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Quantum avoidance of Gödel's closed timelike curves

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Abstract In a large class of nonlocal as well as local higher derivative theories minimally coupled to the matter sector, we investigate the exactness of two different classes of homogeneous Gödel-type solutions, which may or may not allow closed time-like curves (CTC). Our analysis is limited to spacetimes solving the Einstein's EoM, thus we can not exclude the presence of other Gödel-type solutions solving the EoM of local and nonlocal higher derivative theories but not the Einstein's EoM. It turns out that the homogeneous Gödel spacetimes without CTC are basically exact solutions for all theories, while the metrics with CTC are not exact solutions of (super-)renormalizable local or nonlocal gravitational theories. Hence, the quantum renormalizability property excludes theories suffering of the Gödel's causality violation. We also comment about nonlocal gravity non-minimally coupled to matter. In this class of theories, all the Gödel's spacetimes, with or without CTC, are exact solutions at classical level. However, the quantum corrections, although perturbative, very likely spoil the exactness of such solutions. Therefore, we can state that the Gödel's Universes with CTC and the super-renormalizability are mutually exclusive.

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1 Introduction

In a very inspiring and relevant paper [1], the authors investigated the presence of homogeneous cosmological Gödel-type solutions in a quite general class of nonlocal gravitational theories. We here expand on the analysis in [1] looking for a principle of mutual exclusion between unitarity (as already suggested in [1]) and/or renormalizability and the presence of homogeneous Gödel-type exact solutions with CTC in nonlocal gravity. We will mainly study a class of nonlocal theories characterized by the minimal coupling to gravity [2,3]. However, in the end we will also comment on the same issue in a class of nonlocal gravitational theories with non-minimal coupling to matter [4,5]. Let us now introduce the theory and its main properties. Nonlocal quantum gravity has been extensively studied since 2011 as a consistent proposal for quantum gravity in the quantum field theory framework. The minimal action for pure gravity was studied in [6], but subsequent further searches revealed the existence of two cornerstone papers on nonlocal quantum gravity by Krasnikov [7] and Kuz'min [8]. In the former paper it was proposed a tree-level unitary action with exponential non locality, while in the latter the power counting super-renormalizability [9] was rigorously proved for a class of asymptotically polynomial nonlocal theories. In [10] the theory was extended to any dimension and proved to be finite at quantum level in odd dimension [11] (see also [12] for a more recent review). In [13], the theories proposed in [6–8] were extended by adding few other local operators in order to achieve the quantum finiteness in any dimension. The other crucial property satisfied by these theories is the perturbative unitarity which has been rigorously proved in [14–16]. In short, the main idea is to evaluate all the loop-integral along the imaginary axis for the energy, assuming also the external energy to be purely imaginary, and, afterwards make the analytic continuation to the real physical energy. It turns out that the Cutkosky rules are the same of the local theory with which the nonlocal theory shares the perturbative spectrum, namely the asymptotic degrees of freedom.

We come now to the main topic of the paper, namely the Gödel's spacetimes. In 1949, Gödel [17] discovered an exact solution of the Einstein's field equations sourced by a negative cosmological constant and a pressure-free perfect fluid. The line element of the Gödel Universe reads:

$$\begin{aligned} ds^2 &= -[dt + H(x)dy]^2 + D(x)dy^2 + dx^2 + dz^2, \\ H(x) &= e^{mx}, \quad D(x) = \frac{e^{mx}}{\sqrt{2}}, \end{aligned} \tag{1}$$

and it is sourced by the following energy–momentum tensor,

$$\begin{aligned} T_{ab} &= \rho V_a V_b, \quad V^a = (\partial_t)^a, \\ m^2 &= -2\Lambda_{cc} = \kappa^2 \rho = 2\omega^2, \end{aligned} \tag{2}$$

where ρ is the constant density of matter, V^a is the four-velocity of fluid, Λ_{cc} is the negative cosmological constant, κ is related to the Newton's gravitational constant, ω is the rotation velocity of matter.

The Gödel spacetime is geodetically complete, but it presents CTC that could lead to a violation of macro-causality. The latter statement is actually debated because the CTC are not geodesics and can not be traveled in a finite amount of time. However, in this paper we will only focus on the presence of Gödel's solutions in a large class of gravitational theories without investigating the geodesic motion of probe particles.

More recently, in 1982, a class of homogeneous space-times of Gödel-type [18,19] was proposed by Reboucas and Tiomno [20]. The latter metric is homogeneous [21] and causal (no CTC) whether the parameters in the metric take specific values. This was the first proposal for a completely causal homogeneous and rotating Universe. In 2021 the homogeneous Gödel-type without CTC has been shown to be an exact causal solution of a special class of classical nonlocal gravitational theories [1]. Moreover, in [1] the authors did not find Gödel-type solutions with CTC. In this paper, we generalize the result in [1] deriving the general constrain that a local or nonlocal higher derivative theory has to satisfy whether we want the Gödel-type metrics to be exact solutions of such theory. In this project, we will assume the Gödel-type metrics to be exact solutions of the Einstein's two-derivative theory. Therefore, we can not exclude other Gödel-type metrics that do not solve the Einstein's equations, but solve the full theory.

Let us here list the issues addressed in each section of the paper. In Sect. 2, we review the homogeneous Gödel-type metrics: the classification and the conditions for the existence of CTC. In Sect. 3, we introduce the notations and the implicit form of a general action for local or nonlocal gravitational theory minimally coupled to matter. In Sect. 3.1, we explicitly introduce the action, while in Sect. 3.2, we derive the equations to be solved in order to prove or disprove the presence of exact homogeneous Gödel-type solutions in local or nonlocal higher derivative theories minimally coupled to matter. In Sect. 3.3, we explicitly state in compact form the equations of motion and study under which restrictions the homogeneous Gödel-type spacetimes (including the metrics with CTC) are exact solutions in higher derivative or nonlocal gravitational theories. In Sects. 3.4 and 3.5, we show that the homogeneous Gödel-type spacetimes without CTC are exact solutions of basically all local or nonlocal theories with higher derivative operators.

In Sect. 4, we show that the conditions for unitarity and renormalizability exclude theories having solutions with CTC. Finally, in Sect. 5, we shortly review nonlocal gravity non-minimally coupled to matter and show that all Gödel-type spacetimes are exact solutions by construction. However, the quantum corrections very likely will exclude such solutions.

2 Gödel-type metrics and CTC

In this section, we briefly review the homogeneous Gödel-type metrics and provide the condition for having CTC (for more details we invite the reader to consult [20]).

The homogeneous Gödel-type metrics are defined by the following line element in cylindrical coordinates,

$$d^2s = -[dt + H(r)d\theta]^2 + D^2(r)d\theta^2 + dr^2 + dz^2, \tag{3}$$

where $H(r)$ and $D(r)$ are functions of the radial coordinate r and satisfy the following conditions,

$$\frac{H'(r)}{D(r)} = 2\omega \in \mathbb{R}\backslash\{0\} \quad \text{and} \quad \frac{D''(r)}{D(r)} = m^2 \in \mathbb{R}, \tag{4}$$

where the prime stays for the derivative with respect to the radial coordinate r.

