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Semi-global fixed-time state estimation and unknown input reconstruction via first-order sliding mode observers with delay

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***Abstract*—This paper addresses the state estimation and unknown input reconstruction in strongly observable linear systems under the so-called matching conditions. Our proposal extends previous approaches to predefined-time state estimation, which involved the use of a pair of time-delay observers, adapting them to the framework of sliding mode observation in the presence of unknown inputs. As a result, a novel sliding mode-based observation algorithm is developed, which exhibits desirable properties such as global finite-time convergence or semi-global fixed-time convergence. Notably, the method is also capable of reconstructing the unknown inputs without requiring any low-pass filtering or continuous approximations of the discontinuous observer signals. The convergence properties are proven through a formal stability analysis and validated through numerical simulations.**

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I. Introduction

STATE ESTIMATION in the presence of *unknown inputs* (UIs) has been extensively studied and it has a well-developed theoretical foundation [1]–[3]. However, design techniques for *Unknown Input Observers* (UIOs) continue to evolve due to their variegate range of applications [4], including fault detection [5], robust output-feedback control [6], cyber-physical systems monitoring [7], and large-scale systems [8], [9], among others. The majority of existing results are focused on developing UIOs for *linear time-invariant* (LTI) systems [4]–[9]. The existence of a UIO generally requires that: i) the plant must be *strongly detectable* with respect to the UIs, meaning that it cannot possess any unstable invariant zeros, and ii) the *rank* (or *matching*) *condition* holds, implying the existence of a transformation that expresses the UIs as a function of the first derivatives of the plant outputs. If these conditions are met then the resulting UIO will exhibit a convergence rate that cannot be faster than exponential for each system's mode associated with a stable invariant zero. This is because the convergence rate of an *unobservable mode* cannot be adjusted by feedback.

Instead, if the invariant zeros set is empty, the system is referred to as *strongly observable*, and finite-time or fixed-time converging UIOs can be designed [4].

With reference to strongly observable LTI systems, the milestone work [10] offers a comprehensive overview of linear (full-order, reduced-order, or functional) UIO design techniques which aim to identify a change of coordinates allowing to decouple the dynamics of the states to be estimated from the UIs [1].

More recently, a novel approach to finite-time observation has been proposed in [11], which employs a pair of time-delay observers to guarantee the prescribed time UIO stability in a *deadbeat-like* fashion. In [12] the approach was further developed to address the state estimation problem within a fixed time despite the presence of UIs. However, these approaches have some limitations: i) the state estimates in [11], [12] are meaningless at any time instant preceding such predefined convergence time, and ii) neither [12] nor [1], [10] allow for the unknown input reconstruction (UIR), which is a desirable feature in fault detection and fault tolerant control applications [5].

To achieve UIR, first-order sliding mode observers (FOSMOs) such as the Walcott-Zak [13] and unit-vector FOSMOs [5], [14] have been utilized under the matching condition [3, Chapter 6]. While FOSMOs ensure finite-time UIR, the convergence of the unmeasurable component z of the system state $\mathrm{x} \in \mathbb{R}^n$, where $\mathrm{x} \mapsto (z; y)$ and $y \in \mathbb{R}^p$ denotes the system output, is only guaranteed with an exponential rate.

As far as we know, the only available techniques ensuring at the same time finite-time state estimation and UIR are those based on *higher-order sliding mode observers* (HOSMOs) [15]–[19]. It is also worth noting that these approaches are often combined with (bi)-homogeneous feedback terms to provide fixed-time convergence [20].

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A. Objective of the study and main contributions

Based on the previous discussion, to the best of our knowledge there are currently no FOSMO design techniques that can address the state estimation in finite or fixed time while also allowing the finite-time UIR. To achieve this goal, we propose a FOSMO scheme to be integrated into the prescribed-time estimation architecture introduced by [11]. Our method involves developing a pair of ad-hoc FOSMOs, whose estimates are combined to ensure that the resulting estimation error converges within a predetermined time, provided that sliding modes along their local output estimation error $\hat{y}_i - y$ are established. Moreover, our proposal uses a combination of discontinuous and polynomial output injection terms, resulting in a UIO featuring global finite-time convergence or semi-global fixed-time convergence concerning state estimation and UIR, thus yielding a step beyond the current FOSMOs literature [3, Chapter 6].

Additional major contributions of this study can be summarized as follows: i) Compared with [12], with which we share the main basic ideas, our method further enables the UIR; ii) Compared with [11], [12], our method relaxes the limitation that the estimate of the state is meaningless for all time instants preceding the finite or fixed convergence time, thus making the scheme well-suited for output feedback applications, see [12, Remark 2].

We also remark that the proposed design differs from known FOSMO designs in the following aspects: i) The change of coordinates at the basis of our design is consistent as it does not rely on specific observer gain matrices (differently from existing results such as, e.g., the transformations (6.10) or (6.71) of [3]). This consistency is necessary to integrate our FOSMOs within the design framework of [11]; ii) The type of output injection terms adopted. While existing methods (e.g. [5], [13], [14]) employ unit-vector based injection terms of the type $P(\hat{y}_i - y)/\|P(\hat{y}_i - y)\|$, where $P > 0$ is a design parameter, we consider component-wise discontinuous output injection terms. Indeed, the inherent coupling of the unit-vector-based output injection terms, due to the normalization factor $\|P(\hat{y}_i - y)\|$, prevents the possibility of integrating them into the decoupled design framework of [11].

B. Structure of the paper

Preliminary concepts of finite-time and fixed-time stability are discussed in Section II, while the problem statement is formulated in Section III. The proposed UIO and the key results of this study are presented in Section IV, followed by numerical simulations that corroborate the theoretical findings in Section V. Finally, Section VI provides concluding remarks.

C. Mathematical notation

The integer, strictly positive real, and complex numbers are denoted by $\mathbb{Z}, \mathbb{R}_{>0}, \mathbb{C}$. For $\mathrm{x}_i \in \mathbb{R}$, $\mathrm{x} = (\mathrm{x}_1; \mathrm{x}_2; \cdots; \mathrm{x}_\mathrm{n})^T \in \mathbb{R}^\mathrm{n}$ is a column vector, and $\mathrm{x}^\mathsf{T} \in \mathbb{R}^{1\times \mathrm{n}}$ its transpose. The component-wise absolute-value operator is $|\mathrm{x}|$, and $\mathrm{sign}(\mathrm{x}) : \mathbb{R}^n \mapsto [-1, 1]^n$ is the component-wise set-valued sign-operator, which i-th output is 1 if $\mathrm{x}_i > 0$, -1 if $\mathrm{x}_i < 0$, otherwise $[-1, 1]$, whereas the i-th entry of $\mathrm{sig}^\alpha(\mathrm{x})$ is $|\mathrm{x}_i|^\alpha \mathrm{sign}(\mathrm{x}_i)$, and $\alpha \geq 0$. The p-norm of x is $\|\mathrm{x}\|_\mathrm{p} = (\sum_{i=1}^{\mathrm{n}} |\mathrm{x}_i|^\mathrm{p})^{\frac{1}{\mathrm{p}}}$ with $\mathrm{p} \geq 1$, whereas $\|\mathrm{x}\|_\infty = \max_i |\mathrm{x}_i|$. For $\mathrm{x}, \mathrm{y} \in \mathbb{R}^\mathrm{n}$, $\mathrm{p}, \mathrm{q} \in \mathbb{Z}_{>0} : 1/\mathrm{p} + 1/\mathrm{q} = 1$, then $\mathrm{x}^\mathsf{T}\mathrm{y} \leq |\mathrm{x}^\mathsf{T}\mathrm{y}| \leq \|\mathrm{x}\|_\mathrm{p}\|\mathrm{y}\|_\mathrm{q}$. Note that $\|\mathrm{x}\|_1 \geq \|\mathrm{x}\|_p \geq \|\mathrm{x}\|_\infty$. For $\mathrm{M} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$, its j-th eigenvalue is $\lambda_j(\mathrm{M}) \in \mathbb{C}$, where the eigenvalues are ordered by their real parts such that $\Re(\lambda_1) \geq \Re(\lambda_2) \geq \cdots \geq \Re(\lambda_n)$. If $\mathrm{M} = \mathrm{M}^\mathsf{T}$ and $\lambda_j(\mathrm{M}) \in \mathbb{R}_{>0}\ \forall\ j$, then M is positive definite ($\mathrm{M} \succ 0$). The matrix exponential operator of M is $\mathrm{e}^\mathrm{M} = \sum_{k=0}^{\infty} \frac{1}{k!}\mathrm{M}^k$. $\mathrm{diag}(\mathrm{M}_1, \mathrm{M}_2)$ returns a block-diagonal matrix with their arguments, in order of occurrence, on the main diagonal. $0_{\mathrm{n},\mathrm{h}}$ is the all-zero $\mathrm{n} \times \mathrm{h}$ matrix. If $\mathrm{h} = 1$ we simply write 0_n. For a full rank $\mathrm{R} \in \mathbb{R}^{\mathrm{n}\times\mathrm{h}}$, with $\mathrm{n} > \mathrm{h}$, its pseudo-inverse is $\mathrm{R}^+ = (\mathrm{R}^\mathsf{T}\mathrm{R})^{-1}\mathrm{R}^\mathsf{T} : \mathrm{R}^+\mathrm{R} = I_\mathrm{h}$, and I_h is the $\mathrm{h} \times \mathrm{h}$ identity matrix. For $\mathrm{n} < \mathrm{h}$, the column null-space of R is N_R where $\mathrm{R} \perp \mathrm{N}_\mathrm{R} \implies \mathrm{R}\mathrm{N}_\mathrm{R} = 0_{n,h-n}$. Given two scalar functions $f_1(\tau)$ and $f_2(\tau)$, the notation $f_2(\tau) = o(f_1(\tau))$ means that $\lim_{\tau\to+\infty} \frac{|f_2(\tau)|}{f_1(\tau)} = 0$.

