Quaternion algebra approach to nonlinear Schrödinger equations with nonvanishing boundary conditions

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Abstract

In this article we apply quaternionic linear algebra and quaternionic linear system theory to develop the inverse scattering transform theory for the nonlinear Schrödinger equation with nonvanishing boundary conditions. We also determine its soliton solutions by using triplets of quaternionic matrices.

1 Introduction

The initial-value problem for the focusing nonlinear Schrödinger (NLS) equation

$$i\tilde{q}_t + \tilde{q}_{xx} - 2|\tilde{q}|^2\tilde{q} = 0 \tag{1.1}$$

with nonvanishing boundary conditions $\tilde{q}(x,t) \rightarrow \tilde{q}_{r,l}(t)$ as $x \rightarrow \pm\infty$, where $\tilde{q}_{r,l}(t) = \mu\, e^{-2i\mu^2 t + i\theta_{r,l}}$ for a positive constant μ and phases $\theta_{r,l} \in \mathbb{R}$, has been abundantly studied using the inverse scattering transform (IST) technique [9, 10, 14, 20]. In [8] the IST with full account of the spectral singularities has led to rogue wave solutions of the focusing NLS with nonvanishing boundary conditions. Throughout

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this article we study instead of (1.1) the NLS-like equation

$$iq_t + q_{xx} - 2|q|^2 q + 2\mu^2 q = 0, \tag{1.2}$$

obtained from (1.1) by applying the gauge transformation

$$\tilde{q}(x, t) = e^{-2i\mu^2 t} q(x, t),$$

where $q(x, t)$ tends to the time invariant limits $q_{r,l} = \mu e^{i\theta_{r,l}}$ as $t \to \pm\infty$. We also write $\mathcal{Q} = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}$ to convert (1.2) into the 2×2 matrix NLS-like equation

$$i\sigma_3 \mathcal{Q}_t + \mathcal{Q}_{xx} - 2\mathcal{Q}^3 - 2\mu^2 \mathcal{Q} = 0_{2 \times 2}, \tag{1.3}$$

where $\mathcal{Q}^\dagger = -\mathcal{Q}$. Here we write I_p for the identity matrix of order p, $0_{p \times r}$ for the $p \times r$ matrix with zero entries, the dagger for the complex conjugate matrix transpose, and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for the third Pauli matrix. The nonlinear Schrödinger equations have served as mathematical models for surface waves on deep waters [1, 2, 41], signals along optical fibers [24–26, 35], plasma oscillations [39], magnetic spin waves [11, 40], and particle states in Bose–Einstein condensates [29, 32, 33].

In [17] a new method to solve the initial-value problem of the matrix NLS equation by means of the inverse scattering transform technique was introduced. Instead of determining the time evolution of the scattering data associated with the Zakharov–Shabat system $v_x = (-ik\sigma_3 + \mathcal{Q})v$ and solving the Marchenko integral equations associated with the time dependent scattering data (as in [14]), we determined the time evolution of the scattering data associated with the matrix Schrödinger equation $-\psi_{xx} + \boldsymbol{Q}\psi = \lambda^2 \psi$, where $\boldsymbol{Q} = \mathcal{Q}^2 + \mathcal{Q}_x + \mu^2 I_2$ and $\lambda = \sqrt{k^2 + \mu^2}$ is the conformal mapping defined for all complex k cut along $[-i\mu, i\mu]$ and satisfying $\lambda \sim k$ at infinity. Since this conformal mapping $k \mapsto \lambda$ is 1, 1 for k in the upper half-plane $\mathbb{C}^+$ cut along $(i0, i\mu]$ and $\lambda \in \mathbb{C}^+$, this has led to a great simplification compared to the treatment based on the Zakharov–Shabat system $v_x = (-ik\sigma_3 + \mathcal{Q})v$ given in [9, 10, 14].

In this article we restrict ourselves to solving the initial-value problem for the $1 + 1$ focusing NLS equation. The advantage of this restriction is that the potential $\boldsymbol{Q}$ satisfies the symmetry relation

$$\boldsymbol{Q}^* = \sigma_2 \boldsymbol{Q} \sigma_2, \tag{1.4}$$

where the asterisk denotes complex conjugation without transposition and $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the second Pauli matrix. Using the algebra isomorphism between the algebra $\boldsymbol{\Sigma}$ of complex 2×2 matrices S satisfying $S^* = \sigma_2 S \sigma_2$ and the division ring $\mathbb{H}$ of quaternions [23], we can reduce the resolution of the Marchenko integral equations to solve the inverse scattering problem for the matrix Schrödinger equation $-\psi_{xx} + \boldsymbol{Q}\psi = \lambda^2 \psi$ to calculations involving quaternions.

In this article we rely significantly on the direct and inverse scattering theory for the matrix Schrödinger equation developed for $\boldsymbol{Q}^\dagger = \boldsymbol{Q}$, in [3, 5, 6] on the half-line and in [4, 31, 38] on the full line, albeit with some modifications due to the

symmetry relation (1.4). For technical reasons we assume throughout this article that the integral $\int_{-\infty}^{\infty} dx\,(1 + |x|)\|\boldsymbol{Q}(x)\|$ converges. For the various applications of the matrix Schrödinger equation with selfadjoint potential we refer to [6].

Let us discuss the contents of this article. In Sect. 2 we review the direct and inverse scattering theory of the matrix Schrödinger equation with symmetry relation (1.4), where we essentially rely on the more general scattering theory given in [16, 17]. In Sect. 3 we discuss the time evolution of the scattering theory. In Sect. 4 we discuss matrices having quaternion elements and their isomorphic images of double matrix order. Here we rely on the seminal monograph on quaternionic matrices by Rodman [34]. Section 5 is devoted to the multisoliton solutions of the AKNS system with nonvanishing boundary conditions parametrized by choosing minimal triplets of quaternionic matrices. Results on the invertibility of the Sylvester solutions $\boldsymbol{P}_r$ and $\boldsymbol{P}_l$ appearing in the multisoliton solutions are relegated to Appendix A.

2 Direct and inverse scattering

In this article we discuss the direct and inverse scattering theory for the matrix Schrödinger equation

$$-\psi_{xx} + \boldsymbol{Q}\psi = \lambda^2\psi, \tag{2.1}$$

where the complex 2×2 potential $\boldsymbol{Q}$ satisfies the symmetry relation

$$\boldsymbol{Q}^* = \sigma_2\boldsymbol{Q}\sigma_2 \tag{2.2}$$

and hence belongs to the algebra $\boldsymbol{\Sigma} = \left\{\begin{pmatrix} S_1 & -S_2^* \\ S_2 & S_1^* \end{pmatrix} : S_1, S_2 \in \mathbb{C}\right\}$. Then this potential $\boldsymbol{Q}$ also satisfies the more restrictive adjoint symmetry relation

$$\boldsymbol{Q}^\dagger = \sigma_3\boldsymbol{Q}\sigma_3, \tag{2.3}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the third Pauli matrix. Hence, by virtue of (2.3), all of the results on the direct and inverse scattering theory of (2.1) developed in [16, 17] go though in the present situation, although we need to discuss the impact of the more restrictive symmetry relation (2.2) on the results separately.

Let us define the *Jost solution from the left* $F_l(x, \lambda)$ and the *Jost solution from the right* $F_r(x, \lambda)$ as those solutions of the matrix Schrödinger equation (2.1) which satisfy the asymptotic conditions

$$F_l(x, \lambda) = e^{i\lambda x}\left[I_2 + o(1)\right], \qquad\qquad x \to +\infty, \tag{2.4a}$$
$$F_r(x, \lambda) = e^{-i\lambda x}\left[I_2 + o(1)\right], \qquad\qquad x \to -\infty. \tag{2.4b}$$

Calling $m_l(x, \lambda) = e^{-i\lambda x} F_l(x, \lambda)$ and $m_r(x, \lambda) = e^{i\lambda x} F_r(x, \lambda)$ *Faddeev functions*, we easily define them as the unique solutions of the Volterra integral equations

$$m_l(x, \lambda) = I_2 + \int_x^\infty dy \, \frac{e^{2i\lambda(y-x)} - 1}{2i\lambda} \, \mathbf{Q}(y) m_l(y, \lambda), \tag{2.5a}$$

$$m_r(x, \lambda) = I_2 + \int_{-\infty}^x dy \, \frac{e^{2i\lambda(x-y)} - 1}{2i\lambda} \, \mathbf{Q}(y) m_r(y, \lambda). \tag{2.5b}$$

Then, for each $x \in \mathbb{R}$, $m_l(x, \lambda)$ and $m_r(x, \lambda)$ are continuous in $\lambda \in \mathbb{C}^+ \cup \mathbb{R}$, are analytic in $\lambda \in \mathbb{C}^+$, and tend to I_2 as $\lambda \to \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$. For $0 \neq \lambda \in \mathbb{R}$ we can reshuffle (2.5) and arrive at the asymptotic relations

$$F_l(x, \lambda) = e^{i\lambda x} A_l(\lambda) + e^{-i\lambda x} B_l(\lambda) + o(1), \qquad x \to -\infty, \tag{2.6a}$$

$$F_r(x, \lambda) = e^{-i\lambda x} A_r(\lambda) + e^{i\lambda x} B_r(\lambda) + o(1), \qquad x \to +\infty, \tag{2.6b}$$

where

$$A_{r,l}(\lambda) = I_2 - \frac{1}{2i\lambda} \int_{-\infty}^\infty dy \, \mathbf{Q}(y) m_{r,l}(y, \lambda), \tag{2.7a}$$

$$B_{r,l}(\lambda) = \frac{1}{2i\lambda} \int_{-\infty}^\infty dy \, e^{\mp 2i\lambda y} \mathbf{Q}(y) m_{r,l}(y, \lambda). \tag{2.7b}$$

Then $A_{r,l}(\lambda)$ is continuous in $0 \neq \lambda \in \mathbb{C}^+ \cup \mathbb{R}$, is analytic in $\lambda \in \mathbb{C}^+$, and tends to I_2 as $\lambda \to \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$, while $2i\lambda[I_2 - A_{r,l}(\lambda)]$ has the finite limit $-\mathbf{\Delta}_{r,l} = \int_{-\infty}^\infty dy \, \mathbf{Q}(y) m_{r,l}(y, \lambda)$ as $\lambda \to 0$ from within $\mathbb{C}^+ \cup \mathbb{R}$. By the same token, $B_{r,l}(\lambda)$ is continuous in $0 \neq \lambda \in \mathbb{R}$, vanishes as $\lambda \to \pm\infty$, and satisfies $2i\lambda B_{r,l}(\lambda) \to -\mathbf{\Delta}_{r,l}$ as $\lambda \to 0$ along the real λ-axis.

