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SOME CHARACTERIZATIONS OF THE COMPLEX PROJECTIVE SPACE VIA EHRHART POLYNOMIALS

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ABSTRACT. Let $P_{\lambda\Sigma_n}$ be the Ehrhart polynomial associated to an integral multiple λ of the standard simplex $\Sigma_n \subset \mathbb{R}^n$. In this paper we prove that if (M, L) is an n-dimensional polarized toric manifold with associated Delzant polytope Δ and Ehrhart polynomial P_Δ such that $P_\Delta = P_{\lambda\Sigma_n}$, for some $\lambda \in \mathbb{Z}^+$, then $(M, L) \cong (\mathbb{C}P^n, O(\lambda))$ (where $O(1)$ is the hyperplane bundle on $\mathbb{C}P^n$) in the following three cases: 1. arbitrary n and $\lambda = 1$, 2. $n = 2$ and $\lambda = 3$, 3. $\lambda = n + 1$ under the assumption that the polarization L is asymptotically Chow semistable.

Contents

1. Introduction

A beautiful fundamental result [12] in the theory of toric manifolds states that a pair (M, L) given by a (smooth) compact n-dimensional toric manifold M and a very ample line bundle L on M (i.e. M a polarized toric manifold with polarization L) is described combinatorially by a convex polytope $\Delta \subseteq \mathbb{R}^n$, called the *Delzant polytope*, having vertices in $\mathbb{Z}^n$ (from now on, we will call such a polytope a *lattice polytope*). More precisely, in order to represent a smooth polarized toric manifold a lattice polytope must satisfy the so-called *Delzant condition* (cfr. [13]):

(i) there are n edges meeting at each vertex p;
(ii) each edge is of the form $p + tv_i$, $i = 1, \ldots, n$, where $v_i \in \mathbb{Z}^n$;
(iii) $v_1, \ldots, v_n$ can be chosen to be a basis of $\mathbb{Z}^n$.

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Notice that if $\Delta, \tilde{\Delta} \subseteq \mathbb{R}^n$ are two equivalent Delzant polytopes, i.e. there exists a matrix $A \in GL_n(\mathbb{Z})$ and an integral vector $c \in \mathbb{Z}^n$ such that $\tilde{\Delta} = A(\Delta) + c$, then they represent, up to isomorphism, the same polarized toric manifold.

In order to find some numerical invariants of a polarized toric manifold (M, L) one considers the polarized toric manifolds (M, L^m), where L^m denotes the m-th tensor power of L (m is a natural number). In this case (M, L^m) corresponds to the dilated polytope $m\Delta$ and the dimension of the space $H^0(M, L^m)$ of global holomorphic sections of L^m equals the number $\sharp(m\Delta \cap \mathbb{Z}^n)$ of the lattice points belonging to $m\Delta$. By the celebrated Ehrhart theory of convex polytopes, it is known that this number is given by a polynomial

$$P_\Delta(m) = A_n m^n + A_{n-1} m^{n-1} + \cdots + A_1 m + 1 \tag{1}$$

of degree n in m called the *Ehrhart polynomial of* Δ. We refer the reader to [5, Chapter 3] for the meaning of the coefficients A_j (which are independent of m and depend only on the non-dilated Δ). Here we just recall that the leading coefficient A_n equals the euclidean volume $\mathrm{Vol}(\Delta)$ of Δ (see [5, Corollary 3.20]) which in turn is related to the Riemannian volume $\mathrm{Vol}(M)$ of M (with respect to any Kähler metric $\omega \in c_1(L)$) by

$$A_n = \mathrm{Vol}(\Delta) = (2\pi)^{-n}\,\mathrm{Vol}(M). \tag{2}$$

The prototype and fundamental example of polarized toric manifold is the complex projective space equipped with its hyperplane bundle described in the following example.