II. Notions on Finite-time and Fixed-time stability

Consider the nonlinear dynamical system

$$\dot{\mathrm{z}}(t) = \mathrm{g}(t, \mathrm{z}(t)), \quad \mathrm{z}(0) = \mathrm{z}_0, \tag{1}$$

with state $\mathrm{z} : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^\mathrm{n}$, and the locally measurable nonlinear function $\mathrm{g} : \mathbb{R}_{\geq 0} \times \mathbb{R}^\mathrm{n} \mapsto \mathbb{R}^\mathrm{n}$ may be discontinuous with respect to z. The initial condition is $\mathrm{z}_0 \in \mathbb{R}^n$, and the origin $\mathrm{z} = 0_n$ is assumed to be an equilibrium of (1).

Since the right-hand side of (1) may be discontinuous, their solutions will be understood in the *Sense of Filippov* [21]. Basics of finite-time and fixed-time stability are now recalled.

Lemma 1 (Lyapunov Stability [22]) *Let $V : \mathbb{R}^\mathrm{n} \mapsto \mathbb{R}_{\geq 0}$ be a differentiable radially unbounded function and satisfies $V(\mathrm{z}) = 0 \Leftrightarrow \mathrm{z} = 0_\mathrm{n}$. If $\dot{V}(\mathrm{z}(t)) < 0\ \forall\ \mathrm{z} \neq 0_n$. Then the origin is globally asymptotically stable. Additionally:*

1) *If $\dot{V}(\mathrm{z}(t)) \leq -\kappa_\alpha\ V^\alpha(\mathrm{z}(t))$ for some $\kappa_\alpha > 0, \alpha \in (0, 1)$. Then, the origin is globally finite-time stable, namely $\mathrm{z}(t, \mathrm{z}_0) = 0_\mathrm{n}\ \forall\ \mathrm{z}_0 \in \mathbb{R}^\mathrm{n}, t \geq \mathrm{T}(\mathrm{z}_0)$, where $\mathrm{T}(\mathrm{z}_0) : \mathbb{R}^\mathrm{n} \mapsto \mathbb{R}_{\geq 0}$ is called settling-time function.*
2) *If $\dot{V}(\mathrm{z}(t)) \leq -\kappa_\alpha\ V^\alpha(\mathrm{z}(t)) - \kappa_\beta\ V^\beta(\mathrm{z}(t))$ for some $\kappa_\alpha > 0, \kappa_\beta > 0, \alpha \in (0, 1), \beta > 1$. Then, the origin is globally fixed-time stable and $\mathrm{T}(\mathrm{z}_0)$ has a uniform upper-bound $\mathrm{T}_{\max}$ such that*

$$\mathrm{T}(\mathrm{z}_0) \leq \mathrm{T}_{\max} = \frac{1}{\kappa_\alpha(1-\alpha)} + \frac{1}{\kappa_\beta(\beta-1)} \quad \forall\ \mathrm{z}_0 \in \mathbb{R}^\mathrm{n}. \tag{2}$$

□

III. Problem formulation and preliminaries

Consider an uncertain linear dynamical system of the form

$$\dot{\mathrm{x}}(t) = \mathrm{A}\mathrm{x}(t) + \mathrm{B}u(t) + \mathrm{D}\xi(t, \mathrm{x}, u), \quad \mathrm{x}(0) = \mathrm{x}_0, \tag{3}$$

$$y(t) = \mathrm{C}\mathrm{x}(t), \tag{4}$$

with state $\mathrm{x} \in \mathbb{R}^n$, known input $u \in \mathbb{R}^m$, and measured output $y \in \mathbb{R}^p$, where $p < n$. The unknown input vector $\xi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^q$ represents exogenous or parametric perturbations, or actuator faults. Similarly to several related papers on fixed-time UIO's such as [11], [12], [15]–[19] we make the following assumptions:

A1 Matrices $\mathrm{C} \in \mathbb{R}^{p\times n}$ and $\mathrm{D} \in \mathbb{R}^{n\times q}$ are full-rank, with $p > q$, and such that $\mathrm{rank}(\mathrm{CD}) = q$;

A2 The triplet $(\mathrm{A}, \mathrm{D}, \mathrm{C})$ does not possess invariant zeros;

A3 The unknown input $\xi(t, \mathrm{x}, u)$ fulfills the inequality $\|\xi(t, \mathrm{x}, u)\|_\infty \leq \mu\|u\|+\rho(t, y)$ where μ is a known scalar and $\rho : \mathbb{R}_{\geq 0} \times \mathbb{R}^p \mapsto \mathbb{R}_{>0}$ is a known function.

A. Objective A: Global Finite-time state estimation

The goal is to design a FOSMO that is able to reconstruct both the state $\mathrm{x} \in \mathbb{R}^n$ and the unknown input $\xi \in \mathbb{R}^q$ of (3)-(4) in finite time, using only the known input $u \in \mathbb{R}^m$ and output $y \in \mathbb{R}^p$. More formally, let $z \in \mathbb{R}^{n-p}$ be the unmeasurable component of x such that $\mathrm{x} \mapsto (z; y)$. Our **first objective** is to ensure that the estimates of z and y, resp. $\hat{z}$ and $\hat{y}$, satisfy

$$\begin{array}{rcl} \hat{z}(t) - z(t) &=& 0 \\ \hat{y}(t) - y(t) &=& 0 \end{array} \quad \forall\, \mathrm{x}_0,\ \xi,\ t \geq \mathrm{T}(\mathrm{x}_0), \tag{5}$$

where $\mathrm{T}(\mathrm{x}_0) : \mathbb{R}^n \mapsto \mathbb{R}_{>0}$ denotes the estimation settling time. In addition, the UIR, namely $\hat{\xi}(t) \to \xi(t)$, is also desired.