Using the transformation $F(x, \lambda) \mapsto F(x, -\lambda^*)^*$ in the matrix Schrödinger equation (2.1), we easily prove the symmetry relations

$$F_l(x, \lambda) = \sigma_2 F_l(x, -\lambda^*)^* \sigma_2, \qquad F_r(x, \lambda) = \sigma_2 F_r(x, -\lambda^*)^* \sigma_2. \tag{2.8}$$

With the help of (2.6) we then obtain the symmetry relations

$$A_l(\lambda) = \sigma_2 A_l(-\lambda^*)^* \sigma_2, \; A_r(\lambda) = \sigma_2 A_r(-\lambda^*)^* \sigma_2, \quad 0 \neq \lambda \in \mathbb{C}^+ \cup \mathbb{R}, \tag{2.9a}$$

$$B_l(\lambda) = \sigma_2 B_l(-\lambda)^* \sigma_2, \; B_r(\lambda) = \sigma_2 B_r(-\lambda)^* \sigma_2, \quad 0 \neq \lambda \in \mathbb{R}. \tag{2.9b}$$

Introducing the *reflection coefficients*

$$R_{l,r}(\lambda) = B_{l,r}(\lambda) A_{l,r}(\lambda)^{-1} = -A_{r,l}(\lambda)^{-1} B_{r,l}(-\lambda), \tag{2.10}$$

we easily obtain the symmetry relations

$$R_l(\lambda) = \sigma_2 R_l(-\lambda)^* \sigma_2, \qquad R_r(\lambda) = \sigma_2 R_r(-\lambda)^* \sigma_2, \quad 0 \neq \lambda \in \mathbb{R}, \tag{2.11}$$

provided $\det A_{l,r}(\lambda) \neq 0$.

Above we have defined $\mathbf{\Delta}_{l,r}$ as follows:

$$\mathbf{\Delta}_{l,r} = \lim_{\lambda \to 0} 2i\lambda A_{l,r}(\lambda) = \lim_{\lambda \to 0^{\pm}} 2i\lambda B_{l,r}(\lambda),$$

where the first limit may be taken from the closed upper half-plane. Then the matrices $\mathbf{\Delta}_{l,r}$ have the same determinant. If $\mathbf{\Delta}_{l,r}$ is nonsingular, we are said to be in the *generic case*; if instead $\mathbf{\Delta}_{l,r}$ is singular, we are said to be in the *exceptional case* (cf. [4]). We are said to be in the *superexceptional case* if $\mathbf{\Delta}_{l,r} = 0_{2\times2}$ and $A_{l,r}(\lambda)$ tends to a nonsingular matrix, $A_{l,r}(0)$ say, as $\lambda \to 0$ from within $\mathbb{C}^+ \cup \mathbb{R}$. It is clear that $\mathbf{\Delta}_{l,r} = \sigma_2 \mathbf{\Delta}_{l,r}^* \sigma_2$. Throughout this article (as well as in [17]) we assume the absence of *spectral singularities*, i.e., the absence of nonzero real λ for which $\det A_{l,r}(\lambda) = 0$. Under this condition the reflection coefficients $R_{l,r}(\lambda)$ are continuous in $0 \neq \lambda \in \mathbb{R}$. For general potentials $\mathbf{Q}$ satisfying (2.2) or (2.3) there may very well be spectral singularities (see [8, 30] for focusing AKNS examples), even though spectral singularities do not occur if $\mathbf{Q}^\dagger = \mathbf{Q}$ [6, 17, 30].

The Jost solutions allow the triangular representations

$$F_l(x,\lambda) = e^{i\lambda x} I_2 + \int_x^\infty dy\, e^{i\lambda y} K(x,y), \tag{2.12a}$$

$$F_r(x,\lambda) = e^{-i\lambda x} I_2 + \int_{-\infty}^x dy\, e^{-i\lambda y} J(x,y), \tag{2.12b}$$

where for every $x \in \mathbb{R}$

$$\int_x^\infty dy\, \|K(x,y)\| + \int_{-\infty}^x dy\, \|J(x,y)\| < +\infty.$$

Then the potential $\mathbf{Q}(x)$ can be found from the auxiliary functions $K(x,y)$ and $J(x,y)$ as follows:

$$K(x,x) = \frac{1}{2}\int_x^\infty dy\, \mathbf{Q}(y), \qquad J(x,x) = \frac{1}{2}\int_{-\infty}^x dy\, \mathbf{Q}(y). \tag{2.13}$$

Equations (2.8) and (2.12) imply the symmetry relations

$$K(x,y) = \sigma_2 K(x,y)^* \sigma_2, \qquad J(x,y) = \sigma_2 J(x,y)^* \sigma_2. \tag{2.14}$$

Thus the auxiliary functions $K(x,y)$ and $J(x,y)$ belong to the algebra $\mathbf{\Sigma}$.

Let us write the reflection coefficients in the form

$$R_l(\lambda) = \int_{-\infty}^\infty d\alpha\, e^{i\lambda\alpha} \hat{R}_l(\alpha), \qquad R_r(\lambda) = \int_{-\infty}^\infty d\alpha\, e^{-i\lambda\alpha} \hat{R}_r(\alpha), \tag{2.15}$$

where $\hat{R}_{l,r} \in L^1(\mathbb{R})^{2\times2}$. Although this Fourier representation has only been proved under the absence of spectral singularities assumption and in the generic case (for

$Q \in L^1(\mathbb{R}; (1 + |x|)dx)^{2 \times 2})$ and in the superexceptional case (for $Q \in L^1(\mathbb{R}; (1 + |x|)^2 dx)^{2 \times 2})$ [16], we assume it to be also true in the most general exceptional case. We then easily prove the symmetry relations

$$\hat{R}_l(\alpha) = \sigma_2 \hat{R}_l(\alpha)^* \sigma_2, \qquad \hat{R}_r(\alpha) = \sigma_2 \hat{R}_r(\alpha)^* \sigma_2. \tag{2.16}$$

Thus the functions $\hat{R}_l(\alpha)$ and $\hat{R}_r(\alpha)$ belong to the algebra $\boldsymbol{\Sigma}$.

So far we have only discussed the direct scattering problem for (2.1). The inverse scattering problem can be solved by computing one of the auxiliary functions $K(x, y)$ or $J(x, y)$ as the solutions of one of the Marchenko integral equations

$$K(x, y) + \Omega_r(x + y) + \int_x^\infty dz\, K(x, z)\Omega_r(z + y) = 0_{2 \times 2}, \tag{2.17a}$$

$$J(x, y) + \Omega_l(x + y) + \int_{-\infty}^x dz\, J(x, z)\Omega_l(z + y) = 0_{2 \times 2}, \tag{2.17b}$$

followed by an application of one of (2.13). Here the *Marchenko integral kernels* $\Omega_{l,r}(w)$ are given by

$$\Omega_r(w) = \hat{R}_r(w) + \sum_{s=1}^N e^{i\lambda_s w} N_{r;s}, \tag{2.18a}$$

$$\Omega_l(w) = \hat{R}_l(w) + \sum_{s=1}^N e^{-i\lambda_s w} N_{l;s}, \tag{2.18b}$$

where we assume the poles λ_s ($s = 1, \ldots, N$) of the *transmission coefficients* $A_{l,r}(\lambda)^{-1}$ to be simple; in that case the so-called *norming constants* $N_{r;s}$ and $N_{l;s}$ are defined by

$$F_r(x, \lambda_s)\tau_{r;s} = i\, F_l(x, \lambda_s)N_{r;s}, \tag{2.19a}$$
$$F_l(x, \lambda_s)\tau_{l;s} = i\, F_r(x, \lambda_s)N_{l;s}, \tag{2.19b}$$

where $\tau_{r;s}$ and $\tau_{l;s}$ are the residues of $A_r(\lambda)^{-1}$ and $A_l(\lambda)^{-1}$ at the simple pole $\lambda_s \in \mathbb{C}^+$ ($s = 1, \ldots, N$). If there exist multiple poles of $A_{l,r}(\lambda)^{-1}$ in $\mathbb{C}^+$, then the expressions for $\Omega_{r,l}(w) - \hat{R}_{r,l}(w)$ can be derived in a straightforward way as a finite sum of polynomials times exponentials which obviously are entire analytic functions of x. We can then prove the symmetry relations

$$\Omega_r(w) = \sigma_2 \Omega_r(w)^* \sigma_2, \qquad \Omega_l(w) = \sigma_2 \Omega_l(w)^* \sigma_2. \tag{2.20}$$

Thus the Marchenko kernels $\Omega_r(w)$ and $\Omega_l(w)$ belong to the algebra $\boldsymbol{\Sigma}$. The proof can be based on (a) the unique solvability of the Marchenko equations (for $\Omega_{r,l}$ as unknowns with the auxiliary functions assumed to be known) for large enough $\pm x$, (b) the symmetry relations (2.16), and (c) the analyticity of the functions $\Omega_{r,l}(w) - \hat{R}_{r,l}(w)$ in $x \in \mathbb{R}$. We refer to [15] for the rather technical details.

