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SOME MIXED PROBLEMS FOR SEMILINEAR PARABOLIC TYPE EQUATION*

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Abstract In this paper, some mixed problem with third type boundary value for a semilinear parabolic equation is investigated. Here the solvability theorems for considered problem and the uniqueness theorem for a model case of the problem are showed.

Keywords Semilinear parabolic equation, third type boundary value problem, existence and uniqueness theorems, sublinear case, linear case, super linear case.

MSC(2000) 35D30, 35K58, 35M12.

1. Introduction

We consider the problem

$$\frac{\partial u}{\partial t} - \Delta u + g(x,t,u) = h(x,t),\ (x,t) \in Q_T \equiv \Omega \times (0,T), \tag{1.1}$$

$$u(x,0) = 0, \quad x \in \Omega, \tag{1.2}$$

$$\left(\frac{\partial u}{\partial \eta} + a(x',t)u\right)\bigg|_{\Sigma_T} = \varphi(x',t), \quad (x',t) \in \ \Sigma_T \equiv \partial\Omega \times [0,T], T > 0. \tag{1.3}$$

Here $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded domain with sufficiently smooth boundary $\partial\Omega$; Δ denotes the Laplace operator with $n-$dimension ($\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$); $g : Q_T \times \mathbb{R}^1 \longrightarrow \mathbb{R}^1$ and $a : \Sigma_T \longrightarrow \mathbb{R}^1$ are given functions; h and φ are given generalized functions.

In this article we investigate nonhomogenous third type boundary value problem for equation (1.1) with mapping g in general form. Elliptic part of equation (1.1) is an Emden-Fowler type equation, since it becomes Emden-Fowler equation for a special case of mapping g (see [10, 11]). Equation (1.1) has been studied mostly in homogeneous form by taking mapping g in special cases with Dirichlet or Neumann boundary conditions. For instance, in [6], existence of positive solutions of homogenous form of (1.1) when $g(x,t,u) := \frac{u}{1-u}$ with initial and homogenous Dirichlet condition was studied. In [8], global existence of positive solutions of equation (1.1) by taking $g(x,t,u) := -|u|^p$ with initial and Robin boundary condition was studied in $\Omega \times \mathbb{R}^+$. In [7], global existence of solution of homogenous form of equation (1.1) by taking $g(x,t,u) := g(u)$ with initial and third type boundary value was investigated in a bounded star-shaped region. In [5], existence of global positive solutions

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of homogenous form of (1.1) when $g(x,t,u) := -|u|^{p-1}u$ for special cases of p with initial homogenous Dirichlet condition was investigated.

We investigate problem (1.1)-(1.3) in sublinear, linear and super linear cases, by depending on mapping g, i.e. the form of g creates these cases depending on u. For the existence of generalized solution of problem (1.1)-(1.3) and for the uniqueness in a model case, we obtained sufficient conditions for function a and mapping g. And under these conditions we obtained that problem (1.1)-(1.3) is solvable and we showed the uniqueness of the solution for a model case in corresponding spaces.

2. Formulation and the main conditions of problem (1.1)-(1.3)

For problem (1.1)-(1.3), we shall assume $h \in L_2(0,T;(W_2^1(\Omega))^*) + L_q(Q_T)$ (generally $q > 1$) and $\varphi \in L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$.
We consider the following conditions:

(1) g is a Caratheodory function in $(Q_T \times \mathbb{R}^1)$ and there exist a number $\alpha \geq 0$ and functions $c_1 \in L_{s_1}(0,T;L_{r_1}(\Omega))$, $c_0 \in L_{s_2}(0,T;L_{r_2}(\Omega))$ such that g satisfies the following inequality for a.e. $(x,t) \in Q_T$ and for any $\xi \in \mathbb{R}^1$:

$$|g(x,t,\xi)| \leq c_1(x,t)\,|\xi|^{\alpha} + c_0(x,t),$$

($r_1,\ r_2, s_1,\ s_2 > 1$ will be defined later).

(2) $a \in L_\infty(0,T;L_{n-1}(\partial\Omega))$.

We understand the solution of considered problem in the following sense:

Definition 2.1. Let $P_0 := L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T) \cap W_2^1(0,T;(W_2^1(\Omega))^*) \cap \{u : u(x,0) = u_0\}$. A function $u \in P_0$ is called generalized solution of problem (1.1)-(1.3) if it satisfies the equality

$$\begin{aligned}
&-\int_0^T\int_\Omega u\frac{\partial v}{\partial t}dxdt + \int_\Omega u(x,T)v(x,T)dx + \int_0^T\int_\Omega Du.Dvdxdt \\
&+\int_0^T\int_\Omega g(x,t,u)vdxdt + \int_0^T\int_{\partial\Omega} a(x',t)uvdx'dt \\
&= \int_0^T\int_\Omega hvdxdt + \int_0^T\int_{\partial\Omega}\varphi vdx'dt
\end{aligned}$$

for all $v \in L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T) \cap W_2^1(0,T;(W_2^1(\Omega))^*)$.