As discussed in [20], the homogeneous Gödel-type spaces can be organized in four classes depending on the values of the parameters ω and m^2:

1. the *hyperbolic class*: $m^2 > 0, \omega \neq 0$:

$$H(r) = \frac{2\omega}{m^2}[\cosh(m\,r) - 1] \quad \text{and} \quad D(r) = \frac{1}{m}\sinh(m\,r), \tag{5}$$

2. the *trigonometric class*: $-\mu^2 = m^2 < 0, \omega \neq 0$:

$$H(r) = \frac{2\omega}{\mu^2}[1 - \cos(\mu\,r)] \quad \text{and} \quad D(r) = \frac{1}{\mu}\sin(\mu\,r), \tag{6}$$

3. the *linear class*: $m^2 = 0, \omega \neq 0$:

$$H(r) = \omega r^2 \quad \text{and} \quad D(r) = r, \tag{7}$$

4. the *degenerate class*: $m^2 \neq 0, \omega = 0$:

$$H(r) = 0. \tag{8}$$

The reader can verify that the above four classes of metrics are exact solutions of the Einstein's EoM. As a particular example, the Gödel original metric (1) corresponds to $m^2 = 2\omega^2 > 0$.

Closed time-like curves

The CTC in the spacetime (3) are characterized by the following curve,

$$C = \Big\{(t, r, \theta, z);\; t, r, z = \text{const},\;\; \theta \in [0, 2\pi]\Big\}. \tag{9}$$

Notice that for $t, r, z = \text{const}$, the θ coordinate is time-like. Therefore, the curve C is a CTC because θ is a periodic angular coordinate.

It turns out that the condition for having a CTC in an homogeneous Gödel-type metric is:

$$\boxed{\exists\, r_0 \text{ s.t. } G(r_0) \equiv D^2(r_0) - H^2(r_0) < 0 \iff \boxed{4\omega^2 > m^2 > -\infty}}. \tag{10}$$

It is worth noting that there are no CTC only for the degenerate class of solutions.

3 Gödel-type solutions in local and nonlocal gravity

In order to fix the notation we here define the action and the equations of motion (EoM). The action for a general gravitational theory reads:

$$S = \int d^4x\sqrt{-g}\mathcal{L}_g + S_m[g_{\mu\nu}, \Psi] \quad \text{and} \quad \mathcal{L}_g = \frac{1}{2\kappa^2}(R - 2\Lambda_{\rm cc}) + \mathcal{L}, \tag{11}$$

where $2\kappa^2 = 16\pi G$, G is Newton's constant, $\Lambda_{\rm cc}$ is the cosmological constant. $\mathcal{L}$ is the Lagrangian beyond the Einstein-Hilbert one, and S_m is the action for matter. Taking the variation respect to the metric $g_{\mu\nu}$, the EoM can be written as:

$$-\frac{2}{\sqrt{-g}}\frac{\partial S}{\partial g_{ab}}=0 \implies \frac{1}{\kappa^2}(G_{ab}+\Lambda_{cc}g_{ab}+Q_{ab})=T_{ab}, \tag{12}$$

where Q_{ab} is the extra term coming from the variation of $\mathcal{L}$.

We now look for Gödel-type solutions in several nonlocal and local theories.

3.1 Nonlocal gravity

In order to extend the result in [1], in this section we consider the nonlocal gravitational action in the Ricci–Weyl basis,

$$S = \int d^4x\sqrt{-g} \times \left[\frac{1}{2\kappa^2}(R - 2\Lambda_{\rm cc}) + R\gamma_0(\Box)R + R_{ab}\gamma_2(\Box)R^{ab} + C_{abcd}\gamma_4(\Box)C^{abcd}\right] + S_m[g_{ab}, \Psi], \tag{13}$$

where the analytic form factors γ_i are infinite power series of the dimensionless d'Alembert operator $\Box_\Lambda \equiv \Box/\Lambda^2$, i.e.,

$$\gamma_i(\Box) = \sum_{n=0}^{\infty}\gamma_{i,n}\Box_\Lambda^n \quad \text{and} \quad i = \{0, 2, 4\}, \tag{14}$$

$\gamma_{i,n}$ are the coefficients (dimensionless) of the power series in $\Box_\Lambda$, Λ is an invariant fundamental mass scale of the theory, namely in this case the non locality scale.

Notice that the unitarity of the theory has been investigate in the Riemann–Ricci basis [3], hence, we will later in Sect. 4 change basis in order to make a connection between renormalizable and unitary theories with the presence of Gödel solutions.

3.2 Equations of motion with the Gödel's ansatz

In this project we do not look for new solutions, but we investigate under which conditions the Gödel-type metrics of the Sect. 2, which solve the Einstein's EoM, are solutions of the general nonlocal or local higher derivative theory (11).

Therefore, in order to solve the modified EoM (12) the Gödel's metric $\bar{g}_{ab}$ should satisfy:

$$Q^{\alpha\beta}(\bar{g}) \equiv 0 \quad \forall\, m,\; \omega. \tag{15}$$

The latter equation turns out to be an algebraic constraint on the space of theories, namely on the constant coefficients $\gamma_{i,n}$. Indeed,

$$\cancel{G^{\alpha\beta}(\bar{g})+\Lambda_{\rm cc}\,\bar{g}^{\alpha\beta}}+Q^{\alpha\beta}(\bar{g}) = \cancel{\kappa^2 T^{\alpha\beta}(\bar{g})} \implies Q^{\alpha\beta}(\bar{g})=0. \quad (16)$$

For the sake of simplicity from now on we will identify $\bar{g}_{ab}$ with g_{ab}.

3.3 General solutions

We now investigate under which conditions the Gödel metric is an exact solution of the nonlocal theory (13) for general values of the parameters m and ω. For convenience, we introduce the following truncation of the action (13),

$$S_n=\int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}(R-2\Lambda_{\rm cc})+\gamma_{0,n}R\Box^n R +\gamma_{2,n}R_{ab}\Box^n R^{ab}+\gamma_{4,n}C_{abcd}\Box^n C^{abcd}\right]+S_m[g_{ab},\Psi], \quad (17)$$

where n stays for the n-th order of the theory (13). The nonlocal theory is obtained summing on n from zero to infinity the higher derivative operators.

Let us start with the action S_0. It turns out that the theory S_0 reduces to Einstein's gravity for $\gamma_{i,0}=0$. Therefore, for $\gamma_{i,0}=0$ $(i=0,2,4)$, the Gödel metric is an exact solution of the theory. Solutions for $\gamma_{i,0}\neq 0$ consistent with causality were found in [22].

