reduce the effect of exogenous disturbances, or parameter uncertainties, respectively (Shi & Johansson (2013); Zhang et al. (2021)) and (Xu et al. (2020)). Then, nonlinear control techniques have also been employed to handle nonlinearities as well as achieve disturbances rejection. For example, event-triggered control has been used to achieve semi-global robust consensus characteristics (Meng et al. (2019)), and the theory of nonlinear output regulation has been applied to output synchronization problems (Isidori et al. (2014)). Finally, the Variable Structure Control (VSC) with Sliding Modes (SM) has proven to be particularly effective due to its simplicity of implementation, robustness, and finite-time convergence properties, and without requiring any a-priory disturbance model (except for the knowledge of their raw bounds). In addition, it has been shown that VSC with Sliding Modes (SM) can address either more challenging or new coordination problems. Examples are: achieving the leader-follower consensus without requiring the global availability of the leader trajectory (cf. Song et al. (2010) with Cao & Ren (2010)), or in the leaderless context perform robust distributed optimization tasks (Pilloni et al. (2020)), and also outlining new coordination problems as for the *midrange* (Cortés (2006)) and the *median consensus* tasks in either closed (Franceschelli et al. (2016) and open (Sanai Dashti et al. (2019)) networks. Additionally, it is worth noting the diverse range of applications where sliding mode consensus controllers have been implemented, including UAV control (Wu et al. (2020)), freeway traffic control (Ferrara et al. (2022)), DC (Cucuzzella et al. (2018)) and AC (Pilloni et al. (2018)) microgrid control, even in the presence of communication delays (Gholami et al. (2021)), averaging problems in sensor networks (Franceschelli et al. (2014b)), and distributed tracking in infinite-dimensional thermal systems (Orlov et al. (2016)) as well as Pan-Tilt camera sensor networks (Kameoka et al. (2022)).

1.1. The VSC approach with SMs

The beginnings of Variable Structure Control (VSC) can be traced back to the studies conducted by Russian researchers, with its initial applications appearing in the 1930s in Russia (Utkin & Chang (2002)). Subsequently, Utkin (1977) can be considered the promoter of Sliding Mode Control (SMC), a specific type of VSC, to the global scientific community.

The idea behind the VSC can easily be explained by comparing it with the linear regulator design for the single-input system $\dot{x} = Ax + bu$. There, the regulator is $u = r^\intercal x$, where $r \in \mathbb{R}^d$ can be selected accordingly to various techniques, and its structure is fixed. Contrarily, the VSC is allowed to change its *structure*, that is, to *switch*, at any instant from one to another member of a set of possible continuous functions of the state. As an example, and following Lyapunov arguments, the switching function can be picked to be, e.g. a linear ($\sigma = s^\intercal x$ (Utkin (2013))) or nonlinear ($\sigma(x)$ (Polyakov et al. (2015))) state function, and

$$u = \begin{cases} u^+(x) & \text{if } \sigma > 0 \\ u^-(x) & \text{if } \sigma < 0 \end{cases} \; : \; \frac{1}{2}\frac{d\sigma^2}{dt} = \sigma \cdot \dot{\sigma} = \sigma\left(\frac{\partial \sigma}{\partial x}\dot{x}\right) < 0, \quad (1)$$

where $u^+(x)$ is some control, possibly extreme, like "on" (or "$+k$") ensuring $\dot{\sigma} < 0$ when $\sigma > 0$; and $u^-(x)$ is possibly extreme as well, like either "off" or "*reverse*" (or "$-k$") ensuring $\dot{\sigma} > 0$ when $\sigma < 0$. This implies that, after an initial *reaching phase*, the system trajectories are forced by the control to start the so-called *sliding motion* along the sliding surface $\sigma(x) = 0$ and remain there, thereby achieving robustness and stability. Therein, desirable reduced-order dynamics can be devised by suitable choices for $\sigma \in \mathbb{R}$ (or $s \in \mathbb{R}^d$) (Polyakov & Fridman (2014)). Thus, thanks to such a versatile, yet simple, design procedure the methodology is proven to be one of the most powerful solutions for rapid and robust stabilization of nonlinear dynamical systems, even in the face of either matched or unmatched uncertainties or disturbances, see (Utkin & Chang (2002)) and (Rubagotti et al. (2011)).

To address some of the limitations of conventional SMCs (e.g.: provide smoother control and achieve more accurate tracking) in the 1990s VSC strategies incorporating higher-order dynamics into the control laws, the so-called higher-order sliding mode controllers (HOSMC), have been proposed (see Levant (1993); Bartolini et al. (1998); Pisano & Usai (2011)). Further developments are in the introduction of Lyapunov (Moreno & Osorio (2012)) and homogeneity (Levant (2005); Bernuau et al. (2014)) arguments in the controller design.

To highlight the potential benefits of the VSC with SM in consensus problems, Section 4 is going to provide, also from a technical viewpoint and with specific references and examples, a detailed overview of some of the most significant and recent results related to its application to MAS with uncertainties. The main features are

- *robust consensus reaching* regardless of the presence of heterogeneously perturbed, possibly nonlinear, agent dynamics. Classical strategies use to adopt linear protocols that result instead ineffective under uncertain environments;

- the *robust finite-time convergence* to the consensus that is an almost unique feature of nonsmooth interaction protocols, as for the VSC and SM control approaches;

- the *combination with other control terms*, possibly nonsmooth, and aimed at characterizing the MAS evolution along a *robust consensus manifold*, to allow for solving more complex objectives, as in *distributed optimization problems*, and in *robustification* of existing protocols to model mismatching and operational uncertainties;

- the *lightweight implementation*, even on cheap hardware, since binary control commands are mostly generated and, depending on the family type of VSC protocols considered, also only binary sensing and binary information exchanging are often requested;

- certain *safeness* and *resilience characteristics* such as the capability of handling *intermittently connected* interaction graphs and the presence of a subset of *outlier* and/or *uncooperative* agents.

These properties are discussed in detail in the following, also through examples, and by using *Filippov's Theory* (Filippov (2013)) and *Nonsmooth Lyapunov arguments* (Paden & Sastry (1987); Shevitz & Paden (1994); Clarke et al. (1998)).

Finally for a thorough review of the results and applications of VSC with Sliding Modes (SM) to consensus problems and networked systems, we recommend referring to the recent survey by (Zambelli & Ferrara (2021)).

1.2. Main contribution

This paper is focused on exploiting some features of the VSC with SM in the MAS consensus problem such that some robustness and finite-time properties are guaranteed. With this aim, a review of the literature highlighting the technicalities of the distributed control design is given, presenting the various faces of the robust consensus problem with simple examples to facilitate the reader's understanding. It also highlighted how including SM-based terms in the consensus algorithms allows for solving more general problems. The treatment and discussion also aim to give hints for future research directions.

1.3. Paper organization

- Section 2 illustrates the mathematical notation while providing the necessary preliminaries and tools on nonsmooth analysis and algebraic graph theory;

- Section 3 motivates the exploration of the VSC approach to reframe the consensus problem as a task of robust coordination over uncertain operational environments. Issues and considerations regarding the major limitations of the classic Laplacian flow control to address both leaderless and leader-following tasks are given;

- Section 4 introduces the two main families of VSC-based consensus controllers, the *signed Laplacian flow* and the *signed deviations' sum flow*; and an overview of the related literature is given. Therein, technicalities behind the analysis and design of multi-variable coupled ordinary differential equations with discontinuous right-hand side are also illustrated by means of several examples;

- Section 5 provides numerical simulations illustrating a detailed comparison of the consensus techniques discussed in the previous sections;
- Section 6 summarizes the benefits of the VSC approach with SM within the control of MASs operating over uncertain environments and provides concluding remarks while outlying the most interesting open research directions.

2. Preliminaries and Notation

2.1. Mathematical notation

The sets *natural*, *integer*, *real*, *positive real*, and *complex numbers* are denoted as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_{\geqslant 0}$, $\mathbb{C}$, respectively. The *real part* of $c \in \mathbb{C}$ is denoted by $\mathfrak{R}\{c\} \in \mathbb{R}$. Let $d \in \mathbb{N}$, $x = [x_i] = (x_1; x_2; \cdots ; x_d) \in \mathbb{R}^d$ denotes a *real column vector*, $x_i \in \mathbb{R}$ is its i-th entry, and $x^\intercal$ is the transpose of x. The *gradient* of $g(x,t) : \mathbb{R}^d \times \mathbb{R} \mapsto \mathbb{R}$ with respect to x is $\nabla_x g$. The *all-one*, and *all-zero column vectors* are denoted by $1_d = (1; 1; \cdots ; 1) \in \mathbb{R}^d$ and $0_d = (0; 0; \cdots ; 0) \in \mathbb{R}^d$, whereas $I_d \in \mathbb{R}^{d \times d}$ stands for the *identity matrix*. For $x, y \in \mathbb{R}^d$, $x^\intercal \cdot y$ and $x \otimes y$ denote the *scalar* and the *Kronecker product*. Let $\mathcal{S}$ be a set of points, possibly disjointed in $\mathbb{R}^d$, card($\mathcal{S}$) denotes its *cardinality*, and $co\{\mathcal{S}\}$ and $\overline{co}\{\mathcal{S}\}$ are, respectively, the *convex hull* and *convex closure* of $\mathcal{S}$. The sign-operator is $\text{sgn}(x)$, which outputs a vector whose i-th entry is 1 if the corresponding entry of x is positive, namely $x_i > 0$, then -1 if $x_i < 0$, 0 otherwise, whereas $\text{SGN}(x)$ is its multi-valued counterpart, which differs from $\text{sgn}(x)$ only at the points where one or more x_i equals 0. If so, its i-th output is $\overline{co}\{-1, 1\} = [-1, 1]$ (*multi-valued*). The 1-, p- and ∞-*norms* of x are $\|x\|_1 = \sum_{i=1}^d |x_i|$, $\|x\|_\text{p} = (\sum_{i=1}^d |x_i|^\text{p})^{1/\text{p}}$, and $\|x\|_\infty = \max_{1 \leq i \leq d}\{\, |x_i| \,\}$, with $\text{p} \in \mathbb{N}$ and $|x_i| = x_i \ \text{sgn}(x_i)$.

For $\text{p}, \text{q} \in \mathbb{N} \ : \ 1/\text{p} + 1/\text{q} = 1$, $x^\intercal \cdot y \leqslant |x^\intercal \cdot y| \leqslant \|x\|_\text{p} \cdot \|y\|_\text{q}$ (*Hölder's inequality*). Moreover, $\|x\|_2 \leq \|x\|_1 \leq \sqrt{d}\, \|x\|_2$.

The operator $\text{diag}(x) : \mathbb{R}^d \mapsto \mathbb{R}^{d \times d}$ returns a square diagonal matrix with the x elements on the principal diagonal. Let $R = [r_{ij}] \in \mathbb{R}^{d \times d}$ be a square matrix, its ij-entry is r_{ij}, its i-th row is $R_i \in \mathbb{R}^{1 \times d}$, and $\lambda_j(R) \in \mathbb{C}$ is the j-th *ordered eigenvalue* of R such that $\mathfrak{R}\{\lambda_1(R)\} \leq \mathfrak{R}\{\lambda_2(R)\} \leq \cdots \leq \mathfrak{R}\{\lambda_d(R)\}$. If $\mathfrak{R}\{\lambda_j(R)\} \in \mathbb{R}_{<0} \ \forall \ j$, R is said to be *Hurwitz*. If $R = R^\intercal$ and $\lambda_j(R) \in \mathbb{R}_{>0}(\mathbb{R}_{\geq 0}) \ \forall \ j$, then R is said to *positive-(semi)definite*, and it is denoted by $R \succ (\succeq)\ 0$. Let $M \in \mathbb{R}^{d \times d}$ and such that $M = M^\intercal$, then $MM^\intercal = M^\intercal M$. Moreover, if $M : \exists \ M^{-1}$ then $MM^\intercal \succ 0$, from which it results $\|Mx\|_1 \geq \|Mx\|_2 > 0 \ \forall \ x \neq 0_d$.

2.2. Preliminaries on nonsmooth analysis

Consider a differential equation

$$\dot{x}(t) = g(x,t), \quad x(t_0) = x_0 \in \mathbb{R}^d \tag{2}$$

where $g : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is measurable and essentially locally bounded. Namely, $g(x,t)$ is defined except on sets of Lebesgue measure zero $\mathcal{N}$ in $\mathbb{R}^d$, and it is measurable in an open region $\mathcal{R} \subset \mathbb{R}^d \times \mathbb{R}$ such that $\forall$ compact $\mathcal{C} \subset \mathcal{R}$, $\exists$ integrable $G(x,t)$ such that $\|g(x,t)\| \overset{\text{a.e.}}{\leqslant} G(x,t)$ almost everywhere (a.e.) in $\mathcal{C}$.

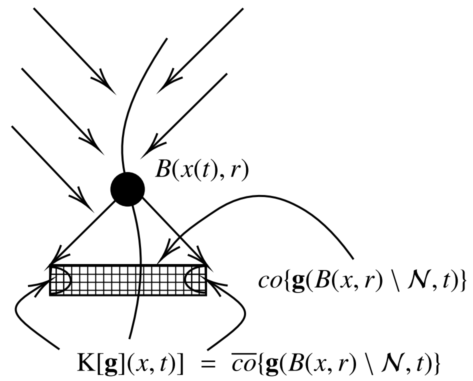

Figure 1: Graphical interpretation of the limiting procedure used in Definition 1 to calculate $\text{K}[g](x,t) = \bigcap_{r>0} \bigcap_{\mu(\mathcal{N})=0} \overline{co}\{g(B(x(t),r) - \mathcal{N}, t)\}$, where $B(x,r)$ is a ball of radius $r > 0$ centered at $x(t)$.

Thus, (2) may also have a discontinuous right-hand side. In the remainder, the solutions of (2) are understood in the so-called *Filippov's sense*, defined next.

Definition 1 (Filippov solution - Filippov (2013)). *A vector $x(t) \in \mathbb{R}^d$ is a Filippov's solution for* (2) *on $T := [t_0, t_1]$, if $x(t)$ is absolutely continuous on T and for almost all $t \in T$ it satisfies the differential inclusion*

$$\dot{x}(t) \in \text{K}[g](x,t) = \bigcap_{r>0} \ \bigcap_{\mu(\mathcal{N})=0} \overline{co}\,\{g(B(x(t),r) \setminus \mathcal{N}, t)\} \tag{3}$$

where $B(x(t),r)$ is an open ball with center $x(t)$, radius $r \in \mathbb{R}_{>0}$, and $\bigcap_{\mu(\mathcal{N})=0}$ is the intersection over all measure zero sets. ■

Notice that since $\text{K}[g](x,t)$ is locally bounded and upper semi-continuous with nonempty compact convex values, then its existence is guaranteed (Filippov (2013)). Filippov's solution can indeed be viewed as the limits of solutions for (2) with the right-hand side averaged over smaller and smaller neighborhoods around the discontinuity points.

Figure 1 helps in understanding the limiting procedure used to define $\text{K}[g](x,t)$. There, the ball $B(x(t),r)$ is the neighborhood around the actual base point $x(t)$ of (3). The drawn open rectangular interface is instead a neglected set of measure zero where $\dot{x}$ may not be defined, and the set $\text{K}[g](x,t)$ reduces to its convex closure of the two limit vectors as $B(x(t),r)$ becomes vanishingly small, namely as the radius $r \to 0$.

Example 1. (Calculus of $\text{K}[-k \ \text{sgn}(x)]$**)** *Consider* (2), *specified for $g(x,t) = -k \ \text{sgn}(x(t))$, with $k > 0$ and $x_0 = 1$. Since $x_0 > 0$, $x(t)$ evolves as $\dot{x}(t) = -k$ until, in a finite time $t_r = x_0/k$, the origin is reached and thereafter remain. $x(t)$ is the Filippov's solution of* (2), *in fact, according to Definition 1, at the discontinuity point $x = 0$, since $B(0,r)$ and $r > 0$, an open interval containing the origin intersect both $(0,+r)$ and $(-r,0)$ on set of positive measure $t \geq t_r$. Thus* (3) *reduces to* $\text{K}[g](x,t) = -k \ \text{SGN}(x(t))$, *namely to the convex closure of the two smooth limit behaviors*

$$\text{K}[g](x,t) = \begin{cases} \{+k\} & \text{for} \quad x > 0 \\ \{-k\} & \text{for} \quad x < 0 \\ \overline{co}\{-k,+k\} = [-k,k\,] & \text{for} \quad x = 0 \end{cases} . \tag{4}$$

□

For further details on the differential inclusions calculus, interested readers may refer to Paden & Sastry (1987).

Let us now consider a function $f(x) : \mathbb{R}^d \mapsto \mathbb{R}$, then $f(x)$ is said to be *locally Lipschitz* at $x \in \mathbb{R}^d$, if there exists a neighborhood $\mathcal{U}$ of x and $C_x \in \mathbb{R}_{\geq 0}$ such that $|f(y) - f(z)| \leq C_x \|y - z\|$ for all $y, z \in \mathcal{U}$. Moreover, if $\mathcal{U} \equiv \mathbb{R}^d$, $f(x)$ is *globally Lipschitz*.

Example 2. (Lipschitz continuity) *Consider $f_1 = x^2$, $f_2 = |x|$, with $x \in \mathbb{R}$. They are respectively locally, and globally Lipschitz because $|y^2 - z^2| \leq |y + z| \cdot |y - z|$, whereas f_2, due to its linear growth, satisfies $\big||y| - |z|\big| \leq (+1)\ |y - z|$, $\forall\ y, z \in \mathbb{R}$.* □

Since from the *Rademacher's Theorem*, a locally Lipschitz function is differentiable a.e., then its gradient exists in the so-called *generalized-sense*, as defined next.

