Robotic-like formulation of the approximated body-guidance problem

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Abstract. A classical problem in the mechanics of mechanisms is the body-guidance synthesis. As first formulated by Burmester, the problem consists of finding the dimensions of a planar four-bar linkage whose coupler link attains a prescribed set of finitely separated poses. The problem is solved either in exact, up to five prescribed poses, or in approximate forms by several methods. Many of them rely on the algebraic geometry to find center- and circle-point loci of the RR dyads composing the mechanism. The method was also used to find the circle-point locus of the PR dyad. In this paper a different approach was followed. We propose a formulation of the problem by using the vector loop equations, usually employed in robotics for kinematic analysis, to obtain the set of nonlinear synthesis equations then solved by advanced and stabilized algorithms. The method allows us to achieve the approximate solution of the body-guidance problem either with RR or PR dyads with high accuracy also including prescribed timing.

Keywords: Burmester's problem · dimensional synthesis · Trust region-reflective least squares algorithm.

1 Introduction

The body-guidance problem has the goal to find the geometric parameters of a four-bar linkage for a prescribed set of finitely separated poses as firstly formulated by Burmester [1]. The problem represents a milestone in the kinematics community that still receives a considerable attention of researchers because of very numerous applications of this linkage. The classical approach reduces the problem to the dyad synthesis (repeated two times to compose the whole linkage), a *dyad* being a rigid link carrying two kinematic pairs.

The problem admits exact real solutions for five poses as roots of a quartic equation. Instead, in the case of four poses it has infinitely many solutions since each RR dyad leads to two cubic curves, *i.e.*, *centrepoint*, *circlepoint*, locus of solutions of the problem. In the light of the approach followed in this paper it is noteworthy to point out here that the five poses problem has to be solved numerically although it has analytical solutions.

As reported in [2], authors in [3–5] solved the five-pose problem by intersecting two centrepoint curves of two four-pose subsets out of the given five-pose set, to obtain the centrepoints. Al-Widyan, *et al.*, developed a robust algorithm based on dyalitic elimination method [6]. Some authors solved the problem by method based either on projective geometry, or via the kinematic mapping [7, 8], Sandor and Erdman used complex numbers in their textbook [9]. Only few works addressed the synthesis of the four-bar linkage in presence of the PR dyad. Angeles provided a comprehensive solution of the Burmester problem including this case [2, 10]. The same author, *et al.*, dealt with the approximate synthesis of the four-bar linkage in order to best approximate a large number of prescribed poses [11]. The approach followed exploited symbolic computations to reduce the normality conditions of the approximate kinematic synthesis problem to a set of two bivariate equations and a refinement by means of the Newton-Raphson method to the desired accuracy.

The idea of this paper is to use the loop equations, usually employed for kinematic analysis, as synthesis equations. That leads to a formulation of the problem that differs from the others as the positional variables appear as unknowns besides the dimensional parameters. The equations turn to be highly nonlinear whose approximate solutions can be found as minimum of a constrained optimization problem by means of modern numerical algorithms.

The formulation can be easily extended to linkage with PR dyad and it is well suited for treating timing problem, as well.

2 Problem Formulation

The body-guidance problem, a.k.a. Burmester problem, can be stated as:

Given a set of discrete set of m poses, of a rigid body attached to the coupler link of a four-bar linkage, the problem consists in finding the geometrical parameters of the linkage such that the poses are attained.

The set of given poses can be parametrized by $\{\mathbf{r}^{(j)}, \phi^{(j)}\}_1^m$, $\mathbf{r}^{(j)}$ is the position

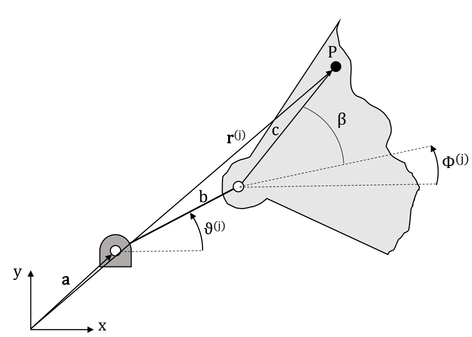

Fig. 1. Notation of the RR dyad.