Example 1.1. Let us denote by Σ_n the n-dimensional standard simplex given by the convex hull of $\bar{0}, e_1, e_2, \ldots, e_n$ (being $e_1, e_2, \ldots, e_n$ the canonical basis of $\mathbb{R}^n$). The corresponding polarized toric manifold is $\mathbb{C}P^n$ endowed with the hyperplane bundle $O(1)$. The Ehrhart polynomial of Σ_n is

$$P_{\Sigma_n}(m) = \frac{(m+1)(m+2)\cdots(m+n)}{n!}$$

In particular, in this case $A_n = \frac{1}{n!}$. Given $\lambda \in \mathbb{Z}^+$, the convex polytope $\lambda\Sigma_n$ representing $(\mathbb{C}P^n, O(\lambda))$ endowed with the λ-th power of $O(1)$, has then associated polynomial

$$P_{\lambda\Sigma_n}(m) = \frac{(\lambda m+1)(\lambda m+2)\cdots(\lambda m+n)}{n!}. \tag{3}$$

It is natural and interesting to analyze to what extent the Ehrhart polynomial $P_\Delta(m)$ determines Δ and hence the corresponding polarized toric manifold (see for example Open Problems in Chapter 3 of [5], and also [11], [14], [16]). For this reason one says that two Delzant polytopes Δ and $\tilde{\Delta}$ are *Ehrhart equivalent* and we write $\Delta \sim \tilde{\Delta}$, if their Ehrhart polynomials are equal, i.e. $P_{\tilde{\Delta}}(m) = P_\Delta(m)$. Of course two unimodularly equivalent Delzant polytopes $\Delta, \tilde{\Delta} \subseteq \mathbb{R}^n$ are Ehrhart equivalent but not viceversa. However, one can prove ([14], Proposition 4.3) that in that case Δ and $\tilde{\Delta}$ are *equidecomposable*: this means that there exists two decompositions of Δ and $\tilde{\Delta}$ as finite union of rational polytopes $\Delta = D_1 \cup \cdots \cup D_k$, $\tilde{\Delta} = \tilde{D}_1 \cup \cdots \cup \tilde{D}_k$ (intersecting at most along their boundaries) such that D_j and $\tilde{D}_j$ are equivalent (with associated matrix A_j and vector c_j depending on j). For more material on equidecomposability in lattice polytopes, see also [6].

In this paper we address the following natural question.

Question 1. *Let $\Delta \subseteq \mathbb{R}^n$ be the Delzant polytope representing a smooth polarized toric n-dimensional manifold (M, L). Assume that $\Delta \sim \lambda\Sigma_n$, for some $\lambda \in \mathbb{Z}^+$, where Σ_n is given in Example 1.1. Under which conditions on the polarization (M, L) can one deduce $\Delta = \lambda\Sigma_n$ or equivalently $(M, L) \simeq (\mathbb{C}P^n, O(\lambda))$?*

The following two examples show the necessity of imposing some extra conditions in order to achieve the conclusion of Question 1.

Example 1.2. For any natural number n, let λ be the smallest common divisor of $2, 3, \ldots, n$ and consider the n-dimensional polarized manifold (M, L) given by the product of n-copies of the one-dimensional complex projective space $\mathbb{C}P^1$ where the j-th factor has polarization $O(\frac{\lambda}{j})$ with $j = 1, \ldots, n$, namely

$$(M, L) = \left(\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1, O(\lambda) \times O\left(\frac{\lambda}{2}\right) \times \cdots \times O\left(\frac{\lambda}{n}\right)\right).$$

Notice that the Ehrhart polynomial of the one-dimensional simplex $\left[0, \frac{\lambda}{j}\right]$, which represents $\left(\mathbb{C}P^1, O\left(\frac{\lambda}{j}\right)\right)$, is given by $\frac{\lambda}{j}m + 1$, $j = 1, \ldots, n$. Thus, if Δ is the polytope representing (M, L) and one uses the general fact that if Δ_1 (resp. Δ_2) is the polytope representing the polarized toric (M_1, L_1) (resp. (M_2, L_2)) then $\Delta_1 \times \Delta_2$ represents $(M_1 \times M_2, L_1 \times L_2)$ one gets

$$P_\Delta(m) = (\lambda m + 1)\left(\frac{\lambda}{2}m + 1\right) \cdots \left(\frac{\lambda}{n}m + 1\right) = \frac{(\lambda m + 1)(\lambda m + 2) \cdots (\lambda m + n)}{n!}$$

and hence, by (3), $\Delta \sim \lambda\Sigma_n$. Notice that by taking $n = 2$ and $\lambda = 2$ we have that the polytope $2\Sigma_2$ representing $(\mathbb{C}P^2, O(2))$ is Ehrhart equivalent to the polytope representing $(\mathbb{C}P^1 \times \mathbb{C}P^1, O(2) \times O(1))$, i.e. the rectangle with vertices $(0, 0), (2, 0), (0, 1), (2, 1)$. In fact, it is easy to show that the polytopes are equidecomposable as defined above.