Definition 2 (Clarke's generalized gradient - Clarke (1990)). *For a locally Lipschitz function $f(x) : \mathbb{R}^d \to \mathbb{R}$, let $\mathcal{N}$ be any set of zero measure in $\mathbb{R}^d$, and let $\mathcal{W}_f \subset \mathbb{R}^d$ be the set of points where $\nabla_x f(x)$ is not defined. The generalized gradient of $f(x)$ is $\partial f(x) = \overline{co}\{\ \lim_{x_i \to x} \nabla_x f(x_i)\ :\ x_i \notin \mathcal{W}_f \cup \mathcal{N}\ \}$.* ■

Notice ∂f is a set-valued map and represents the set of all the convex combinations between all of the possible limits of $\nabla_x f$ at the neighboring points of x where $f(x)$ is differentiable. Moreover, if $f(x)$ is regular at x, provided that it is also Lipschitz near x, then it admits directional derivatives at x for all the possible directions. Further, since the above definition does not consider discontinuities in t, which are not allowed, the subscript x by ∂f is dropped, in the remainder, for brevity.

Example 3. (Generalized gradient of $\|x\|_1$) *Consider the globally Lipschitz, nonsmooth, function $f(x) = \|x\|_1$ with $x \in \mathbb{R}^d$. By applying Definition 2, $\partial f(x) = \mathrm{SGN}(x)$, whose i-th entry is multi-valued and belongs to the closed convex set $[-1, 1]$, if $x_i = 0$. Further note that $\partial f(x)$ equals $\mathrm{K}[\mathrm{sgn}(x)]$.* □

Note that if $f(x)$ is *continuously differentiable* at x, it is also regular at x, whereas a convex function which is Lipschitz near x, is also regular at x (Clarke et al., 1998, Chapter 2). Next, we recall two fundamental results for Filippov's solutions.

Theorem 1 (Chain rule - Shevitz & Paden (1994)). *Let $x(t)$ be a Filippov solution of* (3)*, on an interval containing t and let $V(x) : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz regular function. Then $V(x)$ is absolutely continuous, i.e. $(d/dt)\,[V(x(t))]$ exists a.e., and*

$$\frac{d}{dt}\,[V(x(t))] \overset{\text{a.e.}}{\in} \bigcap_{\zeta \in \partial V(x)} \zeta^\mathsf{T} \cdot \dot{x}. \tag{5}$$

Theorem 2 (Lyapunov Theorem - Paden & Sastry (1987)). *If 1) $V(x) : \mathbb{R}^d \to \mathbb{R}$, $V(0) = 0$, and $V(x) > 0\ \forall\ x \neq 0$, and 2) $x(t) : \mathbb{R} \to \mathbb{R}^d$ and $V(x(t))$ is absolutely continuous on $[t_0, \infty)$ with the generalized time-derivative satisfying*

$$\frac{d}{dt}\,[V(x(t))] \overset{\text{a.e.}}{\leq} -\epsilon < 0 \quad a.e. \quad on \quad \Big\{\ t \mid x(t) \neq 0\ \Big\},$$

then x converges to zero in a finite time interval.

Example 4. (Sliding-mode control robustness) *Consider* (2)*, with $g(x, t) = -k\ \mathrm{sgn}(x(t)) + d(t)$ and $x \in \mathbb{R}$. Let $|d(t)| \leq \Pi$ be any uniformly bounded exogenous perturbation and $k - \Pi > 0$. Let $V = |x|$, from the associative property of differential inclusions, and the* chain rule*, it yields $\dot{x} \overset{\text{a.e.}}{\in} -\ k\ \partial V + \mathrm{K}[d](t)$. Thus it results*

$$\frac{d}{dt}\,[V(x(t))] \overset{\text{a.e.}}{\in} \bigcap_{\zeta \in \partial V} \zeta \cdot \dot{x} \overset{\text{a.e.}}{\leq} \bigcap_{\zeta \in \partial V} \Big(-k|\zeta|^2 + |\zeta|\ |\xi|\Big), \tag{6}$$

with $\zeta \in \partial V$ and $\xi \in \mathrm{K}[d](t) = [-\Pi, \Pi]$.
Let $\zeta = \arg\min\{\ |\zeta|^2\ :\ \zeta \in \partial V\}$, and $k - \Pi \geq \epsilon > 0$, then

$$\frac{d}{dt}\,[V(x(t))] \overset{\text{a.e.}}{\leq} \bigcap_{\zeta \in \partial V} \Big[-\epsilon|\zeta|^2 - \Pi\ \Big(|\zeta|^2 - |\zeta|\Big)\Big]. \tag{7}$$

Now note that $\partial V = [-1, 1]$. Then, due to the convexity of $V(x)$, $\partial V(x)|_{x \neq 0} \cap (-1, 1) = \emptyset$ (this follows from (Clarke, 1990, Proposition 2.2.9). Thus, since $\arg\min_{\zeta \in \partial V} |\zeta| = 1\ \forall\ x \neq 0$*, then $\dot{V} \overset{\text{a.e.}}{\leq} -\ \epsilon$. Moreover, $V(x)$ is radially unbounded, and thus the global finite-time stability of $x = 0$ can be concluded.*

Specifically, all the sets of (possibly disjoint) time intervals of measure zero for which $x(t) = 0$, according to (Filippov (2013)), can be disregarded. Whereas if $x(t) = 0$ on compact time intervals $\mathcal{T}$ of positive measure, due to the absolute continuity of Filippov's solutions, in the same positive time intervals the derivative of the corresponding Filippov solution is zero as well, namely $\dot{x}(t) \overset{\text{a.e.}}{=} \{0\}$, almost everywhere $\forall\ t \in \mathcal{T}$. □

2.3. Preliminaries on algebraic graph theory

The results in this section are mainly taken from Godsil & Royle (2001), Biyikoglu et al. (2007), and Horn & Johnson (2012). A *directed graph*, or *digraph*, is a pair $\mathsf{G}(\mathsf{V}, \mathsf{E})$ comprising of a *nodes set* $\mathsf{V} = \{1, \ldots, n\}$, and an *edges set* $\mathsf{E} \subseteq \{\mathsf{V} \times \mathsf{V}\}$, where $(j, i) \in \mathsf{E}$ if there exists an *edge* (or *link*) from j to i ($i \leftarrow j$) (cf. Figure 2a). $\mathsf{A} = [\mathsf{a}_{ij}] \in \mathbb{R}^{n \times n}_{\geq 0}$ is called *adjacency matrix*, where $\mathsf{a}_{ij} = 1$ if $(j, i) \in \mathsf{E}$, 0 otherwise. Also, since self-loops are generally not considered $\mathsf{a}_{ii} = 0\ \forall\ i$. Let $\mathsf{N}_i = \{j \in \mathsf{V}/\{i\} : (j, i) \in \mathsf{E}\} \subseteq \mathsf{V}$ be the *neighbors set* of i, $\mathsf{d}_i = \mathrm{card}(\mathsf{N}_i)$ is the *in-degree* of i. A *directed path* $\mathsf{p_d}(u, v)$, consists of an ordered sequence of edges connecting u with $v \in \mathsf{V}$. A *cycle* $\mathsf{c}(u) = \mathsf{p_d}(u, u)$ is a non-empty directed path that starts and ends at the same node. A *rooted directed spanning tree* is a graph with no cycles such that there exists a node, called *root*, from which any other node can be reached by walking a directed path. The *Laplacian matrix* encodes the pattern of information flows in G as

$$\mathsf{L}(\mathsf{G}) = \mathsf{D} - \mathsf{A} = [\ell_{ij}], \quad \text{with} \quad \|\mathsf{L}(\mathsf{G})\|_\infty \leq 2\|\mathsf{D}\|_\infty \tag{8}$$

where $\mathsf{D} = \mathrm{diag}((\mathsf{d}_1; \mathsf{d}_2; \ldots; \mathsf{d}_n))$ is called *in-degree matrix*. Since L is weakly diagonally dominant, see Figure 2b, then $\Re\{\lambda_j(\mathsf{L})\} \geq 0\ \forall\ j$. Also, $\mathsf{L}1_n = 0_n$ implying L has at least one eigenvalue in 0 and 1_n is the corresponding *right-eigenvector*.

A digraph is *balanced* if $\sum_i \mathsf{a}_{ij} = \sum_j \mathsf{a}_{ij}$. Thus 1_n is the *left-eigenvector* corresponding to $\lambda_1(\mathsf{L}) = 0$ (cf. Figure 2b with 2d).

A *cut* over G partitions V into two sets joined by at least one edge. A *minimum cut* is such that the two sets are joined by the minimum edges' number among all the possible cuts. A *digraph* may admit more minimum cuts and, if the minimum cuts contain at least $\mathsf{k} \in \mathbb{N}$ edges, G is said to be k-*connected*.

An *undirected digraph*, or simply a *graph*, satisfies $(j, i) \in \mathsf{E} \Rightarrow \exists (i, j) \in \mathsf{E}$ (cf. Figure 2g), thus $\mathsf{L} = \mathsf{L}^\mathsf{T} \geq 0$, and $1_n^\mathsf{T} \mathsf{L} = 0_n^\mathsf{T}$.

Let $h \leq n_\mathsf{E} = \mathrm{card}(\mathsf{E})$ be an ordered index identifying each $(j, i) \in \mathsf{E}$, then $\mathsf{B}(\mathsf{G}) = [\mathsf{b}_{hi}] \in \mathbb{R}^{n_\mathsf{E} \times n}$ denotes the so-called *edge-vertex incidence matrix* where, accordingly with Figure 2f, $\mathsf{b}_{hi} = 1$ if the edge corresponding to the given h starts at i (e.g. $j \leftarrow i$), whereas $\mathsf{b}_{hj} = -1$ if the edge corresponding to h ends at j (e.g. $j \leftarrow i$), 0 otherwise. Thus, by construction, it yields $\mathsf{B} 1_n = 0_{n_\mathsf{E}}$. Note that B differs from the more common *vertex-edge incidence matrix*, or simply *incidence matrix*, denoted by $\mathsf{H} = [\mathsf{h}_{ih}] \in \mathbb{R}^{n \times n_\mathsf{E}}$ with $\mathsf{h}_{ih} = \mathsf{b}_{hi}$ (Bullo (2019)), because H satisfies $1_n^\mathsf{T} \mathsf{H} = 0_{n_\mathsf{E}}$. However, since for our purposes is of interest to analyze the behavior, along the manifold $x_i = x_j \ \forall\ i, j \in \mathsf{V}$, of non-smooth fields of the form $[\sum_{j \in \mathsf{N}_i} \mathrm{sgn}(x_i - x_j)]$, then consider B guarantees there $\exists\ \bar{\mathsf{B}} : \bar{\mathsf{B}} \mathrm{sgn}(\mathsf{B} x) = [\sum_{j \in \mathsf{N}_i} \mathrm{sgn}(x_i - x_j)]$, thus providing not only a more compact representation of such a field but also, since $\mathsf{B} 1_n = 0_{n_E}$, to draw conclusions about the network dynamics along the consensus manifold. Moreover, if G is either weighed-balanced (namely $\sum_i \ell_{ij} = \sum_j \ell_{ji}$) or simply undirected, such a matrix $\bar{\mathsf{B}}$ corresponds to $\mathsf{B}^\mathsf{T} = \mathsf{H}$, thus further yielding $\mathsf{B}^\mathsf{T} \mathrm{diag}([\mathsf{a}_h]) \mathsf{B} = \mathsf{L} \geq 0$, where a_h is edge weights corresponding to h (Bullo, 2019, Lemma 9.1).

For an undirected *graph*, strong connectivity is referred to as simply *connectivity*. If so, $1 \geq \lambda_2(\mathsf{L}) \geq 1/(n \bar{\mathsf{d}})$, where $\bar{\mathsf{d}} = \max_{u \in \mathsf{V}} \max_{v \in \mathsf{V}} \mathsf{d}(u, v)$ is the *graph's diameter*, and $\mathsf{d}(u, v)$ denotes the *distance* (or *path length*) from u to $v \in \mathsf{V}$ ((Gross & Yellen, 2003, pag. 571)). $\lambda_2(\mathsf{L})$ is called *algebraic connectivity* and provides a measure of the *graph* connectedness.

A digraph G is said to be *weakly connected* if the corresponding underlying *graph* is connected, whereas it is *strongly connected* if a directed path connecting any two arbitrary nodes of G exists. If so, $\mathrm{rank}\{\mathsf{L}\} = n - 1$, $\lambda_2(\mathsf{L}) > 0$, and $\tilde{\mathsf{L}} = (\mathsf{L} + \mathsf{L}^\mathsf{T})/2 \geq 0 : 1_n^\mathsf{T} \tilde{\mathsf{L}} = 0_n^\mathsf{T}$. Strongly connected balanced *digraphs*, and connected *graphs* (thus such that $\tilde{\mathsf{L}} = \mathsf{L}$), satisfy

$$\chi^\mathsf{T}\, \tilde{\mathsf{L}} \chi \ \geq\ \lambda_2(\tilde{\mathsf{L}})\, \|\, \chi\, \|_2^2 > 0 \ \text{ with } \ \chi \in \mathbb{R}^n \ :\ 1_n^\mathsf{T} \cdot \chi \ =\ 0. \quad (9)$$

Let G be a digraph containing at least one rooted spanning tree, and let $x = [x_i] \in \mathbb{R}^n$, (Franceschelli et al., 2014b, Proposition 2.1) states that if $\exists (j, i) \in \mathsf{V} \times \mathsf{V}$ such that $x_i \neq x_j$, then

$$\|\mathrm{sgn}(\mathsf{L}x)\|_1 \geq 1 = \|\mathrm{sgn}(\mathsf{L}x)\|_\infty, \ \ \mathrm{sgn}(\mathsf{L}x)^\mathsf{T} \mathsf{L} \mathrm{sgn}(\mathsf{L}x) \geq 1. \quad (10)$$

Finally, for a time-varying graph, if $\mathsf{G}(t) \in \{\mathsf{G}_1, \ldots, \mathsf{G}_p\}$, then we let $s(t) : \mathbb{R} \mapsto \{1, \cdots, p\}$ be the *switching sequence*, we refer to $\mathsf{G}(t)$ as $\mathsf{G}_{s(t)}$. Indeed, for graphs with a finite number of nodes, the number of possible configurations is finite as well.

3. Formulations of Consensus Problems

Different and popular formulations of the *consensus problem*, in the *leaderless* and *leader-following* setting, are now outlined, and a discussion highlighting their limits, and considerations justifying the necessity to explore the VSC approach in the *protocol design*, are illustrated.

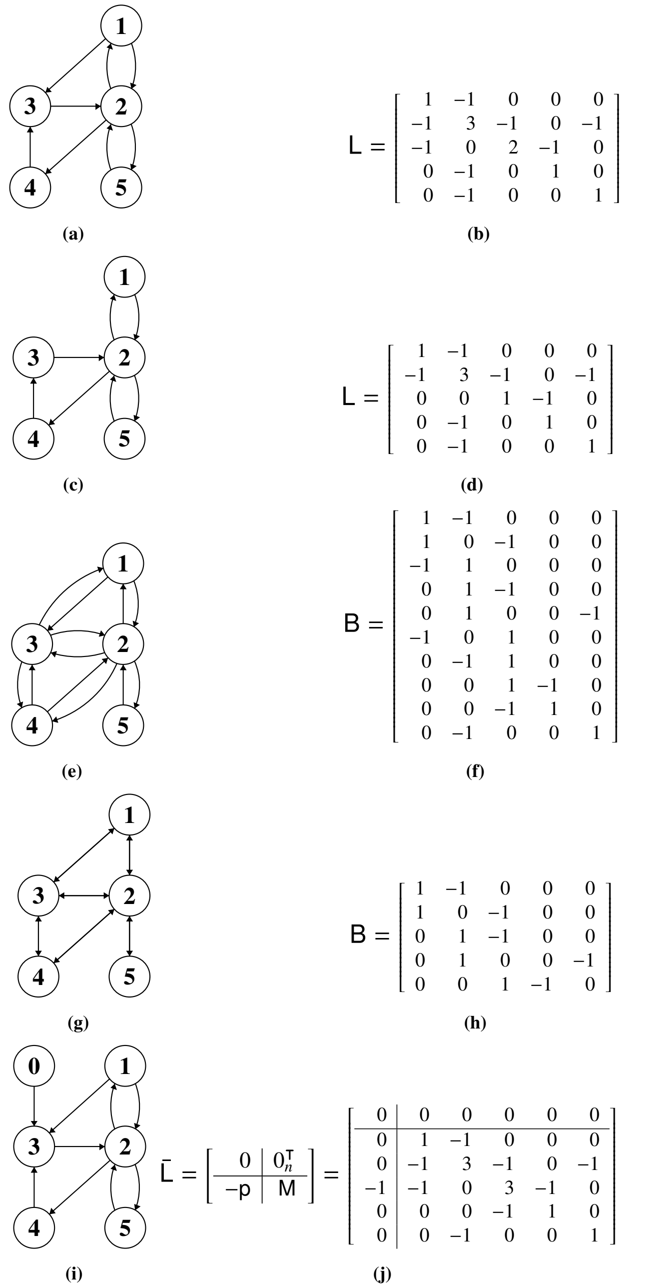

Figure 2: (a): A strongly connected digraph $\mathsf{G}_1(\mathsf{V}, \mathsf{E})$ with vertex set $\mathsf{V} = \{1, 2, 3, 4, 5\}$ and edge set $\mathsf{E} = \{(1, 2), (1, 3), (2, 1), (2, 4), (2, 5), (3, 2), (4, 3), (5, 2)\} \subseteq \mathsf{V} \times \mathsf{V}$; (b): Laplacian matrix $\mathsf{L}(\mathsf{G}_1)$; (c): A strongly connected weighted-balanced digraph G_2; (d): Laplacian matrix $\mathsf{L}(\mathsf{G}_2)$; (e): A strongly connected digraph G_3; (f): Edge-to-vertex incidence matrix $\mathsf{B}(\mathsf{G}_3)$; (g): A connected graph G_4; (f): Edge-to-vertex incidence matrix $\mathsf{B}(\mathsf{G}_4)$; (i): A connected digraph $\bar{\mathsf{G}}_5$ with 0 as root node; (j): Laplacian matrix $\mathsf{L}(\bar{\mathsf{G}}_5)$ where, according to Example 10, $\mathsf{M}(\bar{\mathsf{G}}_5) = \mathsf{P} + \mathsf{L}(\mathsf{G}_1)$ and $\mathsf{P} = \mathrm{diag}(\mathsf{p})$, with $\mathsf{p} = (0; 0; 1; 0; 0)$.