Next, we study the action S_n for any finite value of the integer n. Making use of (A11)–(A15) in (A7), (A8), (A9), we can find the tensor $Q_{\mu\nu}$ defined in (15) for the local theory (17). The following result is obtained by mathematical induction for each n for the action S_n with $\boxed{n\geq 1}$ and assuming $\gamma_{i,0}=0$,

$$Q^{(n)}_{\mu\nu}=\frac{2\kappa^2}{\Lambda^{2n}}(\gamma_{2,n}+2\gamma_{4,n})(m^2-4\omega^2)2\omega^2(6\omega^2)^{n-1} \times\begin{pmatrix}(2n+1)m^2-(20+8n)\omega^2 & & & \\ & (2n+1)m^2-(12+8n)\omega^2 & & \\ & & (2n+1)m^2-(12+8n)\omega^2 & \\ & & & 4\omega^2-m^2\end{pmatrix}. \quad (18)$$

The EoM for the nonlocal theory (13) are obtained taking the sum on the integer n from $n=1$ to $n=+\infty$ of the terms (18), namely for the moment we assume $\gamma_{i,0}=0$. The result takes the following compact form after resummation of the form factors,

$$\begin{aligned}Q_{\mu\nu}=\sum_{n=1}^{\infty}Q^{(n)}_{\mu\nu}&=2\kappa^2\frac{4\omega^2}{\Lambda^2}(m^2-4\omega^2)^2(\gamma_2'(6\omega^2/\Lambda^2)\\&+2\gamma_4'(6\omega^2/\Lambda^2))\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&0\end{pmatrix}\\&+2\kappa^2\frac{m^2-4\omega^2}{3}(\gamma_2(6\omega^2/\Lambda^2)\\&+2\gamma_4(6\omega^2/\Lambda^2))\begin{pmatrix}m^2-20\omega^2&&&\\&m^2-12\omega^2&&\\&&m^2-12\omega^2&\\&&&4\omega^2-m^2\end{pmatrix},\end{aligned} \quad (19)$$

where $\gamma_i(z)$ is defined in (14) and $\gamma_i'(z)$:

$$\gamma_i'(z)=\sum_{i=1}^{\infty}\gamma_{i,n}\,n\,z^{n-1},\quad z\equiv\frac{6\,\omega^2}{\Lambda^2}. \quad (20)$$

Finally, the modified EoM (12), namely

$$G_{\mu\nu}+\Lambda_{\rm cc}g_{\mu\nu}+Q_{\mu\nu}=\kappa^2 T_{\mu\nu}, \quad (21)$$

for $\underline{\gamma_{i,0}=0}$ read:

$$\begin{aligned}&3\omega^2-m^2-\Lambda_{cc}+2\kappa^2\left[\frac{4\omega^2}{\Lambda^2}(m^2-4\omega^2)^2(\gamma_2'(z)+2\gamma_4'(z))+\frac{m^2-4\omega^2}{3}(\gamma_2(z)+2\gamma_4(z))(m^2-20\omega^2)\right]=\kappa^2T_{00},\\&\omega^2+\Lambda_{cc}+2\kappa^2\left[\frac{4\omega^2}{\Lambda^2}(m^2-4\omega^2)^2(\gamma_2'(z)+2\gamma_4'(z))+\frac{m^2-4\omega^2}{3}(\gamma_2(z)+2\gamma_4(z))(m^2-12\omega^2)\right]=\kappa^2T_{11}=\kappa^2T_{22},\\&m^2-\omega^2+\Lambda_{cc}-2\kappa^2\frac{(m^2-4\omega^2)^2}{3}(\gamma_2(z)+2\gamma_4(z))=\kappa^2T_{33},\end{aligned} \quad (22)$$

where $z\equiv 6\omega^2/\Lambda^2$. while the matter content consists on the electromagnetic field and a real scalar field [19] whose total energy–momentum tensor is:

$$T_{\mu\nu} = \begin{pmatrix} \rho + \frac{e^2+E_0^2}{2} & & & \\ & p + \frac{E_0^2-e^2}{2} & & \\ & & p + \frac{E_0^2-e^2}{2} & \\ & & & p + \frac{e^2-E_0^2}{2} \end{pmatrix}. \tag{23}$$

The parameters e and E_0 were introduced in [19] in order to describe the energy–momentum tensor for a scalar field and for the electromagnetic field respectively, namely

$$T_{\mu\nu}^{(\text{Scalar})} = \text{diag}\left(\frac{e^2}{2}, -\frac{e^2}{2}, -\frac{e^2}{2}, \frac{e^2}{2}\right),$$
$$T_{\mu\nu}^{(\text{EM})} = \text{diag}\left(\frac{E_0^2}{2}, \frac{E_0^2}{2}, \frac{E_0^2}{2}, -\frac{E_0^2}{2}\right). \tag{24}$$

So far we assumed $\gamma_{i,0} = 0$, *but now we move to consider the general case* $\gamma_{i,0} \neq 0$.

According to the explicit calculation of $Q_{\mu\nu}^{(0)}$, which consists in evaluating P_1^{ab}, P_2^{ab}, and P_3^{ab} in (A7) but taking only the order $n = 0$ in the Taylor's expansion of the form factors, we get the following non vanishing diagonal contributions to $Q_{\mu\nu}^{(0)}$,

$$Q_{00}^{(0)} = 2\kappa^2 \left[\frac{10}{3}\omega^4\alpha + m^4\beta - 4m^2\omega^2\gamma\right],$$
$$Q_{11}^{(0)} = Q_{22}^{(0)} = 2\kappa^2 \left[2\omega^4\alpha + m^4\beta - \frac{8}{3}m^2\omega^2\gamma\right],$$
$$Q_{33}^{(0)} = 2\kappa^2 \left[-\frac{2}{3}\omega^4\alpha - m^4\beta + \frac{4}{3}m^2\omega^2\gamma\right], \tag{25}$$

where we have defined the following parameters that take into account of the order zero in the Taylor expansion of the form factors,

$$\alpha = 3\gamma_{0,0} + 9\gamma_{2,0} + 16\gamma_{4,0},$$
$$\beta = 2\gamma_{0,0} + \gamma_{2,0} + \frac{2}{3}\gamma_{4,0},$$
$$\gamma = 3\gamma_{0,0} + 3\gamma_{2,0} + 4\gamma_{4,0}. \tag{26}$$

In order to find the full tensor $Q_{\mu\nu}$ for the nonlocal theory we have to sum the contributions $Q_{\mu\nu}^{(n)}$ from $n = 0$ to infinity. The final result reads:

$$Q_{\mu\nu} = Q_{\mu\nu}^{(0)} + \sum_{n=1}^{+\infty} Q_{\mu\nu}^{(n)} = \delta_{\mu\nu}\,\omega^4 \left(a_\mu \frac{m^4}{\omega^4} + b_\mu \frac{m^2}{\omega^2} + c_\mu\right) \tag{27}$$

where $Q_{\mu\nu}$ is a diagonal matrix and, hence, there is no sum on the index μ. In (27) the first term is given in (25) and second term in (19).

Therefore, looking at the above expression (27) for $Q_{\mu\nu}$ as a polynomial in $y \equiv m^2/\omega^2$, one can easily figure out by linear independence of the monomials y^0, y^1, and y^2 that:

$$Q_{\mu\nu} = 0\,\forall\, m, \omega \implies a_\mu = b_\mu = c_\mu = 0\,\forall\,\omega$$
$$(\mu = 0, 1, 2, 3). \tag{28}$$

Now we rewrite the expressions for a_μ, b_μ, and c_μ in terms of the form factors. Indeed, after performing the sum over n, a_μ, b_μ, and c_μ will depend on a special linear combination of the form factors γ_2 and γ_4, namely $\gamma_2 + 2\gamma_4$ and its derivative respect to the argument $z = 6\omega^2/\Lambda^2$, i.e. $\gamma_2' + 2\gamma_4'$. Notice that the form factors in (19) do not include the order zero of their Taylor's expansion because we have taken into account of the terms $\gamma_{i,0}$ in $Q_{\mu\nu}^{(0)}$, namely in (25) or (26). Therefore, the sum on n in $\gamma_2 + 2\gamma_4$ and $\gamma_2' + 2\gamma_4'$ starts from $n = 1$.