3 Time evolution

Straightforward calculations imply [17]

$$
\begin{aligned}
i\sigma_3 \boldsymbol{Q}_t + \boldsymbol{Q}_{xx} - 2\mathcal{Q}\boldsymbol{Q}_x - 2\mathcal{Q}_x \boldsymbol{Q} &= (i\sigma_3 \mathcal{Q}_t + \mathcal{Q}_{xx} - 2\mathcal{Q}^3 - 2\mu^2\mathcal{Q})\mathcal{Q} \\
&\quad - \mathcal{Q}(i\sigma_3 \mathcal{Q}_t + \mathcal{Q}_{xx} - 2\mathcal{Q}^3 - 2\mu^2\mathcal{Q}) \\
&\quad + (i\sigma_3 \mathcal{Q}_t + \mathcal{Q}_{xx} - 2\mathcal{Q}^3 - 2\mu^2\mathcal{Q})_x.
\end{aligned}
\tag{3.1}
$$

Thus any solution of the matrix NLS-like equation (1.3) with nonvanishing time invariant limits $\mathcal{Q}_{r,l}$ for $\mathcal{Q}(x; t)$ as $x \to \pm\infty$ is a solution of the nonlinear evolution equation

$$
i\sigma_3 \boldsymbol{Q}_t + \boldsymbol{Q}_{xx} - 2\mathcal{Q}\boldsymbol{Q}_x - 2\mathcal{Q}_x \boldsymbol{Q} = 0_{2\times 2},
\tag{3.2}
$$

where $\mathcal{Q}_x = \frac{1}{2}(\boldsymbol{Q} - \sigma_3 \boldsymbol{Q}\sigma_3)$.

The pair of 4×4 matrices $(\boldsymbol{X}, \boldsymbol{T})$, where

$$
\boldsymbol{X}(x, t, \lambda) = \begin{pmatrix} 0_{2\times 2} & I_2 \\ \boldsymbol{Q}(x; t) - \lambda^2 I_2 & 0_{2\times 2} \end{pmatrix},
\tag{3.3a}
$$

$$
\boldsymbol{T}(x, t, \lambda) = \begin{pmatrix} i\sigma_3(\boldsymbol{Q} - 2\lambda^2 I_2) & -2i\sigma_3 \mathcal{Q} \\ i\sigma_3(\boldsymbol{Q}_x - 2\mathcal{Q}\boldsymbol{Q} + 2\lambda^2 \mathcal{Q}) & i\sigma_3(\boldsymbol{Q} - 2\lambda^2 I_2 - 2\mathcal{Q}_x) \end{pmatrix},
\tag{3.3b}
$$

is an AKNS pair for the nonlinear evolution equation (3.2) in the sense that the zero curvature condition

$$
\boldsymbol{X}_t - \boldsymbol{T}_x + \boldsymbol{X}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{X} = 0_{4\times 4}
$$

is satisfied iff $\boldsymbol{Q}$ satisfies (3.2) (see [17]). Then it is easily verified that $\boldsymbol{T}(x, t, \lambda)$ tends to the limits

$$
\boldsymbol{T}_{\pm\infty} = \begin{pmatrix} -2i\lambda^2 \sigma_3 & -2i\sigma_3 \mathcal{Q}_{r,l} \\ 2i\sigma_3 \mathcal{Q}_{r,l} & -2i\lambda^2 \sigma_3 \end{pmatrix}
\tag{3.4}
$$

as $x \to \pm\infty$. Note that $\det \boldsymbol{T}_{\pm\infty} = 16(\lambda^4 + \mu^2)^2$.

Following [17], we introduce the Jost solutions $\boldsymbol{F}_{r,l}(x, \lambda; t)$ of the first order system

$$
\begin{pmatrix} V \\ V' \end{pmatrix}' = \begin{pmatrix} 0_{n\times n} & I_n \\ \boldsymbol{Q}(x) - \lambda^2 I_n & 0_{n\times n} \end{pmatrix} \begin{pmatrix} V \\ V' \end{pmatrix}
$$

defined by

$$
\boldsymbol{F}_l(x, \lambda) = \begin{pmatrix} F_l(x, -\lambda) & F_l(x, \lambda) \\ F_l'(x, -\lambda) & F_l'(x, \lambda) \end{pmatrix}, \quad \boldsymbol{F}_r(x, \lambda) = \begin{pmatrix} F_r(x, \lambda) & F_r(x, -\lambda) \\ F_r'(x, \lambda) & F_r'(x, -\lambda) \end{pmatrix},
$$

where the prime denotes differentiation with respect to x. Letting $V(x, \lambda; t)$ be a nonsingular 4×4 matrix solution of the pair of first order equations

$$V_x = XV, \qquad V_t = TV, \tag{3.5}$$

the fact that $F_{r,l}(x, \lambda; t)$ satisfies the first of (3.5) implies the existence of nonsingular matrices $C_{F_{r,l}}(\lambda; t)$ not depending on x such that

$$F_{r,l}(x, \lambda; t) = V(x, \lambda; t) C_{F_{r,l}}(\lambda; t)^{-1}.$$

Then a simple differentiation yields

$$\left[C_{F_{r,l}}(\lambda; t) \right]_t C_{F_{r,l}}(\lambda; t)^{-1} = F_{r,l}^{-1} T F_{r,l} - F_{r,l}^{-1} [F_{r,l}]_t,$$

where the left-hand side does not depend on x and hence equals the limits of the right-hand side as $x \to \pm\infty$. Using (3.4) we easily get

$$\left[C_{F_{r,l}}(\lambda; t) \right]_t C_{F_{r,l}}(\lambda; t)^{-1} = \begin{pmatrix} -\Lambda_{r,l}^{\text{up}}(\lambda) & 0_{2\times 2} \\ 0_{2\times 2} & -\Lambda_{r,l}^{\text{dn}}(\lambda) \end{pmatrix}, \tag{3.6}$$

where

$$\Lambda_{r,l}^{\text{up}}(\lambda) = 2i\lambda^2 \sigma_3 + 2\lambda \sigma_3 \mathcal{Q}_{r,l}, \tag{3.7a}$$

$$\Lambda_{r,l}^{\text{dn}}(\lambda) = 2i\lambda^2 \sigma_3 - 2\lambda \sigma_3 \mathcal{Q}_{r,l}, \tag{3.7b}$$

are time invariant. Then we easily verify the symmetry relations

$$\Lambda_{r,l}^{\text{up}}(\lambda) = \sigma_2 \Lambda_{r,l}^{\text{up}}(-\lambda^*)^* \sigma_2, \qquad \Lambda_{r,l}^{\text{dn}}(\lambda) = \sigma_2 \Lambda_{r,l}^{\text{dn}}(-\lambda^*)^* \sigma_2. \tag{3.8}$$

Using that

$$F_r(x, \lambda; t) = F_l(x, \lambda; t) A_r(\lambda; t), \qquad F_l(x, \lambda; t) = F_r(x, \lambda; t) A_l(\lambda; t),$$

where

$$A_r(\lambda; t) = \begin{pmatrix} A_r(\lambda; t) & B_r(-\lambda; t) \\ B_r(\lambda; t) & A_r(-\lambda; t) \end{pmatrix}, \quad A_l(\lambda; t) = \begin{pmatrix} A_l(-\lambda; t) & B_l(\lambda; t) \\ B_l(-\lambda; t) & A_l(\lambda; t) \end{pmatrix},$$

for $0 \neq \lambda \in \mathbb{R}$ we easily compute

$$[A_{r,l}]_t = A_{r,l}(\lambda; t) \begin{pmatrix} \Lambda_{r,l}^{\text{up}}(\lambda) & 0_{2\times 2} \\ 0_{2\times 2} & \Lambda_{r,l}^{\text{dn}}(\lambda) \end{pmatrix} - \begin{pmatrix} \Lambda_{l,r}^{\text{up}}(\lambda) & 0_{2\times 2} \\ 0_{2\times 2} & \Lambda_{l,r}^{\text{dn}}(\lambda) \end{pmatrix} A_{r,l}(\lambda; t). \tag{3.9}$$

Then the reflection coefficients satisfy

$$[R_r]_t = R_r(\lambda; t) \Lambda_l^{\text{up}}(\lambda) - \Lambda_l^{\text{dn}}(\lambda) R_r(\lambda; t), \tag{3.10a}$$

$$[R_l]_t = R_l(\lambda; t)\Lambda_r^{\text{dn}}(\lambda) - \Lambda_r^{\text{up}}(\lambda)R_l(\lambda; t). \tag{3.10b}$$