B. Objective B: Semi-global fixed-time estimation

Under the additional assumption that z evolves into a known, and possibly arbitrarily large in size, open domain $\mathcal{Z} \subset \mathbb{R}^{n-p}$, such that

$$\exists\ Z\ :\ \|z\| \leq Z \quad \forall\, z \in \mathcal{Z}, \tag{6}$$

our **second objective** is to demonstrate that the proposed observer, designed to guarantee (5), is such that

$$\exists\ \bar{\mathrm{T}}_{\max} > 0\ :\ \mathrm{T}(\mathrm{x}_0) \leq \bar{\mathrm{T}}_{\max} \quad \forall\, \mathrm{x}_0,\ \xi, \tag{7}$$

where $\bar{\mathrm{T}}_{\max}$ is a design parameter that can be chosen freely.

It is worth noting that relation (7) implies that the designed observer exhibits semi-global fixed-time convergence concerning the state estimation errors in (5). Then, the UIR is wanted.

Remark 1 *Assumptions **A1** and **A3**, coupled with the requirement that the triplet* $(\mathrm{A}, \mathrm{D}, \mathrm{C})$ *has no unstable invariant zeros, imply that* (3)-(4) *is strongly detectable. These assumptions are standard in the design of FOSMOs [3, Chapter 6.3.2]. Additionally, we introduced Assumption **A2**, which is more stringent than the simpler requirement of the absence of unstable invariant zeros but is necessary for the design of finite-time or fixed-time observers [11], [12] and Higher-Order Sliding Mode Observers (HOSMOs) [4, Section 5.3]. Specifically, if the triple* (A, D, C) *had invariant zeros then an unobservable subspace would exist (refer to Lemma 2 for details). However, Assumption **A2** ensures the absence of such unobservable subspace.* ■

C. An Instrumental Lemma

The next Lemma, which is instrumental to several subsequent developments, is recalled.

Lemma 2 (cf. Lemma 6.1 of [3]) *Consider a linear system of the form* (3)-(4) *with $p > q$ and rank*$(\mathrm{CD}) = q$*. Then a change of coordinates exists such that the transformed triplet* (A, D, C) *in the new coordinates has the following structure:*

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline \begin{matrix} A_{211} \\ A_{212} \end{matrix} & A_{22} \end{array}\right], \tag{8}$$

where $A_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$ and $A_{211} \in \mathbb{R}^{(p-q)\times(n-p)}$. Moreover it results that

$$A_{11} = \begin{bmatrix} A_{11}^{no} & A_{12}^{o} \\ 0_{n-p-r,r} & A_{22}^{o} \end{bmatrix}, \quad A_{211} = \begin{bmatrix} 0 & A_{21}^{o} \end{bmatrix}, \tag{9}$$

where $A_{11}^{no} \in \mathbb{R}^{r\times r}$, with $r \geq 0$, corresponds to the unobservable subspace, while $A_{21}^{o} \in \mathbb{R}^{(p-q)\times(n-p-r)}$ is such that the pair (A_{22}^{o}, A_{21}^{o}) is observable. Furthermore, the eigenvalues $\lambda_j(A_{11}^{o})$, with $1 \leq j \leq r$, correspond to the invariant zeros of $(\mathrm{A}, \mathrm{D}, \mathrm{C})$.

$$D = \left[\begin{array}{c} 0_{n-q,q} \\ \hline 0_{p-q,q} \\ D_2 \end{array}\right] \tag{10}$$

where $D_2 \in \mathbb{R}^{q\times q}$ is non-singular.

$C = \begin{bmatrix} 0_{n-q,q} & T \end{bmatrix}$, where $T \in \mathbb{R}^{p\times p}$ is an orthogonal, namely $T^{-1} = T^\mathsf{T}$. □

D. Preliminary manipulations

We are going to illustrate the essential steps to map the state x of (3)-(4) to $(z; y)$ through the mapping $\bar{T} = T_3T_2T_1$, where

$$\mathrm{x} \xmapsto{T_1} (\bar{\mathrm{z}}; y) \xmapsto{T_2} (\bar{z}; \bar{y}) \xmapsto{T_3} (z; y).$$

Let $T_1 = (\mathrm{N}_\mathrm{C}^\mathsf{T}; \mathrm{C})$ with $\mathrm{CN}_\mathrm{C} = 0_{p,n-p}$. We begin by deriving the $(\bar{\mathrm{z}}; y)$ dynamics as follows

$$\begin{bmatrix} \dot{\bar{\mathrm{z}}} \\ \dot{y} \end{bmatrix} = \overbrace{\begin{bmatrix} \bar{\mathrm{A}}_{11} & \bar{\mathrm{A}}_{12} \\ \bar{\mathrm{A}}_{21} & \bar{\mathrm{A}}_{22} \end{bmatrix}}^{\bar{\mathrm{A}}=T_1\mathrm{A}T_1^{-1}} \begin{bmatrix} \bar{\mathrm{z}} \\ y \end{bmatrix} + \overbrace{\begin{bmatrix} \bar{\mathrm{B}}_1 \\ \bar{\mathrm{B}}_2 \end{bmatrix}}^{\bar{\mathrm{B}}=T_1\mathrm{B}} u + \overbrace{\begin{bmatrix} \bar{\mathrm{D}}_1 \\ \bar{\mathrm{D}}_2 \end{bmatrix}}^{\bar{\mathrm{D}}=T_1\mathrm{D}} \xi, \tag{11}$$

$$y = \underbrace{\begin{bmatrix} 0_{p,n-p} & I_p \end{bmatrix}}_{\bar{\mathrm{C}}=\mathrm{C}T_1^{-1}} \begin{bmatrix} \bar{\mathrm{z}} \\ y \end{bmatrix}. \tag{12}$$

Then, under Assumption **A1**, and by exploiting Lemma 2, a second transformation is performed with the associated matrix

$$T_2 = \begin{bmatrix} I_{n-p} & -\bar{\mathrm{D}}_1\bar{\mathrm{D}}_2^+ \\ 0_{n-p,p} & T \end{bmatrix}, \tag{13}$$

such that (11)-(12) is rewritten as

$$\begin{bmatrix} \dot{\bar{z}} \\ \dot{\bar{y}} \end{bmatrix} = \overbrace{\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline \begin{bmatrix} A_{211} \\ A_{212} \end{bmatrix} & A_{22} \end{array}\right]}^{A=T_2\bar{A}T_2^{-1}} \begin{bmatrix} \bar{z} \\ \bar{y} \end{bmatrix} + \overbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}^{B=T_2\bar{\mathrm{B}}} u + \overbrace{\left[\begin{array}{c} 0_{n-p,q} \\ \hline D_2 \end{array}\right]}^{D=T_2\bar{\mathrm{D}}} \xi,$$

$$y = \underbrace{\left[\begin{array}{c|c} 0_{p,n-p} & T \end{array}\right]}_{C=\bar{\mathrm{C}}T_2^{-1}} \begin{bmatrix} \bar{z} \\ \bar{y} \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0_{p-q,q} \\ \hat{D}_2 \end{bmatrix}, \tag{14}$$

where $A_{21} = [A_{211}; A_{212}]$ and the pair (A_{11}, A_{211}) is observable. The first row of T_2 enables us to decouple the unknown input ξ from the unmeasurable state dynamics in

(14). Meanwhile, the second row performs the resulting lower decomposition of $D_2 = [0_{p-q,q}; \bar{D}_2]$[1].