Consider a set V of agents whose *information flow* is encoded by a *digraph* $\mathsf{G}(\mathsf{V}, \mathsf{E})$. Let $x_i(t) \in \mathbb{R}^d$ be a state vector representing a set of agent i characteristics, e.g., physical quantities such as attitude, temperature, voltage, etc. The *consensus problem* consists in reaching a group agreement in a *dynamic way*. Namely for each pair $(i, j) \in \{\mathsf{V} \times \mathsf{V}\}$ the condition

$$\|\delta_{ij}(t)\| = \|x_i(t) - x_j(t)\| \to 0 \Rightarrow x_i(t) \to c(t),\ \forall\ i \in \mathsf{V} \quad (11)$$

is verified for some *consensus vector* $c(t) \in \mathbb{R}^d$, or equivalently such that the so-called *disagreement vector* $\delta(t)$ satisfies

$$\delta(t) = \mathsf{L_c} x(t) \to 0_{nd}, \quad \text{where} \quad \mathsf{L_c} = \left(I_{nd} - \frac{1}{n} 1_{nd} 1_{nd}^\mathsf{T} \right) \quad (12)$$

has the same form as the Laplacian matrix of a *complete graph*, and satisfies $\mathsf{L}\mathsf{L_c} = \mathsf{L}$ (and, if G is a *graph*, also $\mathsf{L_c}\mathsf{L} = \mathsf{L}$), whereas $x = (x_1; x_2; \cdots; x_n) \in \mathbb{R}^{nd}$ is stacking the agents states.

Let agents' dynamics be represented by $\dot{x}_i(t) = u_i(t)$, thus defining $u_i(t) \in \mathbb{R}^d$ as the agent's *local control*. Due to the information flow along G, relation (11) could feasibly be achieved using some feedback terms proportional to the *neighborhood state deviations*, as

$$u_i(t) = \sum_{j \in \mathsf{N}_i} \mathsf{a}_{ij} \left(x_j(t) - x_i(t) \right) = -\mathsf{L}_i \otimes I_d \; x \, . \quad (13)$$

Notice that (13) is the so-called *Laplacian flow*, and it may imply either that agent i receives x_j on an *absolute reference frame*, or that it autonomously senses the *neighbor state's deviation*

$$\delta_{ij}(t) = x_j(t) - x_i(t) \quad \forall \; j \in \mathsf{N}_i \quad (14)$$

in a local frame, depending on the *sensors* and measurement/connection *facilities* the agents are equipped with. When considering a VSC approach for designing control $u_i(t)$, either the so-called *signed Laplacian flow*

$$u_i(t) = k \; \mathrm{sgn}\Big(\sum_{j \in \mathsf{N}_i} \mathsf{a}_{ij} \cdot (x_j(t) - x_i(t)) \Big) = -k \; \mathrm{sgn}(\mathsf{L}_i \otimes I_d \; x), \quad (15)$$

with $k > 0$, or the *signed deviations sum flow*

$$u_i(t) = k \sum_{j \in \mathsf{N}_i} \mathrm{sgn}\big(x_j(t) - x_i(t)\big) = -k \; \mathsf{B}_i' \otimes I_d \; \mathrm{sgn}(\mathsf{B} \otimes I_d x) \quad (16)$$

can be considered. The choice of (15) or (16) would make great theoretical and technical differences. For instance, in a *multi-robot flocking problem* where no GPS information is available and where the agents sense only their respective *orientation deviation*, e.g., because they are equipped with low-accuracy proximity sensors, then (16) is the only realistic solution.

Remark 1. In consensus problems assuming the existence of *global information* at the local level, as for the whole topology G, is generally discouraged. The exception is made sometimes for offline tuning purposes, for the *agents number upper-bound* or for the *graph's algebraic connectivity* (for which distributed estimation algorithms exist (Franceschelli et al. (2013a))). Moreover, *multi-hop information*, namely from neighbors of a neighbor, is also not allowed at the local level since unfeasible. △

Let us now illustrate the two main classes of consensus problems, which mainly differ on the topological constraints on G.

3.1. Leaderless consensus problems

If G is strongly connected, (11)-(12) imply that all agents are committed to dynamically rendezvous on a configuration $c(t)$ everyone actively supports, even if it is implicitly *a-priory unknown* to all, at the startup. Thus, since all agents behave as *peers*, this is called *leaderless consensus problem*, or simply *consensus problem*. For understanding the fundamentals at the basis of *leaderless consensus*, let us now introduce the next example.

Remark 2. *If the interaction graph* G *is arbitrary, following (Olfati-Saber et al., 2007, Section II.B and Lemma 2), the consensus analysis is generally studied by exploiting the* Gershgorin's theorem *to provide a spectral characterization of* $\mathsf{L}(\mathsf{G})$. *Instead, if* G *is balanced or undirected, thanks to property* (9)*, the Lyapunov method can be easily applied. To provide the reader with a Lyapunov-based consensus background, which is useful to understand Section 4, Section 3.1 is focused on directed balanced graphs and undirected graphs. Finally, in Section 4, where VSC approaches are illustrated, graphs containing a spanning tree are analyzed with no exceptions.* △

Example 5. **(Average-consensus problem)** *Consider a set of agents whose dynamics are represented by simple integrators*

$$\dot{x}_i(t) = u_i(t) \quad \text{with} \quad x(0) \in \mathbb{R}. \quad (17)$$

Let G *be strongly connected balanced, and* $u_i(t)$ *is defined as in* (13)*, the network dynamics results into*

$$\dot{x}(t) = -\mathsf{L} x(t) \quad (18)$$

and (12) *the consensus subspace is globally exponentially asymptotically stable.*

To study (18)*, consider* $V(\delta) = \frac{1}{2}\|\delta\|_2^2$ *as a Lyapunov function for the disagreement vector* (12)*. By differentiating along* (18) *it results* $\dot{\delta} = -\mathsf{L}\delta$*, and by means of* (9) *one can conclude that function* $\dot{V} = -\delta^\mathsf{T} \tilde{\mathsf{L}} \delta \leq -2\lambda_2(\tilde{\mathsf{L}}) V(\delta) < 0$ *is exponentially stable.*

Thus, $\delta(\infty) \to 0_n \Rightarrow x(\infty) \to c 1_n$*, for some* $c \in \mathbb{R}$*. Now, let* $1_n : 1_n^\mathsf{T} \mathsf{L} = 0_n^\mathsf{T}$ *be the left-eigenvector of* L*, clearly* $y(t) = 1_n^\mathsf{T} x(t)$ *is a time-invariant quantity because* $\dot{y}(t) = -1_n^\mathsf{T} \mathsf{L} x(t) = 0 \;\forall\; t$*.*

Thus, $y(t) = 1_n^\mathsf{T} x(0) \Rightarrow 1_n^\mathsf{T} x(0) = 1_n^\mathsf{T}(c 1_n)$ *and the consensus value* c *reduces to the initial states average*

$$c = \mathrm{ave}(x(0)) = \frac{1_n^\mathsf{T} \cdot x(0)}{1_n^\mathsf{T} \cdot 1_n} = \frac{1}{n} 1_n^\mathsf{T} \cdot x(0). \quad (19)$$

For completeness, if G *was simply a strongly connected digraph, since* $\Re\{\lambda_j(\mathsf{L})\} \geq 0 \;\forall\; j \geq 2$*, then* (18) *is stable as well. From that, and by using the Jordan matrix decomposition it can similarly prove* $c = \gamma^\mathsf{T} x(0) / (\gamma^\mathsf{T} 1_n)$ *is reached, where* γ *is the left-eigenvector of* L *corresponding to* $\lambda_1(\mathsf{L}) = 0$*.* □

Further, (18) achieves (11) also for *switching networks* if the union of all the underlying G_s has a *spanning tree* (Moreau (2005)); and also in the presence of communication delays if G is a fixed connected graph (Olfati-Saber & Murray (2004)).

From a robust control perspective Example 5, which numerical simulations will be illustrated later in Figure 5, gives rise to some considerations:

(i) Protocol (13) achieves the consensus only because the overall network is governed by the unperturbed dynamics (18). If the network is affected by some *unknown inputs* w_i, due to, e.g., uncertainties or faults, such that (17) is modified into

$$\dot{x}_i(t) = u_i(t) + w_i(t), \tag{20}$$

and, by applying (13), the following controlled dynamics result

$$\dot{x}(t) = -\mathsf{L}x(t) + w(t). \tag{21}$$

Then, due to the unknown input $w = (w_1; w_2; \cdots ; w_n)$, (12) becomes no longer practicable using (13). In fact in the presence, for instance, of nonzero mean disturbances $w_i(t)$, the solutions of (21) will clearly diverge while keeping the relative consensus error (11) bounded (cf. the results illustrated in Figure 5a with that on Figure 5b). Those considerations lead to the definition of the so-called *practical consensus* condition

$$\|x_i(t) - x_j(t)\| \leq \Gamma \quad \text{where} \quad \Gamma \in \mathbb{R}_{>0}, \quad \forall\, i, j \in \mathsf{V}, \tag{22}$$

where Γ generally depends on the uncertainties' upper bounds. Practical consensus concepts are resorted, not only to deal with zero-mean measurement noises Li & Zhang (2009) but also to study the consensus over switching, intermittently connected, networks (Franceschelli et al. (2013b)); and in the presence of multiple root nodes, as in opinion spreading problems (Vasca et al. (2021)). (ii) Protocol (13) achieves the consensus on (19) only because (18) has well-defined *left-* and *right-eigenvectors* associated to $\lambda_1(\mathsf{L}) = 0$. Suppose the uncertainty consists of a stable state-dependent dynamic as $w_i = \epsilon_i x_i$ with $\epsilon_i < 0$. Thus (18) reduces to $\dot{x} = \Theta x$, with $\Theta = -\mathsf{L} - \mathrm{diag}([\epsilon_i])$ where $\Re\{\lambda_i(\Theta)\} < 0\ \forall\ i$, implying agents will collapse to the origin, thereby compromising the achievement to task the MAS has been devised, be either the average or f-consensus value estimation.

Said that above, and to provide some robustness characteristics, at least those to treat the aspect described by Section 3.1:(i) in the case of constant disturbances, or zero mean stochastic noisy channels, PI-based consensus protocols can also be applied to (20) (Bullo (2019)), as next illustrated.

Example 6. (PI robust consensus) *Let* G *be a connected graph. Consider n agents as in* (20)*, each one perturbed by a step disturbance* $w_i = \text{const.}$*, and* $u_i(t) = -\mathsf{L}_i x(t) - \int_0^{\mathsf{T}} \mathsf{L}_i x(t) dt$*,*

$$\dot{x}(t) = -\mathsf{L}x(t) - \int_0^{\mathsf{T}} \mathsf{L}x dt + w = -\mathsf{L}x(t) - \mathsf{L}q(t) \tag{23}$$

$$\dot{q}(t) = x(t) \qquad \text{with} \quad q(0) = w. \tag{24}$$

Let $V = \frac{1}{2}\|\delta_1\|_2^2 + \frac{1}{2}\delta_2^{\mathsf{T}}\mathsf{L}\delta_2$ *be a candidate Lyapunov function for the disagreement vectors* $\delta_1 = \mathsf{L}_{\mathsf{c}}x$ *and* $\delta_2 = \mathsf{L}_{\mathsf{c}}q$*. After time-differentiating,* $\dot{V} \leq -\delta_1^{\mathsf{T}}\mathsf{L}\delta_1 \leq -\lambda_2(\mathsf{L})\|\delta_1\|_2^2 \leq 0$ *which implies* $\delta_1 \to 0_n$*. Then, it results* $\dot{\delta}_2 = \delta_1 \to 0_n$ *and* $x(t) \to c(t)1_n$*, from which we can further conclude that* $\dot{\delta}_1 = 0_n$ *showing* $0_n = -\mathsf{L}\delta_1 - \mathsf{L}\delta_2 \Rightarrow -\mathsf{L}\delta_2 = 0_n \Rightarrow \delta_2 \to 0_n$*. Thus the consensus is guaranteed by the* LaSalle's Invariance Principle.

Nevertheless, since $y(t) = 1_n^{\mathsf{T}} x(t)$ *is still time-invariant along* (23) *because* $\dot{y}(t) = 0\ \forall\ t$*, the group decision is no longer* (19) *but a disturbance-dependent* ramp *of the form*

$$c(t) = \frac{1_n^{\mathsf{T}} \cdot (x(0) + t\, q(0))}{n} = \frac{1_n^{\mathsf{T}} \cdot x(0)}{n} + \frac{1_n^{\mathsf{T}} \cdot w}{n} t. \tag{25}$$

$\square$

From Example 6 a third consideration arises:

(iii) The PI Laplacian flow (23)-(24) guarantees the consensus achievement only against constant unknown inputs, whereas if w is time-varying only a practical agreement along the manifold $\mathsf{L}x$ could be achieved. However, since w changes the network structural properties of (20), an unexpected network emergent behavior arises. This reinforces what is stated in Section 3.1:(i).

Indeed, (23)-(24) may achieve the *average consensus* if and only if, at the same time, $w = 0_n$ and $q(0) = 0_n$, showing the integral control component initialization is a critical aspect independently by the uncertainties presence.

3.2. Leader-following consensus problems

If G has a *root node*, referred to as the *leader* and indexed by 0, then achieving the consensus condition (11) is equivalent to

$$\|x_i(t) - x_0(t)\| = 0 \ \ \forall\ i \in \mathsf{V}. \tag{26}$$

As graphically illustrated on Figure 2i, since $\mathsf{N}_0 = \emptyset$, a hierarchy between the *leader* 0, which behaves independently, and the remaining agents, called *followers*, exist. The major difference to the leaderless case is that $c(t)$ is now a-priory given and communicated at least to a few of *followers*. Thus, the coordination task reduces simply to the *distributed tracking* of $x_0(t)$.

The following example aims to highlight the limitations and problems behind the solution of the leader-following task through the Laplacian flow protocol (13).

Example 7. (Leader-following via Laplacian flow) *Let* $\mathsf{G}(\mathsf{V}, \mathsf{E})$ *be the digraph describing the followers' interaction (e.g. see Figure 2a), and let* $\bar{\mathsf{G}}(\bar{\mathsf{V}}, \bar{\mathsf{E}})$ *be an augmented digraph where* $\bar{\mathsf{V}} = \{0\} \cup \mathsf{V}$ *and* 0 *is the only* root *of* $\bar{\mathsf{G}}$ *(e.g. see Figure 2i). Then* $\bar{\mathsf{L}}(\bar{\mathsf{G}}) = [0_n^{\mathsf{T}}; -\mathsf{p}, \mathsf{M}]$*, where* $\mathsf{M} = \mathsf{L}(\mathsf{G}) + \mathsf{P} > 0$ *and* $\mathsf{P} = \mathrm{diag}(\mathsf{p})$ *is the so-called* pinning matrix*, with* $\mathsf{p}_i = 1$ *if* $(0, i) \in \bar{\mathsf{E}}$ *(cf. Figure 2i with Figure 2j).*

Let us consider (13) *with* $d = 1$*, and suppose the followers have the* integrator dynamic *in* (17)*, whereas the leader state* $x_0(t) \in \mathbb{R}$*, possibly virtually generated, is given by*

$$\dot{x}_0(t) = u_0(t) \quad \textit{where} \quad |u_0(t)| \leq \Pi, \tag{27}$$

where Π *can be either a time-varying or a uniform, raw, bound depending on the available leader's information. Now, to study the MAS dynamics, introduce the tracking error*

$$e(t) = x(t) - 1_n \otimes x_0(t) \quad : \quad \dot{e}(t) = -\mathsf{M}e(t) - 1_n \otimes u_0(t). \tag{28}$$

Let $V = \frac{1}{2}\|e\|_2^2$ *be a candidate Lyapunov function, by differentiating along the solutions of* (28)*, yields*

$$\dot{V} \leq -\lambda_1(\tilde{\mathsf{M}})\|e\|_2^2 + |u_0|\ \|e\|_1 \leq -\left[\lambda_1(\tilde{M})\|e\|_2 - \Pi\sqrt{n}\right]\sqrt{2V} \tag{29}$$

where $\tilde{\mathsf{M}} = (\mathsf{M} + \mathsf{M}^\mathsf{T})/2 > 0$. *From that it results* $e(\infty) \to 0_n$ *if and only if either* $u_0(t)$ *is vanishing, or* $x_0(t)$ *is a piece-wise constant function, i.e.,* $\dot{x}_0(t) \overset{\text{a.e.}}{=} 0$. □

From Example 7, two more considerations arise:

(iv) In the case of a *non-autonomous leader* (when $u_0(t)$ in (27) drove $x_0(t)$ arbitrarily) and without compromising the distributed nature of interactions over G by assuming $u_0(t)$ globally known (as made in Song et al. (2010); Zhou et al. (2012)), only the *practical consensus* can be guaranteed through (13), as also confirmed by the numerical results illustrated in Figure 9a. Specifically, and in accordance with (22), it results an accuracy upper-bound that, from (29), results to be equal to $\|e\|_2 \leq \Gamma = \Pi\sqrt{n}/\lambda_1(\tilde{\mathsf{M}})$.