vector of a reference point of the coupler link at the j-pose and $\phi^{(j)}$ is the corresponding angle of a line of the coupler link.
We formulate the *synthesis equations* from the kinematic equations (loop equations) of each chain of the linkage. Therefore, we treat the kinematic equations as a system of $2m$ nonlinear equations (or $4m$ if the timing problem is taken into account):

$$\mathcal{F}(\mathbf{P}^{(j)}, \mathbf{X}^{(j)}, \boldsymbol{\Pi}) = \mathbf{0}, \quad j = 1, \ldots, 2m, \tag{1}$$

where $\mathbf{X}^{(j)}$ is the array of the linkage positional variables, robot-like *joints variables*, at the j-pose, $\boldsymbol{\Pi}$ is the array of the linkage dimensional parameters and $\mathbf{P}^{(j)}$ is the array with $\mathbf{r}^{(j)}$ and $\phi^{(j)}$. Eq. 1 has both the linkage positional variables and dimensional parameters as unknowns. Dimensions of the unknowns arrays depend on the problem treated leading either to a determined or to a overdetermined system.

2.1 Numerical Algorithm

The idea of this paper is to solve the synthesis problem with advanced and stabilized methods based on the numerical solution of a set of nonlinear equations. By grouping the unknowns in an unique array $\mathbf{x}$, Eq. 1 can be expressed as $\{f_j(\mathbf{x}) = 0\}_1^{2m}$ whose solution is computed by solving the nonlinear least-squares problem stated as:

$$F(\mathbf{x}) := (f_1(\mathbf{x})^2 + \ldots + f_j(\mathbf{x})^2 + \ldots + f_{2m}(\mathbf{x})^2).$$

$$\underset{\mathbf{x}}{\text{minimize}}\{F(\mathbf{x})\}$$

$$\text{with} \qquad l_i \leq x_i \leq u_i,\ i = 1, \ldots, n+m.$$

where n is the dimension of $\boldsymbol{\Pi}$.
The minimization problem is implemented in the built-in function *lsqnonlin* of `Matlab`. The function uses the *trust-region-reflective-algorithm* briefly recalled here for an unconstrained minimization problem.
First the Taylor approximation of the function in the neighborhood $\mathbf{s}$ of the point $\mathbf{x}_k$ (k-iterate) is considered:

$$q(\mathbf{s}) \simeq F(\mathbf{x}_k) + \frac{1}{2}\mathbf{s}^T\mathbf{H}\mathbf{s} + \mathbf{s}^T\mathbf{g} \tag{2}$$

In Eq. (2) $\mathbf{g}$ is the gradient of $F(\mathbf{x}_k)$, $\mathbf{H}$ is the Hessian matrix. $\mathbf{s}$ is the trial step to be sought such that the the trust-region subproblem is solved:

$$\begin{array}{ll} \underset{\mathbf{s}}{\text{minimize}} & \{q(\mathbf{s})\} \\ \text{s.t.} & \|\mathbf{D}\mathbf{s}\| \leqslant \Delta \end{array}$$

with $\mathbf{D}$ is a diagonal scaling matrix, Δ is a positive scalar. The minimization problem is solved by means of a iterative Newton method applied to the secular equation $\frac{1}{\Delta} - \frac{1}{\|\mathbf{s}\|} = 0$, [12]. This approach typically needs computations of the $\mathbf{H}$

eigenvalues. However, the algorithm implemented in the built-in `Matlab` function reduces the problem to a two-dimensional subspace $\mathcal{S}$ such that only a (2×2) matrix has to be dealt with. The subspace $\mathcal{S}$ is defined as the linear space spanned by $\mathbf{s}_1$ and $\mathbf{s}_2$. $\mathbf{s}_1$ is in the direction of the gradient (scaled gradient direction) of $\mathbf{g}$ whereas $\mathbf{s}_2$ is obtained from a conjugate gradient process returning either:

$$\mathbf{H}\mathbf{s}_2 = -\mathbf{g}: \text{ approx. Newton direction}$$
$$\text{or}$$
$$\mathbf{s}_2^T\mathbf{H}\mathbf{s}_2 < 0: \text{ negative curvature direction.}$$

Then, the trial step is chosen as one of three: *i)* the scaled gradient solution; *ii)* $\mathcal{S}$ trust region solution; and *iii)* reflected $\mathcal{S}$ trust region solution. The choice is made by comparing the approximation functions $q(\mathbf{s})$ and picking that one producing its lowest value.
Once the trial step is computed it may possible to compare $F(\mathbf{x}_k + \mathbf{s})$ with $F(\mathbf{x}_k)$. If $F(\mathbf{x}_k + \mathbf{s}) < F(\mathbf{x}_k)$ then the point is uploaded such that $\mathbf{x}_k = (\mathbf{x}_k + \mathbf{s})$ otherwise the current point remains unchanged, Δ is shrunk and the trial step computation is repeated.