Finally, taking advantage of the straightforward relation $TT^\mathsf{T} = I_p$ the mapping $(\bar{z};\bar{y}) \overset{T_3}{\longmapsto} (z;y)$ is further applied, with $T_3 = \mathrm{diag}(I_{n-p}, T^\mathsf{T})$, thus obtaining

$$\begin{bmatrix}\dot{z}\\ \dot{y}\end{bmatrix} = \underbrace{\left[\begin{array}{c|c} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \hline \begin{bmatrix}\mathcal{A}_{211}\\ \mathcal{A}_{212}\end{bmatrix} & \mathcal{A}_{22}\end{array}\right]}_{\mathcal{A}=T_3AT_3^{-1}} \begin{bmatrix}z\\ y\end{bmatrix} + \underbrace{\begin{bmatrix}\mathcal{B}_1\\ \mathcal{B}_2\end{bmatrix}}_{\mathcal{B}=T_3B} u + \underbrace{\begin{bmatrix}0_{n-p,q}\\ \mathcal{D}_2\end{bmatrix}}_{\mathcal{D}=T_3D} \xi$$

$$y = \underbrace{\left[\begin{array}{c|c} 0_{p,n-p} & I_p\end{array}\right]}_{\mathcal{C}=CT_3^{-1}} \begin{bmatrix}z\\ y\end{bmatrix}, \qquad \mathcal{D}_2 = \begin{bmatrix}0_{p-q,q}\\ \hat{\mathcal{D}}_2\end{bmatrix} \tag{15}$$

Note that $\bar{T}$ has been developed ad-hoc to meet our design needs and it differs from traditional FOSMOs design schemes like the Utkin's and the Walcott-Zak's observers. Only T_1 and T_2 are shared with the unit-vector SMO design of [3, Chapters 6.2 and 6.3.2].

IV. The proposed scheme

Consider the pair of FOSMOs

$$\dot{r}_i = A_{11}r_i + \mathcal{A}_{12}w_i - R_i(w_i - y) + \mathcal{B}_1 u + \tilde{L}_i v_i \tag{16}$$
$$\dot{w}_i = \mathcal{A}_{21}r_i + \mathcal{A}_{22}w_i - W_i(w_i - y) + \mathcal{B}_2 u + v_i \tag{17}$$

with $i = 1,2$, $\mathcal{A}_{21} = [\mathcal{A}_{211}; \mathcal{A}_{212}]$, and the output injection vector

$$v_i = -a\,\mathrm{sig}^\alpha(w_i - y) - b\,\mathrm{sig}^\beta(w_i - y) - c(t,y,u)\,\mathrm{sign}(w_i - y), \tag{18}$$

with parameters $a > 0, \kappa_\beta > 0, \alpha \in (0,1), \beta > 1$ and $b > \kappa_\beta / p^{\frac{\beta-1}{2}}$, where $c(t,y,u)$ is any scalar function satisfying

$$c(t,y,u) \geq \|\mathcal{D}_2\|(\mu\|u\| + \rho(t,y) + \eta). \tag{19}$$

To ensure that (5) is satisfied, it is enough to select any $\eta > 0$. If the more strict tuning inequality $\eta > \|\mathcal{A}_{211}\|Z/\|\mathcal{D}_2\|_1$ is in force, where Z is the upper-bound to the norm of z as defined in (6), then the fixed-time convergence condition (7) additionally holds. The design matrices are

$$R_i = \mathcal{A}_{12} + \tilde{L}_i(\mathcal{A}_{21}\tilde{L}_i - \mathcal{A}_{22}^*) + F_i\tilde{L}_i,\ \tilde{L}_i = \begin{bmatrix}L_i & 0_{n-p,q}\end{bmatrix},$$
$$F_i = A_{11} - L_i\mathcal{A}_{211}, \quad W_i = \mathcal{A}_{22} + \mathcal{A}_{21}\tilde{L}_i - \mathcal{A}_{22}^*, \tag{20}$$

where the gain matrices $\mathcal{A}_{22}^*$ and $L_i \in \mathbb{R}^{(n-p)\times(n-q)}$ are selected such that the next constraints are satisfied

$$\Re(\lambda_j(F_2)) < \Re(\lambda_\ell(F_1)) \in \mathbb{R}_{<0}, \forall\, j,\ell = 1,\dots,n-p, \tag{21}$$
$$\Re(\lambda_j(\mathcal{A}_{22}^*)) \in \mathbb{R}_{<0}, \forall\, j = 1,\dots,p. \tag{22}$$

Finally, by stacking the states and the output injections of (16)-(17) as $r = (r_1; r_2)$, $w = (w_1; w_2)$, and $v = (v_1; v_2)$, the estimations of z, y and ξ are constructed as follows:

$$\hat{z}(t) = K\left[r(t) - \mathrm{e}^{F\tau}r(t-\tau)\right] \tag{23}$$
$$\hat{y}(t) = Qw(t) \tag{24}$$
$$\hat{\xi}(t) = \mathcal{D}_2^+ Qv(t) \tag{25}$$

[1]The MATLAB instructions to determine T in (13) are "`[T,~]=qr(T1*D);T=flipud(T');`" where D and T_1 are assumed to be defined in accordance with the treatment of Section III-D.

where $\tau > 0$ is a time-delay parameter and the constant matrices $K \in \mathbb{R}^{(n-p)\times 2(n-p)}$, $F \in \mathbb{R}^{2(n-p)\times 2(n-p)}$ and $H \in \mathbb{R}^{2(n-p)\times(n-p)}$ take the form

$$K = \begin{bmatrix}I_{n-p} & 0_{n-p,n-p}\end{bmatrix}\begin{bmatrix}H & \mathrm{e}^{F\tau}H\end{bmatrix}^{-1}, \tag{26}$$
$$F = \begin{bmatrix}F_1 & 0_{n-p,n-p}\\ 0_{n-p,n-p} & F_2\end{bmatrix}, H = \begin{bmatrix}I_{n-p}\\ I_{n-p}\end{bmatrix}, Q = \frac{1}{2}\begin{bmatrix}I_p & I_p\end{bmatrix}.$$

A. Main results

A preliminary Lemma is proven before of investigating the convergence properties of the proposed observer.

Lemma 3 *Consider system* (3)-(4) *and let the Assumptions **A1-A2** be satisfied. Let the observer matrices L_1 and L_2 in* (20) *be such that relation* (21) *holds. Then, matrix $\begin{bmatrix}H & \mathrm{e}^{F\tau}H\end{bmatrix}$, where F and H are defined in* (26)*, is invertible for almost all $\tau > 0$.* □

Proof: It follows from Lemma 2 that the pair $(A_{11}, \mathcal{A}_{211})$ is observable. Therefore, the eigenvalues of $F_i = A_{11} - L_i\mathcal{A}_{211}$ can be arbitrarily placed by an appropriate choice of L_i in (20). Consider now (26), which yields that

$$\begin{bmatrix}H & \mathrm{e}^{F\tau}H\end{bmatrix} = \begin{bmatrix}I_{n-p} & \mathrm{e}^{F_1\tau}\\ I_{n-p} & \mathrm{e}^{F_2\tau}\end{bmatrix} \tag{27}$$
$$= \begin{bmatrix}I_{n-p} & 0_{n-p,n-p}\\ I_{n-p} & -I_{n-p}\end{bmatrix}\begin{bmatrix}I_{n-p} & \mathrm{e}^{F_1\tau}\\ 0_{n-p} & \mathrm{e}^{F_1\tau} - \mathrm{e}^{F_2\tau}\end{bmatrix}.$$

Straightforward manipulations of (27) yield the next relation

$$\det\left(\begin{bmatrix}H & \mathrm{e}^{F\tau}H\end{bmatrix}\right) = G(\tau) = (-1)^{n-p}\det\left(\mathrm{e}^{F_1\tau} - \mathrm{e}^{F_2\tau}\right)$$
$$= (-1)^{n-p}G_1(t)G_2(t) \tag{28}$$

with $G_1(\tau) = \det\left(\mathrm{e}^{F_1\tau}\right)$ and $G_2(\tau) = \det\left(I_{n-p} - \mathrm{e}^{-F_1\tau}\mathrm{e}^{F_2\tau}\right)$. Note that $G_2(\tau)$ is zero at $\tau = 0$ whence $G(0) = 0$. Due to (21), $\mathrm{e}^{F_2\tau} = o(\mathrm{e}^{F_1\tau})$, and thus $\lim_{\tau\to\infty}\mathrm{e}^{F_2\tau}\mathrm{e}^{-F_1\tau} = 0_{n-p,n-p}$. It means that $\lim_{\tau\to\infty}G_2(t) = 1$. As a result, (28) is a non-identically vanishing polynomial function, thus it is an analytic function. By applying the *Principle of isolated zeros* [23] it follows that (28) possesses only isolated zeros. Thus, $\begin{bmatrix}H & \mathrm{e}^{F\tau}H\end{bmatrix}^{-1}$ exists for almost all $\tau > 0$. ■

From the previous statement, it follows that any value of the time delay τ far enough from the isolated zeros of $G(t)$ can be taken to avoid too large entries in the matrix K, which is inversely proportional to $G(t)$.