Example 1.3. For another polytope Ehrhart equivalent to $2\Sigma_2$, take the convex hull Δ of the vertices $(0, 0), (0, 1), (1, 1), (3, 0)$. In fact, it is easy to see that the $P_\Delta = 2m^2 + 3m + 1 = P_{2\Sigma_2}$, where the last equality follows by (3). This polytope is well-known in the theory of toric surfaces as it represents a very ample line bundle on the so-called *Hirzebruch surface* F_2, defined as the projectivized line bundle $\mathbb{P}(\mathbb{C}+O(-2))$ on $\mathbb{C}P^1$ (the general Hirzebruch surface F_n[1] is just the projectivized line bundle $\mathbb{P}(\mathbb{C}+O(-n))$ on $\mathbb{C}P^1$, represented by the polytope having vertices $(0,0),(0,1),(1,1),(n+1,0)$). Notice that the polarization is given by the pull-back of the hyperplane bundle on $\mathbb{C}P^5$ via the embedding $(z_1,z_2)\mapsto[1,z_1,z_2,z_2^2,z_2^3,z_1z_2]$ of the dense complex torus $(\mathbb{C}^*)^2\subset F_2$ into $\mathbb{C}P^5$.

The main result of this paper is the following theorem where we prove that Question 1 has an affirmative answer in three interesting cases: arbitrary n and $\lambda=1$ (hyperplane bundle), $n=2$ and $\lambda=3$ (namely a polarization of a toric surface with the Ehrhart polynomial of the anticanonical bundle over $\mathbb{C}P^2$) and finally arbitrary n, $\lambda=n+1$ (anticanonical bundle) under the additional assumption that the polarization (M,L) is asymptotically Chow semistable (in the sequel ACsemiS) (see Section 3 for details).

Theorem 1.4. *Let $\Delta\subseteq\mathbb{R}^n$ be the Delzant polytope representing a polarized toric n-dimensional manifold (M,L). The then following facts hold true.*

(i) *if $\Delta\sim\Sigma_n$ then $(M,L)\simeq(\mathbb{C}P^n,O(1))$;*
(ii) *if $\Delta\sim 3\Sigma_2$ then $(M,L)\simeq(\mathbb{C}P^2,O(3))$;*
(iii) *if $\Delta\sim(n+1)\Sigma_n$ and (M,L) is a ACsemiS then $(M,L)\simeq(\mathbb{C}P^n,O(n+1))$.*

Notice that the assumption of ACsemiS of (M,L) in (iii) of Theorem 1.4 cannot be dropped (see Example 3.4 in Section 3 below, for $n=5$).

The paper is organized as follows. In Section 2 we recall some basic facts on reflexive polytopes and barycenters needed in the proof of Theorem 1.4. In Section 3 after recalling various definitions of stability for a given polarization, their links with the regular quantization of Kähler manifolds and the theory of balanced metrics in Donaldsons' terminology, we prove Theorem 1.4.

2. Reflexive polytopes and barycenters

An important notion in the theory of lattice poytopes is that of *dual polytope*. Let us recall that given a convex polytope $\Delta\subseteq\mathbb{R}^n$ such that the origin $\bar{0}$ is contained in its interior, the dual polytope Δ^* is defined as

$$\Delta^*=\{x\in\mathbb{R}^n\mid\langle x,y\rangle\leq 1\text{ for any }y\in\Delta\}$$

Equivalently, one has

[1]The importance of Hirzebruch surfaces in toric geometry lies in the result that all compact toric surfaces are obtained from $\mathbb{C}P^2$ or a Hirzebruch surface by blowing ups (in points fixed by the torus action). Let us notice that Hirzebruch surfaces F_n, $n\geq 2$, are not Fano. Indeed the only toric Fano surfaces are $\mathbb{C}P^2,\mathbb{C}P^1\times\mathbb{C}P^1$ and the blowups of $\mathbb{C}P^2$ at one, two or three points fixed by the torus action; F_1 coincides with the blow up of $\mathbb{C}P^2$ at one point.

$$\Delta^* = \{x \in \mathbb{R}^n \mid \langle x, v\rangle \leq 1 \text{ for any vertex } v \text{ of } \Delta\}$$

(the equivalence is easily seen by using the fact that Δ is the convex hull of its vertices). Notice that $(\Delta^*)^* = \Delta$.