(v) Finally, compared to the leaderless case where the consensus is generally concluded through consideration on $\mathsf{L}x$ and by exploiting property (9) (since $\nexists\ \mathsf{L}^{-1}$), for the *leader-following* the analysis is often easier since $\exists\ \mathsf{M}^{-1}$ and $\Re\{\lambda_j(\mathsf{M})\} < 0\ \forall\ j$. Moreover, contrarily to Section 3.1:(ii), agents with stable dynamics do not compromise the consensus reaching since the agreement value $c(t)$ is provided by the leader. This further shows the *leader-following* problem can be viewed, in some sense, as a special case of the *leaderless* one.

Later, it is shown that the VSC approach allows making the leader-follower consensus problem (26) robust against both the $\dot{x}_0(t)$ effects and the influence of additional time-varying local uncertainties $w_i(t)$ corrupting the agents' dynamics.

Remark 3. *The leader-following paradigm plays a fundamental role also in mobile robotics* displacement-based *flocking and formation control. There, a team of followers, either linear or nonholonomic agents, have to track the position of one or more leaders with some prescribed offsets to avoid collisions while maintaining a desired formation shape. See (Oh et al., 2015, Section 5) for an overview of techniques designed to modify the local Laplacian flow* (13) *to account for the desired displacement respecting each neighbor based on the available setup and the presence of absolute or relative measurements.* △

3.3. Extension to higher-order consensus problems

The *higher-order consensus problem* has been formalized (Cao et al. (2012)). Nevertheless, since reaching a consensus on a high-order manifold $\mathsf{L}(d^r x(t)/dt^r) = 0_n \forall\ r \leq h$, where $h \geq 2$, returns an undamped agreement trajectory in the absences of a leader and/or additional damping terms, except for some specific applications (e.g., coordination of nonholonomic systems (Xie et al. (2022)), synchronization of distributed parameter systems (Pilloni et al. (2015), or higher-order distributed tracking (Du et al. (2017))), the *second-order* case ($h = 2$),

$$\begin{array}{rcl} \dot{x}(t) & = & v(t) \\ \dot{v}(t) & = & u(t) \\ u(t) & = & -k_x\mathsf{L}x - k_v\mathsf{L}v - \eta_x x(t) - \eta_v v(t) \end{array} \tag{30}$$

results to be the only of particular interest (Bullo, 2019, Table 8.1). This is mainly due to the large number of applications involving constant speed operation, such as for Lagrangian systems, mobile robots, and synchronized oscillators such as clocks or power generators, see e.g. Mei et al. (2015); Lin et al. (2016); Smith et al. (2021), to mention a few.

Example 8. (PD-based consensus via Laplacian flow) *Let* G *be a connected graph. Consider a set of n double-integrator agents* $\ddot{x}_i(t) = u_i(t)$ *subjected to the PD-consensus control protocol* $u_i = -\mathsf{L}_i x - \mathsf{L}_i\dot{x}$, *where* $x = [x_i] \in \mathbb{R}^n$. *Let* $v = \dot{x}$, *by simple manipulation* (30) *represents the considered controlled dynamics, with* $k_x = k_v = 1$, $\eta_x = \eta_v = 0$.

Let $\delta_1 = \mathsf{L}_\mathsf{c}x$, *and* $\delta_2 = \mathsf{L}_\mathsf{c}v$ *be the position and the velocity disagreement vectors where* L_c *is as in* (12). *Now, by differentiating* $V = \frac{1}{2}\delta_1^\mathsf{T}\mathsf{L}\delta_1 + \frac{1}{2}\|\delta_2\|_2^2$, *and thank to* (9), *it yields*

$$\dot{V} = -\delta_2^\mathsf{T}\mathsf{L}\delta_2 \leq -\lambda_2(\mathsf{L})\|\delta_2\|_2^2 \Rightarrow \delta_2(t) \to 0_n \Rightarrow v(t) \to c'(t)1_n.$$

Then, by invoking the LaSalle's Invariance Principle, *it results that* $\delta_1 \to 0_n \Rightarrow x(t) = c(t)1_n$. *However, because* $y(t) = 1_n^\mathsf{T}v(t)$ *is time-invariant* $\forall\ t \geq 0$, *then* $y(t) = 1_n^\mathsf{T}v(0)$. *Thus, all the velocities are constant and equal to* $c' = 1_n^\mathsf{T}v(0)/n$. *Moreover, as anticipated, the positions asymptotically approach an undamped agreement trajectory of the form*

$$c(t) = \frac{1_n^\mathsf{T} \cdot x(0)}{n} + c't. \tag{31}$$

Further note $c(t)$ *is as in* (25). □

4. A tutorial overview on VSC-based consensus control

Motivated by the above-outlined considerations, the effective implementation of the VSC approach in a distributed fashion to guarantee the robustness of the consensus condition despite local perturbations and model mismatching, while inducing finite-time convergence, is discussed.

Furthermore, the VSC approach is exploited to decouple, thanks to its finite-time convergence characteristics, simultaneous control objectives in which the *consensus* is only a part of more complex decision tasks, as in distributed optimization.

In the following, it is assumed that the agents are characterized by the first-order perturbed dynamics

$$\dot{x}_i(t) = m_i(x_i, x_{j\in\mathsf{N}_i}, t) + u_i(t), \quad x_i(0) \in \mathbb{R}, \quad i \in \mathsf{V}, \tag{32}$$

where $x_i \in \mathbb{R}$ is the local state, $u_i \in \mathbb{R}$ the control devised over G, and $m_i(x_i, x_j \in \mathsf{N}_i, t)$ does not to satisfy any particular constraint other then being simply *essentially locally bounded* (thus it can also be nonlinear and possibly discontinuous), i.e.

$$|m_i(x_i, x_{j\in\mathsf{N}_i}, t)| \overset{\text{a.e.}}{\leq} \bar{m}_i(x_i, x_{j\in\mathsf{N}_i}, t) \tag{33}$$

where $m_i(x_i, x_{j\in\mathsf{N}_i}, t)$ considers every mismatch to the nominal decision-making model (17), e.g., it accounts for uncertain physical couplings propagated through the interaction graph G, perturbations and/or any model uncertainty. In the following, couplings at either physical or software level are described, without loss of generality, by the same digraph G.

Remark 4. *In physically coupled systems an exact characterization of the node model and couplings is unrealistic since they are not software entities. Model* (32)-(33) *thus describes cyber-physical networks whose behavior is wanted to be orchestrated through the* consensus*, as for generators on a microgrid (Pilloni et al. (2018)), coupled oscillators (Bauso (2017)), mobile robots (Ajina et al. (2020)), etc. Let $\hat{m}_i$ be the known part of m_i, assuming that $\tilde{m}_i = m_i - \hat{m}_i$ is at least locally bounded by a known norm-bound $\bar{m}_i$ is widely accepted. Thus, if m_i is well characterized, $\bar{m}_i$ would be small, otherwise possibly unnecessary control gains result. This may lead to "chattering" in the control loop. Chattering in VSC can however be alleviated, under certain system characteristics and at the price of losing the ideal accuracy, by approximating discontinuous control terms as smooth functions (Lee & Utkin (2007)).* △

Following Remark 4, and taking into account that the known part of the drift term of (32) can always be feed-forwardly compensated, in the remainder, m_i in (32) is unknown with known norm-bound $\bar{m}_i$, to keep the notation simple.

By the mentioned structural distinction between VSC-based consensus control protocols, in the following, respectively, the *signed Laplacian flow* in (15) and the *sum of signed deviations* in (16) are discussed, separately.

4.1. Signed Laplacian flow control protocols

The first attempt to exploit the VSC to solve consensus problems is presented in Cortés (2006). There, by non-smooth gradient-flow arguments, it is proved that (15) reaches consensus in finite-time if G is a *connected graph*, and agents are *unperturbed* as in (17). Then Rao & Ghose (2011), using the *Utkin's equivalent control*, extend the result to *time-varying topologies*. Then, in Xiao et al. (2009), the switching control (15) is replaced by its signed square root counterpart ($u_i = -k|\mathsf{L}_i x|^\alpha \otimes I_d \mathrm{sgn}(\mathsf{L}_i \otimes I_d x)$, $\alpha \in (0,1)$), obtaining, with continuous control and in the unperturbed case, the finite-time consensus as well. Uncertain couplings or disturbances are instead studied in Franceschelli et al. (2014b).

Example 9. (Robust consensus via signed Laplacian flows) *Let* G *containing at least a rooted directed spanning tree and let agents be uncertain as in* (32)-(33). *Consider the* signed Laplacian flows (15) *with* $k > 2\|\mathsf{D}\|_\infty \|\bar{m}\|_1$*, and* $\bar{m} = [\bar{m}_i]$*. Let* $\xi \in [-\|\bar{m}\|_\infty, \|\bar{m}\|_\infty]$*, then from Definitions 1 and 2, it yields*

$$\dot{x} \overset{\text{a.e.}}{\in} \mathrm{K}[-k\,\mathrm{sgn}(\mathsf{L}x) + m] = -k\,\partial\|\mathsf{L}x\|_1 + \xi. \tag{34}$$

Let $V(\delta) = \|\mathsf{L}\delta\|_1$ *be a candidate nonsmooth Lyapunov function for the disagreement vector* (12)*, which dynamic satisfies*

$$\dot{\delta} \overset{\text{a.e.}}{\in} -k\mathsf{L}_{\mathsf{c}}\partial\|\mathsf{L}\delta\|_1 + \mathsf{L}_{\mathsf{c}}\xi = -k\,\partial\|\mathsf{L}\delta\|_1 + \mathsf{L}_{\mathsf{c}}\xi + \frac{k 1_n^\mathsf{T}\partial\|\mathsf{L}\delta\|_1}{n} 1_n. \tag{35}$$

Then, by invoking Definitions 2, (8), (10), $\mathsf{L}1_n = 0_n$*, and* $\mathsf{L}\mathsf{L}_{\mathsf{c}} = \mathsf{L}$,

$$\begin{aligned}\dot{V} &\overset{\text{a.e.}}{\in} \bigcap_{\zeta\in\partial\|\mathsf{L}\delta\|_1} \zeta^\mathsf{T}\mathsf{L}\dot{\delta} \overset{\text{a.e.}}{\leq} \bigcap_{\zeta\in\partial\|\mathsf{L}\delta\|_1} -k\,\zeta^\mathsf{T}\mathsf{L}\zeta + \|\zeta\|_\infty\|\mathsf{L}\bar{m}\|_1 \\ &\overset{\text{a.e.}}{\leq} -k + 2\|\mathsf{D}\|_\infty\|\bar{m}\|_1 \overset{\text{a.e.}}{\leq} -\epsilon < 0, \quad \text{with} \quad \epsilon > 0. \end{aligned} \tag{36}$$

This implies $\dot{V}$ decreases until all the entries of δ are zero on sets of positive measure, and thus all the solutions of (34) *converge to an invariant set $x(t) \to c(t)1_n$. More formally, and reminding discontinuities at measure zero time interval can be disregarded, it results that, if one or more entries of $\mathsf{L}\delta$ are zero on a time interval of positive measure, since Filippov solutions are absolutely continuous, and $\delta(t)$ is the Filippov solution of* (35)*, then in the same positive time intervals the derivative of the corresponding Filippov solution for those elements which are zero is zero as well, until $\|\delta\| = 0$ for $t \geq t_r$.*

However, the agreement value $c(t)$ is time-varying since influenced by uncertainties, thus it is unpredictable. As a special case, if $m = 0_n$ and G *is either* strongly connected balanced*, or a* connected graph*, then (Cortés, 2006, Theorem 11)*

$$c(t) = \mathrm{mid}(x(0)) = \frac{1}{2}\left(\max_{\forall\, i\in\mathsf{V}} x_i(0) + \min_{\forall\, i\in\mathsf{V}} x_i(0)\right), \tag{37}$$

that is, the so-called average max-min value *is established, known as* mid-range *of* $x(0)$. *Dynamics* (34) *do not converge to* $1_n^\mathsf{T}x(0)/n$ *because, differently to* (18), $1_n^\mathsf{T}\dot{x}(t) = 0_n$ *is not time-invariant along* (34)*, by design.* □

The *signed Laplacian flow* (15) solves the leaderless consensus problem (11) in a finite-time, and in the presence of uncertainty, while giving some robustness property to the closed loop dynamics (34), even if the resulting behavior still depends by the local uncertainties, namely robustness is not fully achieved.

It is also worth remarking that in the absence of perturbation, it solves, in a finite-time as well, a novel consensus problem that was not solved before and that gives information on the central tendency of the agents' initial conditions, or *mid-range*. Numerical simulations confirming the above statements are also provided in Figure 6a, where $m = 0_n$, and in Figure 6b.
Instead, concerning the leader-following problem, note that:

(i) the *leader-following* and the *leaderless* problems differ only for the digraph topology since the leader is not influenced by the followers and it is the only root for G;

(ii) moreover by relating Section 3.1:(i), with Section 3.2:(v), it results that u_0 in (28) plays the role of an unknown drift term as w_i in (20), or as m_i in (32).

From these considerations, and because of Section 3.2:(iv), one can conclude the leader-following consensus, can be viewed as a particular case of the leaderless problem, which proof can be easily extended, if the leaderless case-study is still solved. Besides, it is not true the vice-versa since the Laplacian matrix is singular, thus making the corresponding Lyapunov analysis more general and difficult. To confirm these statements, in the following, the leader-following problem for the *signed Laplacian flow* is illustrated in Example 10. Moreover, for sake of completeness, their corresponding numerical simulations are provided in Figure 9b.

Example 10. (Leader-following via signed Laplacian flow) *Consider the digraphs* G *and* $\bar{\mathsf{G}}$*, the matrices* P, L, $\mathsf{M} = \mathsf{L} + \mathsf{P}$*, the vectors* p, x*, and* $|u_0| \leq \Pi$*, as in Example 7. Let* x_0 *be the leader state governed by* (27)*. Let the followers be governed by the dynamic* (16), (20)*, with* $k > (2\|\mathsf{D}\|_\infty + 1)\|_\infty(\|\bar{m}\|_1 + \Pi)$.

Let $e = x - 1_n x_0$, since $\mathsf{M}x - \mathsf{p}x_0 = \mathsf{M}x - \mathsf{p}x_0 \pm \mathsf{M}1_n x_0 = \mathsf{M}e + \mathsf{L}(\mathsf{G})1_n x_0 = \mathsf{M}e$, it yields $\dot{e} = -k\mathrm{sgn}(\mathsf{M}e) + (m - 1_n u_0)$, which has the same structure of (34)*, except for the* M *in place of* L*, where* M *is invertible and Hurwitz. Let us now consider the Lyapunov function of Example 9, with* M *in place of* L*, then it results*

$$V(e) = \|\mathsf{M}e\|_1 \Rightarrow \dot{V} \overset{\text{a.e.}}{\leq} -k + \|\mathsf{M}\|_\infty(\|\bar{m}\|_1 + \|\Pi\|_1) < -\epsilon < 0,$$

for some $\epsilon > 0$, where, due to (8)*, $\|\mathsf{M}\|_\infty = \|\mathsf{L}(\mathsf{G}) + \mathsf{P}(\bar{\mathsf{G}})\|_\infty \leq 2\|\mathsf{D}(\mathsf{G})\|_\infty + 1$, and $\|\mathsf{P}(\bar{\mathsf{G}})\|_\infty = 1$. From that, and because of $\exists\ \mathsf{M}^{-1}$, it yields $\mathsf{M}e = 0_n \Rightarrow e = 0_n$ a.e. $\forall\ t \geq t_r = V(0)/\epsilon$, regardless both the followers model mismatching $m_i(x_i, x_j \in \mathsf{N}_i, t)$ and the unknown $u_0(t)$, cf. with Section 3.2:*(iii). □

Finally, to complete the *signed Laplacian flow* review, let us now introduce its robust second-order extension, which can be interpreted as a distributed reformulation of the well-known second-order SM control algorithm, referred to as *twisting algorithm* (Levant (1993)). This problem has been studied in the leaderless setting in Pilloni et al. (2013), for a set of perturbed double-integrator agents

$$\ddot{x}_i(t) = m_i(x_i, x_j \in \mathsf{N}_i, t) + u_i(t) \quad \text{with} \quad |m_i| \leq \bar{m}_i, \tag{38}$$

operating over a connected graph, under the non-smooth protocol

$$u_i(t) = -k_x\ \mathrm{sgn}(\mathsf{L}_\mathsf{i}\ x(t)) - k_v\ \mathrm{sgn}(\mathsf{L}_\mathsf{i}\ \dot{x}(t)) \tag{39}$$

with k_x and k_v sufficiently large and in a proper ratio to counteract the uncertainty.