The final result for the zero components of the three vectors a_μ, b_μ, and c_μ reads:

$$a_0 = \frac{4\omega^2}{\Lambda^2}(\gamma_2' + 2\gamma_4') + \frac{1}{3}(\gamma_2 + 2\gamma_4) + \beta,$$
$$b_0 = -\frac{32\omega^2}{\Lambda^2}(\gamma_2' + 2\gamma_4') - 8(\gamma_2 + 2\gamma_4) - 4\gamma,$$
$$c_0 = \frac{64\omega^2}{\Lambda^2}(\gamma_2' + 2\gamma_4') + \frac{80}{3}(\gamma_2 + 2\gamma_4) + \frac{10}{3}\alpha, \tag{29}$$

while for the first and the second components we have:

$$a_1 = a_2 = \frac{4\omega^2}{\Lambda^2}(\gamma_2' + 2\gamma_4') + \frac{1}{3}(\gamma_2 + 2\gamma_4) + \beta,$$
$$b_1 = b_2 = -\frac{32\omega^2}{\Lambda^2}(\gamma_2' + 2\gamma_4') - \frac{16}{3}(\gamma_2 + 2\gamma_4) - \frac{8}{3}\gamma,$$
$$c_1 = c_2 = \frac{64\omega^2}{\Lambda^2}(\gamma_2' + 2\gamma_4') + 16(\gamma_2 + 2\gamma_4) + 2\alpha. \tag{30}$$

Finally, for the third components we get:

$$a_3 = -\frac{1}{3}(\gamma_2 + 2\gamma_4) - \beta,$$
$$b_3 = \frac{8}{3}(\gamma_2 + 2\gamma_4) + \frac{4}{3}\gamma,$$
$$c_3 = -\frac{16}{3}(\gamma_2 + 2\gamma_4) - \frac{2}{3}\alpha. \tag{31}$$

Replacing the expression (31) into (30) and imposing (28) we get:

$$f'(6\omega^2/\Lambda^2) = 0\,\forall\,\omega,$$
$$\text{where } f'(6\omega^2/\Lambda^2) = \gamma_2'(6\omega^2/\Lambda^2) + 2\gamma_4'(6\omega^2/\Lambda^2). \tag{32}$$

Simplifying the equations $a_3 = b_3 = c_3 = 0$ taking into account of (26) we get:

$$f(6\omega^2/\Lambda^2) + \gamma_{2,0} + 2\gamma_{4,0} = \sum_{n=1}^{+\infty}(\gamma_{2,n} + 2\gamma_{4,n})\left(\frac{6\omega^2}{\Lambda^2}\right)^n$$
$$+\gamma_{2,0} + 2\gamma_{4,0} = 0\,\forall\omega, \quad \text{and} \quad 3\gamma_{0,0} + \gamma_{2,0} = 0. \tag{33}$$

Including the constant term again in the sum we can rewrite (33) as follows,

$$\sum_{n=0}^{+\infty}(\gamma_{2,n}+2\gamma_{4,n})\left(\frac{6\omega^2}{\Lambda^2}\right)^n \equiv \gamma_2+2\gamma_4=0\ \forall\omega,$$
$$\text{and}\quad 3\gamma_{0,0}+\gamma_{2,0}=0. \tag{34}$$

Hence, the final result for the nonlocal theory is:

$$\boxed{Q_{\mu\nu}=0\ \forall m,\omega \iff \gamma_2+2\gamma_4=0\ (\text{sum from 0 to infinity})\ \text{and}\ 3\gamma_{0,0}+\gamma_{2,0}=0.} \tag{35}$$

For a general local theory defined by the action,

$$S_N=\int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}(R-2\Lambda_{\rm cc})+\sum_{n=0}^{N}\gamma_{0,n}R\Box^nR\right.$$
$$\left.+\sum_{n=0}^{N}\gamma_{2,n}R_{ab}\Box^nR^{ab}+\sum_{n=0}^{N}\gamma_{4,n}C_{abcd}\Box^nC^{abcd}\right]+S_m, \tag{36}$$

the condition (35) holds whether we include the extra condition $\gamma_{i,n}=0$ for $n>N$, namely the condition (35) applies to the coefficients $\gamma_{i,n}$ for $0\leqslant n\leqslant N$, i.e.

$$\boxed{Q_{\mu\nu}=0\ \forall m,\omega \iff (\gamma_{2,n}+2\gamma_{4,n})=0\quad \text{for}\ 0\leqslant n\leqslant N\ \text{and}\ 3\gamma_{0,0}+\gamma_{2,0}=0.} \tag{37}$$

3.4 Exact solutions without CTC in nonlocal gravity

Another class of particular Gödel exact solutions for local as well as nonlocal gravitational theories is obtained for $m^2=4\omega^2$ [1] regardless of the explicit form of the form factors as long as the constant term in the form factors γ_i is zero, namely $\gamma_{i,0}=0$. Indeed, both $Q^{(n)}_{\mu\nu}$ and $Q_{\mu\nu}$ are identically zero for $m^2-4\omega^2=0$ whether $\gamma_{i,0}=0$ (see (18) and (19)).

However, if the conditions (35) and (37) on the nonlocal or local form factors are satisfied, then the rotating spacetimes for which $m^2=4\omega^2$ are exact solutions although $\gamma_{i,0}\neq 0$.

It deserves to be notice that according to (10) for $m^2=4\omega^2$ there are no CTC. Therefore, rotating causal Universes are exact solutions of nonlocal and local theories.

3.5 A more general action

We can extend the class of solutions with $m^2-4\omega^2=0$ found in the previous section to a more general class of theories. Let us consider the action (13) augmented by other operators whose prototype is the following one,

$$\mathcal{L}_{a,b}=A_{a_1\cdots a_n}\nabla^{a_n}B^{a_1\cdots a_{n-1}}, \tag{38}$$

where $A_{a_1\cdots a_n}$ and $B_{a_1\cdots a_{n-1}}$ are general tensors made of any finite number of derivatives (including no derivatives) and one or more curvature tensors (for example: $A_{abcd}=(\nabla^f R)R_a{}^eR_{eb}\nabla_f R_{cd}$). Now performing an explicit computation, it turns out that

$$\nabla_a R_{bcde}\propto(m^2-4\omega^2),\quad \text{or in short :}$$
$$\nabla\,\text{Riem}\propto(m^2-4\omega^2), \tag{39}$$

which vanishes for $m^2-4\omega^2=0$. Taking into account of (39) the variation of the action operator for the Lagrangian term (38) with respect to the metric gives:

$$\delta\int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_{a,b}=\int \mathrm{d}^4x\sqrt{-g}$$
$$\times\left[\mathcal{L}_{a,b}\frac{g^{ab}}{2}\delta g_{ab}+\delta(A_{a_1\cdots a_n})\nabla^{a_n}B^{a_1\cdots a_{n-1}}\right.$$
$$\left.+A_{a_1\cdots a_n}\delta(\nabla^{a_n})B^{a_1\cdots a_{n-1}}+A_{a_1\cdots a_n}\nabla^{a_n}\delta(B^{a_1\cdots a_{n-1}})\right]. \tag{40}$$