Example 2.1. Take the simplex[2] Σ in $\mathbb{R}^2$ given by the convex hull of its vertices $v_1 = (-1,-1), v_2 = (\lambda-1,-1), v_3 = (-1,\lambda-1)$. It is easy to see that the vertices of Σ^* are then given by the points $(-1,0)$, $(0,-1)$, $\left(\frac{1}{\lambda-2}, \frac{1}{\lambda-2}\right)$. Notice that Σ^* is in general a rational polytope and it is a lattice polytope if and only if $\lambda = 3$.

A lattice polytope Δ for which the dual Δ^* is again a lattice polytope is called *reflexive*. It is well-known that reflexive polytopes correspond to toric (Fano) varieties with the anticanonical polarization (see, e.g. [4]). For example, the above example shows that $3\Sigma_2 \subseteq \mathbb{R}^2$ is reflexive and, in general, the simplex $\lambda\Sigma_n \subseteq \mathbb{R}^n$ described in the introduction is reflexive if and only if $\lambda = n+1$ (indeed the anticanonical bundle of $\mathbb{C}P^n$ is given by $O(n+1)$). An interesting criterion for a lattice polytope $\Delta \subseteq \mathbb{R}^n$ in order to be reflexive in terms of the Ehrhart polynomial is given by the following theorem, due to Hibi [15] (see also Theorem 3 and Remark 2.1 in [23]).

Theorem 2.2. *A lattice polytope $\Delta \subseteq \mathbb{R}^n$ with $\bar{0} \in \Delta$ is reflexive if and only if*

$$P_\Delta(m) = (-1)^n P_\Delta(-m-1)$$

for every $m \in \mathbb{N}$.

The Ehrhart polynomial can be considered a particular case in a more general class of functions defined on the dilations of a polytope. Let ϕ be a polynomial map homogeneous of degree k on $\mathbb{R}^n$ and consider the function

$$P^\phi_\Delta(m) := \sum_{x \in (m\Delta)\cap\mathbb{Z}^n} \phi(x) \tag{4}$$

The Ehrhart polynomial corresponds to choosing ϕ constant equal to 1. One can show that, analogously to the Ehrhart case, one always gets a polynomial on m of degree $n+k$ and that the following property holds

$$(-1)^{n+k} P^\phi_\Delta(-m) = P^\phi_{\Delta^0}(m) := \sum_{x \in (m\Delta)^0 \cap \mathbb{Z}^n} \phi(x), \tag{5}$$

where $(m\Delta)^\circ$ denotes the interior of $m\Delta$ (see [7, Proposition 4.1]). In particular when $\phi(x) = 1$ one gets

$$(-1)^n P_\Delta(-m) = \sharp((m\Delta)^\circ \cap \mathbb{Z}^n) \tag{6}$$

[2]It is just the simplex $\lambda\Sigma_2$ having as vertices $(0,0), (\lambda, 0), (0, \lambda)$ translated by the vector $(-1,-1)$ in order to have the origin in its interior as required by the definition of dual.

This is called *Ehrhart-MacDonald reciprocity* (for a direct proof, see e.g. [23, Theorem 2]).

Remark 2.3. Notice that by (6) and Theorem 2.2 with $m = 0$ one deduces that a reflexive polytope has exactly one lattice point in its interior.