We investigate problem (1.1)-(1.3) in three different sections according to the values of α (see condition **(1)**): Sublinear Case, Linear Case and Super Linear Case.

3. Solvability of problem (1.1)-(1.3) in sublinear case

Let $0 \leq \alpha < 1$. In this case, since $L_2(0,T;W_2^1(\Omega)) \subset L_{\alpha+1}(Q_T)$, then

$$P_0 \equiv L_2(0,T;W_2^1(\Omega)) \cap W_2^1(0,T;(W_2^1(\Omega))^*) \cap \{u : u(x,0) = 0\}.$$

We consider the following conditions:

(1)′ Condition **(1)** is satisfied with nonnegative functions c_1, c_0 and parameters: $s_1 := \frac{2}{1-\alpha}$, $r_1 := \frac{p_0 q_0}{p_0 - \alpha q_0}$, $s_2 := 2$, $r_2 := q_0$, where $p_0 := \frac{2n}{n-2}$, $q_0 := (p_0)'$.

(3) There exists a number $a_0 > 0$ such that $a(x', t) \geq a_0$ for a.e. $(x', t) \in \Sigma_T$.

Theorem 3.1. *Let conditions (1)′, (2), (3) be fulfilled for* $0 \leq \alpha < 1$. *Then problem (1.1)-(1.3) is solvable in* P_0 *for any*

$$(h, \varphi) \in L_2(0, T; (W_2^1(\Omega))^*) \times L_2(0, T; W_2^{-\frac{1}{2}}(\partial\Omega)).$$

The proof is based on a general result of Soltanov [9] that is given below:

Theorem 3.2. *Let* X *and* Y *be Banach spaces with duals* X^* *and* Y^* *respectively,* Y *be a reflexive Banach space,* $\mathcal{M}_0 \subseteq X$ *be a weakly complete "reflexive" pn-space,* $X_0 \subseteq \mathcal{M}_0 \cap Y$ *be a separable vector topological space. Let the following conditions be fulfilled:*

(i) $f : P_0 \to L_q(0, T; Y)$ *is a weakly compact (weakly continuous) mapping, where*

$$P_0 \equiv L_p(0, T; \mathcal{M}_0) \cap W_q^1(0, T; Y) \cap \{x(t) \mid x(0) = 0\},$$

$1 < \max\{q, q'\} \leq p < \infty$, $q' = \frac{q}{q-1}$;

(ii) *there is a linear continuous operator* $A : W_m^s(0, T; X_0) \to W_m^s(0, T; Y^*)$, $s \geq 0$, $m \geq 1$ *such that* A *commutes with* $\frac{\partial}{\partial t}$ *and the conjugate operator* A^* *has* $ker(A^*) = \{0\}$;

(iii) *operators* f *and* A *are derivative, in generalized sense, a coercive pair on space* $L_p(0, T; X_0)$, *i.e. there exist a number* $r > 0$ *and a function* $\Psi : R_+^1 \to R_+^1$ *such that* $\Psi(\tau)/\tau \nearrow \infty$ *as* $\tau \nearrow \infty$ *and for any* $x \in L_p(0, T; X_0)$ *under* $[x]_{L_p(\mathcal{M}_0)} \geq r$ *following inequality holds:*

$$\int_0^T \langle f(t, x(t)), Ax(t) \rangle dt \geq \Psi\left([x]_{L_p(\mathcal{M}_0)}\right);$$

(iv) *there exist some constants* $C_0 > 0$, $C_1, C_2 \geq 0$, $\nu > 1$ *such that the inequalities*

$$\begin{aligned} \int_0^T \langle \xi(t), A\xi(t) \rangle dt &\geq C_0 \|\xi\|_{L_q(0,T;Y)}^{\nu} - C_2, \\ \int_0^t \langle \frac{dx}{d\tau}, Ax(\tau) \rangle d\tau &\geq C_1 \|x\|_Y^{\nu}(t) - C_2, \quad \text{a.e. } t \in [0, T] \end{aligned}$$

hold for any $x \in W_p^1(0, T; X_0)$ *and* $\xi \in L_p(0, T; X_0)$.