3 Synthesis equations for a Four-bar linkage

We address the synthesis equations for a four-bar linkage formed by either two RR dyads or a RR dyad with a PR dyad.
In the case of the RR dyad we have Eq. (3), (Figure 1).

$$\mathbf{a} + \mathbf{b}^{(j)} + \mathbf{c}^{(j)} - \mathbf{r}^{(j)} = \mathbf{0}, \quad j = 1, \ldots, m. \tag{3}$$

Eq. (3) is nothing but that the loop equation of the RR chain at an arbitrary configuration. It leads to $2m$ scalar equations.

$$\begin{aligned} a_x + b\cos\theta^{(j)} + c\cos(\phi^{(j)} + \beta) - r_x^{(j)} &= 0, \\ a_y + b\sin\theta^{(j)} + c\sin(\phi^{(j)} + \beta) - r_y^{(j)} &= 0. \end{aligned} \tag{4}$$

Eqs. (4) have five dimensional and m positional parameters as unknowns:
$a_x,\ a_y,\ b,\ c,\ \beta,\ \theta^{(j)}$. Thus, to have a determined system of nonlinear equations we can select five arbitrary poses, *i.e.*, $5 + m = 2m$.
It is noteworthy that the approach followed is completely different from those used in literature mainly based on the algebraic geometry with the goal to find center- and circle-point loci of the RR dyads composing the mechanism. Furthermore, the method has the great advantage to be well suited for solving the motion generation problems with prescribed timing. In this case m poses are known as well as the velocity of the reference point and the angular velocity of the coupler link at the m poses: $\{\dot{\mathbf{r}}^{(j)}, \dot{\phi}^{(j)}\}_1^m$ leading to a system formed by Eqs. (4) and their derivatives with respect the time:

$$\begin{aligned} b\sin\theta^{(j)}\dot{\theta}^{(j)} + c\sin(\phi^{(j)} + \beta)\dot{\phi}^{(j)} + \dot{r}_x^{(j)} &= 0, \\ b\cos\theta^{(j)}\dot{\theta}^{(j)} + c\cos(\phi^{(j)} + \beta)\dot{\phi}^{(j)} - \dot{r}_y^{(j)} &= 0. \end{aligned} \tag{5}$$

Eqs. (4) and Eqs. (5) form a system of $4m$ nonlinear equations with $5 + 2m$ unknowns as $\{\dot{\theta}^{(j)}\}_1^m$ has to be computed, too. Therefore, in this case, it may be possible to obtain approximate solutions with at least $m = 3$ poses.
A caveat is in order here. There exists an alternative method to deal with the synthesis of RR four-bar linkage. Indeed Eq. (3) can be reshaped to obtain one scalar equation removing $\theta^{(j)}$ as unknown.

$$\mathbf{b}^T\mathbf{b} - \mathbf{r}^T\mathbf{r} + 2(\mathbf{r}^T\mathbf{a} + \mathbf{r}^T\mathbf{c} + \mathbf{a}^T\mathbf{c}) = 0. \tag{6}$$

Therefore we can form a determined system imposing $m = 5$ poses as in the general method.
Also, nothing changes when the prescribed timing problem is dealt with. Indeed in this case two scalar equations can be written and only an overdetermined system can be cast imposing at least $m = 3$.
The method presented can be easily extended to synthesize four-bar linkages with the presence of one PR dyad. In this case (Figure 2), the kinematic equation of