On the other hand, by choosing $\phi(x) = x$, one can deduce the following:

$$P^x_\Delta(m) := \sum_{x \in (m\Delta) \cap \mathbb{Z}^n} x = \left(\int_\Delta x dv \right) m^n + \left(\frac{1}{2} \int_{\partial\Delta} x da \right) m^{n-1} + \cdots , \tag{7}$$

where da is defined on each facet (i.e. $n-1$-dimensional face) of P by the equality $dx_1 \cdots dx_n = \pm da \wedge dl$, where $l = l(x) = x \cdot v + c$ is the affine function defining the facet such that the normal $v \in \mathbb{Z}^n$ is a primitive vector of the lattice (see, for example, Section 3.2 in [17]) and the sign depends on the orientation of the facet. Alternatively (see Section 5.4 in [5]), for any face F of Δ, the lattice structure of $\mathbb{Z}^n$ in $\mathbb{R}^n$ induces on the affine subspace A_F containing F a sublattice $\mathbb{Z}^n \cap A_F$, and then a decomposition of A_F as union of "fundamental parallelepipeds": the measure da is the one for which these fundamental parallelepipeds have unitary volume.

Let us recall that the barycenter b_Δ of a polytope $\Delta \subseteq \mathbb{R}^n$ is the point of $\mathbb{R}^n$ given by:

$$b_\Delta = \frac{\int_\Delta x dv}{\mathrm{Vol}(\Delta)}$$

so the leading coefficient of $P^x_\Delta(m)$ is $\mathrm{Vol}(\Delta) \cdot b_\Delta$. The coefficient of m^{n-1} is a multiple of the barycenter of the boundary $\partial\Delta$ of Δ. It is interesting to notice (even if it is not used in the rest of the paper) that these two barycenters are in general different: they coincide if and only if the so-called *Futaki invariant* of (M, L) vanishes (see [18]).

We conclude this section with the following result needed in the proof of (iii) of Theorem 1.4.

Lemma 2.4. [21, Theorem 1.4] *Let $\Delta \subseteq \mathbb{R}^n$ be a convex polytope such that:*

(a) Δ is contained in the dual D^* of a lattice polytope D;

(b) the barycenter b_Δ of Δ is the origin, i.e. $b_\Delta = \bar{0}$.

Then $\mathrm{Vol}(\Delta) \leq \frac{(n+1)^n}{n!}$ *and the equality is attained if and only if* $\Delta = (n+1)\Sigma_n$.

3. The proof of Theorem 1.4

Let (M, L) be a polarization of a compact complex manifold M. In order to analyze the existence of a constant scalar curvature Kähler (cscK) metric $\omega \in c_1(L)$ many concepts of *stability* of (M, L) have been introduced. Here we recall the following (starting from the weaker to the stronger assumption) 1. *asymptotically*

Chow semistability (ACSemiS), 2. *asymptotically Chow polystability (ACpolS)*. and 3. *asymptotically Chow stability (ACS)*. The definitions of these notions are omitted since they require some technical tools such as those of test configuration and Chow weight not treated in the present paper (the interested reader is referred to [9] for details). From the point of view of Geometric Invariant Theory, the ACpolS of (M, L) is equivalent to the existence of a balanced metric $\omega_m \in c_1(L^m)$ for any $m >> 0$. Following Donaldson [10] (see also [1]) one says that a Kähler metric $\omega_m \in c_1(L^m)$ on a compact polarized manifold (M, L) is said to be *balanced* if the so called *Kempf distortion function*

$$T_{\omega_m}(p) = \sum_{j=0}^{N_m} h_m(s_j(p), s_j(p))$$

is a positive constant, where h_m is a hermitian metric on L^m such that $Ric(h_m) = \omega_m$ and $s_0, \dots, s_N$ form a orthonormal basis of $H^0(M, L^m)$ with respect to the L^2-scalar product $\langle s, t\rangle = \int_M h_m(s(p), t(p))\frac{\omega^n}{n!}$. Another notion stronger than ACpolS is the following. A polarization (M, L) of a compact complex manifold M is said to be a *regular quantization* (originally introduced in [8], see also [2]) if there exists $\omega \in c_1(L)$ such that $m\omega$ is balanced for $m >> 0$. Of course a regular quantization is ACpolS and hence ACsemiS. Hence Theorem 1.4 immediately implies the following:

Corollary 3.1. *Let $\Delta \sim (n+1)\Sigma_n$ and assume that (M, L) is a regular quantization. Then $(M, L) \simeq (\mathbb{C}P^n, O(n+1))$.*

Remark 3.2. Actually an alternative proof of Corollary 3.1 can be obtained as follows. By $\Delta \sim (n+1)\Sigma_n$ we deduce as in the proof below of (iii) in Theorem 1.4 that the polytope Δ is reflexive and hence L is the anticanonical bundle. Let $\omega \in c_1(L)$ be a Kähler metric such that $m\omega$ is balanced for $m >> 0$. Then $\omega \in c_1(L)$ is a cscK metric (see [20] for a proof) and hence Kähler -Einstein (being a cscK metric in the first Chern class of the anticanonical bundle). Then, the conclusion follows immediately by Proposition 1.3 of [21].