Defining $\hat{R}_{r,l}(\alpha; t)$ by (2.15), we easily derive the PDEs

$$[\hat{R}_r]_t = -2i\left([\hat{R}_r]_{\alpha\alpha}\sigma_3 - \sigma_3[\hat{R}_r]_{\alpha\alpha} + [\hat{R}_r]_\alpha\sigma_3\mathcal{Q}_l - \mathcal{Q}_l\sigma_3[\hat{R}_r]_\alpha\right), \tag{3.11a}$$

$$[\hat{R}_l]_t = -2i\left([\hat{R}_l]_{\alpha\alpha}\sigma_3 - \sigma_3[\hat{R}_l]_{\alpha\alpha} + [\hat{R}_l]_\alpha\sigma_3\mathcal{Q}_r - \mathcal{Q}_r\sigma_3[\hat{R}_l]_\alpha\right), \tag{3.11b}$$

provided $\int_{-\infty}^\infty d\alpha\,(1 + \alpha^2)\|\hat{R}_{r,l}(\alpha; t)\|$ converges for every $t \in \mathbb{R}$. Using (3.11) and time evolution properties of the norming constants [17, (4.4)] we obtain

$$[\Omega_r]_t = -2i\left([\Omega_r]_{ww}\sigma_3 - \sigma_3[\Omega_r]_{ww} + [\Omega_r]_w\sigma_3\mathcal{Q}_l - \mathcal{Q}_l\sigma_3[\Omega_r]_w\right), \tag{3.12a}$$

$$[\Omega_l]_t = -2i\left([\Omega_l]_{ww}\sigma_3 - \sigma_3[\Omega_l]_{ww} + [\Omega_l]_w\sigma_3\mathcal{Q}_r - \mathcal{Q}_r\sigma_3[\Omega_l]_w\right). \tag{3.12b}$$

Hence, the reflection kernels $\hat{R}_{r,l}(\alpha; t)$ and the Marchenko integral kernels $\Omega_{r,l}(w; t)$ satisfy the same PDEs. We have also seen before that $\hat{R}_{r,l}(\alpha; t)$ and $\Omega_{r,l}(w; t)$ belong to the algebra $\boldsymbol{\Sigma}$.

4 Quaternionic matrix algebra

Let $\boldsymbol{\Sigma}$ stand for the (noncommutative) division ring of complex 2×2 matrices S satisfying $S^* = \sigma_2 S \sigma_2$. Then it is easily verified [34] that $\boldsymbol{\Sigma}$ is isomorphic (as a real unital algebra) to the noncommutative division ring of quaternions $\mathbb{H}$ by means of the isomorphism

$$S = \begin{pmatrix} S_1 & -S_2^* \\ S_2 & S_1^* \end{pmatrix} = (\text{Re }S_1)I_2 + i(\text{Im }S_1)\sigma_3 - i(\text{Re }S_2)\sigma_2 + i(\text{Im }S_2)\sigma_1$$

$$\xrightarrow{\varphi} (\text{Re }S_1)\mathbf{1} + (\text{Im }S_1)\boldsymbol{i} - (\text{Re }S_2)\boldsymbol{j} + (\text{Im }S_2)\boldsymbol{k}, \tag{4.1}$$

where $\{\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is the standard quaternion basis. Thus, letting $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ stand for the first Pauli matrix, we see that $\{I_2, i\sigma_3, i\sigma_2, i\sigma_1\}$ is the basis of the real vector space $\boldsymbol{\Sigma}$ that corresponds to the quaternion basis $\{\mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ by means of φ. If $x = a\mathbf{1} + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \in \mathbb{H}$ for $a, b, c, d \in \mathbb{R}$, then the quaternion squared length is defined by $|x|^2 = a^2 + b^2 + c^2 + d^2$. Thus, for each $S = \begin{pmatrix} S_1 & -S_2^* \\ S_2 & S_1^* \end{pmatrix} \in \boldsymbol{\Sigma}$ we see that $\det S$ coincides with the squared quaternion length of $\varphi(S)$.

The map φ has a natural extension as a real algebra isomorphism from $\boldsymbol{\Sigma}^{p \times p}$ onto $\mathbb{H}^{p \times p}$, the algebras of $p \times p$ matrices with entries in $\boldsymbol{\Sigma}$ and $\mathbb{H}$, respectively. For $p \neq r$ there also exists a natural extension from the real linear subspace $\boldsymbol{\Sigma}^{p \times r}$ onto $\mathbb{H}^{p \times r}$.

For later use we introduce the *similarity orbit* of $S = \begin{pmatrix} S_1 & -S_2^* \\ S_2 & S_1^* \end{pmatrix} \in \boldsymbol{\Sigma}$ as the set [34, Thm. 2.2.6]

$$\mathrm{Sim}(S) = \left\{ X^{-1}SX : 0_{2\times 2} \neq X \in \mathbf{\Sigma} \right\}$$
$$= \left\{ \begin{pmatrix} T_1 & -T_2^* \\ T_2 & T_1^* \end{pmatrix} : \mathrm{Re}\, S_1 = \mathrm{Re}\, T_1 \text{ and } (\mathrm{Im}\, S_1)^2 + |S_2|^2 = (\mathrm{Im}\, T_1)^2 + |T_2|^2 \right\}. \tag{4.2}$$

4.1 Determinants and quaternionic linear algebra

Since multiplication of quaternion numbers is noncommutative, there is no obvious way to define the determinant of square quaternion matrices. Fortunately, the map φ allows one to define the determinant of a quaternion $p \times p$ matrix $M \in \mathbb{H}^{p\times p}$ as the determinant of the complex $2p \times 2p$ matrix $\varphi^{-1}(M)$ (cf. [34, Ch. 5]). For alternative ways to define determinants of square quaternionic matrices we refer to [12, 18, 36] and references therein.

The following theorem has been proved by Rodman [34, Th. 5.9.2] using the quaternionic Jordan normal form. Below we present an independent proof based on Schur complements (cf. [19, Sec. 1.7] and references therein).

Theorem 4.1 *For $p = 1, 2, 3, \ldots$ the matrices $S \in \mathbf{\Sigma}^{p\times p}$ have a nonnegative determinant.*

Proof For $p = 1$ the theorem is obviously true. For $p \geq 2$ we define the Schur complement

$$\mathcal{S} = \begin{pmatrix} S_{22} & \ldots & S_{2p} \\ \vdots & & \vdots \\ S_{p2} & \ldots & S_{pp} \end{pmatrix} - \begin{pmatrix} S_{21} \\ \vdots \\ S_{p1} \end{pmatrix} S_{11}^{-1} \begin{pmatrix} S_{12} & \ldots & S_{1p} \end{pmatrix}, \tag{4.3}$$

provided $\det(S_{11}) = \|S_{11}\|^2 > 0$. Then

$$\det(S) = \|S_{11}\|^2 \det \mathcal{S}. \tag{4.4}$$

Under the induction hypothesis that all matrices $S \in \mathbf{\Sigma}^{(p-1)\times(p-1)}$ have a nonnegative determinant, we see from (4.4) that any matrix $S \in \mathbf{\Sigma}^{p\times p}$ satisfying $\|S_{11}\| > 0$ has a nonnegative determinant. If one of $\|S_{j1}\| > 0$, we switch the first and j-th double rows without changing the determinant and repeat the above Schur complement argument to conclude that $\det(S) \geq 0$. If $\|S_{11}\| = \cdots = \|S_{p1}\| = 0$, then obviously $\det(S) = 0$. $\square$

4.2 Jordan normal form and matrix triplets

The following theorem can be obtained from [34, Thm. 5.5.3] upon application of φ^{-1}.

Theorem 4.2 *For every $S \in \Sigma^{p \times p}$ there exist positive integers $m_1, \ldots, m_k$ adding up to p and matrices $A^{[1]}, \ldots, A^{[k]} \in \Sigma$ such that S is similar to the direct sum*

$$J_{m_1}(A^{[1]}) \oplus \ldots \oplus J_{m_k}(A^{[k]}) \tag{4.5}$$

by means of a similarity transformation belonging to $\Sigma^{p \times p}$. The Σ-Jordan normal form (4.5) is unique up to changing the order in the direct sum and replacing the matrices $A^{[1]}, \ldots, A^{[k]}$ by matrices in the same similarity orbit.

It should be noted that the Σ-Jordan normal form (or: the quaternionic Jordan normal form discussed at length in [34]) differs from the usual complex Jordan normal form. Since $A = \begin{pmatrix} A_1 & -A_2^* \\ A_2 & A_1^* \end{pmatrix}$ is a diagonalizable 2×2 matrix with eigenvalues $\text{Re } A_1 \pm i\sqrt{(\text{Im } A_1)^2 + |A_2|^2}$, the corresponding complex Jordan normal form is obtained from (4.5) below as follows:

1. If $A^{[s]}$ is the diagonal matrix $\left(\text{Re } A_1^{[s]}\right) I_2$, we replace $J_{m_s}(A^{[s]})$ by the direct sum of Jordan blocks $J_{m_s}(\text{Re } A_1) \oplus J_{m_s}(\text{Re } A_1)$.
2. If $A^{[s]}$ is not a real multiple of I_2, we replace $J_{m_s}(A^{[s]})$ by the direct sum of the Jordan blocks of order m_s at the complex conjugate eigenvalues $\text{Re } A_1 \pm i\sqrt{(\text{Im } A_1)^2 + |A_2|^2}$.