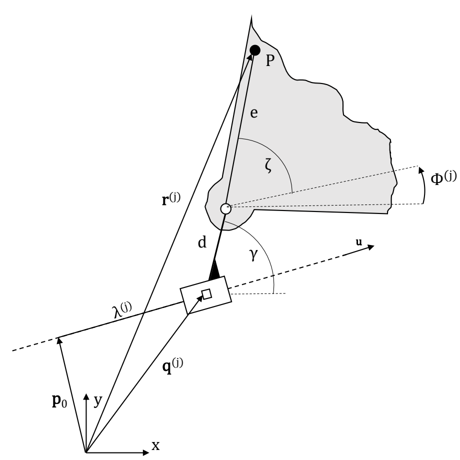

Fig. 2. Notation of the PR dyad.

the PR chain takes the following form:

$$\begin{aligned} \mathbf{q}^{(j)} + \mathbf{d} + \mathbf{e}^{(j)} &= \mathbf{r}^{(j)}, \qquad j = 1, \ldots, m. \\ \text{with} \quad \mathbf{q}^{(j)} &= \mathbf{p}_0 + \lambda^{(j)}\mathbf{u}, \quad \mathbf{u}^T\mathbf{p}_0 = 0, \quad \|\mathbf{u}\| = 1. \end{aligned} \tag{7}$$

Eq. 7 lead to 2m+2 scalar equations:

$$\begin{aligned} p_{0,x} + \lambda^{(j)}u_x + d\cos\gamma + e\cos(\phi^{(j)} + \zeta) - r_x^{(j)} &= 0, \\ p_{0,y} + \lambda^{(j)}u_y + d\sin\gamma + e\sin(\phi^{(j)} + \zeta) - r_y^{(j)} &= 0, \\ p_{0,x}u_x + p_{0,y}u_y &= 0, \\ u_x^2 + u_y^2 &= 1. \end{aligned} \tag{8}$$

Eqs. (8) have eight dimensional and m positional parameters as unknowns: $p_{0,x}$, $p_{0,y}$, u_x, u_y, d, e, γ, ζ, $\lambda^{(j)}$. Thus, to have a determined system of nonlinear equations we can select six arbitrary poses, *i.e.*, $8 + m = 2m + 2$.
Whenever the synthesis of the PR chain is associated to the synthesis of the RR chain such to form the linkage, six prescribed poses must be selected leading to either a determined or overdetermined system of nonlinear equations. Some simplified cases can be dealt with, as well. For example e can be selected to be null or γ can take a convenient value such to have the link normal to the P-joint ditection. In this cases the determined system of equations can be obtained with five arbitrary poses.
A caveat is in order here. We identified either determined or overdetermined systems depending on the number of prescribed poses. However, the algorithm deals with both the systems with no differences reaching in both cases the optimal solutions in the least-square sense.
Thus, the method proposed is general, it can be directly applied to synthesize any planar linkages, and potentially any spatial mechanisms. It is only required to solve the position equations including position variables. Solutions can be driven either by an opportune choice of the guess vector or limiting the solution range. Applications of the method to six-bar linkages will be presented in a future work by the same authors.

4 Numerical Examples

In this section we present four numerical examples for the four-bar linkage with RR dyads and one with a PR dyad: *i)* a classical problem proposed in the textbook [9] leading to a determinated system of nonlinear equations, *ii)* an overdetermined problem, *iii)* a problem with prescribed timing, *iv)* the synthesis of a lift-assist chair linkage, *v)* the synthesis of the RR-PR linkage.
The synthesis method is based on an optimization algorithm that approximates the exact solution in the least square sense. For this reason, it is noteworthy to define an error metrics to evaluate the accuracy of the solution. We select two parameters, *i.e.*, $\tilde{\epsilon}$, R, defined as follows:

$$\epsilon_k := (f_k - \overline{f}_k),\ k = 1, \dots 2m,\ \text{with}\ \ \overline{f}_k \equiv f(\overline{x}_k) = 0:\ \text{exact solution};$$
$$\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_k, \dots, \epsilon_{2m})^T;$$
$$\tilde{\epsilon} := \frac{1}{2m} \sum_{k}^{2m} \epsilon_k \text{ and } R := \|\boldsymbol{\epsilon}\|^2.$$

Thus, $\tilde{\epsilon}$ represents the mean of the residuals, whilst R is the squared 2-norm of the residuals at $\mathbf{x}$.