Assume that conditions (i)-(iv) are fulfilled. Then the Cauchy problem

$$\frac{dx}{dt} + f(t, x(t)) = y(t), \quad y \in L_q(0, T; Y); \quad x(0) = 0$$

is solvable in P_0 *in the following sense*

$$\int_0^T \left\langle \frac{dx}{dt} + f(t, x(t)), y^*(t) \right\rangle dt = \int_0^T \langle y(t), y^*(t)\rangle\, dt, \quad \forall y^* \in L_{q'}(0,T;Y^*),$$

for any $y \in L_q(0,T;Y)$ *satisfying the inequality*

$$\sup\left\{ \frac{1}{[x]_{L_p(0,T;\mathcal{M}_0)}} \int_0^T \langle y(t), Ax(t)\rangle\ dt \ \mid x \in L_p(0,T;X_0) \right\} < \infty.$$

Proof. [Proof of Theorem 3.1:] To apply Theorem 3.2 to problem (1.1)-(1.3), firstly we define corresponding mappings and acting spaces for the problem using the spaces that mentioned before:

$$f = \{f_1, f_2\}$$

such that

$$f_1(u) := -\Delta u + g(x,t,u), \tag{3.1}$$

$$f_2(u) := \frac{\partial u}{\partial \eta} + a(x',t)u, \tag{3.2}$$

$$A \equiv Id. \tag{3.3}$$

Here,

$$f : P_0 \to L_2(0,T;(W_2^1(\Omega))^*) \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega)); \quad A : P_0 \to P_0.$$

Now we shall give the following lemmas to see that the conditions of Theorem 3.2 are satisfied: □

Lemma 3.1. *f is bounded and weakly continuous from P_0 to $L_2(0,T;(W_2^1(\Omega))^*)$, under the assumptions of Theorem 3.1.*

Proof. It is obvious that linear parts of f are bounded. Using condition $\mathbf{(1)'}$, we obtain that

$$\begin{aligned}
&\|g\|_{L_2(0,T;L_{q_0}(\Omega))} \le \gamma(\|u\|_{L_2(0,T;L_{p_0}(\Omega))}),\\
&\gamma(\|u\|_{L_2(0,T;L_{p_0}(\Omega))}) = c[\|c_1\|^2_{L_{\frac{2}{1-\alpha}}(0,T;L_{\frac{p_0 q_0}{p_0-\alpha q_0}}(\Omega))}\|u\|^{2\alpha}_{L_2(0,T;L_{p_0}(\Omega))}\\
&\qquad\qquad + \|c_0\|^2_{L_2(0,T;L_{q_0}(\Omega))}]^{\frac{1}{2}},
\end{aligned}$$

$c > 0$ is a constant. This means, g is a bounded mapping from P_0 to $L_2(0,T;L_{q_0}(\Omega))$, since $P_0 \subset L_2(0,T;W_2^1(\Omega)) \subset L_2(0,T;L_{p_0}(\Omega))$.

Since linear parts of f are bounded, they are already weakly continuous. It is enough to investigate the nonlinear part of f, i.e. mapping g. Let $\{u_m\} \subset P_0$ and $u_m \rightharpoonup u_0$ in P_0. Then $u_m \rightharpoonup u_0$ in $L_2(0,T;L_{p_0}(\Omega))$. Since $L_2(0,T;W_2^1(\Omega)) \cap W_2^1(0,T;(W_2^1(\Omega))^*) \circlearrowleft L_2(Q_T)$, then $\exists\ \{u_{m_l}\} \subset \{u_m\}$ such that $u_{m_l} \longrightarrow u_0$ almost everywhere in Q_T.

Using condition $\mathbf{(1)'}$ we can say that

$$g(x,t,\cdot) : \mathbb{R}_1 \longrightarrow \mathbb{R}_1$$

is a continuous function and we also obtained that g is bounded.