We are now in a position to state our first main result.

Theorem 1 (Global finite-time state estimation) *Consider system* (3)-(4) *and let **A1-A3** be satisfied. Consider the observer* (16)-(18) *with outputs* (23)-(24)*, and parameters chosen as in* (19)-(21)*. Then* (5) *is verified $\forall\ a > 0, b > \kappa_\beta/p^{\frac{\beta-1}{2}}, \kappa_\beta > 0, \alpha \in (0,1), \beta > 1, \eta > 0$ and almost all $\tau > 0$.* □

Proof: The proof of Theorem 1 is divided into three parts.

Part A (Global quadratic stability): Consider $\mathrm{x} \overset{\bar{T}}{\longmapsto} (z;y)$ with $\bar{T} = T_3T_2T_1$, then (3)-(4) is rewritten as in (15). Let

$$e_{r,i} = r_i - z, \quad e_{w,i} = w_i - y. \tag{29}$$

Using (15)-(17), and (20) one derives the error dynamics

$$\dot{e}_{r,i} = A_{11}e_{r,i} - (\tilde{L}_i(\mathcal{A}_{21}\tilde{L}_i - \mathcal{A}_{22}^*) + F_i\tilde{L}_i)e_{w,i} + \tilde{L}_i v_i$$
$$\dot{e}_{w,i} = \mathcal{A}_{21}e_{r,i} + (\mathcal{A}_{22}^* - \mathcal{A}_{21}\tilde{L}_i)e_{w,i} + \begin{bmatrix} v_{i,1} \\ v_{i,2} - \hat{\mathcal{D}}_2\xi \end{bmatrix} \quad (30)$$

where v_i is decomposed into $v_{i,1} \in \mathbb{R}^{p-q}$, and $v_{i,2} \in \mathbb{R}^q$. Let us further define the auxiliary error vector

$$\tilde{e}_{r,i} = e_{r,i} - \tilde{L}_i e_{w,i}. \quad (31)$$

Using (20) and (30) one obtains the corresponding auxiliary error dynamics

$$\dot{\tilde{e}}_{r,i} = \dot{e}_{r,i} - \tilde{L}_i\dot{e}_{w,i} = A_{11}e_{r,i} - F_i\tilde{L}_i e_{w,i} - \tilde{L}_i\mathcal{A}_{21}e_{r,i}$$
$$= (A_{11} - L_i\mathcal{A}_{211})(e_{r,i} - \tilde{L}_i e_{w,i}) = F_i\tilde{e}_{r,i}. \quad (32)$$

Replacing (31) into (30) one obtains

$$\dot{e}_{w,i} = \mathcal{A}_{21}\tilde{e}_{r,i} + \mathcal{A}_{22}^* e_{w,i} + \begin{bmatrix} v_{i,1} \\ v_{i,2} - \hat{\mathcal{D}}_2\xi \end{bmatrix}. \quad (33)$$

where, due to (21)-(22), $\Re(\lambda_j(F_i)) \in \mathbb{R}_{<0}$, $\Re(\lambda_j(\mathcal{A}_{22}^*)) \in \mathbb{R}_{<0}$, $\forall\ j,\ i = 1,2$. Let $J_{i,1} \in \mathbb{R}^{(n-p)\times(n-p)}$ be a positive definite matrix, such that

$$\mathcal{A}_{22}^* + (\mathcal{A}_{22}^*)^\mathsf{T} = -J, \quad \tilde{J}_i = \mathcal{A}_{21}^\mathsf{T} J^{-1}\mathcal{A}_{21} + J_{i,1}, \quad (34)$$

then, notice that, $\tilde{J}_i = \tilde{J}_i^\mathsf{T} \succ 0$. Moreover, let $P_{i,1} \succ 0$ be the solution to the Lyapunov equation

$$P_{i,1}F_i + (F_i)^\mathsf{T} P_{i,1} = -\tilde{J}_{i,1}. \quad (35)$$

Now, by differentiating along (32)-(33) the candidate Lyapunov function $V_i(\tilde{e}_{r,i}, e_{w,i}) = \tilde{e}_{r,i}^\mathsf{T} P_{i,1}\tilde{e}_{r,i} + e_{w,i}^\mathsf{T} e_{w,i}$, yields

$$\dot{V}_i \le -\tilde{e}_{r,i}^\mathsf{T}\tilde{J}_i\tilde{e}_{r,i} + \tilde{e}_{r,i}^\mathsf{T}\mathcal{A}_{21}^\mathsf{T} e_{w,i} - e_{w,i}^\mathsf{T}\mathcal{A}_{21}\tilde{e}_{r,i}$$
$$- e_{w,i}^\mathsf{T} J e_{w,i} + 2e_{w,i}^\mathsf{T}\begin{bmatrix} v_{i,1} \\ v_{i,2} - \|\hat{\mathcal{D}}_2\|_1\|\xi\|_\infty \end{bmatrix} \quad (36)$$

Define $\tilde{e}_{w,i} = (e_{w,i} - J^{-1}\mathcal{A}_{21}\tilde{e}_{r,i})$. Then, the next relation holds

$$\tilde{e}_{w,i}^\mathsf{T} J\tilde{e}_{w,i} \equiv e_{w,i}^\mathsf{T} J e_{w,i} - 2\tilde{e}_{r,i}^\mathsf{T}\mathcal{A}_{21}^\mathsf{T} e_{w,i} + e_{r,i}^\mathsf{T}\mathcal{A}_{21}^\mathsf{T} J^{-1}\tilde{e}_{r,i}.$$

By the assumption **A3** and relations (18)-(19), the right-hand side of (36) is upper-estimated as

$$\dot{V}_i \le -\tilde{e}_{r,i}^\mathsf{T} J_i\tilde{e}_{r,i} - \tilde{e}_{w,i}^\mathsf{T} J\tilde{e}_{w,i} - 2ae_{w,i}^\mathsf{T}\mathrm{sig}^\alpha(e_{w,i})$$
$$- 2be_{w,i}^\mathsf{T}\mathrm{sig}^\beta(e_{w,i}) - 2\eta\|\hat{\mathcal{D}}_2\|_1\|e_{w,i}\|_1 < 0. \quad (37)$$

Hence, the Lyapunov function V_i is non increasing, and thus uniformly bounded, and system (30) is at least globally quadratically stable.

Part B (Finite-time output error stability) Consider the new Lyapunov function $W_i(e_{w,i}) = \|e_{w,i}\|_2^2$. By differentiating W_i then substituting (18) and (33), and upper-estimating, it follows that

$$\dot{W}_i \le 2\|\mathcal{A}_{21}\tilde{e}_{r,i}\|_\infty\|e_{w,i}\|_1 - 2\eta\|\hat{\mathcal{D}}_2\|_1\|e_{w,i}\|_1. \quad (38)$$

It follows from (32) that $\tilde{e}_{r,i} \to 0_{n-p}$. In the domain $\mathcal{R} = \{(\tilde{e}_{r,i}, e_{w,i}) : \|\mathcal{A}_{21}\tilde{e}_{r,i}\| < \|\hat{\mathcal{D}}_2\|_1\eta - \gamma\}$, where $\gamma > 0$ is arbitrary, the estimation $\dot{W}_i < -2\eta\|e_{w,i}\|_1 \le -2\eta\sqrt{W_i}$ holds. The domain $\mathcal{R}$ is finite-time attracting and invariant. Then, W_i tends to zero in finite time. This implies that a sliding motion along the manifold $e_{w,i} = 0_p$ takes place for all $t \ge \mathrm{T}_i(\mathrm{x}_0)$ and $i = 1,2$.