Let (A, B, C) be a triplet consisting of the $p \times p$ matrix A with entries in Σ, the $p \times 1$ matrix B with entries in Σ, and the $1 \times p$ matrix C with entries in Σ. Then this matrix triplet is called *minimal* if the matrix order of A is minimal among all triplets for which $Ce^{-zA}B$ is the same Σ-valued function of $z \in \mathbb{R}$. According to Theorem 4.2, given a minimal triple (A, B, C) of matrices with entries in Σ there exists an invertible $S \in \Sigma^{p \times p}$ such that SAS^{-1} has the Jordan normal form (4.5) and the triplet (SAS^{-1}, SB, CS^{-1}) is minimal.

Theorem 4.3 *Suppose (A, B, C) is a triplet of size compatible matrices with entries of Σ, where the eigenvalues of A all have positive real part. Let us assume that A has been brought to the Σ-Jordan normal form (4.5). Then no pair of matrices $A^{[1]}, \ldots, A^{[k]}$ belongs to the same similarity orbit and among the Σ-entries $B_s^{[j]}$ of B and $C_s^{[j]}$ of C $(j = 1, \ldots, k, s = 1, \ldots, m_j$ the Σ-entries $B_{m_j}^{[j]}$ and $C_1^{[j]}$ $(j = 1, 2, \ldots, k)$ are nontrivial matrices.*

Proof Consider the matrix triplet $(J_m(A), B, C)$, where $A \in \Sigma$ is not the zero matrix, B is the column with entries $B_1, \ldots, B_m \in \Sigma$ and C is the row with entries $C_1, \ldots, C_m \in \Sigma$. Then for $n = 0, 1, 2, \ldots$ we get

$$\left[J_m(A)^n\right]_{j,l} = \begin{cases} A^n, & j = l, \\ \binom{n}{l-j}A^{n+j-l}, & j < l \le \min(m-1, n+j), \\ 0_{2 \times 2}, & j > l \text{ or } l > \min(m-1, n+j), \end{cases}$$

which is an upper triangular Toeplitz matrix with entries in Σ. Letting X be the column with entries $X_1, \ldots, X_m \in \Sigma$, the identity

$$C J_m(A)^n X = 0_{2 \times 2}, \qquad n = 0, 1, \ldots, m-1,$$

allows a solution X with $X_1 \neq 0_{2 \times 2}$ if $C_1 = 0_{2 \times 2}$. Thus assuming $C_1 \neq 0_{2 \times 2}$ in Σ, we get the equality

$$\sum_{j=1}^{m} C_j \left(A^n X_j + \sum_{l=j+1}^{\min(m-1,n+1)} \binom{n}{l-j} A^{n+j-l} X_l \right) = 0_{2 \times 2}, \quad n = 0, 1, 2, \ldots,$$

allowing us to express each X_j into $X_{j+1}, \ldots, X_m$ ($j = 1, 2, \ldots, m - 1$) linearly and to conclude that $X_m = 0_{2 \times 2}$. Thus $X_1 = \cdots = X_m = 0_{2 \times 2}$. In other words, if $C_1 \neq 0_{2 \times 2}$, then

$$\bigcap_{n=0}^{m-1} \operatorname{Ker} (C J_m(A)^n) = (0_{2 \times 2}).$$

In the same way we prove that

$$\bigvee_{n=0}^{m-1} \operatorname{Im} (J_m(A)^n B) = \mathbb{C}^{2m}$$

if $B_m \neq 0_{2 \times 2}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

5 Soliton solutions using matrix triplets

Let us now solve the right and left Marchenko equations (2.17a) and (2.17b) for reflectionless Marchenko kernels (2.18a) and (2.18b), where the reflection coefficients $R_{r,l}(\lambda; t)$ vanish.

5.1 Minimal matrix triplet representations

Since the Marchenko kernels $\Omega_{l,r}(w; t)$ are finite linear combinations of the exponentials $e^{\pm i \lambda_s w}$ ($n = 1, 2, \ldots, N$) and polynomials of w multiplied by such exponentials with time dependent coefficients, there exist a square matrix A of even order $2p$ whose eigenvalues have positive real parts, $2p \times 2$ matrices B_r and B_l, $2 \times 2p$ matrices C_r and C_l, and a $2p \times 2p$ matrix H commuting with A such that

$$\Omega_r(z, t) = C_r e^{-zA} e^{tH} B_r, \qquad \Omega_l(z, t) = C_l e^{zA} e^{tH} B_l. \tag{5.1}$$

The representations (5.1) are chosen in such a way that the order of the complex matrix A is minimal among all representations (5.1) for the same Marchenko kernels $\Omega_r(z, t)$ and $\Omega_l(z, t)$. In that case $2p$ coincides with the sum of the algebraic multiplicities of the discrete eigenvalues in $\mathbb{C}^+$ (which is N if the discrete eigenvalues are algebraically simple, as assumed so far). Moreover, for any pair of minimal representations (5.1) [where the matrices in the second pair carry a prime or double prime, respectively],

there exist unique nonsingular $2p \times 2p$ complex matrices S and $\overline{S}$ such that [7, Ch. 1]

$$A' = SAS^{-1}, \quad B_r' = SB_r, \quad C_r' = C_rS^{-1}, \quad H' = SHS^{-1}, \tag{5.2a}$$

$$A'' = \overline{S}A\overline{S}^{-1}, \quad B_l'' = \overline{S}B_l, \quad C_l'' = C_l\overline{S}^{-1}, \quad H'' = \overline{S}H\overline{S}^{-1}. \tag{5.2b}$$

In other words, choosing the primed and double primed matrix quadruplets to be $(A^*, B_{r,l}^*\sigma_2, \sigma_2 C_{r,l}^*, H^*)$, the symmetry relations (2.20) for the Marchenko kernels imply the existence of unique nonsingular $2p \times 2p$ matrices S and $\overline{S}$ such that

$$A^* = SAS^{-1}, \quad B_r^*\sigma_2 = SB_r, \quad \sigma_2 C_r^* = C_rS^{-1}, \quad H^* = SHS^{-1}, \tag{5.3a}$$

$$A^* = \overline{S}A\overline{S}^{-1}, \quad B_l^*\sigma_2 = \overline{S}B_l, \quad \sigma_2 C_l^* = C_l\overline{S}^{-1}, \quad H^* = \overline{S}H\overline{S}^{-1}. \tag{5.3b}$$

Taking complex conjugates we get

$$\begin{cases} A^* = S^{*-1}AS^*, \quad B_r^*\sigma_2 = -S^{*-1}B_r, \\ \sigma_2 C_r^* = -C_rS^*, \quad H^* = S^{*-1}HS^*, \end{cases} \tag{5.4a}$$

$$\begin{cases} A^* = \overline{S}^{*-1}A\overline{S}^*, \quad B_l^*\sigma_2 = -\overline{S}^{*-1}B_l, \\ \sigma_2 C_l^* = -C_l\overline{S}^*, \quad H^* = \overline{S}^{*-1}H\overline{S}^*. \end{cases} \tag{5.4b}$$

The uniqueness of the similarity transformations S and $\overline{S}$ then implies that

$$S^* = -S^{-1}, \quad \overline{S}^* = -\overline{S}^{-1}. \tag{5.5}$$

We observe that the minimal matrix triplets (A_r, B_r, C_r) and (A_l, B_l, C_l) need not consist of matrices having their entries in Σ, even though the expressions $C_re^{-zA_r}B_r$ and $C_le^{zA_l}B_l$ belong to Σ for each $z \in \mathbb{R}$.

Let us now apply a similarity transformation to the triplets (A_r, B_r, C_r) and (A_l, B_l, C_l) such that the newly found triplets consist of matrices having their entries in Σ. Indeed, letting $T = \lambda^{-1}(\Sigma_2 + \lambda^2 S^*)$ where Σ_2 is the direct sum of p copies of σ_2, $|\lambda| = 1$, and $\Sigma_2 + \lambda^2 S^*$ is nonsingular, we obtain

$$ST\Sigma_2 = \lambda^{-1}S + \lambda SS^*\Sigma_2 = \lambda^{-1}S - \lambda\Sigma_2 = (\lambda^{-1}\Sigma_2 + \lambda S^*)^* = T^*,$$

and hence $S = T^*\Sigma_2T^{-1}$ (see [37] for a similar argument involving the Ansatz $S^* = S^{-1}$). Substituting the latter into (5.4a) we get

$$\begin{cases} (T^{-1}AT)^* = \Sigma_2(T^{-1}AT)\Sigma_2, \\ (T^{-1}B)^* = \Sigma_2(T^{-1}B)\sigma_2, \\ (CT)^* = \sigma_2(CT)\Sigma_2, \end{cases}$$

where we have omitted the subscripts r and l. Hence, the matrix triplet $(T^{-1}AT, T^{-1}B, CT)$ consists of matrices having their entries in Σ. In the same way, by replacing H with $T^{-1}HT$ we arrive at a matrix belonging to $\Sigma^{p \times p}$.