4.1 Optimal Solution with five prescribed poses (RR dyads)

The linkage must guide its coupler link through the five poses in Table 1. Thus, as first, the level of approximation reached has to be checked. In this case, solutions obtained approximate very well the exact solutions for both the linkage chains as $\tilde{\epsilon} = \mathcal{O}(-7)$, $R = \mathcal{O}(-15)$. Accordingly, the five prescribed poses are not reached exactly. To evaluate the error in position and orientation, Eqs. (4) are considered all together for chains 1 and 2 forming a linear system of four equations for each pose with $a_{x,i},\ a_{y,i},\ b_i,\ c_i,\ \beta_i,\ \theta^{(j)}$, $(i = 1,\ 2,\ j = 1, \ldots m)$, as known values and $\tilde{r}_x^{(j)},\ \tilde{r}_y^{(j)},\ \cos\tilde{\phi}^{(j)},\ \sin\tilde{\phi}^{(j)}$ as unknowns. The absolute errors obtained between the prescribed and calculated values are shown in Table 2. It can be noticed

Table 1. RR-RR linkage: Five prescribed poses for rigid-body guidance

j	$\mathbf{r}^j$ (m)	ϕ^j $(°)$
1	$[0, 0]$	0
2	$[1.5, 0.8]$	10
3	$[1.6, 1.5]$	20
4	$[2.0, 3.0]$	60
5	$[2.3, 3.5]$	90

that the absolute errors are lower than 0.4 mm. Figure 3 shows the synthesized linkage at the first pose.

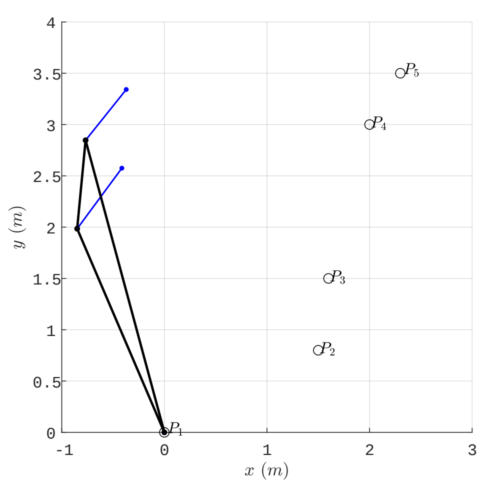

Fig. 3. Five prescribed poses: sinthesized linkage at the first pose.

Table 2. Five prescribed poses: Absolute errors in position and orientation

j	$\lvert\epsilon_{rx}^{(j)}\rvert$ (mm)	$\lvert\epsilon_{ry}^{(j)}\rvert$ (mm)	$\lvert\epsilon_{\phi}^{(j)}\rvert$ (degree)
1	0.189	0.059	0.003
2	0.386	0.192	0.009
3	0.361	0.339	0.010
4	0.159	0.334	0.006
5	0.091	0.213	0.004

Optimal Solution with eleven prescribed poses (RR dyads)

The linkage must guide its coupler link through the eleven poses in Table 3. Also

Table 3. Eleven prescribed poses for rigid-body guidance

j	$\mathbf{r}^j$ (m)	ϕ^j $(^\circ)$
1	$[3.988, 4.848]$	7.620
2	$[3.624, 5.803]$	12.089
3	$[2.996, 6.660]$	16.673
4	$[2.122, 7.348]$	21.657
5	$[1.045, 7.802]$	27.330
6	$[-0.174, 7.971]$	33.919
7	$[-1.462, 7.819]$	41.596
8	$[-2.732, 7.330]$	50.477
9	$[-3.891, 6.509]$	60.618
10	$[-4.839, 5.391]$	71.791
11	$[-5.491, 4.038]$	83.938

in this case $\tilde{\epsilon}$ and R are used to check the level of approximation reached. We obtained $\tilde{\epsilon} = \mathcal{O}(-4)$ and $R = \mathcal{O}(-6)$ as average of both the chains solutions. The absolute errors obtained between the prescribed and calculated values are shown in Table 4. The errors are well below of 1 mm except for one pose. Figure 4 shows the synthesized linkage at the poses.