In the above variation the first and the second term contain ∇ Riem. Therefore, according to (39) they are zero when evaluated for $m^2=4\omega^2$. So far the variation reads:

$$\delta\int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_{a,b}\Big|_{m^2=4\omega^2}$$
$$=\int \mathrm{d}^4x\sqrt{-g}[A_{a_1\cdots a_n}\delta(\nabla^{a_n})B^{a_1\cdots a_{n-1}}$$
$$-\delta(B^{a_1\cdots a_{n-1}})\nabla^{a_n}A_{a_1\cdots a_n}], \tag{41}$$

where the second term resulting from the integration by parts is again zero because of (39). Let us now compute the variation of the covariant derivative,

$$\delta\int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_{a,b}\Big|_{m^2=4\omega^2}$$
$$=\int \mathrm{d}^4x\sqrt{-g}[A_{a_1\cdots a_n}\delta(g^{a_na})\nabla_aB^{a_1\cdots a_{n-1}}$$
$$+A_{a_1\cdots a_{n-1}}{}^a\delta(\nabla_a)B^{a_1\cdots a_{n-1}}]$$
$$=\int \mathrm{d}^4x\sqrt{-g}[A_{a_1\cdots a_{n-1}}{}^a$$
$$\sum_{i=1}^{n-1}(\delta\Gamma^{a_i}{}_{ab})B^{a_1\cdots a_{i-1}ba_{i+1}\cdots a_{n-1}}], \tag{42}$$

where the variation of the connection is:

$$\delta\Gamma^c{}_{ab} = \frac{1}{2}(\nabla_a h_b{}^c + \nabla_b h_a{}^c - \nabla^c h_{ab}), \quad \delta g_{ab} \equiv h_{ab}. \quad (43)$$

Integrating by parts the derivatives present in the variation of the connection, we end up with expression containing derivatives of the tensors A and B. Hence, the variation (42) is zero among using one more time (39) for $m^2 = 4\omega^2$.

Except for the last step of Eq. (40), the other terms vanish because these terms contain $\nabla_a R_{bcde}$. Such terms will vanish when $m^2 - 4\omega^2 = 0$ due to (39) and integration by parts in the last step of Eq. (40) will introduce the derivative operator in $A_{a_1\cdots a_{n-1}}{}^a B^{a_1\cdots a_{i-1}ba_{i+1}\cdots a_{n-1}}$, so the result is zero when $m^2 - 4\omega^2 = 0$.

We can conclude that if a general Lagrangian contains operators with at least one derivative, besides the Einstein-Hilbert term in presence of cosmological constant, then, the metrics for which $m^2 - 4\omega^2 = 0$ (without CTC) are exact solutions of the theory.

The above statement include the result at the end of the previous section relative to the case $m^2 - 4\omega^2 = 0$. Indeed, the Gödel metric with $m^2 = 4\omega^2$ is a solution in nonlocal gravity if all the constant terms in the Taylor expansion of the form factors $\gamma_{i,0}$ are zero regardless of the explicit form of the form factors.

4 Quantum renormalizability and CTC

We here investigate the presence of CTC in a special class of classical nonlocal theories compatible with unitary and (super-)renormalizability. Indeed, in the previous section we did not assume any relation between the form factors γ_0, γ_2, and γ_4 and we found that the Gödel spacetimes exact solutions if the conditions (35) are satisfied by the theory. In this section we compare the result (35) with the relations that the form factors should satisfy in order to have theories compatible with unitary and renormalizability.

The latter properties had been extensively studied in literature for the following Lagrangian [3] written in the Ricci–Riemann bases,

$$\mathcal{L} = \frac{1}{2\kappa^2}(R - 2\Lambda_{cc}) + R\tilde{\gamma}_0(\Box)R + R_{ab}\tilde{\gamma}_2(\Box)R^{ab} + R_{abcd}\tilde{\gamma}_4(\Box)R^{abcd}, \quad (44)$$

$$\tilde{\gamma}_0(\Box) = -\frac{(D-2)(e^{\mathrm{H}_0(\Box)}-1)+D(e^{\mathrm{H}_2(\Box)}-1)}{2\kappa^2\, 4(D-1)\Box} + \tilde{\gamma}_4(\Box), \quad (45)$$

$$\tilde{\gamma}_2(\Box) = \frac{e^{\mathrm{H}_2(\Box)}-1}{2\kappa^2\,\Box} - 4\tilde{\gamma}_4(\Box), \quad (46)$$

where the form factors have been properly selected in order to end up with the most general propagator (we here remind only the gauge invariant parto of the propagator) consistent with unitarity and (super-)renormalizability [2,3], namely

$$\mathcal{O}^{-1}(k) = -\frac{1}{k^2}\left[\frac{P^{(2)}}{e^{\mathrm{H}_2(k^2)}} - \frac{P^{(0)}}{(D-2)e^{\mathrm{H}_0(k^2)}}\right], \quad (47)$$

where $\{P^{(i)} |\ i = 0, 2\}$ are the projectors [10]. H_0 and H_2 are non-zero entire functions asymptotically approaching the same logarithm of a polynomial in the variable k^2 (at least in the simplest version of the theory). The entire functions H_2 and H_0 must have the same asymptotic behaviour in order to achieve renormalizability, while according to tree-level [23–26] and perturbative Unitarity [14–16] $H_2(0) = H_0(0) = 0$.

For completeness, we remind the Kuzmin's form factor $H(z)$ [8]. As mention above, $H(z)$ is an entire analytic function with asymptotic logarithmic behavior defined one possible explicit construction of such function is [6,8]:

$$H(z) = \int_0^{p(z)} \mathrm{d}w\, \frac{1-e^{-w}}{w} = \gamma_E + \Gamma[0, p(z)] + \log[p(z)], \quad (48)$$

where $p(z)$ is the most general polynomial of degree $n+1$ in the variable z, namely

$$p(z) = a_0 + a_1 z + a_2 z + \cdots + a_{n+1} z^{n+1}, \quad a_i \in \mathbb{R}. \quad (49)$$

The Gödel solution has been studied for the theory in the Ricci–Weyl basis (13), thus, in order to infer about a possible relation with unitarity and renormalizability we have to change basis. In the Appendix B we derived the relation between the form factors $\tilde{\gamma}_i$ (44) and the form factors γ_i (13). The outcome is:

$$\gamma_2 + 2\gamma_4 = \tilde{\gamma}_2 + 4\tilde{\gamma}_4. \quad (50)$$

Therefore, according to (35) the metrics for general m and ω are exact solutions of the theory (44) if:

$$\tilde{\gamma}_2 = -4\tilde{\gamma}_4. \quad (51)$$

Replacing the above identity in (46) we get: $H_2 = 0$, which is inconsistent with the renormalizability of the theory. Hence we can make the following statement,

Gödel spacetimes with CTC are not exact solutions in nonlocal (super-)renormalizable gravitational theories.