The equivalent control concept [3] entails the computation of the equivalent control by nullifying the right-hand side of (30). It follows that $v_{i,2} \xrightarrow{\mathrm{eq}} \hat{\mathcal{D}}_2\xi$ asymptotically, as the error $\tilde{e}_{r,i}$ approaches the origin.

Part C (Finite-time unmeasurable error stability) Consider (16), (31) and (32) and $t \ge \mathrm{T}(\mathrm{x}_0)$, then $w_i(t)=y(t)$. By using the equivalent control concept, we can further deduce that $\tilde{L}_i v_i = L_i v_{i,1} \overset{\mathrm{eq}}{=} -L_i\mathcal{A}_{211}e_{r,i}$, $\forall\ i = 1,2$. Thus,

$$\dot{r}_i \overset{\mathrm{eq}}{=} A_{11}r_i + \mathcal{A}_{12}y + \mathcal{B}_1 u - L_i\mathcal{A}_{211}e_{r,i} \ \ \forall\ t \ge \mathrm{T}(\mathrm{x}_0). \quad (39)$$

Consider now (15), (20), (39), and let $E = (L_1; L_2)\mathcal{A}_{211}$, then

$$\frac{d}{dt}(r - Hz) \overset{\mathrm{eq}}{=} Fr + H(\mathcal{A}_{12}y + \mathcal{B}_1 u) + Ez \quad (40)$$
$$- H\left[A_{11}z + L\mathcal{A}_{211}y + \mathcal{B}_1 u\right] + (FHz - FHz)$$
$$= F(r - Hz) + \left[FH - HA_{11} + E\right]z = F(r - Hz),$$

where $r = (r_1; r_2)$. Therefore it results

$$r(t) - Hz(t) = \mathrm{e}^{F\tau}\left[r(t-\tau) - Hz(t-\tau)\right], t \ge \mathrm{T}(\mathrm{x}_0) + \tau \quad (41)$$

Now, consider the definitions of K and H in (26). By applying the block matrix inversion rule it yields

$$\begin{bmatrix} H & \mathrm{e}^{F\tau}H \end{bmatrix}^{-1} = \begin{bmatrix} I_{n-p} & -\mathrm{e}^{F_1\tau}(\mathrm{e}^{F_1\tau} - \mathrm{e}^{F_2\tau})^{-1} \\ 0_{n-p,n-p} & (\mathrm{e}^{F_1\tau} - \mathrm{e}^{F_2\tau})^{-1} \end{bmatrix} \times \begin{bmatrix} I_{n-p} & 0_{n-p,n-p} \\ I_{n-p} & -I_{n-p} \end{bmatrix}. \quad (42)$$

By (23), (26) and (42), and noticing that $KH = I_{n-p}$ and $K\mathrm{e}^{F\tau}H = 0_{n-p,n-p}$, one derives that

$$\hat{z}(t) = K\left[r(t) - \mathrm{e}^{F\tau}r(t-\tau)\right] + z(t) - KHz(t) \quad (43)$$
$$= z(t) + K\left[r(t) - Hz(t)\right] - K\mathrm{e}^{F\tau}\left[r(t-\tau) - Hz(t-\tau)\right].$$

By (41) and (43) it yields that $\hat{z}(t) = z(t)\ \forall\ t \ge \mathrm{T}(\mathrm{x}_0) + \tau$, irrespective of the initial conditions of $\hat{z}(t)$ for $t \in [-\tau, 0]$. ■

Drawing from the problem statement and assumptions outlined in Section III-B, we will now proceed to present the second main result of this study.

Theorem 2 (Semi-global fixed-time state estimation) *Consider* (3)-(4) *and let **A1**-**A3** be satisfied. Further assume* $z(t) \in \mathcal{Z} \subset \mathbb{R}^{n-p}$, *and there exists a known* $Z > 0$ *such that* (6) *holds. Consider the observer* (16)-(18) *with outputs* (23)-(24) *and parameters chosen as in* (19)-(21) *where* $\eta > \|\mathcal{A}_{211}\|_\infty Z/\|\hat{\mathcal{D}}_2\|_1$. *Then* (7) *is verified* $\forall$ $a > 0, b > \kappa_\beta/p^{\frac{\beta-1}{2}}, \kappa_\beta > 0, \alpha \in (0,1), \beta > 1$, *and almost all* $\tau > 0$, *and almost all* $\tau > 0$. □

Proof: The proof of Theorem 2 is divided into three parts.

Part A (Global quadratic stability): All development and results derived in the Part A of the proof of Theorem 1 apply. Thus, global quadratic stability of system (30) is at least in force

Part B (Semi-global fixed-time output error stability) Consider the quadratic Lyapunov function $W_i(e_{w,i}) = \|e_{w,i}\|_2^2$.

By evaluating its time derivative, substituting in the resulting expression (18), (33) and $\eta = \|\mathcal{A}_{211}\|Z/\|\hat{\mathcal{D}}_2\|_1 + \epsilon$, for some small scalar $\epsilon > 0$, and then upper-estimating the resulting right-hand side, one obtains

$$\begin{aligned}\dot{W}_i \leq &- e_{w,i}^\mathsf{T} J e_{w,i} - 2a e_{w,i}^\mathsf{T} \mathrm{sig}^\alpha(e_{w,i}) - 2b e_{w,i}^\mathsf{T} \mathrm{sig}^\beta(e_{w,i}) \\ &- 2\|\mathcal{A}_{21}\tilde{e}_{r,i}\|_\infty \|e_{w,i}\|_1 - 2\eta\|\hat{\mathcal{D}}_2\|_1\|e_{w,i}\|_1 \\ \leq &- 2a\|e_{w,i}\|_{1+\alpha}^{1+\alpha} - 2b\|e_{w,i}\|_{1+\beta}^{1+\beta} - 2\epsilon\|e_{w,i}\|_1 \end{aligned} \tag{44}$$

where, due to the exponential stability of $\tilde{e}_{r,i}$, it yields $\|\mathcal{A}_{211}\tilde{e}_{r,i}(t)\|_\infty \leq \|\mathcal{A}_{211}\tilde{e}_{r,i}(0)\|_\infty \leq \|\mathcal{A}_{211}\|_\infty Z$. Moreover,

$$\begin{aligned}\|e_{w,i}\|_1 \geq \|e_{w,i}\|_{1+\alpha} \geq &\|e_{w,i}\|_2 = \sqrt{W_i}, \quad \forall\, \alpha \in (0,1), \\ \sqrt{p}\|e_{w,i}\|_{1+\beta} \geq &\|e_{w,i}\|_2 = \sqrt{W_i}, \quad \forall\, \beta > 1. \end{aligned} \tag{45}$$

Thus, from (45), and because of $b > \kappa_\beta / p^{\frac{\beta-1}{2}}$ with $\kappa_\beta > 0$, it turns out that (44) can be further manipulated as follows

$$\dot{W}_i \leq -2aW_i^{\frac{1+\alpha}{2}} - 2bp^{\frac{\beta-1}{2}} W_i^{\frac{1+\beta}{2}} = -2aW_i^{\bar{\alpha}} - 2\kappa_\beta W_i^{\bar{\beta}}, \tag{46}$$

where $\bar{\alpha} = \frac{1+\alpha}{2} \in (0,1)$, and $\bar{\beta} = \frac{1+\beta}{2} > 1$. Using Lemma 1, we can establish the existence of a fixed time $\mathrm{T}_{\max} = 1/(a(1-\bar{\alpha})) + 1/(\kappa_\beta(\bar{\beta}-1))$ after which a *sliding motion* along $e_{w,i} = 0_p$ is established. This implies $w_i \overset{\text{eq}}{=} y$, $\forall\, t \geq \mathrm{T}_{\max}$, $i = 1, 2$, and $z \in \mathcal{Z} \subset \mathbb{R}^p$.