4.2 Optimal Solution with five prescribed poses and timing (RR dyads)

Five, *i.e.* $m = 5$, of the eleven poses of section 4.1 were picked to synthesize the linkage with prescribed velocities of coupler link, as well. Thus, for each RR dyad, Eqs. (4) and Eqs. (5) form a system of 20 nonlinear equations with 15 unknowns as $\{\dot{\theta}^{(j)}\}_1^5$ has to be computed besides $\{\theta^{(j)}\}_1^5$ and $a_x,\ a_y,\ b,\ c,\ \beta$. To evaluate the approximation reached by the computation with respect to the prescribed velocities, we used Eqs. (5) for both the dyads forming a nonlinear system with $\{\phi^{(j)},\ \dot{\phi}^{(j)},\ \dot{r}_x^{(j)},\ \dot{r}_y^{(j)}\}_1^5$ as unknowns. The system was solved by the built-in

Table 4. Eleven prescribed poses: Absolute errors in position and orientation

j	$\lvert\epsilon_{r_x}^{(j)}\rvert$ (mm)	$\lvert\epsilon_{r_y}^{(j)}\rvert$ (mm)	$\lvert\epsilon_{\phi}^{(j)}\rvert$ (degree)
1	0.363	0.238	0.001
2	0.578	0.182	0.000
3	0.142	0.839	0.006
4	0.247	0.029	0.004
5	0.160	0.057	0.004
6	0.581	0.320	0.009
7	0.176	0.286	0.002
8	0.859	0.452	0.002
9	1.489	0.838	0.029
10	0.605	0.869	0.018
11	0.170	0.397	0.006

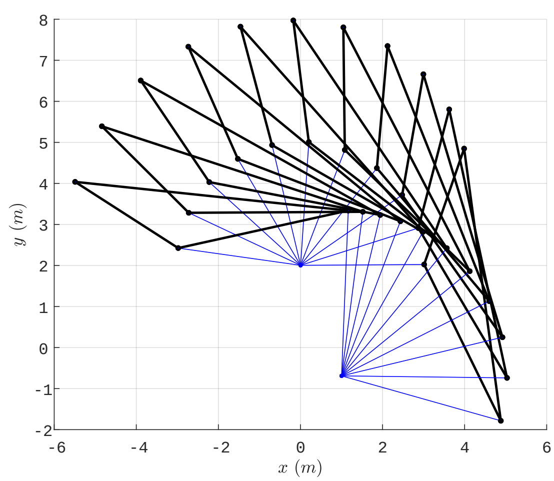

Fig. 4. Eleven prescribed poses: sinthesized linkage at the poses.

function *fsolve* of `Matlab` with high accuracy, $\mathcal{O}(-10)$. Table 5 shows the absolute errors obtained between the prescribed and calculated values. Table 5 shows

Table 5. Five prescribed poses with timing: Absolute errors in point and link velocities

j	$\lvert\epsilon_{v_x}^{(j)}\rvert$ (mm/s)	$\lvert\epsilon_{v_y}^{(j)}\rvert$ (mm/s)	$\lvert\epsilon_{\phi}^{(j)}\rvert$ (degree/s)
1	0.789	0.134	0.001
2	0.050	0.931	0.014
3	0.033	0.275	0.001
4	0.649	0.861	0.022
5	0.363	0.127	0.002

that, in addition to very good pose approximation (not shown), the method allows us to synthesize a linkage with a very good approximation of the velocities parameters as well.

4.3 Syntesis of a lift-assist chair linkage

We present a practical application of the synthesis method inspired by the biomechanical problem of the sit-to-stand movement [13, 14]. The goal is to synthesize a four-bar linkage guiding the chair seat according to the movement required to pass from a sitting to standing pose. The definition of the prescribed poses was possible by using the frames sequence reported in [15]. Links dimensions and joints locations are constrained as the mechanism must be contained in the area identified by the legs (front and back) and the seat, a square with side of 40 cm. The synthesized linkage takes an area of 38 $cm \times 20$ cm. The seat is rotated from about 6° to 80° and lifted of about 40 cm.
The linkage at the prescribed poses is shown in Figure 5.