In other words, super-renormalizability and the Gödel's spacetimes with CTC are incompatible. Notice that the Unitarity is consistent with the Gödel's spacetimes with CTC because $H_2 = 0$ does not change the residue at the Cutkosky cuts [14,16].

For the case a local theories, one have to replace exp H_2 with a polynomial that must be zero for consistency with (51)) whether we want the Gödel's spacetimes to be solutions. Hence, the local theories that predict spacetimes with CTC are non-renormalizable.

According to Sect. 3.4, it is doubly surprising that metrics with CTC are not solutions while those without CTC are so in practically all theories.

Finally, we would like to comment on the case of nonlocal exponential form factors, exp $p(z)$, very often regarded and extensively studied in literature. From the classical point of view, the simplest form factor $H = -\Box$ is perfectly fine and it defines theories in which the Gödel Universes with CTC are not exact solutions according to (51) or $H_2(0)$. In other words, theories with exponential form factors are consistent with the claim in the box of above.

However, till now we are not able to quantize a diffeomorphisms invariant or a gauge invariant theory defined by such exponential for factors. Indeed, in gravity or gauge theories we have the same form factor and in both the kinetic operator and the vertexes. Therefore, we have to evaluate loops' integrals that are ratios of exponentials. Such integrals are not only divergent, but ill defined. Thus, at the moment we are not able to deal with such kind of non locality. On the other hand, the form factors exp H, with $H(z)$ defined in (48) introduced for the first time by Kuzmin, are asymptotically polynomial and the power counting turns out to be the same that in local higher derivatives theories. This is in short the main reason why Einstein's gravity can be safely extended to a nonlocal theory consistently with Unitarity and renormalizability.

Therefore, we can state that (super-)renormalizable theories are free of CTC (this is a short way to say that in such theories there are no Gödel solutions with CTC), but we can have other local or nonlocal theories inconsistent with the (super-)renormalizability property though free of CTC. In other words: the (super-)renormalizability implies no-CTC, but no-CTC does not imply (super-)renormalizability. The (super-)renormalizability condition is a sufficient condition, but not a necessary condition for avoiding CTC.

5 Nonlocal gravity non-minimally coupled to matter

In this section, we very briefly review the nonlocal gravitational theory coupled to matter proposed in [4,5]. The classical action is:

$$S[\Phi] = \int d^D x \sqrt{|g|} \left[\mathcal{L}_{\rm loc} + E_i F^{ij}(\hat{\Delta}) E_j \right], \tag{52}$$

$$S_{\rm loc} = \int d^D x \sqrt{|g|}\, \mathcal{L}_{\rm loc}, \quad \mathcal{L}_{\rm loc} = \frac{1}{2\kappa^2} R + \mathcal{L}_{\rm m}(g_{\mu\nu}, \phi, \psi, A^\mu), \tag{53}$$

$$E_i(x) = \frac{\delta S_{\rm loc}}{\delta \Phi^i(x)}, \quad \Delta_{ij}(x,y) = \frac{\delta E_i(x)}{\delta \Phi^j(y)} = \frac{\delta^2 S_{\rm loc}}{\delta \Phi^j(y) \delta \Phi^i(x)} = \hat{\Delta}_{ij}\, \delta(x,y), \quad (\hat{\Delta}_\Lambda)_{ij} = \frac{\hat{\Delta}_{ij}}{(\Lambda)^{[\hat{\Delta}_{ij}]}}, \tag{54}$$

$$2\hat{\Delta}_{ik}\, F^k{}_j(\hat{\Delta}) = \left[e^{\mathrm{H}(\hat{\Delta}_\Lambda)} - 1 \right]_{ij}, \tag{55}$$

where by $\Phi^i \equiv (g_{\mu\nu}, \phi, \psi, A^\mu)$ we mean any field, F^{ij} is a symmetric tensorial entire function whose argument is the Hessian operator $\hat{\Delta}_{ij}$, and $\mathrm{H}(\hat{\Delta}_{\Lambda_*})$ is an entire analytic function whose argument is the dimensionless Hessian. In the above formula, we used the notation $[X]$ to indicate the dimensionality of the quantity X in powers of mass units, i.e., X has dimension of $(\text{mass})^{[X]}$. Since $[\Lambda] = 1$, it follows that $[(\hat{\Delta}_\Lambda)_{ij}] = 0$, as claimed.

The theory (52), enjoys the following essential properties. (i) All the solutions of Einstein's gravity coupled to matter, namely the solutions of the local theory (53), are solutions of the nonlocal theory too. Indeed, the equations of motion of (52) read:

$$\left[e^{\mathrm{H}(\hat{\Delta}_{\Lambda_*})} \right]_{kj} E_j + O(E^2) = 0, \tag{56}$$

where E_i are the EoM of the local theory. (ii) The nonlocal theory gives the same tree-level scattering amplitudes of the local theory [25,26]. This latter property guarantees macro-causality [27], namely the Shapiro's time delay evaluated in the eikonal approximation is the same of the one in Einstein's local theory coupled to the standard model of particle physics. (iii) The stability properties are the same at linear and non linear level of the local theory whether we perturb an exact solution of the local theory [23,24]. (iv) The theory is super-renormalizable or finite at quantum level [6,8,10], and unitary at any perturbative order in the loop expansion [23].

Among the above four properties the most important one for what concerns the topic of this paper is the first one. Indeed, if we include in the local theory a cosmological term, all the Gödel-type metrics are exact solutions of the theory (52). In particular, the Gödel-type spacetimes with CTC turn out to be solutions of (52). However, the quantum effective action will include perturbative corrections to the form factor that very likely will not satisfy the condition (51). Therefore, the Gödel-type metrics with CTC will not be solution of the quantum effective equations of motion.

Conclusions

We have investigated whether the homogeneous Gödel-type metrics can be exact solutions of a general class of nonlocal and local gravitational theories minimally coupled to matter.

It tuned out that the Gödel's metrics without CTC are basically solutions of all nonlocal as well as local higher derivatives theories, while the Gödel's spacetimes with CTC that solve the Einstein's theory do not solve the EoM of (super-)renormalizable gravitational theories. It turns out that the super-renormalizability property is the guiding to select out theories consistent with the Gödel-type causality. Indeed, we showed that unitarity alone is not enough to guarantee such kind of cosmological causality. In particular, the spacetimes with CTC are exact solutions of a large class of nonlocal ghost-free but non-renormalizable theories.

In another class of nonlocal gravitational theories with non-minimal coupling to matter, all the Gödel's spacetimes are exact solutions of the classical theory, and, thus the causality violation is manifest. However, very likely the quantum corrections will spoil the above statement that needs a very special relation between the R^2's and the Ric^2's quantum form factors, namely the relation between the following two operators,

$$R f_0(\Box) R \quad \text{and} \quad \mathrm{Ric}\, f_2(\Box)\mathrm{Ric}. \tag{57}$$

Therefore, we are entitled to state that (super-) renormalizability and Gödel's causality violation exclude each other.