Since the Zakharov–Shabat system $v_x = (-ik\sigma_3 + Q)v$ is $1 + 1$, every discrete eigenvalue $k_s \in \mathbb{C}^+$ is *geometrically* simple. Because the conformal mapping $k \mapsto \lambda = \sqrt{k^2 + \mu^2}$ is 1, 1 on $\mathbb{C}^+$ cut along the segment $(i0^+, i\mu]$, the eigenvalues λ_s of the matrix Schrödinger equation (2.1) in $\mathbb{C}^+$ are geometrically simple. Thus the matrix $A_{r,l}$ in the minimal representations (5.1) has a Σ-Jordan structure with exactly two Jordan blocks of the same order per positive eigenvalue, one Jordan block per complex eigenvalue with positive real part, and Jordan blocks of the same order corresponding to complex conjugate eigenvalues (which have positive real part). As a result, there exist quadruplets (A_r, B_r, C_r, H_r) and (A_l, B_l, C_l, H_l) consisting of matrices having their entries in Σ such that A_r and A_l have the above Σ-Jordan normal form and have minimal matrix order among all quadruplets leading to the same Marchenko integral kernels (5.1).

5.2 Inverse scattering implemented

Let us depart from the representations (5.1) of the Marchenko integral kernels, where the quadruplets (A_r, B_r, C_r, H_r) and (A_l, B_l, C_l, H_l) consist of matrices having their entries in Σ such that A_r and A_l have the above Σ-Jordan normal form and have minimal matrix order among all quadruplets leading to the same Marchenko integral kernels (5.1).

Substituting the first of (5.1) into the right Marchenko equation (2.17a), we obtain using the commutativity of A and H

$$
\begin{aligned}
K(x, y; t) &= -\left[C_r e^{-xA} + \int_x^\infty ds\, K(x, s; t) C_r e^{-sA} \right] e^{-yA} e^{tH} B_r \\
&= -W_r(x; t) e^{-yA} e^{tH} B_r,
\end{aligned} \tag{5.6}
$$

where $W_r(x; t) = C_r e^{-xA} - W_r(x; t) e^{-xA} e^{tH} P_r e^{-xA}$ and

$$
P_r = \int_0^\infty ds\, e^{-sA} B_r C_r e^{-sA} \tag{5.7}
$$

is the unique solution of the Sylvester equation $AP_r + P_r A = B_r C_r$. Hence,

$$
K(x, y; t) = -C_r e^{-xA} \left[I_{2p} + e^{-xA} e^{tH} P_r e^{-xA} \right]^{-1} e^{-yA} e^{tH} B_r, \tag{5.8}
$$

provided the inverse matrix exists. Then Theorem A.1 implies that P_r is invertible. Moreover, Theorem A.3 implies that the inverse matrix in (5.8) exists for all but finitely many $x \in \mathbb{R}$. Similarly, substituting the second of (5.1) into the left Marchenko

equation (2.17b), we obtain

$$J(x, y; t) = -\left[C_l e^{xA} + \int_{-\infty}^{x} ds \, J(x, s; t) C_l e^{sA} \right] e^{yA} e^{tH} B_l$$

$$= -W_l(x; t) e^{yA} e^{tH} B_l, \tag{5.9}$$

where $W_l(x; t) = C_l e^{xA} - W_l(x; t) e^{xA} e^{tH} P_l e^{xA}$ and

$$P_l = \int_{0}^{\infty} ds \, e^{-sA} B_l C_l e^{-sA} \tag{5.10}$$

is the unique solution of the Sylvester equation $A P_l + P_l A = B_l C_l$. Then Theorem A.1 implies that P_l is invertible. Analogously,

$$J(x, y; t) = -C_l e^{xA} \left[I_{2p} + e^{xA} e^{tH} P_l e^{xA} \right]^{-1} e^{yA} e^{tH} B_l, \tag{5.11}$$

provided the inverse matrix exists. Moreover, Theorem A.3 implies that the inverse matrix in (5.11) exists for all but finitely many $x \in \mathbb{R}$. Furthermore, P_r and P_l belong to $\Sigma^{p \times p}$.

Using (2.13) in (5.8) and (5.11) and differentiating with respect to x we obtain

$$Q(x; t) = -4 C_r \left[e^{2xA} e^{-tH} + P_r \right]^{-1} A e^{2xA} e^{-tH} \left[e^{2xA} e^{-tH} + P_r \right]^{-1} B_r, \tag{5.12a}$$

$$Q(x; t) = -4 C_l \left[e^{-2xA} e^{-tH} + P_l \right]^{-1} A e^{-2xA} e^{-tH} \left[e^{-2xA} e^{-tH} + P_l \right]^{-1} B_l. \tag{5.12b}$$

Since P_r and P_l are nonsingular, these expressions are exponentially decaying as $x \to \pm\infty$. Writing $B_r = \begin{pmatrix} B_{r,1} & B_{r,2} \end{pmatrix}$ and $C_r = \begin{pmatrix} C_{r,1} \\ C_{r,2} \end{pmatrix}$ and similarly for B_l and C_l, we obtain the following expressions relating the potentials to the asymptotic potentials q_r and q_l

$$q(x, t) = q_r + 2 C_{r,1} \left[e^{2xA} e^{-tH} + P_r \right]^{-1} B_{r,2}$$

$$= q_l - 2 C_{l,1} \left[e^{-2xA} e^{-tH} + P_l \right]^{-1} B_{l,2}, \tag{5.13a}$$

$$q^*(x, t) = q_r^* + 2 C_{r,2} \left[e^{2xA} e^{-tH} + P_r \right]^{-1} B_{r,1}$$

$$= q_l^* - 2 C_{l,2} \left[e^{-2xA} e^{-tH} + P_l \right]^{-1} B_{r,1}, \tag{5.13b}$$

provided $e^{\pm 2xA} e^{tH} + P_{r,l}$ (for each $x \in \mathbb{R}$) are nonsingular matrices. Since $P_{r,l}$ are nonsingular, we get

$$q_l = q_r + 2 C_{r,1} P_r^{-1} B_{r,2}, \qquad q_r = q_l - 2 C_{l,1} P_l^{-1} B_{l,2}, \tag{5.14a}$$

$$q_l^* = q_r^* + 2C_{r,2} P_r^{-1} B_{r,1}, \qquad q_r^* = q_l^* - 2C_{l,2} P_l^{-1} B_{r,1}. \tag{5.14b}$$

Since $\mu = |q_l| = |q_r|$, the right and left matrix triplets cannot be chosen arbitrarily. The first of (5.14a) implies that

$$C_{r,1} P_r^{-1} B_{r,2} = \tfrac{1}{2}\mu(e^{i\theta_l} - e^{i\theta_r}).$$

Since $|e^{i\theta_l} - e^{i\theta_r}| \leq 2$, we see that the matrix triplet is to satisfy

$$|C_{r,1} P_r^{-1} B_{r,2}| = \mu \left|\sin[\tfrac{1}{2}(\theta_r - \theta_l)]\right| \leq \mu, \tag{5.15}$$

where $e^{i(\theta_l - \theta_r)}$ and hence $e^{i\theta_l}$ can be evaluated from known μ and $e^{i\theta_r}$. This means that the triplet and μ are not independent. Once μ has been chosen to satisfy $\mu \geq |C_{r,1} P_r^{-1} B_{r,2}|$, it is possible to determine θ_l uniquely up to an additive multiple of 2π. Moreover, we have established the following

Proposition 5.1 *If $0 < \mu < |C_{r,1} P_r^{-1} B_{r,2}|$, no soliton solution exists.*

The matrix H commuting with A is easily seen to be given by

$$H = \frac{1}{2\pi i} \oint_\Gamma d\lambda \, [2i\lambda\sqrt{\lambda^2 - \mu^2} - i\mu^2](\lambda I_{2p} - iA)^{-1}, \tag{5.16}$$

where $k(\lambda) = \sqrt{\lambda^2 - \mu^2}$ is the conformal mapping from $\mathbb{C}^+$ onto $\mathbb{C}^+$ satisfying $k(\lambda) \sim \lambda$ at infinity and Γ is a closed rectifiable Jordan contour in the upper half-plane which has winding number $+1$ with respect to each eigenvalue of iA. Then

$$e^{tH} = \frac{1}{2\pi i} \oint_\Gamma d\lambda \, e^{i[2\lambda\sqrt{\lambda^2 - \mu^2} - \mu^2]t}(\lambda I_{2p} - iA)^{-1}. \tag{5.17}$$

Let us finally derive the expressions for the transmission coefficients. Substituting (5.8) into (2.12a) and (5.9) into (2.12b) we get

$$F_l(x, \lambda; t) = e^{i\lambda x} \left(I_2 - iC_r \left[e^{2xA} e^{-tH} + P_r\right]^{-1} (\lambda I_{2p} + iA)^{-1} B_r \right),$$

$$F_r(x, \lambda; t) = e^{-i\lambda x} \left(I_2 - iC_l \left[e^{-2xA} e^{tH} + P_l\right]^{-1} (\lambda I_{2p} + iA)^{-1} B_l \right).$$

Dividing by $e^{\pm i\lambda x}$, taking the limits of the resulting equalities as $x \to \mp\infty$, and using (2.6a) and (2.6b) we arrive at the identities

$$A_l(\lambda) = I_2 - iC_r P_r^{-1}(\lambda I_{2p} + iA)^{-1} B_r, \tag{5.18a}$$

$$A_r(\lambda) = I_2 - iC_l P_l^{-1}(\lambda I_{2p} + iA)^{-1} B_l, \tag{5.18b}$$

where we have used the nonsingularity of $P_{r,l}$. Using the Sylvester equations for $P_{r,l}$ we obtain the transmission coeffients

$$A_l(\lambda)^{-1} = I_2 + i C_r (\lambda I_{2p} - i A)^{-1} P_r^{-1} B_r, \tag{5.19a}$$

$$A_r(\lambda)^{-1} = I_2 + i C_l (\lambda I_{2p} - i A)^{-1} P_l^{-1} B_l. \tag{5.19b}$$

Observe that the transmission coefficients are time-invariant. Using the Sherman-Morrison-Woodbury formula $\det(I - TS) = \det(I - ST)$ [cf. [22]] and the Sylvester equations for $P_{r,l}$ we easily obtain

$$\det[A_{l,r}(\lambda)^{-1}] = \frac{\det(\lambda I_{2p} + i A)}{\det(\lambda I_{2p} - i A)}.$$

6 Examples

In this section we work out various examples of multisoliton solutions based on the minimal quadruplet (A, B, C, H), where $H = \phi(A)$ for some function ϕ that is analytic in a neighborhood of the eigenvalues of A. In fact [cf. (5.17)], $\phi(\lambda) = i[2\lambda\sqrt{\lambda^2 - \mu^2} - \mu^2]$, where $k(\lambda) = \sqrt{\lambda^2 - \mu^2}$ is the conformal mapping from $\mathbb{C}^+$ onto $\mathbb{C}^+$ satisfying $k(\lambda) \sim \lambda$ at infinity.