Part C (Fixed-time unmeasurable error stability) Finally, by analogous developments as those in Proof of Theorem 1:Part C, it follows $\hat{z}(t) = z(t)$ for all $t \geq \mathrm{T}_{\max} + \tau$, ■

B. Performance discussion and Implementation hints

The results presented in Theorem 2 demonstrate the effectiveness of observer (16)-(20) as a semi-global fixed-time UIO for (3)-(4). Semi-globality derives from the working assumption (6), dictating that z is confined within a known domain. However, as proven by Theorem 1, the unmeasured state estimation error in (41) is globally uniformly finite-time stable at the origin even if the system exceeds the specified range at the startup for any reason. More precisely, convergence within a fixed time remains guaranteed once the unmeasured estimation error decreases below Z in norm. Acknowledging the presence of input/output port saturation constraints in controller devices, as well as state limitations across various output feedback scenarios [24], we emphasize that the observer's semi-global stability does not pose a limitation in practice.

Let us now discuss some implementation aspects. First, notice that our UIO is numerically well-posed for any initial condition $r(t)$ in the interval $t \in [-\tau, 0]$. However, undesired overshoots may occur in $\hat{z}(t)$ until convergence is achieved. This is because the unmeasured state estimation inherits major limitations from [11], [12]. Indeed, $\hat{z}(t)$ in (23) is meaningless for any time instant preceding the convergence time. To address this, we propose a simple and effective switching rule to replace (23). Since both the estimations $r_i(t)$ are meaningful, we suggest replacing (23) with

$$\hat{z}(t) : \begin{cases} Q'r(t), & \text{if } [\ \|e_{w,1} + e_{w,2}\|_1 \geq \lambda\ \wedge\ t < \mathrm{T}_{\max} + \tau\ ] \\ K\left[r(t) - \mathrm{e}^{F\tau} r(t-\tau)\right], & \text{otherwise.} \end{cases} \tag{47}$$

Here, $Q' = [I_p, I_p]/2$ is an averaging matrix, as Q in (26), and $\lambda > 0$ is a small threshold detecting the sliding regime. See Fig. 3 for a comparison between (23) and (47).

Referring instead to the chattering problem, its effect is known to have a lesser impact on estimation problems than control. However, it can still introduce distortions in the estimations, potentially compromising the control system performance. To address this issue, we propose incorporating Q from (26) into the output estimation $\hat{y}$, as shown in (24). Specifically, $\hat{y}$ is calculated by averaging the two estimations $\hat{w}_1$ and $\hat{w}_2$ provided by our pair of FOSMOs. This allows for mitigating the chattering and improving the accuracy of the estimation by inherently smoothing out any high-frequency distortion through averaging [25].

Similar ideas will also be applied to the UIR process using (25). First, we recall that in SMOs the UIR is achieved by extracting the *equivalent control* from the output injection vector (18). One common approach is to low-pass filter the signal v_i to retrieve $v_i^{\text{eq}} = \mathcal{D}_2\xi$ and derive $\hat{\xi} = \mathcal{D}_2 v_i^{\text{eq}} = \hat{\mathcal{D}}_2^+ v_{i,2}^{\text{eq}} \to \xi$. Nevertheless, this introduces complexity and necessitates ad-hoc tuning of the filter's time constant. Another strategy involves utilizing smooth sigmoidal approximations for v_i, but this may limit the observer robustness [5]. Alternatively, we suggest using (25), which is rewritten here for clarity:

$$\hat{\xi}(t) = Q\mathcal{D}_2^+ v(t) = (\hat{\mathcal{D}}_2^\intercal \hat{\mathcal{D}}_2)^{-1} \hat{\mathcal{D}}_2^\intercal \left(\frac{v_{1,2}(t) + v_{2,2}(t)}{2} \right). \tag{48}$$

Relation (48) implements a static averaging between the two output injections provided by the two observers and it represents the instantaneous *sample mean* between two realizations of the same noisy estimation [25]. This method filters out the high-frequency noisy components corrupting $v_{eq}(t)$ obtaining similar results of low pass filtering without requiring filter dynamics and careful ad-hoc time-constant tuning. Figures 4 and 5 provide a comparison between the UIR process using (48) and that using a low pass filter with a cutoff frequency of 1kHz, in the case of a smooth and noisy UIs. Simulations show that (48) exhibits remarkable filtering capabilities.

TABLE I: Pendulum-cart system parameters [5, Chapter 5.6.3].

M	m	J	l	$\mathrm{F_q}$	F_θ	g
3.2 kg	0.534 kg	0.062 kg m^2	0.365 m	6.2 kg/sec	0.009 kg m^2	9.807 m/sec^2

V. Simulations

To evaluate the UIO performance, we conducted numerical simulations in MATLAB/Simulink using the Euler fixed-step solver with a sampling time of $\mathrm{T}_s = 10^{-4}$.

Consider the pendulum-cart system depicted in Fig. 1, with parameters specified in Table I. The manipulable horizontal force $u \in \mathbb{R}$ is applied to the cart to maintain the rod in the upright position while controlling the cart position $\mathrm{q} \in \mathbb{R}$. A disturbance force ξ also affects the system's motion. The corresponding motion equations are

$$(\mathrm{M}+\mathrm{m})\ddot{\mathrm{q}} + \mathrm{F_q}\dot{\mathrm{q}} + \mathrm{ml}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = u + \xi \quad (49)$$

$$\mathrm{J}\ddot{\theta} + \mathrm{F}_\theta\dot{\theta} - \mathrm{mlg}\sin\theta + \mathrm{ml}\ddot{\mathrm{q}}\cos\theta = 0 \quad (50)$$

Denote the state vector by $\mathrm{x} = (\mathrm{q};\theta;\dot{\mathrm{q}};\dot{\theta})$, and the output by $y = (\mathrm{q};\theta;\dot{\mathrm{q}})$. Linearizing around the unstable equilibrium point 0_4 yields the system dynamics in the form (3)-(4) where

$$\left[\begin{array}{c|c} \mathrm{A} & \mathrm{D} \\ \hline \mathrm{C} & - \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1.933 & -1.987 & 0.0091 & 0.321 \\ 0 & 36.977 & 6.259 & -0.174 & -1.011 \\ \hline 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & - \\ 0 & 0 & 1 & 0 & \end{array}\right],$$

and $\mathrm{B} = \mathrm{D}$. Following Section III-D, the system dynamics given by (3)-(4) are rewritten in the form (15), where

$$\left[\begin{array}{c|c} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \hline \begin{bmatrix} \mathcal{A}_{211} \\ \mathcal{A}_{212} \end{bmatrix} & \mathcal{A}_{22} \end{array}\right] = \left[\begin{array}{c|ccc} -0.1453 & 0 & 30.8890 & 0.4586 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -3.1495 \\ \hline 0.0091 & 0 & -1.9330 & -2.0157 \end{array}\right],$$

$$\mathcal{D} = \begin{bmatrix} 0 \\ \hline 0 \\ 0 \\ \hat{\mathcal{D}}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \hline 0 \\ 0 \\ 0.3205 \end{bmatrix},\ \bar{T} = \begin{bmatrix} 0 & 0 & 3.149 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and $\mathcal{C} = \begin{bmatrix} 0_3 & I_3 \end{bmatrix}$. The UIO (16)-(20) is implemented with $a = 100$, $\kappa_\beta = 3$, $\rho = 2$, $\alpha = 0.2$, $\beta = 3$, $\tau = 0.01$, $\mathcal{A}_{22}^* = \mathrm{diag}(-20,-21,-22)$, and $L_1 = \begin{bmatrix} 0 & 19.8547 \end{bmatrix}$, $L_2 = \begin{bmatrix} 0 & 20.8547 \end{bmatrix}$ such that $F_1 = -20 > F_2 = -21$.

Note that, in accordance with (2), the given tuning provides a $\mathrm{T_{max}} = 0.125$, and $\mathrm{T}(\mathrm{x}_0) \leq \mathrm{T_{max}} + \tau = 0.135$.