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Appendix A: Equations of motion for local and analytic nonlocal theories

As a general operator, we consider the following general action:

$$S_{a,m} = \int d^4x\sqrt{-g}\,\mathcal{L}_{a,m} = \int d^4x\sqrt{-g}\,\gamma_{a,m} A_{a_1\cdots a_n}\Box^m B^{a_1\cdots a_n}, \tag{A1}$$

where $\gamma_{a,m}$ are the coefficients of the power series for the form factors.

Taking the variation of the action respect to the metric, we get:

$$\begin{aligned}
&\delta\int d^4x\sqrt{-g}A_{a_1\cdots a_n}\Box^m B^{a_1\cdots a_n}\\
&\quad=\int d^4x\sqrt{-g}[\frac{g_{ab}}{2}A_{a_1\cdots a_n}\Box^m B^{a_1\cdots a_n}h_{ab}\\
&\quad+\delta(A_{a_1\cdots a_n}\Box^m B^{a_1\cdots a_n})]\\
&\quad=\int d^4x\sqrt{-g}\Bigg[\frac{g_{ab}}{2}A_{a_1\cdots a_n}\Box^m B^{a_1\cdots a_n}h_{ab}\\
&\quad+\delta(A_{a_1\cdots b_n})\Box^m B^{a_1\cdots a_n}\\
&\quad+A_{a_1\cdots a_n}\delta(\Box^m)B^{a_1\cdots a_n}+A_{a_1\cdots a_n}\Box^m\delta(B^{a_1\cdots a_n})\Bigg]\\
&\quad=\int d^4x\sqrt{-g}\Bigg[\frac{g_{ab}}{2}A_{a_1\cdots a_n}\Box^m B^{a_1\cdots a_n}h_{ab}\\
&\quad+\delta(A_{a_1\cdots a_n})\Box^m B^{a_1\cdots a_n}\\
&\quad+\sum_{i=0}^{m-1}(\Box^i A_{a_1\cdots a_n})\delta(\Box)(\Box^{m-1-i}B^{a_1\cdots a_n})\\
&\quad+\delta(B^{a_1\cdots a_n})\Box^m A_{a_1\cdots a_n}\Bigg].
\end{aligned}\tag{A2}$$

Now we list some useful formulas:

$$\Box(AB) = (\Box A)B + 2(\nabla_a A)(\nabla^a B) + A(\Box B), \tag{A3}$$

$$\begin{aligned}
&(\delta\Box)T^{\cdots}{}_{\cdots} = \delta(g^{ab}\nabla_a\nabla_b)T^{\cdots}{}_{\cdots} = -(\nabla^a\nabla^b T^{\cdots}{}_{\cdots})h_{ab}\\
&\quad+\left[\sum\delta\Gamma^{\cdot}{}_{\cdot\cdot}(\nabla_a T^{\cdots}{}_{\cdots}) + \sum\nabla_a(\delta\Gamma^{\cdot}{}_{\cdot\cdot}T^{\cdots}{}_{\cdots})\right]g^{ab},
\end{aligned}\tag{A4}$$

where the variation of the connection respect to the metric reads:

$$\delta\Gamma^c{}_{ab} = \frac{1}{2}(\nabla_a h_b{}^c + \nabla_b h_a{}^c - \nabla^c h_{ab}). \tag{A5}$$

Using the formulas of above we can find the EoM for the theory (13), namely

$$\begin{aligned}
E^{ab} &= G^{ab} + \Lambda_{cc}g^{ab} + P_1^{ab} + P_2^{ab} + P_3^{ab}\\
&-2\Omega_1^{ab} + g^{ab}(g_{cd}\Omega_1^{cd} + \bar{\Omega}_1) - 2\Omega_2^{ab}\\
&+g^{ab}(g_{cd}\Omega_2^{cd} + \bar{\Omega}_2) - 4\Delta_2^{ab} - 2\Omega_3^{ab}\\
&+g^{ab}(g_{cd}\Omega_3^{cd} + \bar{\Omega}_3) - 8\Delta_3^{ab} - \kappa^2 T^{ab} = 0,
\end{aligned}\tag{A6}$$

where the tensors P_i^{ab} in (A6) are defined as follows,

$$P_1^{ab} = \kappa^2\left[\left(4G^{ab} + g^{ab}R - 4(\nabla^a\nabla^b - g^{ab}\Box)\right)\gamma_0(\Box)R\right],$$

$$\begin{aligned}
P_2^{ab} = \kappa^2\Bigg[&4R^{(a}_d\gamma_2(\Box)R^{|d|b)} - g^{ab}R^{cd}\gamma_2(\Box)R_{cd}\\
&-4\nabla_d\nabla^{(b}(\gamma_2(\Box)R^{|d|a)}) + 2\Box(\gamma_2(\Box)R^{ab})\\
&+2g^{ab}\nabla_c\nabla_d(\gamma_2(\Box)R^{cd})\Bigg],
\end{aligned}$$

$$P_3^{ab} = \kappa^2\Big[-g^{ab}C^{cdef}\gamma_4(\Box)C_{cdef} + 4C^{(a}_{\ cde}\gamma_4(\Box)C^{b)cde}$$

$$-4(R_{cd}+2\nabla_c\nabla_d)(\gamma_4(\Box)C^{(b|cd|a)})\Big], \tag{A7}$$

while the tensors Ω_i^{ab} and $\tilde{\Omega}_i^{ab}$ in (A6) read:

$$\begin{aligned}
\Omega_1^{ab} &= \kappa^2\sum_{n=1}^{\infty}\gamma_{0,n}\frac{1}{\Lambda^{2n}}\sum_{l=0}^{n-1}\nabla^a R^{(l)}\nabla^b R^{(n-l-1)},\\
\bar{\Omega}_1 &= \kappa^2\sum_{n=1}^{\infty}\gamma_{0,n}\frac{1}{\Lambda^{2n}}\sum_{l=0}^{n-1}R^{(l)}R^{(n-l)},\\
\Omega_2^{ab} &= \kappa^2\sum_{n=1}^{\infty}\gamma_{2,n}\frac{1}{\Lambda^{2n}}\sum_{l=0}^{n-1}(\nabla^a R^{cd(l)})(\nabla^b R_{cd}^{(n-l-1)}),\\
\bar{\Omega}_2 &= \kappa^2\sum_{n=1}^{\infty}\gamma_{2,n}\frac{1}{\Lambda^{2n}}\sum_{l=0}^{n-1}R^{cd(l)}R_{cd}^{(n-l)},\\
\Omega_3^{ab} &= \kappa^2\sum_{n=1}^{\infty}\gamma_{4,n}\frac{1}{\Lambda^{2n}}\sum_{l=0}^{n-1}(\nabla^a C^{c(l)}_{\ def})(\nabla^b C_c^{\ def(n-l-1)})\\
\tilde{\Omega}_3 &= \kappa^2\sum_{n=1}^{\infty}\gamma_{4,n}\frac{1}{\Lambda^{2n}}\sum_{l=0}^{n-1}C^{a(l)}_{\ bcd}C_a^{\ bcd(n-l)}.
\end{aligned} \tag{A8}$$

Finally, Δ_i^{ab} are:

$$\begin{aligned}
\Delta_2^{ab} &= \kappa^2\sum_{n=1}^{\infty}\gamma_{2,n}\frac{1}{\Lambda^{2n}}\sum_{l=0}^{n-1}\nabla^c\Big(R^{(l)}_{dc}\nabla^{(a}R^{b)d(n-l-1)}\\
&\quad-(\nabla^{(a}R_{dc})R^{b)d(n-l-1)}\Big),\\
\Delta_3^{ab} &= \kappa^2\sum_{n=1}^{\infty}\gamma_{4,n}\frac{1}{\Lambda^{2n}}\sum_{l=0}^{n-1}\nabla^c\Big(C^{d(l)}_{\ cef}\nabla^{(a}C_d^{\ b)ef(n-l-1)}\\
&\quad-(\nabla^{(a}C^{|d(l)|}_{\ cef})C_d^{\ b)ef(n-l-1)}\Big),
\end{aligned} \tag{A9}$$

here we are used the notation $A^{(l)}\equiv\Box^l A$.