Example 6.1 (One-soliton solution with real eigenvalue) Consider the minimal triplet

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & -b_2^* \\ b_2 & b_1^* \end{pmatrix}, \qquad C = \begin{pmatrix} c_1 & -c_2^* \\ c_2 & c_1^* \end{pmatrix},$$

where $a > 0$ and B and C have positive determinants. Then

$$P = \frac{1}{2a} B C = \frac{1}{2a} \begin{pmatrix} d_1 & -d_2^* \\ d_2 & d_1^* \end{pmatrix},$$

where $d_1 = b_1 c_1 - b_2^* c_2$ and $d_2 = b_2 c_1 + b_1^* c_2$. Then (5.8) implies that

$$K(x, y; t) = -e^{-a(x+y)} e^{t\phi(a)} \begin{pmatrix} c_1 & -c_2^* \\ c_2 & c_1^* \end{pmatrix} \times$$

$$\times \begin{pmatrix} 1 + \frac{1}{2a} e^{-2ax} e^{t\phi(a)} d_1 & -\frac{1}{2a} e^{-2ax} e^{t\phi(a)} d_2^* \\ \frac{1}{2a} e^{-2ax} e^{t\phi(a)} d_2 & 1 + \frac{1}{2a} e^{-2ax} e^{t\phi(a)} d_1^* \end{pmatrix}^{-1} \begin{pmatrix} b_1 & -b_2^* \\ b_2 & b_1^* \end{pmatrix},$$

where for any $x \in \mathbb{R}$ the matrix to be inverted has the nonnegative determinant

$$D(x; t) = \left| 1 + \frac{1}{2a} e^{-2ax} e^{t\phi(a)} d_1 \right|^2 + \left| \frac{1}{2a} e^{-2ax} e^{t\phi(a)} d_2 \right|^2.$$

We assume this determinant to be positive for each $(x, t) \in \mathbb{R}^2$. In fact, the determinant $D(x; t)$ vanishes at some $x \in \mathbb{R}$ for given $t \in \mathbb{R}$ [namely, at $x = \frac{1}{2a} \ln(-\frac{d_1}{2a} e^{t\phi(a)})$]

iff $d_1 < 0$ and $d_2 = 0$, i.e., iff $\boldsymbol{BC}$ is a negative multiple of I_2. Therefore,

$$
\begin{aligned}
K(x, y; t) = -\frac{e^{-a(x+y)}e^{t\phi(a)}}{D(x; t)} & \begin{pmatrix} c_1 & -c_2^* \\ c_2 & c_1^* \end{pmatrix} \\
\times & \begin{pmatrix} 1 + \frac{1}{2a}e^{-2ax}e^{t\phi(a)}d_1^* & \frac{1}{2a}e^{-2ax}e^{t\phi(a)}d_2^* \\ -\frac{1}{2a}e^{-2ax}e^{t\phi(a)}d_2 & 1 + \frac{1}{2a}e^{-2ax}e^{t\phi(a)}d_1 \end{pmatrix} \begin{pmatrix} b_1 & -b_2^* \\ b_2 & b_1^* \end{pmatrix}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
q(x) = q_r + 2\frac{e^{-2ax}e^{t\phi(a)}}{D(x; t)} & \begin{pmatrix} c_1 & -c_2^* \end{pmatrix} \\
\times & \begin{pmatrix} 1 + \frac{1}{2a}e^{-2ax}e^{t\phi(a)}d_1^* & \frac{1}{2a}e^{-2ax}e^{t\phi(a)}d_2^* \\ -\frac{1}{2a}e^{-2ax}e^{t\phi(a)}d_2 & 1 + \frac{1}{2a}e^{-2ax}e^{t\phi(a)}d_1 \end{pmatrix} \begin{pmatrix} -b_2^* \\ b_1^* \end{pmatrix} \\
= q_r + \frac{2}{D(x; t)} & \Big[-(c_1 b_2^* + c_2^* b_1^*)e^{-2ax}e^{t\phi(a)} \\
& + \frac{1}{2a}e^{-4ax}e^{2t\phi(a)}\left(-c_1 d_1^* b_2^* + c_1 d_2^* b_1^* - c_2^* d_2 b_2^* - c_2^* d_1 b_1^*\right)\Big].
\end{aligned}
$$

Thus,

$$
q_l = q_r + \frac{4a}{|d_1|^2 + |d_2|^2} \begin{pmatrix} c_1 & -c_2^* \end{pmatrix} \begin{pmatrix} d_1^* & d_2^* \\ -d_2 & d_1 \end{pmatrix} \begin{pmatrix} -b_2^* \\ b_1^* \end{pmatrix} = q_r.
$$

Since $\boldsymbol{P} = \frac{1}{2a}\boldsymbol{BC}$ with $\boldsymbol{B}$ and $\boldsymbol{C}$ nonsingular, we see that

$$
q_l - q_r = 4a \begin{pmatrix} 1 & 0 \end{pmatrix} \boldsymbol{C}(\boldsymbol{BC})^{-1}\boldsymbol{B} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0,
$$

thus conferming our preceding result.

Example 6.2 (one-soliton solution with conjugate eigenvalues) Consider the minimal triplet

$$
\boldsymbol{A} = \begin{pmatrix} a & \omega \\ -\omega & a \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} b_1 & -b_2^* \\ b_2 & b_1^* \end{pmatrix}, \qquad \boldsymbol{C} = \begin{pmatrix} c_1 & -c_2^* \\ c_2 & c_1^* \end{pmatrix},
$$

where $a > 0, 0 \neq \omega \in \mathbb{R}$, and $\boldsymbol{B}$ and $\boldsymbol{C}$ have positive determinants. Then

$$
e^{-x\boldsymbol{A}} = e^{-ax} \begin{pmatrix} \cos(\omega x) & \sin(\omega x) \\ -\sin(\omega x) & \cos(\omega x) \end{pmatrix}.
$$

Using $\int_0^\infty dx\, e^{-2ax}\cos(2\omega x) = \frac{a}{2(a^2+\omega^2)}$ and $\int_0^\infty dx\, e^{-2ax}\sin(2\omega x) = \frac{\omega}{2(a^2+\omega^2)}$, we get the Sylvester solution

$$
\boldsymbol{P} = \begin{pmatrix}
\frac{d_1-d_1^*}{4a} + \frac{a(d_1+d_1^*)}{4(a^2+\omega^2)} + \frac{\omega(d_2+d_2^*)}{4(a^2+\omega^2)} & \frac{d_2-d_2^*}{4a} - \frac{a(d_2+d_2^*)}{4(a^2+\omega^2)} + \frac{\omega(d_1+d_1^*)}{4(a^2+\omega^2)} \\
\frac{d_2-d_2^*}{4a} + \frac{a(d_2+d_2^*)}{4(a^2+\omega^2)} - \frac{\omega(d_1+d_1^*)}{4(a^2+\omega^2)} & -\frac{d_1-d_1^*}{4a} + \frac{a(d_1+d_1^*)}{4(a^2+\omega^2)} + \frac{\omega(d_2+d_2^*)}{4(a^2+\omega^2)}
\end{pmatrix},
$$

where $d_1 = b_1 c_1 - b_2^* c_2$ and $d_2 = b_2 c_1 + b_1^* c_2$. Note that

$$
\begin{aligned}
\det \boldsymbol{P} &= \left[\frac{a(d_1+d_1^*)}{4(a^2+\omega^2)} + \frac{\omega(d_2+d_2^*)}{4(a^2+\omega^2)}\right]^2 + \left[\frac{d_1-d_1^*}{4ia}\right]^2 \\
&\quad + \left[\frac{a(d_2+d_2^*)}{4(a^2+\omega^2)} - \frac{\omega(d_1+d_1^*)}{4(a^2+\omega^2)}\right]^2 + \left[\frac{d_2-d_2^*}{4ia}\right]^2 \\
&= \frac{(d_1+d_1^*)^2 + (d_2+d_2^*)^2}{16(a^2+\omega^2)} - \frac{(d_1-d_1^*)^2 + (d_2-d_2^*)^2}{16a^2}
\end{aligned}
$$

is positive. Therefore,

$$
\begin{aligned}
q_l = q_r &+ \frac{2\left(c_1 \ \ -c_2^*\right)}{\det \boldsymbol{P}} \\
&\times \begin{pmatrix}
-\frac{d_1-d_1^*}{4a} + \frac{a(d_1+d_1^*)}{4(a^2+\omega^2)} + \frac{\omega(d_2+d_2^*)}{4(a^2+\omega^2)} & -\frac{d_2-d_2^*}{4a} + \frac{a(d_2+d_2^*)}{4(a^2+\omega^2)} - \frac{\omega(d_1+d_1^*)}{4(a^2+\omega^2)} \\
-\frac{d_2-d_2^*}{4a} - \frac{a(d_2+d_2^*)}{4(a^2+\omega^2)} + \frac{\omega(d_1+d_1^*)}{4(a^2+\omega^2)} & \frac{d_1-d_1^*}{4a} + \frac{a(d_1+d_1^*)}{4(a^2+\omega^2)} + \frac{\omega(d_2+d_2^*)}{4(a^2+\omega^2)}
\end{pmatrix}
\begin{pmatrix} -b_2^* \\ b_1^* \end{pmatrix}.
\end{aligned}
$$