Then, in Pilloni et al. (2015), by exploiting an artificial state augmentation for the control, it was shown that the smooth integration of (39) achieves synchronization also between infinite-dimensional agents modeled by boundary-controlled heat equations perturbed at the boundaries. Finally, Pilloni et al. (2017a) applies (39) to solve, from a leader-following perspective, the voltage secondary-control problem on a DC-AC microgrid.

Example 11. Twisting 2SMC signed Laplacian flow *Let* G *be a connected graph and let agents be uncertain as in* (38)*. Consider the twisting 2SMC protocol* (39) *with $k_x > k_v + \|\bar{m}\|_\infty$, and $k_v > \|\bar{m}\|_\infty$. A sketch of the proof summarizing the main steps and lines of reasoning necessary to show that* (38)-(39) *achieve the consensus in finite-time is given. For further details refer to Pilloni et al. (2013). The proof is carried on in three steps. Let us first show that the system is uniformly stable. Define δ_1 and δ_2 the disagreement vectors for the $x(t)$ and $\dot{x}(t)$ variables as in* (12)*. By differentiating*

$$V(t) = k_x\ \|\mathsf{L}\delta_1(t)\|_1 + \frac{1}{2}\delta_2(t)^\mathsf{T}\mathsf{L}\delta_2(t) \tag{40}$$

along the Filippov's solutions of the disagreement dynamics

$$\begin{array}{rcl} \dot{\delta}_1 & = & \delta_2 \\ \dot{\delta}_2 & \overset{\text{a.e.}}{\in} & -k_x\mathsf{L}_\mathsf{c}\mathrm{SGN}(\mathsf{L}\delta_1) - k_v\mathsf{L}_\mathsf{c}\mathrm{SGN}(\mathsf{L}\delta_2) + \mathsf{L}_\mathsf{c}\xi \end{array}, \tag{41}$$

where $\xi \overset{\text{a.e}}{\in} [-\|\bar{m}\|_\infty, \|\bar{m}\|_\infty]$, and by letting $\zeta_1 \in \partial\|\mathsf{L}\delta_1\|_1$ and $\zeta_2 \in \partial\|\mathsf{L}\delta_2\|_1$, and thank to $\mathsf{L}\mathsf{L}_\mathsf{c} = \mathsf{L}_\mathsf{c}\mathsf{L} = \mathsf{L}$, after manipulations

$$\begin{array}{rl} \dot{V}(t) \overset{\text{a.e.}}{\in} & \bigcap_{\zeta_1,\zeta_2} k_x\zeta_1^\mathsf{T}\mathsf{L}\delta_2 - k_x\delta_2^\mathsf{T}\mathsf{L}\zeta_1 - k_v\delta_2^\mathsf{T}\mathsf{L}\zeta_2 + \delta_2^\mathsf{T}\mathsf{L}\xi \\ \overset{\text{a.e.}}{\leq} & -(k_v - \|\bar{m}\|_\infty)\,\|\mathsf{L}\delta_2\|_1 \leq 0 \quad \text{if} \quad k_v > \|\bar{m}\|_\infty \end{array} \tag{42}$$

By virtue of (42)*, if* (41) *is initialized in a compact domain $\boldsymbol{\mathcal{D}}_R = \{(\delta_1, \delta_2) \in \mathbb{R}^n \times \mathbb{R}^n : V \leq R\}$, $R > 0$, the perturbed system's trajectories* (41) *cannot leave it. Thus, from* (40)*, it yields $\|\mathsf{L}\delta_1\|_1 \leq R/k_x$, $\delta_2^\mathsf{T}\mathsf{L}\delta_2 \leq 2R$.*
Let $\kappa_R < \min\{2{k_x}^2/R, \lambda_2(\mathsf{L}), (k_v - \|\bar{m}\|_\infty)\sqrt{\lambda_2(\mathsf{L})/(2R)}\}$ be a sufficiently small constant; a well-posed nonsmooth candidate Lyapunov function on $\boldsymbol{\mathcal{D}}_R$ is $V_R(t) = V(t) + \kappa_R\ \delta_1^\mathsf{T}\mathsf{L}\delta_2$, which is radially unbounded and satisfies the Extended Invariance Principle conditions (Orlov (2008)) because

$$V_R(t) \geq V(t) - \frac{\kappa_R}{2}\left[\frac{R}{k_x}\|\mathsf{L}\delta_1\|_1 + \|\delta_2\|_2^2\right] > 0, \tag{43}$$

$$V_R(t) \leq \bar{c}_R(\|\mathsf{L}\delta_1\|_1 + \|\delta_2\|_2^2), \tag{44}$$

$$\bar{c}_R = \max\left\{k_x + \kappa_R\sqrt{2R/\lambda_2(\mathsf{L})}, \sqrt{R/2\lambda_2(\mathsf{L})}\right\}. \tag{45}$$

Finally, by differentiating V_R along (41)*, by suitable upper-bounds (see Pilloni et al. (2013)) and because κ_R in* (43) *can be chosen arbitrarily small, the exponential decay of $V_R(t)$ results if additionally $k_x > k_v + \|\bar{m}\|_\infty$, as it is shown by*

$$\dot{V}_R \overset{\text{a.e.}}{\leq} -\frac{\min\{\kappa_R\,(k_x - k_v - \Pi)\,, k_v - \|\bar{m}\|_\infty - \kappa_R\sqrt{2R/\lambda_2(\mathsf{G})}\}}{\max\{k_x + \kappa_R\sqrt{2R/\lambda_2(\mathsf{G})}, \sqrt{R/(2\lambda_2(\mathsf{G}))}\}}\ V_R \tag{46}$$

Finally, due to (46) *and the perturbation boundedness, and by recalling the* Quasi-Homogeneity Principle *(Orlov (2008)), it can be proved the homogeneous vector $(\delta_2; u(\delta_1, \delta_2)) \in \mathbb{R}^{2n}$ on the right-hand side of* (41) *is, component-wisely, a piecewise continuous function of degree -1 with respect to dilation and perturbations. Thus, both position and velocities consensus is achieved in a finite time regardless of the uncertainties effects.* □

4.2. Signed deviations sum flow protocols

The *signed deviations sum flow* (16) is studied in Hui et al. (2008b) for switching connected undirected networks of unperturbed integrators (17), where since $1_n^\mathsf{T}u(t) = 0\ \forall\ t \geq 0$, it achieves consensus on the average in finite-time. Note that, because of (16)-(17), the system dynamics are studied in the *Filippov's sense*, and $\mathsf{G}(t)$ is assumed connected $\forall\ t \geq 0$, while the switches occur at isolated instants, then those events can be disregarded because of zero measure without further complicating the analysis. By exploiting *homogeneity*, Hui et al. (2008a) found conditions to generalize (16) to arbitrary locally Lipschitz sum of functions of the agent deviations, whereas Xiao et al. (2009), interdependently, study the signed square root counterpart of (16), i.e.,

$$u_i(t) = -k\ \mathrm{diag}((|\mathsf{B}_1 x|^\alpha; \cdots; |\mathsf{B}_n x|^\alpha))\mathsf{B}^\mathsf{T}\mathrm{sgn}(\mathsf{B}x). \tag{47}$$

Both guarantee the finite-time average consensus for the undirected unperturbed case. Worth also mentioning Cao & Ren (2014), showing that over persistently connected switching digraphs, control (47) allows for achieving the consensus in a finite-time among nonlinear agents of the form $\dot{x}_i = \phi(x_i, t) + u_i$ if and only if the Lipschitz condition $|\phi(x_i, t) - \phi(x_j, t)| \leq \mathrm{C}_x|x_i - x_j|$ holds, where $\mathrm{C}_x > 0$ is a-priory known. At last, we

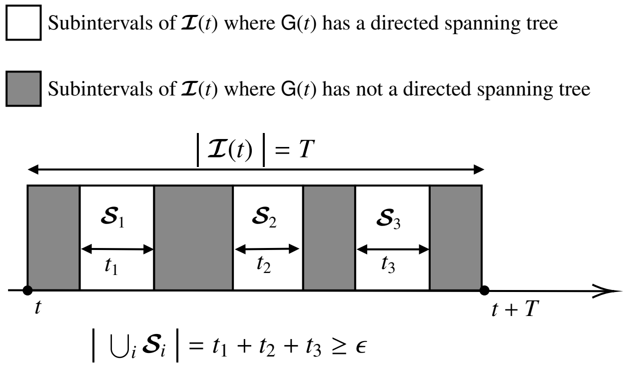

Figure 3: Let $\mathcal{I}(t) = (t, t+T)$ be the sliding time-horizon of Assumption 1. Over $\mathcal{I}(t)$, $\mathsf{G}(t)$ had a directed spanning tree for a positive time $|\bigcup_i \mathcal{S}_i| = t_1 + t_2 + t_3 \geq \epsilon > 0$, whereas it is remained disconnected for $T - (t_1 + t_2 + t_3)$.

mention Franceschelli et al. (2014b), which generalize the results of Hui et al. (2008b); Franceschelli et al. (2013b), demonstrating that a set of uncertain agents of the form (32)-(33), under (16), achieves the exact finite-time consensus (11). Moreover, if G is switching but not persistently connected (intermittent intervals of positive measure where $\mathsf{G}(t)$ remains disconnected may exist), the consensus accuracy in the practical sense (22) is further characterized (Franceschelli et al. (2013b)).

Aimed at illustrating the idea behind this last general result, Assumption 1 and Figure 3 help to understand the switching constraint formulation, while Example 12 illustrates the technicalities to be considered to approach its analysis.

Assumption 1. *Consider a digraph* $\mathsf{G}(t)$ *and a sliding time-horizon* $\mathcal{I}(t) = (t, t+T)$ *with* $T > 0$. $\mathsf{G}(t)$ *should have a directed spanning tree for a set of possibly disjointed subintervals* $\bigcup_i \mathcal{S}_i(t) \subseteq \mathcal{I}(t)$ *whose overall length* $|\bigcup_i \mathcal{S}_i(t)|$ *is at least* $\epsilon > 0$.

Example 12. (Consensus via signed deviations sum over intermittently connected uncertain networks) *Let* $\mathsf{G}(t)$ *be a time-varying digraph satisfying Assumption 1 and let agents be uncertain as in* (32)-(33). *Consider the* signed deviations sum flow (16) *with* $k > 2(T/\epsilon)\|\bar{m}\|_\infty + \mu$, *for some* $\mu > 0$. *Let* $\delta_{ij} = x_i - x_j$ *be as in* (11), *to study the network, the maximum relative deviation over* G, *such that* $(\mathrm{i},\mathrm{j}) = \arg\max_{(i,j)\in\mathsf{V}x\mathsf{V}} |\delta_{ij}|$), *is considered and it is used as a Lyapunov function*

$$V(t) = |\delta_{\mathrm{ij}}(t)| = |x_{\mathrm{i}}(t) - x_{\mathrm{j}}(t)| = \left|\sup_{i\in\mathsf{V}} x_i(t) \; - \inf_{j\in\mathsf{V}} x_j(t)\right|, \quad (48)$$

where $V(t) \to 0$ *implies* (11). *Until* $V(t) \neq 0$, (48) *is continuous, even at those instants when* "i" *or* "j" *change, whereas at the consensus it is not differentiable. Moreover, since a* sum of absolute values *is absolutely continuous,* (48) *can be viewed as Filippov's solutions. Thus, from Definition 1, and by noticing* $\dot{V}(t)$ *exists a.e. in the form of a set-valued map only when* $V(t) = 0$, *following Theorem 1, it results*

$$\begin{aligned}\dot{V}(t) &\overset{\text{a.e.}}{\in} \bigcap_{\zeta\in\partial V} \zeta\,\dot{\delta}_{\mathrm{ij}}(t) = \bigcap_{\zeta\in\partial V} \zeta\Big(\xi - k \sum_{h\in\mathsf{V},h\neq\mathrm{i}} \mathsf{a}_{\mathrm{i}h}(t)\mathrm{sgn}(\delta_{\mathrm{i}h}) \\ &+k \sum_{h\in\mathsf{V},h\neq\mathrm{j}} \mathsf{a}_{\mathrm{j}h}(t)\mathrm{sgn}(\delta_{\mathrm{j}h})\Big) \overset{\text{a.e.}}{\leq} \bigcap_{\zeta\in\partial V} \Big(2\|\bar{m}\|_\infty + \\ &-k \sum_{h\in\mathsf{V},h\neq\mathrm{i}} \mathsf{a}_{\mathrm{i}h}(t)\mathrm{sgn}(\delta_{\mathrm{i}h}) + k \sum_{h\in\mathsf{V},h\neq\mathrm{j}} \mathsf{a}_{\mathrm{j}h}(t)\mathrm{sgn}(\delta_{\mathrm{j}h})\Big),\end{aligned} \quad (49)$$

where $\mathsf{a}_{ij}(t)$ *is made explicit to account* $\mathsf{G}(t)$ *is time-varying,* $\xi \in \mathrm{K}[m_{\mathrm{i}} - m_{\mathrm{j}}] = [-2\|\bar{m}\|_\infty, 2\|\bar{m}\|_\infty]$, *and* $\zeta = 1$ *until* $V(t) \neq 0$.

Since $x_h \in co\{x_{\mathrm{j}}, x_{\mathrm{i}}\}$, $\forall\; h \in \mathsf{V} \setminus \{\mathrm{i},\mathrm{j}\}$ *irrespectively of* $\mathsf{G}(t)$, *all the terms function of the state deviations in* (49) *are non-positive* (≤ 0). *Let* $\bigcup_i \mathcal{S}_i(t)$ *be the union of sub-intervals at which* $\mathsf{G}(t)$ *has a spanning tree, then in the complementary interval where* $\mathsf{G}(t)$ *is potentially disconnected (see Figure 3), due to* (49), *one can easily note that* $\dot{V}(t) \leq 2\|\bar{m}\|_\infty$, $\forall\; t \in \mathcal{I} \setminus \bigcup_i \mathcal{S}_i$.

Besides, if $t \in \bigcup_i \mathcal{S}_i(t)$, *and although* (i,j) *is not unique (if so pick up just one), one of the two holds: a) among the agents carrying* x_{i}, *at least one has a neighbor* $\bar{h} : \delta_{\mathrm{i}\bar{h}} > 0$; *b) among those carrying* x_{j}, *at least one has a neighbor* $\bar{h} : \delta_{\mathrm{i}\bar{h}} < 0$. *Thus:* $\dot{V}(t) \leq 2\|\bar{m}\|_\infty - k$, $\forall\; t \in \bigcup_i \mathcal{S}_i$. *By integrating* (49) *in* $\mathcal{I}$, *and by replacing k, it yields*

$$\begin{aligned} V(t+T) - V(t) &= \int_{\bigcup_i \mathcal{S}_i} \dot{V}(\tau)d\tau + \int_{\mathcal{I}\setminus\bigcup_i \mathcal{S}_i} \dot{V}(\tau)d\tau \\ &\leq \epsilon\,(2\|\bar{m}\|_\infty - k) + 2\,(T-\varepsilon)\,\|\bar{m}\|_\infty < -\mu^2\epsilon < 0, \end{aligned} \quad (50)$$

thereby showing $\exists\; t_r : V(t_r) = V(0) - t_r\epsilon\mu^2 = 0 \Rightarrow$ (11). *However, for* $t \geq t_r$ *fluctuation around* (22) *may occur when* $t \in \mathcal{I} \setminus \bigcup_i \mathcal{S}_i$. *Specifically, once* $V = 0$, *from* (50), *its maximal deviation is at most* $2(T-\varepsilon)\|\bar{m}\|_\infty$. *Instead, if* $\varepsilon = T$, *since* $V = 0 \Rightarrow u_i = 0$, *whereas* $m_i(t)$ *can be nonzero, an infinitesimal deviation around* $V = 0$ *can be observed. However, as soon as* $V \neq 0$, *an arbitrarily small time later* $\bar{\epsilon} > 0$, (16) *restores* $V = 0$. *Thus, despite intermittent disconnections of* $\mathsf{G}(t)$, *and when simultaneously disturbances are acting, only a practical consensus* (22) *is achieved, where* $\Gamma \leq [2(T-\varepsilon) + \bar{\epsilon}]\;\|\bar{m}\|_\infty$ *denotes the consensus accuracy. Further notice that the transient time* t_r *after that* (48) *reaches the origin for the first time can be made arbitrarily small by selecting* μ *large enough. Indeed, a* μ*-dependent majorant curve* $\bar{V}(t)$, *illustrated in Figure 4, and of the form* $V(t) \leq \bar{V}(t) = \max_{t>0}\left[V(0) - \mu^2\epsilon(t/T) + \Gamma \quad \Gamma\right]$ *can always be defined.* ☐

Results of numerical simulations illustrating a performance comparison of the leaderless signed-deviation sum control protocol (16), under different operating conditions (with/without perturbations) are illustrated in Figure 7.

Remark 5. *The result of Example 12 is of particular interest since demonstrating the VSC provides a simple alternative way to deal with intermittent communications (cf. Wen et al. (2014); Xie & Lin (2020)) while preserving robustness to local perturbations, which is not contemplated by the other techniques.* △

Finally, we further mention Gómez-Gutiérrez et al. (2020); Dong et al. (2022) because, even if perturbations are not considered, they discuss modifications for (16) to guarantee the stability in a *fixed-time*, even in the case of actuators' saturation.