Then according to a general result (1. Chapter, 1. Paragraph, Lemma 1.3 of [4]), $\exists\ \{u_{m_j}\} \subset \{u_m\}$ such that

$$g(x,t,u_{m_j}) \underset{L_2(0,T;L_{q_0}(\Omega))}{\rightharpoonup} g(x,t,u_0).$$

Thus g is a weakly continuous mapping from P_0 to $L_2(0,T;(W_2^1(\Omega))^*)$. ☐

Lemma 3.2. *Conditions $(ii), (iii), (iv)$ of Theorem 3.2 are satisfied, under the assumptions of Theorem 3.1.*

Proof. Since A is an identity mapping, it is obvious that condition (ii) is satisfied. Furthermore, for any $u \in W_2^1(0,T;W_2^1(\Omega))$ the following inequalities are satisfied:

$$\int_0^T \langle u,u\rangle_\Omega\,dt = \int_0^T \|u\|^2_{L_2(\Omega)}dt \geq c_6\|u\|^2_{L_2(0,T;(W_2^1(\Omega))^*)},$$

$$\int_0^t \left\langle \frac{\partial u}{\partial \tau}, u\right\rangle_\Omega d\tau = \frac{1}{2}\|u\|^2_{L_2(\Omega)}(t) \geq \frac{1}{2}c_6\|u\|^2_{(W_2^1(\Omega))^*}(t),$$

a.e. $t \in [0,T]$ ($c_6 > 0$ is the constant coming from Sobolev's Imbedding Inequality* [1].)

This means condition (iv) is also satisfied.

It is enough to see that mapping f is coercive on $L_2(0,T;W_2^1(\Omega))$ for condition (iii), since A is an identity mapping:

Using conditions $\mathbf{(1)'}$ and $\mathbf{(3)}$ we obtain,

$$\langle f(u),u\rangle_{Q_T} \geq \Psi(\|u\|_{L_2(0,T;W_2^1(\Omega))}),$$
$$\Psi(\|u\|_{L_2(0,T;W_2^1(\Omega))}) := (\theta c_2 - (c_3)^2\varepsilon)\|u\|^2_{L_2(0,T;W_2^1(\Omega))} - K,$$

here $\theta := \min\{1,\ a_0\}, 0 < \varepsilon < \frac{\theta c_2}{(c_3)^2}$ and $K > 0$ is a constant.

So, $\frac{\Psi(\|u\|)}{\|u\|} \nearrow \infty$ as $\|u\|_{L_2(0,T;W_2^1(\Omega))} \nearrow \infty$. ☐

Proof. [Continuation of the Proof of Theorem 3.1:] We can apply Theorem 3.2 to problem (1.1)-(1.3) by virtue Lemma 3.1 and Lemma 3.2. Hence we obtain that problem (1.1)-(1.3) is solvable in P_0 for any $(h,\varphi) \in L_2(0,T;(W_2^1(\Omega))^*) \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$ satisfying the following inequality

$$\sup\left\{\frac{1}{\|u\|_{L_2(0,T;W_2^1(\Omega))}}\int_0^T \langle h,u\rangle_\Omega + \langle\varphi,u\rangle_{\partial\Omega}\,dt : u \in L_2(0,T;W_2^1(\Omega))\right\} < \infty.$$

If we consider the norm definition of (h,φ) in $L_2(0,T;(W_2^1(\Omega))^*)\times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$, we see that problem (1.1)-(1.3) is solvable in P_0 for any $(h,\varphi) \in L_2(0,T;(W_2^1(\Omega))^*)\times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$. ☐

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* $c_6\,\|u\|^2_{(W_2^1(\Omega))^*} \leq \|u\|^2_{L_2(\Omega)}$

4. Solvability of problem (1.1)-(1.3) in linear case

Let $\alpha = 1$ for condition **(1)**. In this case,

$$P_0 \equiv L_2(0,T;W_2^1(\Omega)) \cap W_2^1(0,T;(W_2^1(\Omega))^*) \cap \{u : u(x,0) = 0\}\,.$$

We consider the following conditions:

(1)$''$ Condition **(1)** is satisfied with nonnegative functions c_1, c_0 and parameters: $s_1 := \infty$, $r_1 := \frac{n}{2}$, $s_2 := 2$, $r_2 := q_0$.

(4) One of the following conditions be satisfied:

I. There exists a number $a_0 > 0$ such that $a(x',t) \geq a_0$ for a.e. $(x',t) \in \Sigma_T$ and $\|c_1\|_{L_\infty(0,T;L_{\frac{n}{2}}(\Omega))} < \frac{\min\{1,\, a_0\}c_2}{(c_3)^2}$ (here c_2 is the constant coming from the inequality[†] [12] and c_3 is the constant of Sobolev's Imbedding inequality[‡] [1]).

II. There exist some numbers $k_0 > 0$ and $k_1 \in \mathbb{R}^1$ such that

$$g(x,t,\xi)\xi \geq k_0 |\xi|^2 - k_1$$

for a.e. $(x,t) \in Q_T$, for any $\xi \in \mathbb{R}^1$ and there exists a number $a_0 > 0$ such that $a(x',t) \geq -a_0$ for a.e. $(x',t) \in \Sigma_T$ and $a_0 < \frac{\min\{1,\, k_0\}}{(c_4)^2}$ (here c_4 is the constant of Sobolev's Imbedding inequality [§] [1]).