Remark 3.3. Notice that the assumption $\Delta \sim (n+1)\Sigma_n$ in (iii) of Theorem 1.4 cannot be dropped even if one assumes the stronger assumptions of ACS or regular quantization. For example, the blow-up of $\mathbb{C}P^2$ at three points endowed with the anticanonical polarization is an ACS compact smooth toric manifold (see Corollary 3.3 in [19]). For another example let us consider Example 1.2 above, where $\Delta \sim \lambda\Sigma_n$ with $\lambda \neq n+1$. It is not hard to see that in this case the polarization (M, L) is indeed regular being a flag manifold, i.e. a homogeneous and simply-connected Kähler manifold (see [1] for a proof). We believe that if M is a compact toric manifold admitting a regular quantization then M is isomorphic to the Kähler product of complex projective spaces.

The following example shows the necessity of ACsemiS in Theorem 1.4.

Example 3.4. Consider the lattice polytope Δ in $\mathbb{R}^5$ with vertices

$$(0,-1,-1,-1,-1),(-1,0,-1,-1,-1),(-1,-1,-1,-1,-1,),(-1,0,0,-1,-1),$$
$$(-1,-1,0,-1,-1),(0,-1,0,-1,-1),(0,-1,0,2,-1),(-1,-1,-1,2,-1),$$
$$(6,-1,-1,-1,2),(-1,6,3,-1,2),(6,-1,3,-1,2),(-1,-1,0,2,-1),$$
$$(-1,-1,3,-1,2),(-1,0,0,2,-1),(-1,0,-1,2,-1),(-1,6,-1,-1,2),$$
$$(0,-1,-1,2,-1),(-1,-1,-1,-1,2).$$

This polytope represents a compact smooth toric manifold M (different from $\mathbb{C}P^5$) polarized by the anticanonical bundle $L=-K_M$ and $\Delta \sim 6\Sigma_5$ (this can be checked by calculating the Ehrhart polynomials of the corresponding polytopes by the software Normaliz, https://www.normaliz.uni-osnabrueck.de/; see also the database on smooth toric Fano varieties http://www.grdb.co.uk/forms/toricsmooth, setting dimension 5 and degree $(n+1)^n = 6^5$). We thank Akiyoshi Tsuchiya for pointing out to us this example. Notice that (M,L) is not ACsemiS by Theorem 1.4.

A necessary condition needed in the proof of (iii) of Theorem 1.4 for the ACsemiS of a polarized toric manifold (M,L) can be expressed as follows in terms of the Delzant polynomial Δ associated to (M,L) and its barycenter.

Lemma 3.5. *([22, Theorem 1.2]) Assume that (M,L) is ACsemiS. Then for any integer $m \in \mathbb{Z}$ one has*

$$P_\Delta^x(m) = P_\Delta(m) b_\Delta, \tag{8}$$

where b_Δ is the barycenter of Δ.

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Proof of (i) The assumption $P_\Delta(m) = P_{\Sigma_n}(m)$ implies in particular that $\mathrm{Vol}(\Delta) = \mathrm{Vol}(\Sigma_n) = \frac{1}{n!}$. But, since Δ is a lattice polytope, it is easy to see that this implies that Δ must coincide with the simplex Σ_n. Indeed, since (M,L) is a toric manifold, Δ is a Delzant polytope and, up to an $SL_n(\mathbb{Z})$ transformation and a translation by a lattice vector we can assume that $\bar{0}$ is a vertex of Δ and that the edges at $\bar{0}$ are given by the directions of $e_1,\dots,e_n$, i.e. Δ must contain $c_1e_1,\dots,c_ne_n$ as vertices, for some $c_1,\dots,c_n \in \mathbb{Z}^+$. Since Δ is a convex lattice polytope, it must contain the simplex Σ given by the convex hull of $\bar{0}, c_1e_1,\dots,c_ne_n$, and then

$$\frac{1}{n!} = \mathrm{Vol}(\Delta) \geq \mathrm{Vol}(\Sigma) = \frac{c_1\cdots c_n}{n!}$$