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

A Invertibility of the Sylvester solutions

Given a matrix triplet (A, B, C), where A is a $2p \times 2p$ matrix whose eigenvalues have positive real parts, B is a $2p \times r$ matrix, and C is an $r \times 2p$ matrix, we define the *controllability subspace* and the *observability subspace* of $\mathbb{C}^{2p}$ as follows:

$$\operatorname{Im}(A, B) = \bigvee_{j=0}^{\infty} \operatorname{Im}(A^j B), \tag{A.1a}$$

$$\operatorname{Ker}(C, A) = \bigcap_{j=0}^{\infty} \operatorname{Ker}(C A^j), \tag{A.1b}$$

where Im T and Ker T stand for the range and the null space of a matrix T, respectively. The V-symbol in (A.1a) denotes the set of finite linear combinations of vectors in the union of Im $(A^j B)$ ($j = 0, 1, 2, \ldots$) and the intersection in (A.1b) is finite. We observe that $\operatorname{Im}(A, B)$ is the smallest A-invariant subspace containing Im B and $\operatorname{Ker}(C, A)$ is the largest A-invariant subspace contained in Ker C. We call the matrix pair (A, B) *controllable* if $\operatorname{Im}(A, B) = \mathbb{C}^{2p}$. We call the matrix pair (C, A) *observable* if $\operatorname{Ker}(C, A)$ is the zero subspace. The matrix triplet (A, B, C) is called *minimal* if (A, B) is controllable and (C, A) is observable [or: if A has minimal matrix order among the triplets (A, B, C) leading to the same $\Omega(z) = C e^{-zA} B$]. A comprehensive account of controllability and observability can be found in any textbook on linear control theory [7, 13, 28].

For the above matrix triplets we obviously have in mind (A, B_r, C_r) and (A, B_l, C_l). In most of this subsection we drop the subscripts r and l and consider the triplets (A, BC, I_{2p}) and (A, I_{2p}, BC) with $r = 2p$ as well.

The next result relies on arguments provided by Hearon [27] for triplets (A, B, C) of complex matrices. Here Hearon's arguments are adapted to matrix triplets (A, B, C), where $A \in \Sigma^{p \times p}$, $B \in \Sigma^{p \times 1}$, and $C \in \Sigma^{1 \times p}$.

Theorem A.1 *Let (A, B, C) be a matrix triplet, where $A \in \Sigma^{p \times p}$ only has eigenvalues with positive real part, $B \in \Sigma^{p \times 1}$, and $C \in \Sigma^{1 \times p}$. Then the following statements are equivalent:*

(a) *The unique solution P of the Sylvester equation*

$$AP + PA = BC \tag{A.2}$$

is invertible.
(b) *The pair (A, BC) is controllable.*
(c) *The pair (BC, A) is observable.*

Proof Let us first prove that $\operatorname{Im}(BC)$ is contained in Im P iff Im P is A-invariant. Indeed, if Im P is A-invariant, then for each $h \in \mathbb{C}^{2p}$ there exists $k \in \mathbb{C}^{2p}$ such that $APh = Pk$; then, using (A.2), we get $BCh = P(k + Ah)$, thus proving that $\operatorname{Im}(BC)$ is contained in Im P. Conversely, if $\operatorname{Im}(BC)$ is contained in Im P, then for

each $h \in \mathbb{C}^{2p}$ there exists $k \in \mathbb{C}^{2p}$ such that $BCh = Pk$; then, using (A.2), we get $APh = P(k - Ah)$, thus proving that Im P is A-invariant.

Next, we prove that Ker (BC) contains Ker P iff Ker P is A-invariant. Indeed, if Ker P is contained in Ker (BC), then for each $h \in \mathbb{C}^{2p}$ such that $Ph = 0$ we have $BCh = 0$, which implies that $PAh = BCh - APh = 0$. Conversely, if Ker P is A-invariant, then for each $h \in \mathbb{C}^{2p}$ such that $Ph = 0$, we have $PAh = 0$ and hence $BCh = APh + PAh = 0$.

(b)$\Rightarrow$(a). Let q be a $p \times 1$ matrix with entries in Σ such that $Pq = 0$. Then $PAq = BCq$. Then there are two options:

(i) $Cq = 0$ whenever $Pq = 0$, or
(ii) $Cq \neq 0$ for some q satisfying $Pq = 0$.

In the first case, we see that $PAq = BCq - APq = 0$ and hence the kernel of P is A-invariant. If we then also assume that (BC, A) is observable, then Im P contains the smallest A-invariant subspace containing Im (BC) and hence the controllability of (A, BC) implies that P is invertible. In the second case we see that the Σ-vector B belongs to the range of P, implying that the range of BC is contained in the range of P so that the range of P is A-invariant. If we then also assume that (A, BC) is controllable and hence the smallest A-invariant subspace containing the range of BC is all of $\Sigma^{p \times 1}$, then P is invertible. In either case we conclude that P is invertible.

(c)$\Rightarrow$(a). Using the arguments of the preceding paragraph, we see that the controllability of $(A^\dagger, C^\dagger B^\dagger)$ implies the invertibility of $P^\dagger$.

(a)$\Rightarrow$[(b)+(c)] Let us first assume P to be invertible. To prove the controllability of the pair (A, BC), we take a vector $h \in \mathbb{C}^{2p}$ orthogonal to the controllability subspace Im (A, BC). Then

$$(A^j BC P^{-1} k, h) = 0, \qquad k \in \mathbb{C}^{2p}, \quad j = 0, 1, 2, \ldots.$$

Therefore, using the identity [cf. (A.2)]

$$A = BC P^{-1} - PAP^{-1}, \tag{A.3}$$

for arbitrary $k \in \mathbb{C}^{2p}$ and $j = 0, 1, 2, \ldots$ we have

$$(A^{j+1}k, h) = (A^j BC P^{-1}k, h) - (A^j PAP^{-1}k, h)$$
$$= -(A^j PAP^{-1}k, h).$$

By the arbitrariness of k we get

$$A^{\dagger j+1}h = -P^{\dagger -1}A^\dagger P^\dagger A^{\dagger j}h. \tag{A.4}$$

Repeated application of (A.4) yields

$$A^{\dagger j}h = -P^{\dagger -1}A^\dagger P^\dagger A^{\dagger j-1}h = (P^{\dagger -1}A^\dagger P^\dagger)^2 A^{\dagger j-2}h$$
$$= \cdots = (-P^{\dagger -1}A^\dagger P^\dagger)^j h = P^{\dagger -1}(-A^\dagger)^j P^\dagger h,$$

which implies that

$$q(A^\dagger)h = P^{\dagger^{-1}}q(-A^\dagger)P^\dagger h$$

for any polynomial $q(z)$. If we take $q(z) = \det(zI_{2p} - A^\dagger)$ [the characteristic polynomial of $A^\dagger$], we obtain $q(A^\dagger) = 0$ by the Cayley-Hamilton theorem [21]. Using that $A^\dagger$ and $-A^\dagger$ do not have common eigenvalues [and hence $q(z)$ and $q(-z)$ do not have common zeros], we obtain the invertibility of $q(-A^\dagger)$. Consequently, $h = 0$. As a result, $\mathrm{Im}\,(A, BC) = \mathbb{C}^{2p}$, yielding the controllability of the pair (A, BC). Finally, using the invertibility of $P^\dagger$, we prove the controllability of the pair $(A^\dagger, C^\dagger B^\dagger)$ and hence the observability of the pair (BC, A). □

Corollary A.2 *The matrices P_r defined by* (5.7) *and P_l defined by* (5.10) *are invertible.*

Theorem A.3 *For each $x \in \mathbb{R}$ except at finitely many values, the matrices $e^{2xA} + P_r$ and $e^{-2xA} + P_l$ are invertible.*

Example 6.1 contains a triplet where $\det(e^{2xA} + P_r) = 0$ for some $x \in \mathbb{R}$.

Proof In Theorem 4.1 above we have proved the nonnegativity of the determinants of $P_{r,l}$ and $e^{\pm2xA}e^{-tH} + P_{r,l}$ for each $x \in \mathbb{R}$. Since for each $t \in \mathbb{R}$ the function $\det(e^{\pm2xA}e^{-tH} + P_{r,l})$ is entire analytic in x, is nonnegative on the real x-line, tends to $+\infty$ as $x \to \pm\infty$ along the real line, and tends to $\det P_{r,l} > 0$ as $x \to \mp\infty$, there are at most finitely values of $x \in \mathbb{R}$ for which the matrix $e^{\pm2xA}e^{-tH} + P_{r,l}$ is singular. □

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