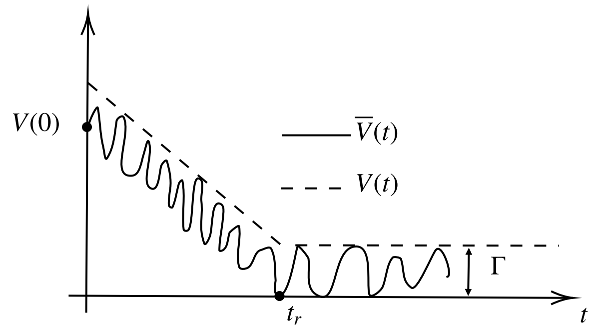

Figure 4: Graphical interpretation of the practical consensus condition encoded through the Lyapunov functional $V(t) = \left|\sup_{i\in\mathsf{V}} x_i(t) - \inf_{j\in\mathsf{V}} x_j(t)\right|$ in (48) of Example 12 and the corresponding majorant curve $V(t) \leq \bar{V}(t) = \max\left[V(0) - \mu^2\epsilon(t/T) + \Gamma \quad \Gamma\right]$, where Γ denotes the consensus accuracy, depending by the perturbation upper-bound $\|\bar{m}\|$ and the proportion of time ϵ/T, $\mathsf{G}(t)$ has a directed spanning tree, accordingly with Assumption 1.

Regarding instead the extension of (16) to second-order perturbed agents, as for (38), although in the literature several non-robust smooth finite-time version of (16) are available, e.g. expressed in terms of the sum of Lipschitz functional of the agents' deviation, and/or based on the sum of the root square agents deviation (see e.g. Wang & Hong (2008); Guan et al. (2012); Zou et al. (2019)), to the best of our knowledge, the nonsmooth twisting counterpart of (39), based on the signed deviations sum flow has not been demonstrated yet, either in the unperturbed case or in the perturbed one, thus remaining an open problem.

Remark 6. *Example 12 requires that* G *has a spanning tree to guarantee* (12). *Thus, one may suppose the leader-following extension of* (16) *is straightforward. But if the leader is non-autonomous as in* (27)*, even in the absence of model mismatching* ($m_i = 0$)*, as for* (17)*, it is not. In fact, let* $\bar{\mathsf{G}}(\{0\}\cup\mathsf{V},\bar{\mathsf{E}})$, $\mathsf{G}(\mathsf{V},\mathsf{E})$, P*,* x*, and* p *be as in Example 7, where* $\mathsf{p}_i = 1$ *if* $\{0\} \in \mathsf{N}_i(\bar{\mathsf{G}})$, 0 *otherwise, and* $\dot{x} = u$*; if* x_0 *is not piece-wise constant but driven by* $|u_0(t)| \leq \Pi$*, since* (16) *reduces to*

$$u_i(t) = \sum_{j\in\mathsf{N}_i(\mathsf{G})} k_{ij}\mathrm{sgn}(x_j(t) - x_i(t)) + k_{i0}\mathsf{p}_i\mathrm{sgn}(x_0(t) - x_i(t)), \quad (51)$$

where no distinction is made between the local gains k_{ij} *and* k_{i0} *(i.e.* $k_{i0} = k_{ij} = k$*), then multiple equilibria may result along the non-trivial error dynamics*

$$\dot{e} = -k\left[\bar{\mathsf{B}}(\mathsf{G})\mathrm{sgn}(\mathsf{B}(\mathsf{G})e) + \mathsf{P}\mathrm{sgn}(e)\right] - \dot{x}_0 1_n, \quad (52)$$

where $e(t) = x(t) - x_0(t)1_n$*. Thus, to make* (16) *applicable to non-autonomous leader-following problems, when only binary sensing orientation deviations are available, the simplest way to force* (26) *(i.e.* $e \to 0$*) is to set either the pinning gains* $\mathsf{p}_i > 1$ *if* $\{0\} \in \mathsf{N}_i(\bar{\mathsf{G}})$*, or by weighting more the signed deviation respecting the leader, through* $k_{i0} = k^\ell > k_{ij} = k > \Pi$ *in* (51)*.*

However, since this request implies the leader is recognizable over $\bar{\mathsf{G}}$ *by suitable tagging procedures, it is not always applicable because depending on the considered sensing and/or communication infrastructure. The validity of the above statements is also confirmed by the comparison of the numerical simulations illustrated, respectively, in Figure 9c and Figure 9d.* △

4.3. Evolution characterization along the consensus manifold

Section 4.1 and Section 4.2 have shown that, if the networked system maintains, through G, the connectivity and perturbations are bounded with a known upper-bound, the robust finite-time consensus can be guaranteed using the VSC approach. We also noticed that disturbances affect the agreement value $c(t)$ (see Section 3.1:(i)-(ii)), which would not be anymore the nominal expected one, namely (37) for (15), (17), or $1_n^\mathsf{T} x(0)/n$ for (16), (17), remaining unspecified and possibly unbounded.

These facts brought attention to the network evolution characterization along the consensus manifold and to two new problems, potentially more challenging than the robust consensus itself, in which the VSC plays a fundamental role again. These problems are illustrated below:

(i) Along the consensus manifold $\mathsf{L_c}x = 0_n$ the uncertainties influences the $c(t)$ of (20) while (12) holds. Thus, considering $m_i(x_i, x_j \in \mathsf{N}_i, t)$ as a part of the control $u_i(t)$ for (17), instead of being a disturbance for (20), would allow for modifying arbitrarily the network emergent behavior, allowing agents to solve more complex problems in which the consensus is only a part of the task, as for the *distributed optimization*.

(ii) Due to model mismatches, the consensus value would no longer be what was expected for the unperturbed case. Thus, understanding how to modify $u_i(t)$ in (20) in such a way the nominal expected emergent behavior is restored, regardless of perturbations, is become of interest.

Both problems are now discussed from a VSC perspective.

4.3.1. From consensus to distributed optimization

The fact that $m_i(x_i, x_j \in \mathsf{N}_i, t)$ in (20) modifies the network equilibrium along the consensus manifold (11), is the key to allow agents to achieve, in a *leaderless setting*, more complex decision tasks by devising $m_i(x_i)$ as an additional control. This problem, referred to as *distributed optimization*, was firstly formalized in discrete-time over a *connected graph* G in Nedic & Ozdaglar (2009), then in continuous-time by Wang & Elia (2011), as

$$\texttt{minimize:}\ 1_n^\mathsf{T}\,\mathsf{f}(x) + \frac{k}{2}x^\mathsf{T}\mathsf{L}(\mathsf{G})x, \quad \text{with}\ \ \mathsf{f}(x) = [\mathsf{f}_i(x_i)], \quad (53)$$

$$\texttt{subject to:}\ \mathsf{L}(\mathsf{G})x = 0_n, \qquad k \geq 0, \quad (54)$$

where, since its *Lagrangian dual problem* is $\max_z \min_x \mathsf{F}(x,z) : \mathsf{F}(x,z) = 1_n^\mathsf{T}\mathsf{f}(x) + \frac{k}{2}x^\mathsf{T}\mathsf{L}x + x^\mathsf{T}Lz$, yields its solution is found by

$$\begin{aligned} \dot{x}(t) &= -\nabla_x\mathsf{F}(x,z) = -m(x) - k\mathsf{L}x(t) - \mathsf{L}z(t) \\ \dot{z}(t) &= \nabla_z\mathsf{F}(x,z) = \mathsf{L}z(t), \end{aligned} \quad (55)$$

and $m(x) = [\nabla_{x_i}\mathsf{f}_i]$. Indeed (53) ask to optimize a sum of local convex differentiable $\mathsf{f}_i(x_i)$, where since decision variables are *separable*, then each agent can optimize, under the consensus constraint, its local objective independently through $m_i(x_i, t)$.

The extra cost in (53) has sum-zero if the constraint is feasible $\forall\ k$. Its role is to improve the solver stability and convergence since *PI-based compensators*, as in (53)-(54), perform better than just an integrator ($k = 0$). Worth also noting the *average consensus* can be formalized as in (53) and solved by (55) with

$$\sum_{i=1}^{n} \mathsf{f}_i(x_i) = \sum_{i=1}^{n} \|x_i(t) - x_i(0)\|_2^2 \Rightarrow m_i(x_i, t) = 2(x_i(t) - x_i(0)). \quad (56)$$

Later, Gharesifard & Cortés (2013) extend this result also to a sum of convex globally Lipschitz $\mathsf{f}_i(x_i)$. When the functions are norm-1,

$$\sum_{i=1}^{n} \mathsf{f}_i(x_i) = \sum_{i=1}^{n} \|x_i(t) - x_i(0)\|_1 \Rightarrow m_i(x_i, t) = \mathrm{sgn}(x_i(t) - x_i(0)), \quad (57)$$

it can be defined as the median value of a set of data as the minimum of an objective function, and thus use distributed optimization approaches to solve the consensus on the median value problem. The consensus on the median problem with finite time convergence using VSC approaches has been studied specifically in Franceschelli et al. (2014a) and later in Franceschelli et al. (2016) where the median value is defined as

$$\mathrm{med}(x(0)) \in \begin{cases} \left\{x_{\frac{n+1}{2}}(0)\right\} & \text{if } n \text{ is odd,} \\ \left[x_{\frac{n}{2}}(0), \quad x_{\frac{n}{2}+1}(0)\right] & \text{if } n \text{ is even.} \end{cases} \quad (58)$$

where in (58), the agents are ordered, without loss of generality, according to their initial conditions such that $x_1(0) \leq x_2(0) \cdots \leq x_n(0)$.

Next, motivated by finite-time convergence arguments, discontinuous solvers are developed. Among them, since VSC-based, it is worth mentioning Pilloni et al. (2016a) which, based on (16) and for quadratic local objectives $\mathsf{f}_i(x_i)$, proposed a discontinuous PI version of (55). Then, for convex differentiable $\mathsf{f}_i(x_i)$, Lin et al. (2016) proved (16), (20), namely

$$\dot{x} = -m(x) - k\, \mathsf{B}^\mathsf{T}\, \mathrm{sgn}(\mathsf{B}x) \quad \text{with} \quad m_i = \rho\, \nabla_{x_i} \mathsf{f}_i(x_i), \quad (59)$$

solves (53) without the need of the Lagrange multiplier auxiliary variable z used in (55), where $\rho > 0$ is a scalar gain aimed to speed up the optimization convergence rate. Then, we mention Franceschelli et al. (2016), that is a step further also to Gharesifard & Cortés (2013), because it proves, for the specific *median* problem (53), (54), (57) (where $m_i(x, t) = \rho\, \mathrm{sgn}(x_i(t) - x_i(0))$), that (59) with $k > \rho$ optimizes, in finite-time, also a *globally Lipschitz functional*. Finally, Li et al. (2017); Chen & Li (2018) extend the results on convex differentiable $\mathsf{f}_i(x_i)$ to the fixed-time setting.

Aimed at illustrating the technicalities behind the use of VSCs for solving distributed optimizations, the following example explains how (16), (20) optimizes the globally Lipschitz functional (57), thus converging to $\mathrm{med}(x(0))$. Notice the reason we focus on that specific application is twofold:

(i) functional (57) is a.e. differentiable and provides uniformly bounded $m_i(x_i)$, thus matching well the nonsmooth and robust consensus characterization of the paper;

(ii) (53), (57) overcomes the major drawback of the average problem (53), (56), the sensitivity to *outliers* (agents deliberately/incidentally features abnormal $x_i(0)$) or to *uncooperative agents* (agents due to *faults* cannot update their state while delivering their state to its neighborhood), thereby disrupting the emergent network behavior in the absence of counter-measures.

Example 13. (Finite-time median consensus problem) *Consider problem* (53)-(54)*, specified for* (57)-(58)*. Consider a network of agents operating over connected graph* G*, and governed by* (59)*, with* $x = (x_1; x_2 \;\cdots; x_n) : x_1(0) \leq x_2(0) \cdots \leq x_n(0)$*, and* $k > \rho > 0$*. To study* (59)*, the analysis is divided into two parts. It is shown that a finite-time consensus on* $c(t)1_n$ *is reached. Then, that* $c(t)$ *drifts to* $\mathrm{med}(x(0))$ *in a finite-time as well. Following Example 12, the consensus achievement can be proved. However, since* G *is static and undirected, a different and elegant nonsmooth Lyapunov analysis is illustrated.*

Consider $V(\delta) = \|\mathsf{B}\delta\|_1$ *for the disagreement vector* $\delta = L_c x$*. Now, since* $\mathsf{B}\mathsf{L}_c = \mathsf{B}$ *holds, then by differentiating it along the disagreement dynamic* $\dot{\delta} = -k\mathsf{L_c}\mathsf{B}^\mathsf{T}\mathrm{sgn}(\mathsf{B}\delta) + \mathsf{L_c}m$*, it results*

$$\begin{aligned} \dot{V} \in \textstyle\bigcap_{\zeta \in \partial V} \zeta^\mathsf{T}\, \mathsf{B}\, \dot{\delta} &\overset{\text{a.e.}}{\leq} -k\, \|\mathsf{B}^\mathsf{T}\zeta\|_2^2 + \|\mathsf{B}^\mathsf{T}\zeta\|_2\, \|m\|_2 \\ &\overset{\text{a.e.}}{\leq} -k\, \|\mathsf{B}^\mathsf{T}\zeta\|_\infty + \|m\|_\infty. \end{aligned} \quad (60)$$

Then, by noticing that if $\delta \neq 0_n$*, at least a pair of agents have a nonzero deviation* $(x_i > x_j)$*, then* $\zeta \in \partial V$ *surely has at least two singleton entries, one equal to* $\{+1\}$*, and the complementary equal to* $\{-1\}$*. From that, due to the* B *structure (see Figure 2h), it results that at least the i-th entry of the term* $\mathsf{B}^\mathsf{T}\zeta$ *(that is the same used in* (59)*) is singleton and equal to* $\{+1\}$*, whereas the j-th is* $\{-1\}$*. This implies* $\|\mathsf{B}^\mathsf{T}\zeta\|_\infty \geq 1$ *if* $\delta \neq 0_n$*. From that, and by noticing* (57) *gives* $\|m\|_\infty \leq \rho < k$*, then* (60) *reduces to* $\dot{V} \overset{\text{a.e.}}{\leq} -\epsilon$ *for some* $\epsilon > 0$*. Thus it further results*

$$\exists\, t_r > 0 : x(t) = c(t)1_n \Rightarrow c(t) = 1_n^\mathsf{T} x(t)/n. \quad (61)$$

Further note that, due to G*, then* $1_n^\mathsf{T}\mathsf{B}$*, and thus*

$$\dot{c}(t) = -\frac{1}{n} 1_n^\mathsf{T}\, m(x) = -\frac{\rho}{n} 1_n^\mathsf{T}\, \mathrm{sgn}(x(t) - x(0)) \quad (62)$$

Define now the disjoint sets $\mathsf{S}_+ = \{k \in \mathsf{V} : x_k(t) < x_k(0)\}$*,* $\mathsf{S}_- = \{k \in \mathsf{V} : x_k(t) > x_k(0)\}$*,* $\mathsf{S}_= = \{k \in \mathsf{V} : x_k(t) = x_k(0)\}$*, which union forms* V*. Let* $W(x) = |c(t) - \mathrm{med}(x(0))|$ *be another nonsmooth Lyapunov function, for* $t \geq t_r \Rightarrow x_i(t) = c(t)\ \forall\ i$*. By differentiating* $W(x)$*, one obtains*

$$\begin{aligned} \dot{W}(x) \overset{\text{a.e.}}{\in} \bigcap_{\xi \in \partial W} \xi\, \dot{c} = -\tfrac{\rho}{n} \bigcap_{\xi \in \partial W} \xi\, &\big[\mathrm{card}(\mathsf{S}^-) - \mathrm{card}(\mathsf{S}^+) \\ &+ \sum_{i \in \mathsf{S}^=} \mathrm{SGN}(c(t) - x_i(0))\big] \end{aligned} \quad (63)$$

where $\sum_{i \in \mathsf{S}^=} \mathrm{SGN}(c(t) - x_i(0)) \overset{\text{a.e.}}{\in} [-\mathrm{card}(\mathsf{S}^=), \mathrm{card}(\mathsf{S}^=)]$*. Now, if* $|\mathrm{card}(\mathsf{S}^-) - \mathrm{card}(\mathsf{S}^+)| > \mathrm{card}(\mathsf{S}^=)$*, then* $\dot{c} \overset{\text{a.e.}}{\geq} \rho/n$ *while* $\xi = \{-1\}$*. If* $|\mathrm{card}(\mathsf{S}^-) - \mathrm{card}(\mathsf{S}^+)| \leq \mathrm{card}(\mathsf{S}^=)$ *then* $\dot{c} \overset{\text{a.e.}}{\leq} -\rho/n$ *while* $\xi = \{+1\}$*. Finally if* $\mathrm{card}(\mathsf{S}^-) = \mathrm{card}(\mathsf{S}^+)$*, due to* (58)*, yields* $c(t) = \mathrm{med}(x(0))$*. From those considerations, and by simple manipulation, it can thus be concluded that for* $c(t)$ *different to* $\mathrm{med}(x(0))$*, results*

$$\dot{W} \overset{\text{a.e.}}{\leq} -\frac{\rho}{n} \bigcap_{\xi \in \partial W} |\xi|^2 \overset{\text{a.e.}}{\leq} -\frac{\rho}{n} < 0. \quad (64)$$

Therefore, $c(t) = \mathrm{med}(x(0))\ \forall\ t \geq t_r + t_r'$ *with* $t' = nW(0)$*.* □

For the sake of completeness, in Figure 8, numerical simulations illustrating a performance comparison of the median consensus protocol (57)-(59) under different operating conditions (with/without exogenous perturbations) is further provided.