Theorem 4.1. *Let conditions (1)$''$, (2), (4) be fulfilled for* $\alpha = 1$. *Then problem (1.1)-(1.3) is solvable in* P_0 *for any* $(h,\varphi) \in L_2(0,T;(W_2^1(\Omega))^*) \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$.

Proof. To prove this theorem we again make use of Theorem 3.2. We define corresponding mappings as (3.1), (3.2), (3.3).

Lemma 4.1. f *is bounded and weakly continuous from* P_0 *to* $L_2(0,T;(W_2^1(\Omega))^*)$, *under the assumptions of Theorem 4.1.*

Proof. It is enough to show that $g : P_0 \subset L_2(0,T;L_{p_0}(\Omega)) \longrightarrow L_2(0,T;L_{q_0}(\Omega))$ is a bounded mapping for $\alpha = 1$:

Using condition **(1)$''$** we obtain,

$$\begin{aligned} &\|g\|_{L_2(0,T;L_{q_0}(\Omega))} \leq \gamma(\|u\|_{L_2(0,T;L_{p_0}(\Omega))}), \\ &\gamma(\|u\|_{L_2(0,T;L_{p_0}(\Omega))}) = \widetilde{c}[\|c_1\|^2_{L_\infty(0,T;L_{\frac{n}{2}}(\Omega))}\|u\|^2_{L_2(0,T;L_{p_0}(\Omega))} \\ &\qquad + \|c_0\|^2_{L_2(0,T;L_{q_0}(\Omega))}]^{\frac{1}{2}}, \end{aligned}$$

$\widetilde{c} > 0$ is a constant. The rest of this proof is similar with the proof of Lemma 3.1. □

Lemma 4.2. *Conditions* $(ii), (iii), (iv)$ *of Theorem 3.2 are satisfied, under the assumptions of Theorem 4.1.*

[†] $c_2 \|u\|^2_{L_2(0,T;W_2^1(\Omega))} \leq (\|Du\|^2_{L_2(Q_T)} + \|u\|^2_{L_2(\Sigma_T)})$

[‡] $\|u\|_{L_2(0,T;L_{p_0}(\Omega))} \leq c_3 \|u\|_{L_2(0,T;W_2^1(\Omega))}$

[§] $\|u\|_{L_2(\Sigma_T)} \leq c_4 \|u\|_{L_2(0,T;W_2^1(\Omega))}$

Proof. This proof is similar with the proof of Lemma 3.2. As a different part, we show that f is coercive on $L_2(0,T;W_2^1(\Omega))$:

If we consider conditions **(1)**$''$ and **(4)-I**, we obtain,

$$\begin{aligned}
&\langle f(u),u\rangle_{Q_T} \geq \Psi(\|u\|_{L_2(0,T;W_2^1(\Omega))}),\\
&\Psi(\|u\|_{L_2(0,T;W_2^1(\Omega))}) := (\theta c_2 - (c_3)^2\varepsilon - (c_3)^2\|c_1\|_{L_\infty(0,T;L_{\frac{n}{2}}(\Omega))})\|u\|^2_{L_2(0,T;W_2^1(\Omega))}\\
&\qquad\qquad - K,
\end{aligned}$$

here $\theta := \min\{1,\ a_0\}$, $0 < \varepsilon < \frac{\theta c_2 - (c_3)^2\|c_1\|_{L_\infty(0,T;L_{\frac{n}{2}}(\Omega))}}{(c_3)^2}$ and $K > 0$ is a constant.

If we consider condition **(4)-II.**, we obtain,

$$\begin{aligned}
&\langle f(u),u\rangle_{Q_T} \geq \Psi(\|u\|_{L_2(0,T;W_2^1(\Omega))}),\\
&\Psi(\|u\|_{L_2(0,T;W_2^1(\Omega))}) := (\widetilde{\theta} - (c_4)^2 a_0)\|u\|^2_{L_2(0,T;W_2^1(\Omega))} - k_1,
\end{aligned}$$

here $\widetilde{\theta} := \min\{1,\ k_0\}$.