Then $c_1 = \cdots = c_n = 1$, i.e. Δ must contain $e_1, \dots, e_n$ as vertices and, since its volume coincide with that of Σ_n, it cannot have other points, i.e. $\Delta = \Sigma_n$ and then (i) is proved.

Proof of (ii) By the assumption $\Delta \sim 3\Sigma_2$ and the reflexivity of $3\Sigma_2$ we deduce that Δ is reflexive (by arguing as at the beginning of the proof of (iii) below). By Section 2, this means that Δ represents a Fano smooth toric surface endowed with the anticanonical polarization. Now, as we have recalled in a footnote above, the toric Fano surfaces are $\mathbb{C}P^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$ and the blowups $Bl_1(\mathbb{C}P^2)$, $Bl_2(\mathbb{C}P^2)$, $Bl_3(\mathbb{C}P^2)$ of $\mathbb{C}P^2$ at one, two or three points fixed by the torus action. It is not hard to see that the Ehrhart polynomials of the polytopes associated to the canonical polarizations are respectively

$$
\begin{aligned}
P_{\mathbb{C}P^2}(m) &= \frac{9}{2}m^2 + \frac{9}{2}m + 1 \\
P_{\mathbb{C}P^1 \times \mathbb{C}P^1}(m) &= 4m^2 + 4m + 1 \\
P_{Bl_1(\mathbb{C}P^2)}(m) &= 4m^2 + 4m + 1 \\
P_{Bl_2(\mathbb{C}P^2)}(m) &= \frac{7}{2}m^2 + \frac{7}{2}m + 1 \\
P_{Bl_3(\mathbb{C}P^2)}(m) &= 3m^2 + 3m + 1
\end{aligned}
$$

, from which the proof of (ii) follows.

Proof of (iii) We show that (a) and (b) in Lemma 2.4 hold for the polytope Δ representing (M, L). By the assumption $\Delta \sim (n+1)\Sigma_n$ and the reflexivity of $(n+1)\Sigma_n$ we deduce that

$$P_\Delta(m) = P_{(n+1)\Sigma_n}(m) = (-1)^n P_{(n+1)\Sigma_n}(-m-1) = (-1)^n P_\Delta(-m-1),$$

which, by Theorem 2.2, yields the reflexivity of Δ. Thus, by the very definition of reflexivity Δ^* is a lattice polytope and hence (a) of Lemma 2.4 is satisfied with $D = \Delta^*$ (since $\Delta = (\Delta^*)^*$).

Notice now that by Remark 2.3 the polytope Δ has only one lattice point in the interior and so, up to an integral translation (which does not change the Ehrhart polynomial), we can assume that this point is the origin $\bar{0}$. Then, by (5)

$$(-1)^{n+1} P^x_\Delta(-1) = P^x_{\Delta^0}(1) = \bar{0}.$$

Using the assumption of ACsemiS and (8) in Lemma 3.5 with $m = -1$ we get

$$\bar{0} = P^x_\Delta(-1) = P_\Delta(-1) b_\Delta.$$

On the other hand, by (6) with $m = 1$, we have

$$P_\Delta(-1) = (-1)^n \sharp(\Delta^0 \cap \mathbb{Z}^n) = (-1)^n \neq 0,$$

and then one concludes $b_\Delta = \bar{0}$. This shows that also (b) of Lemma 2.4 is satisfied. Thus, since $\Delta \sim (n+1)\Sigma_n$ implies $\mathrm{Vol}(\Delta) = \mathrm{Vol}((n+1)\Sigma_n) = \frac{(n+1)^n}{n!}$ (cfr. (2)) Lemma 2.4 yields the desired equality $\Delta = (n+1)\Sigma_n$, i.e. $(M, L) \cong (\mathbb{C}P^n, O(n+1))$. The proof of (iii) and hence of the theorem is concluded. $\square$

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