Finally, and following (Franceschelli et al., 2016, Section V), it is also worth mentioning that, under the assumption G is *k-connected*, exploiting among others also the term of the VSC-based *median consensus protocol* (57), (59) guarantees the cooperative agents converge on a value lying on the convex hull of their initial conditions regardless the effect of (*at most*) a certain number of *uncooperative agents* that persistently influence the network, thus providing *k-safeness* resilience characteristics, exploiting a different non-smooth Lyapunov function. Also, in Sanai Dashti et al. (2019) the consensus on the median value problem was extended to a dynamic setting where consensus is performed over a set of time-varying signals and an *open*-MAS setting, i.e., the case where agents join and leave the network during the algorithm execution.

4.3.2. Nominal consensus robustification despite perturbations

In Section 4.3.1, we show how distributed optimization arguments and the VSC-based consensus control (16) can be combined to obtain the so-called *median consensus protocol*, which possesses some characteristics of robustness against outlier and, or, non-cooperative agents. Moreover, Example 12 shows also (16) is effective in the presence of intermittent interactions.

However, since their convergence is related to the fact (16) in (59) has sum-zero, if agents were heterogeneously perturbed respecting model (17), then the consensus value unpredictably drifts away from med$(x(0))$ (see e.g. Figure 8b). Now, consistently with Section 4.3:(ii), illustrates how to design an additional corrective term within the agent control to compensate for that drift. The approach we are going to illustrate has been formalized in Pilloni et al. (2016b) to address items Section 3.1:(i)-(ii) by robustify the leaderless average consensus control (18) in the case agents have uncertain dynamics as for in (20). The idea takes inspiration from Fridman et al. (2016), where the ISMC (Utkin & Shi (1996)) was used to robustify a linear-quadratic optimal controller. Here, the main difference is that the ISMC requests full state availability, thus to be implemented in a distributed setting, its reformulation to consider the interaction constraints over G has been mandatory since each agent i may access only x_i and $x_j \in \mathsf{N}_i$. Previous attempts to apply the ISMC for distributed control problems were also made in Yu & Long (2015) and Jiang et al. (2016). However, both limit the analysis to the leader-follower case where either (15), or (16), or (39), performs better since they do not add complexity to the control protocol. Worth also mentioning Pilloni et al. (2016a), which shows that the method can positively be applied to the nonsmooth median consensus control, which is also a step further to Franceschelli et al. (2016). Finally, in Pilloni et al. (2020), using non-smooth analysis tools, the approach has been generalized to arbitrarily designed *consensus* algorithms (as for the *average*, *mid-range* or *median* estimation) and *distributed optimization control solvers* (as in (55), and (59)) over *uncertain cyber-physical networked systems*.

The idea behind this approach is fully characterized by Algorithm 1 taken from Pilloni et al. (2020).

Algorithm 1 - Distributed control robustification procedure

a) Consider a set of uncertain agents satisfying (32)-(33);

b) Devise an essentially locally bounded control protocol $\mathsf{g}_i(x_i, x_j \in \mathsf{N}_i, t)$ tasked to solve a desired distributed optimization or control problem over a given G, for *nominal* agents modeled as in (17), i.e. for $\dot{x}_i(t) = \mathsf{g}_i(x_i, x_j \in \mathsf{N}_i, t)$;

c) For each agent, implement the augmented local control

$$\begin{aligned} u_i(t) &= \mathsf{g}_i(x_i, x_{j\in\mathsf{N}_i}, t) - k_i'(x_i, x_{j\in\mathsf{N}_i}, t)\, \nabla\|x_i(t) - q_i(t)\|_1 \\ \dot{q}_i(t) &= \mathsf{g}_i(x_i, , x_{j\in\mathsf{N}_i}, t), \quad q_i(0) = x_i(0) \end{aligned} \tag{65}$$

where $q_i \in \mathbb{R}^d$ is an auxiliary integral control, and $k_i'(x_i, , x_{j\in\mathsf{N}_i}, t) \in \mathbb{R}_{>0}$ is a scalar function of class C^0;

d) If $k_i' > \bar{m}_i \ \forall\ t \geq 0,\ \forall\ i \in \mathsf{V}$, the agents' state trajectories of the uncertain networked system track exactly those of the reference system devised at point **b).**

Last, we mention the preliminary study Ferrara & Zambelli (2019), which proposes the implementation of the so-called *Suboptimal HOSMC* (Bartolini et al. (1998)) in a *High-order ISMC* fashion to solve the robust second-order leader-following distributed tracking problem, by exploiting only neighboring relative position measurements.

Aimed at discussing the main technical aspects behind this methodology, the following example illustrates the median consensus protocol robustification.

Example 14. (VSC-based control protocol robustification) *Consider a set of uncertain agents of the form* (32)-(33)*, where the model mismatching* $m_i(x_i, x_j \in \mathsf{N}_i, t)$ *satisfy* $\|m_i\| \leq \bar{m}_i$. *Suppose it is asked the network to solve the optimization* (53)-(54), (57) *over a connected* G. *As discussed, the median consensus protocol* (57), (59), *alone, is not be able to converge to* (58). *Now, it is shown that Algorithm 1, specified for*

$$\mathsf{g}_i(x_i, x_j \in \mathsf{N}_i, t) = -\mathrm{sgn}(x_i(t) - x_i(0)) - k\, \mathsf{B}^\mathsf{T}\, \mathrm{sgn}(\mathsf{B}x(t)), \tag{66}$$

allows each agent to track exactly the state trajectories of the nominal multi-agent system, satisfying (11) *with* $c(t) =$ med$(x(0))$, *where* $x = (x_1; x_2; \cdots; x_n) : x_1(0) \leq \cdots \leq x_n(0)$.

To study the dynamics described by (20)-(33), (65)-(66)*, let us introduce the sliding variable* $\sigma(t) = [\sigma_i(t)] = [x_i(t) - q_i(t)]$. *Then by applying this change of coordinate, one obtains*

$$\dot{x}(t) = m(x, t) + \mathsf{g}(x, t) - k'(x, t)\, \nabla\|\sigma(t)\|_1 \tag{67}$$

$$\dot{\sigma}(t) = m(x, t) - k'(x, t)\, \nabla\|\sigma(t)\|_1, \tag{68}$$

where $k' = \mathrm{diag}((k_1'; \cdots; k_n'))$. *Now, since* (68) *is homogeneous, it can be studied separately using the nonsmooth Lyapunov function* $V(\sigma) = \|\sigma(t)\|_1$ *where, due to* (65), $\partial\|\sigma(0)\|_1$ *is set-valued component-wisely. By differentiating* $V(t)$ *along the Filippov's solutions of* $\dot{\sigma}(t) \overset{\text{a.e.}}{\in} \mathrm{K}\,[m(x, t) - k'(x, t)\, \nabla\|\sigma(t)\|_1]$, *and* *by exploiting the chain rule* (5)*, one obtains*

$$\begin{aligned}\dot{V} &\overset{\text{a.e.}}{\in} \bigcap_{\zeta\in\partial V} \zeta^{\mathsf{T}}\, \dot{\sigma} \overset{\text{a.e.}}{=} \bigcap_{\zeta\in\partial V} \zeta^{\mathsf{T}}(\xi - k'\zeta) \overset{\text{a.e.}}{\leq} \bigcap_{\zeta\in\partial V} \|\zeta\|_2(\|\xi\|_2 - k'\|\zeta\|_2) \\ &\overset{\text{a.e.}}{\leq} - \bigcap_{\zeta\in\partial V} (k' - \|\bar{m}\|_\infty) \overset{\text{a.e.}}{\leq} -\epsilon < 0 \quad \text{with} \quad \xi \in [-\|\bar{m}\|_\infty, \|\bar{m}\|_\infty],\end{aligned}$$

for some $\epsilon > 0$*, where* $\|\zeta\|_\infty \geq 1\ \forall \sigma \neq 0_n$*. From that it results* $\sigma \overset{\text{a.e}}{=} \dot{\sigma} \overset{\text{a.e}}{=} 0_n$ *a.e. on* $t \geq 0$*. Specifically, since* σ *is absolutely continuous because it is a Filippov solution, whenever one or more of their entries are zero for a measure zero interval, according to Filippov (2013), those discontinuity points on* $\dot{\sigma}_i(t)$ *can be ignored. Whereas, if one or more of their entries are zero on a positive measure set, due to its absolute continuity, in the same intervals,* $\dot{\sigma}_i(t) = 0_n$*. Thus,* $\sigma = x - q \overset{\text{a.e}}{=} 0$ *is an invariant set* $\forall\ t \geq 0$*. Thus, by replacing* $\dot{\sigma} \overset{\text{a.e}}{=} 0$ *into* (67) *it turn out* $\dot{x} \overset{\text{a.e}}{=} \mathsf{g}(x, t)$*, thus confirming, in accordance with Example 13,* $x(t) \to \mathrm{med}(x(0))1_n$ *after a finite-time interval* $t \geq t_r$*.* □

To confirm the generality of Algorithm 1, results of numerical simulations related with the robustification of: the *linear average*; the *midrange finite-time*; the *average finite-time*; and the *median finite-time* consensus control protocols are provided in Figure 5c, Figure 6c, Figure 7c, Figure 8c, respectively.

5. Numerical simulations

Numerical simulations have been devised and implemented on the MATLAB®/Simulink environment with the Euler fixed-step solver and the sampling time was set to 10^{-4} seconds.

The corresponding results are here illustrated in the following figures, from Figure 5 to Figure 9. There a detailed comparison of the consensus approaches and techniques discussed in the previous sections is given. To make the explanation easier, and more readable, the caption of each figure is self-contained, in the sense that it contains all the information necessary to replicate the results, along with the pertinent comments, and pointers to the previous sections, targeted examples, or considerations so that one could immediately return to the mentioned parts of this treatment.

Specifically, Figure 5 shows the limitations of the leaderless Laplacian flow control protocol (13), focusing on a leaderless linear average consensus problem. Then, Figure 6, Figure 7, and Figure 8 illustrate their finite-time robust consensus characteristics by considering, respectively, the finite-time mid-range, average and median nonsmooth consensus problems.

For a clear comparison among the various protocols and the reader's understanding, all simulations consider the connected static graph G_3 drawn in Figure 2g. When considering the leader-following case, the same agents' network is connected to the leader as in $\bar{\mathsf{G}}_5$ in Figure 2i, where node 0 denotes the leader. The perturbation vector m is always modeled as exogenous nonsmooth biased clipped noises, sampled at 0.1 seconds, and such that $m_i \in [0, 1]$ (Van Vleck & Middleton (1966)).

6. Concluding remarks and future challenges

Motivated by the application of the consensus control paradigm to cyber-physical agents operating in uncertain environments, this manuscript reviews, from a tutorial perspective, most of the interesting recent developments related to the application of the VSC paradigm to a wide range of cooperative control problems.

The basic robust coordination issue in uncertain multi-agent systems, i.e., the *finite-time robust consensus problem*, either with or without the presence of leader agents, is presented as a direct application of the VSC robustness features; then, some variants, aimed to provide: *k-safeness* resilience characteristics to intermittent communications, *robust distributed tracking* and *robustification* of arbitrary *optimization* and *consenus problem*, have been presented, with some hints about their high-order extension.

The two main families of consensus-based VSCs have been illustrated, respectively the *signed Laplacian flow* and the *signed deviations' sum flow*, through a set of examples accompanying the reader to understand the main technicalities behind the problem of analysis of multi-variable coupled ordinary differential equations with the discontinuous right-hand side.

The potential benefits of the VSC approach with SM to the control and coordination of MAS look apparent allowing performances that cannot be assured by the classic continuous controllers. The price to pay is related to the usual problems in the implementation of the VSC in real applications, which are even emphasized by the MAS context.

The high robustness performance of VSS with SM is guaranteed by the infinite frequency switching of the control variable, or, as in the case of the application of higher-order sliding mode control algorithms, on one of its time derivatives. The limits of the actuator dynamics prevent the infinite frequency switching characterizing the SM approach, such that the chattering phenomenon appears and has to be taken into account. The design of SMC guaranteeing that the residual oscillations due to the chattering phenomenon fulfill some bound to avoid system's damages in real applications is still to be fully investigated in the MAS context, as well as the design of smoother robust controllers, e.g., those based on higher-order SMs, which are recognized as an approach to attenuate chattering.

Noting that HOSMs constrain the motions of the system on a well-defined higher-order manifold, it would also be interesting to investigate the exploitation of this feature in multi-agent systems characterized by higher-order dynamics.

Indeed, and following what mentioned also in Section 4, for the moment, only the 2SMC twisting-based extension of the *signed Laplacian flow* has been formally demonstrated using Lyapunov arguments. Conversely, the 2SMC *signed deviations' sum flow* counterpart, as well as any h-order nonsmoth flows for $h \geq 3$, have yet to be demonstrated, either in unperturbed or perturbed scenarios. Indeed, it can be stated that the process of discovering a nonsmooth Lyapunov function for a system of multi-variable coupled ordinary differential equations with a discontinuous right-hand side is a very challenging undertaking.

To reduce chattering, an alternative approach, could be also limiting at the minimum necessary the control authority needed to counteract the matching uncertainty. Usually, the tuning of the SMC is based on a worst-case scenario that implies, theoretically, a switching control with a high and conservatively overestimated magnitude. The usual adaptation approaches to online modifying the control magnitude could be considered in the design of VSC to the consensus problem, but the effect of interactions due to the connections among the agents has to be well considered and analyzed. As an example, let think to the fascinating varieties of stable generalized equilibrium the *signed deviations' sum flow* can reach at the consensus.

Finally, since the VSC can fully counteract only the effect of matching uncertainties once it is well-tuned, an interesting problem that deserves future investigation is the design of VSC algorithms and structures that, implemented in a MAS context, allow for the limitation of the propagation of unmatched uncertainties. Such a problem is of particular interest in the case of higher-order cyber-physical systems in which the couplings between agents can appear in an unmatching way. Lastly, an additional challenging avenue for future research would be to the extension of the existing VSC-based consensus developments to the class of stochastic differential inclusions. This would enable to consider asynchronous, and potentially unbounded uncertain stochastic events that are common at communication level of a multi-agent system.

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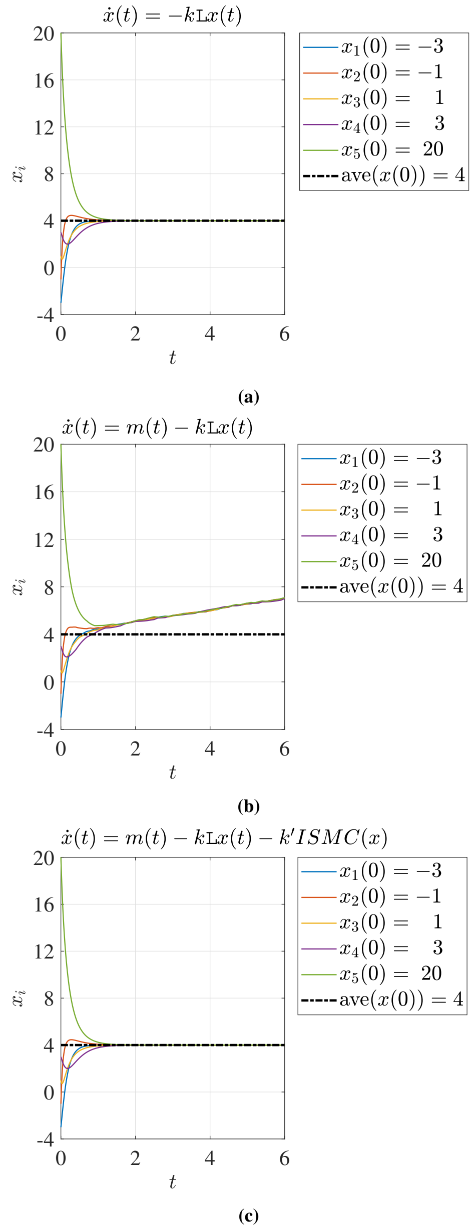
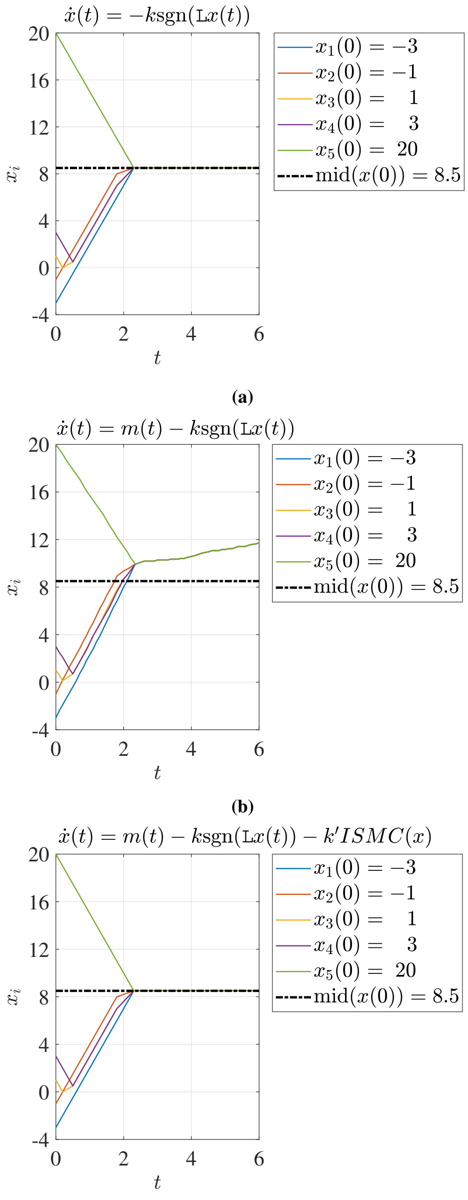