In order to simplify the EoM, it is convenient to rewrite the homogeneous Gödel-type metrics in the orthonormal-tetrad formalism,

$$\begin{aligned}
g_{ab} &= \eta_{\mu\nu}(e^\mu)_a(e^\nu)_b \text{ and } \{(e^\mu)_a\}\\
&= \mathrm{diag}\,(\mathrm{d}t+H(r)\mathrm{d}\theta, \mathrm{d}r, D(r)\mathrm{d}\theta, \mathrm{d}z)\,.
\end{aligned} \tag{A10}$$

Now we are ready to replace the ansatz (A10) in the EoM.

According to (A10), an explicit but tedious computation gives:

$$R_{\mu\nu} = \begin{pmatrix} 2\omega^2 & & & \\ & 2\omega^2-m^2 & & \\ & & 2\omega^2-m^2 & \\ & & & 0\end{pmatrix} \quad\text{and}$$
$$R = 2(\omega^2-m^2). \tag{A11}$$

Therefore, the Einstein's tensor reads:

$$G_{\mu\nu} = \begin{pmatrix} 3\omega^2-m^2 & & & \\ & \omega^2 & & \\ & & \omega^2 & \\ & & & m^2-\omega^2\end{pmatrix}. \tag{A12}$$

In the same vain, we evaluate other invariant tensors present in the EoM, namely

$$\begin{aligned}
&R^{ab}R_{ab} = 2(m^4-4m^2\omega^2+6\omega^4),\\
&R^{ab}\Box R_{ab} = 4\omega^2(4\omega^2-m^2)^2,\\
&C^{abcd}C_{ad} = \frac{4}{3}(m^2-4\omega^2)^2\\
&\quad\text{and}\quad C^{abcd}\Box C_{abcd} = 8\omega^2(m^2-4\omega^2)^2,\\
&\nabla_a\nabla_b\Box^n R^{ab} = 0 \quad\text{and}\quad \nabla_a\Box^n R^{ab} = 0 \quad\text{for}\quad n=0,1.
\end{aligned} \tag{A13}$$

Finally, a very useful formula is:

$$\Box^2 R_{abcd} = 6\omega^2\Box R_{abcd}, \tag{A14}$$

which shows the equivalence of acting with higher derivatives on the Riemann's tensor and the multiplication by $6\omega^2$. Hence, according to (A14), we easily get:

$$R^{ab}\Box^n R_{ab} = \frac{2}{3}(6\omega^2)^n(4\omega^2-m^2)^2 \quad\text{and}\quad C^{abcd}\Box^n C_{abcd} \tag{A15}$$
$$\begin{aligned}
&= \frac{4}{3}(6\omega^2)^n(m^2-4\omega^2)^2 \quad\text{for}\quad n\geq 1,\\
&\nabla_a\nabla_b\Box^n R^{ab} = 0 \quad\text{and}\quad \nabla_a\Box^n R^{ab} = 0 \quad\text{for}\quad n\geq 0.
\end{aligned}$$

Appendix B: Nonlocal gravity in the Riemann–Ricci basis

In this section we derive the relation between the theory in the Weyl–Ricci basis (13) to the theory in the Riemann–Ricci basis (44). We start by recalling the following definition of the Weyl tensor in dimension D,

$$\begin{aligned}
C_{abcd} &= R_{abcd}-\frac{2}{D-2}(g_{a[c}R_{d]b}-g_{b[c}R_{d]a})\\
&\quad+\frac{2}{(D-1)(D-2)}Rg_{a[c}g_{d]b},
\end{aligned} \tag{B1}$$

(notice that C_{abcd} is traceless). Afterwards, we evaluate the Weyl square scalar, namely

$$\begin{aligned}
&C_{abcd}C^{abcd} = R_{abcd}C^{abcd} = C_{abcd}R^{abcd}\\
&= \frac{2}{(D-1)(D-2)}R^2-\frac{4}{D-2}R_{ab}R^{ab}+R_{abcd}R^{abcd}.
\end{aligned} \tag{B2}$$

Since $\nabla_c g_{ab}=0$ and C_{abcd} is a linear function of the Riemann tensor R_{abcd}, we have

$$\begin{aligned}\mathcal{L}_g &= \frac{R-2\Lambda_{cc}}{2\kappa^2} + R\gamma_0(\Box)R + R_{ab}\gamma_2(\Box)R^{ab} \\ &\quad + C_{abcd}\gamma_4(\Box)C^{abcd} \\ &= \frac{R-2\Lambda_{cc}}{2\kappa^2} + R\gamma_0(\Box)R + R_{ab}\gamma_2(\Box)R^{ab} \\ &\quad + \frac{2}{(D-1)(D-2)}R\gamma_4(\Box)R - \frac{4}{D-2}R_{ab}\gamma_4(\Box)R^{ab} \\ &\quad + R_{abcd}\gamma_4(\Box)R^{abcd} = \frac{R-2\Lambda_{cc}}{2\kappa^2} + R[\gamma_0(\Box) \\ &\quad + \frac{2}{(D-1)(D-2)}\gamma_4(\Box)]R \\ &\quad + R_{ab}[\gamma_2(\Box) - \frac{4}{D-2}\gamma_4(\Box)]R^{ab} + R_{abcd}\gamma_4(\Box)R^{abcd},\end{aligned} \tag{B3}$$

which has to be equal to (44), namely

$$\frac{1}{2\kappa^2}(R-2\Lambda_{cc}) + R\tilde{\gamma}_0(\Box)R + R_{ab}\tilde{\gamma}_2(\Box)R^{ab} + R_{abcd}\tilde{\gamma}_4(\Box)R^{abcd}. \tag{B4}$$

Comparing the last step in (B3) with the Lagrangian (B4),

$$\tilde{\gamma}_0 = \gamma_0 + \frac{2}{(D-1)(D-2)}\gamma_4, \quad \tilde{\gamma}_2 = \gamma_2 - \frac{4}{D-2}\gamma_4, \quad \text{and} \quad \tilde{\gamma}_4 = \gamma_4. \tag{B5}$$

For $D = 4$, we have the following relation between form factors γ_i and $\tilde{\gamma}_i$,

$$\gamma_2 + 2\gamma_4 = (\gamma_2 - 2\gamma_4) + 4\gamma_4 = \tilde{\gamma}_2 + 4\tilde{\gamma}_4. \tag{B6}$$

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