To show the effectiveness of the proposed UIO for observer-based control purposes, the estimated state $\hat{\mathrm{x}}$ is fed back to assign closed-loop poles at $\{-4.2, -4.4, -4.6, -4.8\}$, through the static feedback $u(t) = \mathrm{K}\hat{\mathrm{x}}(t)$, with $\hat{\mathrm{x}} = \bar{T}^{-1}(\hat{z};\hat{y})$, and $\mathrm{K} = \begin{bmatrix} -36.0641 & -161.8473 & -39.5484 & -27.6301 \end{bmatrix}$.

Two different classes of UIs ξ are considered: a smooth biased sine wave $\xi = \xi_0$, where $\xi_0 = 0.2 + \sin(2\pi t)$ has a power of $P = 1\mathrm{W}$ (top plot of Fig. 4), and its noisy counterpart $\xi = \xi_0 + \mathcal{N}(0, N_0)$ (top plot of Fig. 5), obtained by passing ξ_0 through an additive white Gaussian noise (AWGN) channel [26] characterized by an SNR of 20dB. The channel bandwidth is $B_0 = 1/(2T_s) = 5\mathrm{kHz}$, as per the sampling theorem. By the definition of $\mathrm{SNR} = P/(N_0 B_0)$, the band-limited white noise $\mathcal{N}$ has a single-sided power spectral density of $N_0 = P/(B_0 10^{20\mathrm{dB}/10}) = 2 \times 10^{-6}$, and zero mean. Note that the finite energy of a band-limited white noise implies ξ is Lebesgue continuous.

Figures 2 and 3 depict the errors in state estimation associated with the proposed UIO, which exhibit convergence to the origin in finite time according to Theorem 1. These figures correspond to a scenario where the unknown input is noisy, such that $\xi = \xi_0 + \mathcal{N}(0, N_0)$. Notably, the errors converge to zero within the fixed time interval $[0, \mathrm{T_{max}} + \tau)$, thereby corroborating Theorem 2, assuming that $|\dot{\theta}| \leq Z$ is constrained by some mechanical upper bounds. More precisely, Fig. 3 compares the unmeasured state estimation error computed using equations (23) and (47). It is evident that (47) eliminates the presence of large overshoot in the unmeasured state estimation error while providing a meaningful estimation for all $t \geq 0$. Furthermore, such estimation converges to the true value within the expected time $\mathrm{T_{max}} + \tau = 0.135$.

Finally, Figures 4 and 5 provide a performance comparison of the UIR process using (48) and the conventional lowpass filtering approach with a cutoff frequency of 1kHz, in the case of a smooth ($\xi = \xi_0$) and noisy ($\xi = \xi_0 + \mathcal{N}(0, N_0)$)

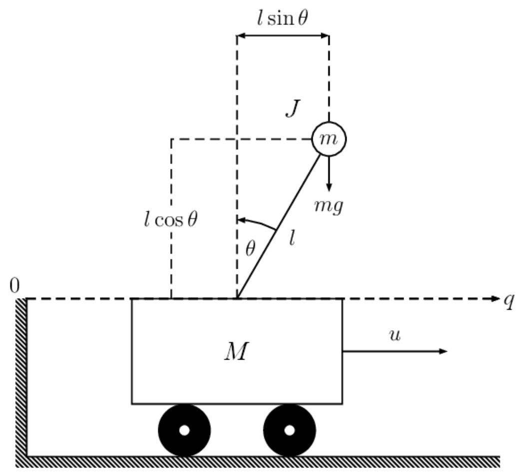

Fig. 1: Schematic of an inverted pendulum with a cart.

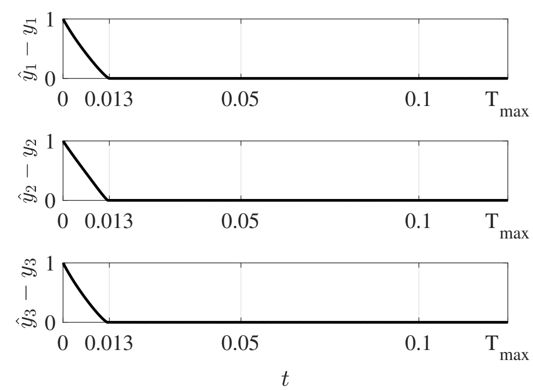

Fig. 2: Output estimation error $\hat{y}(t) - y(t)$ for the test with the noisy UI $\xi = \xi_0 + \mathcal{N}(0, N_0)$.

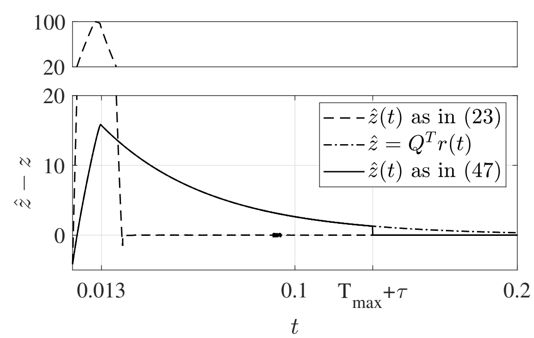

Fig. 3: Unmeasured state estimation error $\hat{z}(t) - z(t)$ for the test with the noisy UI $\xi = \xi_0 + \mathcal{N}(0, N_0)$.

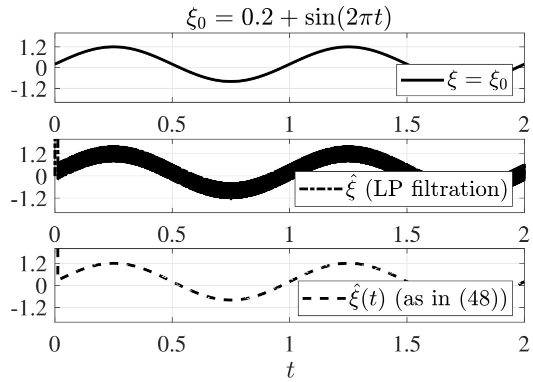

Fig. 4: Smooth unknown input $\xi = \xi_0$ (top-plot), and estimations using (48) and a lowpass filtering (1kHz cutoff).

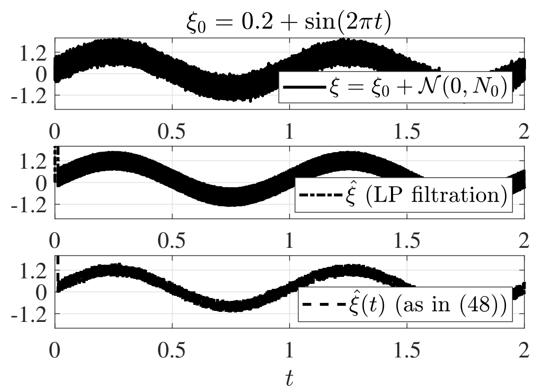

Fig. 5: Noisy unknown input $\xi = \xi_0 + \mathcal{N}(0, N_0)$ (top-plot), and estimations using (48) and a lowpass filtering (1kHz cutoff), with $\xi_0 = 0.2 + \sin(2\pi t)$ and $\mathcal{N}(0, N_0)$ is a band-limited noise

unknown input. Simulations demonstrate that the proposed approach (48) exhibits remarkable filtering capabilities.

VI. Conclusions

Our proposal extends previous approaches to continuous predefined-time state estimation, which involved the use of a pair of time-delay observers, by adapting them to the framework of sliding mode observation. As a result, we propose a novel FOSMO design that allows for finite-time state estimation using first-order sliding-mode algorithms. Moreover, the scheme exhibits semi-global fixed-time convergence and allows for unknown input reconstruction without explicitly implementing low-pass filters or continuous approximations of the discontinuous signals.

Among the problems to be tackled shortly, a challenging and promising direction for further investigations is the adaptive extension of the scheme to guarantee global fixed-time convergence properties. Moreover, we are currently studying extensions such as functional, unit-vector-based, and HOSMO approaches. Also, we are exploring the application of the proposed method for quickly detecting and isolating faults and reconstructing cyber attacks in large-networked applications.

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