So, $\frac{\Psi(\|u\|)}{\|u\|} \nearrow \infty$ as $\|u\|_{L_2(0,T;W_2^1(\Omega))} \nearrow \infty$. □

Continuation of the Proof of Theorem 4.1. We can apply Theorem 3.2 to problem (1.1)-(1.3) by virtue Lemma 4.1 and Lemma 4.2. Hence we obtain that problem (1.1)-(1.3) is solvable in P_0 for any $(h,\varphi) \in L_2(0,T;(W_2^1(\Omega))^*) \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega)$. □

5. Solvability of problem (1.1)-(1.3) in super linear case

Let $\alpha > 1$. In this case,

$$P_0 := L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T) \cap W_2^1(0,T;(W_2^1(\Omega))^*) \cap \{u : u(x,0) = 0\}.$$

We consider the following conditions:

- **(1)**$'''$ Condition **(1)** is satisfied with a positive function c_1, a nonnegative function c_0 and parameters: $s_1 := \infty$, $r_1 := \infty$, $s_2 := \frac{\alpha+1}{\alpha}$, $r_2 := \frac{\alpha+1}{\alpha}$.
- **(5)** There exist some numbers $k_0 > 0$ and $k_1 \in \mathbb{R}^1$ such that
$$g(x,t,\xi)\xi \geq k_0\,|\xi|^{\alpha+1} - k_1$$
for a.e. $(x,t) \in Q_T$, for any $\xi \in \mathbb{R}^1$.
- **(6)** There exists a number $a_0 > 0$ such that $a(x',t) \geq -a_0$ for a.e. $(x',t) \in \Sigma_T$ and $a_0 < \frac{\min\{1,\ k_0\}}{(c_4)^2}$ (here c_4 is the constant of Sobolev's Imbedding inequality¶ [1]).

Theorem 5.1. *Let conditions (1)$'''$, (2), (5), (6) be fulfilled for $\alpha > 1$. Then problem (1.1)-(1.3) is solvable in P_0 for any $(h,\varphi) \in [L_2(0,T;(W_2^1(\Omega))^*)+L_{\frac{\alpha+1}{\alpha}}(Q_T)]$ $\times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$.*

¶ $\|u\|_{L_2(\Sigma_T)} \leq c_4\,\|u\|_{L_2(0,T;W_2^1(\Omega))}$

Proof. To prove this theorem we again make use of Theorem 3.2. We define corresponding mappings as (3.1), (3.2), (3.3). $\square$

Lemma 5.1. *f is bounded and weakly continuous from P_0 to $L_2(0,T;(W_2^1(\Omega))^*)+L_{\frac{\alpha+1}{\alpha}}(Q_T)$, under the assumptions of Theorem 5.1.*

Proof. It is enough to see that mapping g is bounded and weakly continuous from P_0 to $L_2(0,T;(W_2^1(\Omega))^*)+L_{\frac{\alpha+1}{\alpha}}(Q_T)$. Using condition $\mathbf{(1)}'''$, we obtain that

$$\|g\|_{L_{\frac{\alpha+1}{\alpha}}(Q_T)} \leq \gamma(\|u\|_{L_{\alpha+1}(Q_T)}),$$
$$\gamma(\|u\|_{L_{\alpha+1}(Q_T)}) = c[\|c_1\|_{L_\infty(Q_T)}\|u\|^{\alpha+1}_{L_{\alpha+1}(Q_T)} + \|c_0\|^{\frac{\alpha+1}{\alpha}}_{L_{\frac{\alpha+1}{\alpha}}(Q_T)}]^{\frac{\alpha}{\alpha+1}},$$

$c>0$ is a constant. So, g is a bounded mapping from P_0 to $L_{\frac{\alpha+1}{\alpha}}(Q_T)$, since $P_0 \subset L_{\alpha+1}(Q_T)$.

Let $\{u_m\} \subset P_0$ and $u_m \rightharpoonup u_0$ in P_0. Then $u_m \rightharpoonup u_0$ in $L_{\alpha+1}(Q_T)$. Since $L_2(0,T;W_2^1(\Omega)) \cap W^1_{\frac{\alpha+1}{\alpha}}(0,T;(W_2^1(\Omega))^* + L_{\frac{\alpha+1}{\alpha}}(\Omega)) \circlearrowleft L_2(Q_T)$, $\exists \{u_{m_l}\} \subset \{u_m\}$ such that $u_{m_l} \longrightarrow u_0$ almost everywhere in Q_T. Using condition $\mathbf{(1)}'''$ we can say that

$$g(x,t,\centerdot):\mathbb{R}_1 \longrightarrow \mathbb{R}_1$$

is a continuous function and we obtained that g is bounded. Then according to a general result (1. Chapter, 1. Paragraph, Lemma 1.3 of [4]), $\exists \{u_{m_j}\} \subset \{u_m\}$ such that

$$g(x,t,u_{m_j}) \underset{L_{\frac{\alpha+1}{\alpha}}(Q_T)}{\rightharpoonup} g(x,t,u_0).$$

This means g is a weakly continuous mapping from P_0 to $L_2(0,T;(W_2^1(\Omega))^*)+L_{\frac{\alpha+1}{\alpha}}(Q_T)$.