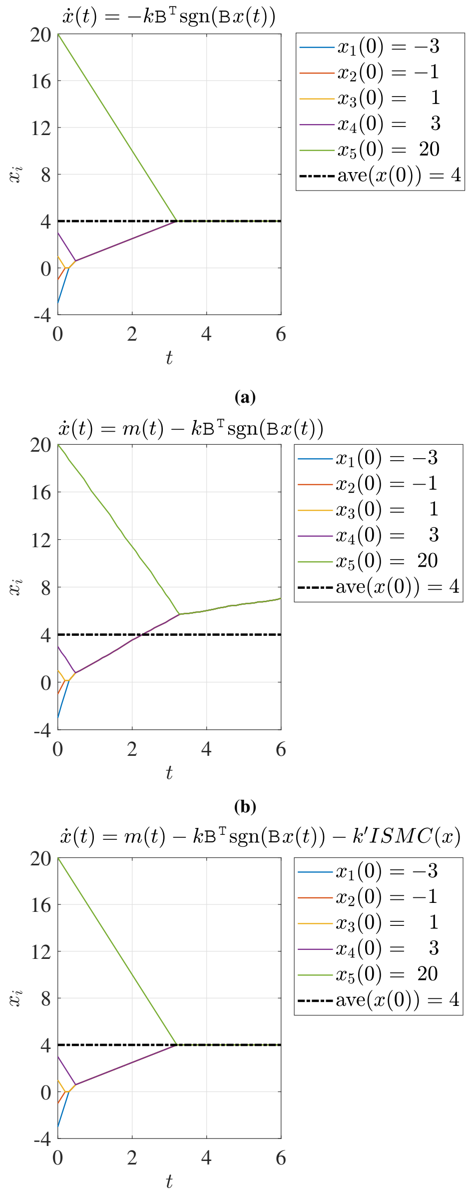
Figure 5: (Leaderless linear average consensus) This figure illustrates the sensitivity of the linear Laplacian flow protocol (13) to the effects of model mismatching and disturbances while highlighting how, thanks to the VSC, it is possible to make the uncertain closed-loop system robust. Some details on the numerical results are given. Consider the connected static graph G_3 drawn in Figure 2g. (a): Depicts the response of the nominal multi-agent system (13), (17), namely $\dot{x} = -k\mathsf{L}x$, for $k = 5$. As expected, and coherently with Example 5, the agents' states converge exponentially to the average of the agents' initial state, namely $\mathrm{ave}(x(0)) = 1_n^\top x(0)/n$; (b) Depicts the response of the uncertain multi-agent system (13), (20), namely $\dot{x} = m - k\mathsf{L}x$, for $k = 5$, and the perturbation vector m is modeled as exogenous nonsmooth biased clipped noises, sampled at 0.1 seconds, and such that $m_i \in [0, 1]$ (Van Vleck & Middleton (1966)). As expected, and coherently with Section 3.1:(i)-(ii), agents do not converge anymore to $\mathrm{ave}(x(0))$ and can achieve only a practical consensus (22) around a perturbation-dependent, unbounded, and unpredictable trajectory; (c) Depicts the response of the uncertain multi-agent system (13), (20), robustified using the VSC-based controller (65). Coherently with Section 4.3.2, the perturbation effects to the consensus trajectory are completely compensated, and the expected average consensus value is restored (cf. with (a)).



(a)

(b)

(c)



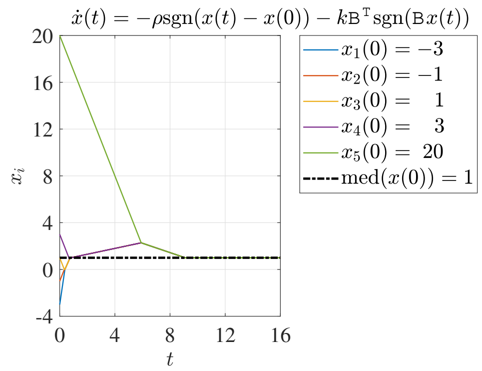
(a)

(b)

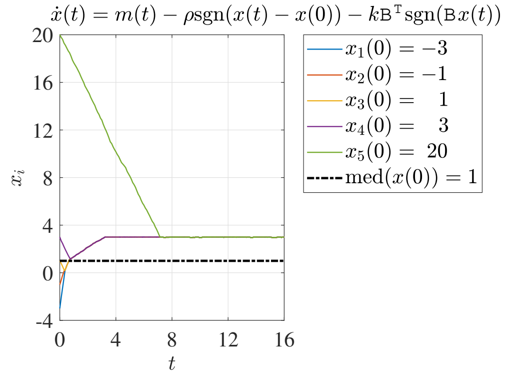
(c)

Figure 6: (Leaderless nonsmooth finite-time mid-range consensus) This figure illustrates a response comparison of the leaderless VSC-based signed Laplacian flow control protocol (15) of Example 9. Some details on the numerical results are given. Consider the connected static graph G_3 drawn in Figure 2g. (a): Depicts the response of the nominal multi-agent system (15), (17), namely $\dot{x} = -k\mathrm{sgn}(\mathsf{L}x)$, for $k = 5$. Coherently with Example 9, the agents' states converge to the mid-range of the agents' initial state, namely $\mathrm{mid}(x(0)) = (\max_i x_i(0)) - \min_i x_i(0))/n$, in a finite-time. It is worth noting that the mid-range gives information on the central tendency of $x(0)$ and that such agreement value can only be reached by exploiting nonsmooth arguments; (b) Depicts the response of the uncertain multi-agent system (15), (20), namely $\dot{x} = m - k(\mathsf{L}x)$, for $k = 5$, and the perturbation vector m is modeled as exogenous nonsmooth biased clipped noises, sampled at 0.1 seconds, and such that $m_i \in [0, 1]$ (Van Vleck & Middleton (1966)). As expected agents reach the consensus in finite-time, however, they do not converge to $\mathrm{mid}(x(0))$ anymore, but to an unpredictable, perturbation-dependent, and possibly unbounded trajectory; (c) Depicts the response of the uncertain multi-agent system (15), (20), robustified using the VSC-based controller (65). Coherently with Section 4.3.2, the perturbation effects to the consensus trajectory are completely compensated, and the expected mid-range consensus value is restored (cf. with (a)).

Figure 7: (Leaderless nonsmooth finite-time average consensus) This figure illustrates a response comparison of the leaderless VSC-based signed deviation sum flow control protocol (16) considered also in Example 12. Some details on the numerical results are given. Consider the connected static graph G_3 drawn in Figure 2g. (a): Depicts the response of the nominal multi-agent system (16), (17), namely $\dot{x} = -k\mathsf{B}^{\mathsf{T}}\mathrm{sgn}(\mathsf{B}x)$, for $k = 5$. Coherently with Example 12, the agents' states converge to the average of the agents' initial state, namely $\mathrm{ave}(x(0)) = 1_n^{\mathsf{T}} x(0)/n$, in a finite-time (cf. with Figure 5a where the convergence rate is only exponential); (b) Depicts the response of the uncertain multi-agent system (16), (20), namely $\dot{x} = m - k\mathsf{B}^{\mathsf{T}}\mathrm{sgn}(\mathsf{B}x)$, for $k = 5$, and the perturbation vector m is modeled as exogenous nonsmooth biased clipped noises, sampled at 0.1 seconds, and such that $m_i \in [0, 1]$ (Van Vleck & Middleton (1966)). As expected agents reach an exact consensus equilibrium in a finite time, however, they do not converge to $\mathrm{ave}(x(0))$ anymore, but to an unpredictable, perturbation-dependent, and possibly unbounded trajectory; (c) Depicts the response of the uncertain multi-agent system (16), (20), robustified using the VSC-based controller (65). Coherently with Section 4.3.2, the perturbation effects to the consensus trajectory are completely compensated, and the expected average consensus value is restored (cf. with (a)).



(a)



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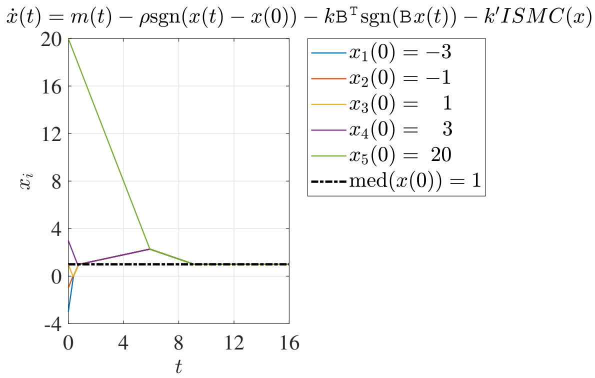

(c)

Figure 8: (Leaderless nonsmooth finite-time median consensus) This figure illustrates a response comparison of the leaderless median consensus control of Example 13. Consider the connected static graph G_3 drawn in Figure 2g. (a): Depicts the response of the nominal multi-agent system $\dot{x}(t) = -\rho\mathrm{sgn}(x(t) - x(0)) - k\mathsf{B}^\mathsf{T}\mathrm{sgn}(\mathsf{B}x(t))$, for $\rho = 1, k = 5$. As expected the network converges to the median value $\mathrm{med}(x(0))$ in (58), in a finite time. Note that, compared to $\mathrm{ave}(x(0))$, since $\mathrm{med}(x(0))$ is statistically robust to outliers, now agent 5 less affects the consensus value (cf. with Figure 5a). Further note $\mathrm{med}(x(0))$ can only be estimated by exploiting nonsmooth arguments; (b) Depicts the response of the uncertain multi-agent system $\dot{x}(t) = m(t) - \rho\mathrm{sgn}(x(t) - x(0)) - k\mathsf{B}^\mathsf{T}\mathrm{sgn}(\mathsf{B}x(t))$, for $\rho = 1, k = 5$, and m is a nonsmooth perturbation vector of biased clipped noises, sampled at 0.1 seconds, and such that $m_i \in [0, 1]$ (Van Vleck & Middleton (1966)). Since (16) is embedded in the control, the consensus is achieved in finite-time, but no more to $\mathrm{med}(x(0))$ but to a perturbation-dependent value. However, compared with Figure 5b, 6b, 7b, such value is now bounded, and lies on the convex hull of $x(0)$ (Franceschelli et al. (2016)); (c) Depicts the response of the uncertain multi-agent system tasked to estimate $\mathrm{med}(x(0))$, robustified using the VSC-based controller (65). Coherently with Section 4.3.2, the perturbation effects to the consensus trajectory are completely compensated, and the expected median consensus value is restored (cf. with (a)).

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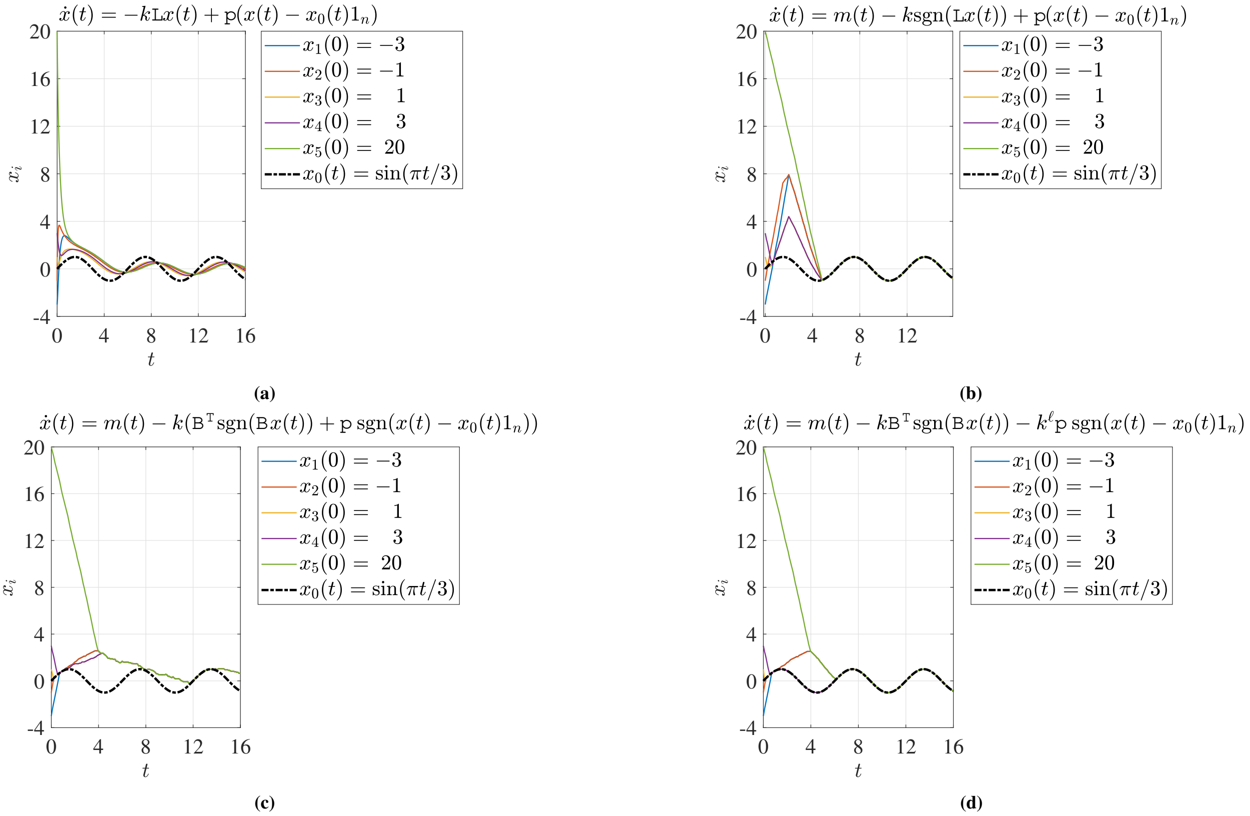

Figure 9: (Leader-following smooth and nonsmooth control protocol response comparison) This figure illustrates a response comparison of the leader-following consensus protocols introduced in the previous sections. Some details on the numerical results are given. Consider $\bar{\mathsf{G}}_5$ in Figure 2i, where node 0 denotes the leader, which state trajectory is $x_0(t) = \sin(\pi t/3)$. Moreover, $\mathsf{p} = [\mathsf{p}_i] = (0; 0; 1; 0; 0)$, in Figure 2j, denotes only follower 3 may access its leader relative deviation. (a): Depicts the response of a network of nominal follower agents of the form (17), controlled by the linear Laplacian flow control protocol (13), and which dynamics are compactly expressed by $\dot{x} = -k\mathsf{L}x + \mathsf{p}(x - x_0)$, for $k = 5$. Coherently with Example 7, and due to the effect of $|\dot{x}_0| \le \Pi = pi/3$ into the tracking error dynamics (28), then followers could only reach a practical consensus condition (22) around $x_0(t)$; (b) Depicts the response of the uncertain follower agents of the form (20), controlled by the nonsmooth signed Laplacian flow control protocol (15) and which dynamics are compactly expressed by $\dot{x} = m - k\mathrm{sgn}\mathsf{L}x + \mathsf{p}(x - x_0)$, for $k = 5$, and the perturbation vector m is modeled as exogenous nonsmooth biased clipped noises, sampled at 0.1 seconds, and such that $m_i \in [0, 1]$ (Van Vleck & Middleton (1966)). Coherently with Example 10, the network of followers can track exactly the leader trajectory regardless the effects of both $\dot{x}_0$ and the model mismatching m; (c) (resp. (d)) Depicts the response of the uncertain follower agents of the form (20), controlled by the nonsmooth signed deviation sum flow (16), and which dynamics are compactly expressed by $\dot{x}(t) = m(t) - k(\mathsf{B}^\intercal \mathrm{sgn}(\mathsf{B}x(t)) + \mathsf{p}\ \mathrm{sgn}(x(t) - x_0(t)1_n)$ for $k = 5$ (resp. $\dot{x}(t) = m(t) - k\mathsf{B}^\intercal \mathrm{sgn}(\mathsf{B}x(t)) - k^\ell \mathsf{p}\ \mathrm{sgn}(x(t) - x_0(t)1_n)$ for $k = 5$ and $k^\ell = 10$), whereas the perturbation vector is as in (b). Coherently with the considerations of Remark 6, one note only the scenarios illustrated in (d) track exactly the leader trajectory regardless of the effect of $\dot{x}_0$ and m. This is simply because, thanks to the fact $\bar{\mathsf{G}}_5$ has a spanning tree, and $k_i^\ell > k > \Pi + 1$, the gain associated with the leader's relative signed deviation in (51), dominates all the other terms in the right-hand side of (52), thus allowing the establishment of a sliding motion along the local manifold $e_3 = x_3 - x_0$. Then, once $e_3 \to 0$ after a finite time, all others also converge to zero one after another, by the given topological constraints encoded by $\bar{\mathsf{G}}_5$, after a finite time.

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