4.4 Syntesis of a RR-PR linkage

The linkage must guide its coupler link through the six poses in Table 6. Also in

Table 6. RR-PR linkage: Six prescribed poses for rigid-body guidance

j	$\mathbf{r}^j$ (m)	ϕ^j (°)
1	[0.100, 2.600]	−12.605
2	[0.506, 3.124]	−9.224
3	[0.868, 3.688]	−2.807
4	[1.160, 4.198]	5.500
5	[1.363, 4.585]	14.954
6	[1.463, 4.802]	25.038

this case $\tilde{\epsilon}$ and R are used to check the level of approximation reached. For the PR

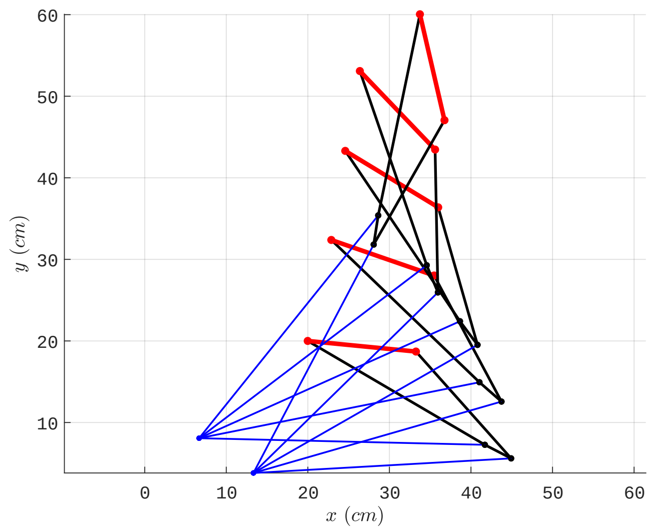

Fig. 5. Sinthesized lift-assist chair linkage at the poses.

chain we obtain $\tilde{\epsilon} = \mathcal{O}(-4)$ and $R = \mathcal{O}(-8)$ while for the RR chain $\tilde{\epsilon} = \mathcal{O}(-4)$ and $R = \mathcal{O}(-9)$. To evaluate the errors of position and orientation, also in this case, a linear system of four equations can be formed directly from Eqs. 4 and 8. The absolute errors obtained between the prescribed and calculated values are shown in Table 7. Also in this case it can be noticed the accuracy reached by

Table 7. RR-PR linkage, six prescribed poses: Absolute errors in position and orientation

j	$\lvert\epsilon_{r_x}^{(j)}\rvert$ (mm)	$\lvert\epsilon_{r_y}^{(j)}\rvert$ (mm)	$\lvert\epsilon_{\phi}^{(j)}\rvert$ (degree)
1	0.043	0.035	0.002
2	0.089	0.075	0.000
3	0.098	0.068	0.001
4	0.008	0.007	0.000
5	0.169	0.015	0.001
6	0.034	0.046	0.000

the method. Figure 6 shows the synthesized linkage at the poses.

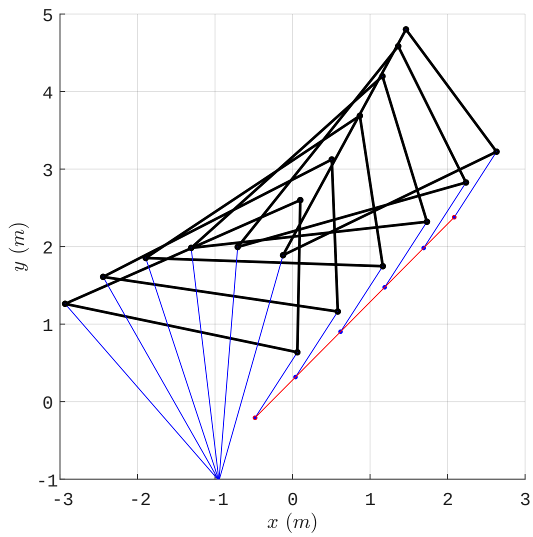

Fig. 6. RR-PR linkage, six prescribed poses: sinthesized linkage at the poses.

5 Conclusions

The paper presents a method for the kinematic synthesis of planar linkages. The method is based on the approximated solutions of the system of nonlinear equations representing the loop equations of the dyads. The method was applied extensively either for RR-RR or RR-PR linkage. In all cases the algorithm proved to be very accurate leading to coupler poses very close to those prescribed. The formulation was easily extended to synthesis with prescibed velocity of the coupler link leading to accurate results, as well. In brief, the method treats the kinematic synthesis as a positional problem of a serial kinematic chain with both geometrical and positional parameters as unknows. It relies on the high accuracy of the modern numerical algorithms able to reach aproximation below the construction errors and tolerances in machining. Results proved that the method works nicely for the cases examined.

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