Lemma 5.2. *Conditions $(ii),(iii),(iv)$ of Theorem 3.2 are satisfied, under the assumptions of Theorem 5.1.*

Proof. Since A is an identity mapping, it is obvious that condition (ii) is satisfied. Furthermore, for any $u \in W_2^1(0,T;W_2^1(\Omega)) \cap W^1_{\alpha+1}(0,T;L_{\alpha+1}(\Omega))$ the following inequalities are satisfied:

$$\int_0^T \langle u,u\rangle_\Omega \, dt = \int_0^T \|u\|^2_{L_2(\Omega)} dt \geq c_6\|u\|^2_{L_2(0,T;(W_2^1(\Omega))^*)+L_{\frac{\alpha+1}{\alpha}}(Q_T)},$$
$$\int_0^t \left\langle \frac{\partial u}{\partial \tau}, u\right\rangle_\Omega d\tau = \frac{1}{2}\|u\|^2_{L_2(\Omega)}(t) \geq \frac{1}{2}c_6\|u\|^2_{(W_2^1(\Omega))^*}(t),$$

a.e. $t \in [0,T]$ ($c_6>0$ is the constant coming from Sobolev's Imbedding Inequality[‖] [1])

This means condition (iv) is also satisfied.

It is enough to see that mapping f is coercive on $L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T)$ for condition (iii), since A is an identity mapping:

[‖] $c_6\,\|u\|^2_{(W_2^1(\Omega))^*} \leq \|u\|^2_{L_2(\Omega)}$

If we consider conditions **(5)**and **(6)** we obtain,

$$\langle f(u), u\rangle_{Q_T} \geq \Psi(\|u\|_{L_2(0,T;W_2^1(\Omega))\cap L_{\alpha+1}(Q_T)}),$$

$\Psi(\|u\|_{L_2(0,T;W_2^1(\Omega))\cap L_{\alpha+1}(Q_T)}) := \frac{1}{4}(\widetilde{\theta}-(c_4)^2 a_0)\|u\|^2_{L_2(0,T;W_2^1(\Omega))\cap L_{\alpha+1}(Q_T)} - K$, here $\widetilde{\theta} := \min\{1,\ k_0\}$ and $K > 0$ is a constant.

So, $\frac{\Psi(\|u\|)}{\|u\|} \nearrow \infty$ as $\|u\|_{L_2(0,T;W_2^1(\Omega))\cap L_{\alpha+1}(Q_T)} \nearrow \infty$. □

Continuation of the Proof of Theorem 5.1. We can apply Theorem 3.2 to problem (1.1)-(1.3) from Lemma 5.1 and Lemma 5.2. Hence we obtain that problem (1.1)-(1.3) is solvable in P_0 for any $(h,\varphi) \in [L_2(0,T;(W_2^1(\Omega))^*) + L_{\frac{\alpha+1}{\alpha}}(Q_T)] \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$ satisfying the following inequality

$$\sup\left\{\frac{1}{\|u\|_{L_2(0,T;W_2^1(\Omega))\cap L_{\alpha+1}(Q_T)}}\int_0^T \langle h,u\rangle_\Omega \right.$$
$$\left. + \langle\varphi,u\rangle_{\partial\Omega}\,dt : u \in L_2(0,T;W_2^1(\Omega)) \cap L_{\alpha+1}(Q_T)\right\} < \infty.$$

If we consider the norm definition of (h,φ) in $[L_2(0,T;(W_2^1(\Omega))^*)+L_{\frac{\alpha+1}{\alpha}}(Q_T)]\times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$, we see that problem (1.1)-(1.3) is solvable in P_0 for any $(h,\varphi) \in [L_2(0,T;(W_2^1(\Omega))^*) + L_{\frac{\alpha+1}{\alpha}}(Q_T)] \times L_2(0,T;W_2^{-\frac{1}{2}}(\partial\Omega))$. □

6. Uniqueness theorem for a model case of problem (1.1)-(1.3)

In this section for problem (1.1)-(1.3), we define mapping g as

$$g(x,t,u) := d(x,t)\,|u|^{\rho-1}\,u + b(x,t)u,\ \ \rho > 0. \tag{6.1}$$

Theorem 6.1. *Let (6.1) and the following conditions be fulfilled for problem (1.1)-(1.3):*

(U_1)

$$d \in \begin{cases} L_\infty(Q_T), & \rho > 1, \\ L_\infty(0,T;L_{\frac{n}{2}}(\Omega)), & \rho = 1, \\ L_{\frac{2}{1-\rho}}(0,T;L_{\frac{p_0}{p_0-\rho-1}}(\Omega)), & \rho < 1. \end{cases}$$

and $d(x,t) \geq 0$ for a.e. $(x,t) \in Q_T$.

(U_2) $a \in L_\infty(0,T;L_{n-1}(\partial\Omega))$ and $b \in L_\infty(0,T;L_{\frac{n}{2}}(\Omega))$ satisfy one of the following conditions:

a. *If there exists a number $a_0 > 0$ such that $a(x',t) \geq a_0$ for a.e. $(x',t) \in \Sigma_T$, then there exists a number $b_0 > 0$ such that*

$$b(x,t) \geq -b_0 \ \ \textit{for a.e.}(x,t) \in Q_T \ \textit{and}\ b_0 < \frac{\min\{1,\ a_0\}\,c_2}{(c_7)^2},$$

(here c_2 is the constant coming from the inequality[**] *[12] and c_7 is the constant of Sobolev's Imbedding inequality*[††] *[1]).*

b. *If there exists a number $b_0 > 0$ such that $b(x,t) \geq b_0$ for a.e. $(x,t) \in Q_T$, then there exists a number $a_0 > 0$ such that*

$$a(x',t) \geq -a_0 \quad \textit{for a.e.}(x',t) \in \Sigma_T \ \textit{ and } a_0 < \frac{\min\{1,\ b_0\}}{(c_4)^2},$$

(here c_4 is the constant of Sobolev's Imbedding inequality[‡‡] *[1]).*

Then the solution of problem (1.1)-(1.3) is unique if it exists in

$$P_1 := L_2(0,T;W_2^1(\Omega)) \cap L_{\rho+1}(Q_T) \cap W_2^1(0,T;(W_2^1(\Omega))^*) \cap \{u : u(x,0) = 0\},$$
$$q = q(\rho) > 1.$$

Proof. Let $u,\ v \in P_1$ be two different solutions of (1.1)-(1.3) (P_1 is defined according to number ρ). If we consider (3.1) and (3.2), we have

$$\begin{cases} f_1(u) - f_1(v) = 0, \\ f_2(u) - f_2(v) = 0. \end{cases}$$

Let $w := u - v$, then

$$\begin{aligned} 0 = &\int_0^T \int_\Omega \frac{\partial w}{\partial t} w dxdt + \int_0^T \int_\Omega Dw.Dwdxdt \\ &+ \int_0^T \int_\Omega d(x,t) \left[|u|^{\rho-1} u - |v|^{\rho-1} v \right] [u - v]\, dxdt \\ &+ \int_0^T \int_\Omega b(x,t) w^2 dxdt + \int_0^T \int_{\partial\Omega} a(x',t) w^2 dx'dt. \end{aligned}$$

If we use condition (U_1) and if we consider $\int_0^T \left\langle \frac{\partial w}{\partial t}, w \right\rangle_\Omega dt = \frac{1}{2} \|w\|^2_{L_2(\Omega)}(T) > 0$, we have

$$0 > \|Dw\|^2_{L_2(Q_T)} + \int_0^T \int_\Omega b(x,t) w^2 dxdt + \int_0^T \int_{\partial\Omega} a(x',t) w^2 dx'dt. \tag{6.2}$$

Now if we consider condition (U_2) for inequality (6.2), we obtain contradiction of $0 > 0$.

Hence, the solution of problem (1.1)-(1.3) is unique if it exists. □

Corollary 6.1. *If g satisfies condition **(1)**$'$ for sublinear case, conditions **(1)**$''$, **(4)** for linear case and conditions **(1)**$'''$, **(5)** for super linear case, then the solution of (1.1)-(1.3) exists and it is unique.*

[**] $c_2 \|u\|^2_{L_2(0,T;W_2^1(\Omega))} \leq (\|Du\|^2_{L_2(Q_T)} + \|u\|^2_{L_2(\Sigma_T)})$

[††] $\|u\|_{L_2(Q_T)} \leq c_7 \|u\|_{L_2(0,T;W_2^1(\Omega))}$

[‡‡] $\|u\|_{L_2(\Sigma_T)} \leq c_4 \|u\|_{L_2(0,T;W_2^1(\Omega))}$

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