Applied Mathematics and Computation 487 (2025) 129094

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Full Length Article

A global approximation method for second-kind nonlinear integral equations

Luisa Fermo [a], Anna Lucia Laguardia [b], Concetta Laurita [b], Maria Grazia Russo [c,*]

[a] Department of Mathematics and Computer Science, University of Cagliari, Via Ospedale 72, Cagliari, 09124, Italy
[b] Department of Basic and Applied Sciences, University of Basilicata, Viale dell'Ateneo Lucano 10, Potenza, 85100, Italy
[c] Department of Engeneering, University of Basilicata, Viale dell'Ateneo Lucano 10, Potenza, 85100, Italy

ARTICLE INFO

MSC:
65R20
45G10
47H30

Keywords:
Nonlinear integral equations
Hammerstein-type integral equations
Nyström method
Boundary integral equations

ABSTRACT

A global approximation method of Nyström type is explored for the numerical solution of a class of nonlinear integral equations of the second kind. The cases of smooth and weakly singular kernels are both considered. In the first occurrence, the method uses a Gauss-Legendre rule whereas in the second one resorts to a product rule based on Legendre nodes. Stability and convergence are proved in functional spaces equipped with the uniform norm and several numerical tests are given to show the good performance of the proposed method. An application to the interior Neumann problem for the Laplace equation with nonlinear boundary conditions is also considered.

1. Introduction

The goal of this paper is to develop a numerical method for the following integral equation

$$f(y)-\int_{-1}^{1} k_1(x,y)f(x)dx-\int_{-1}^{1} k_2(x,y)h(x,f(x))dx=g(y), \quad y\in[-1,1], \tag{1}$$

where f is to be determined, k_1, k_2 and g are given functions, and $h(x,v)$ is a known function which is assumed to be nonlinear in v.

Integral equations of type (1) have wide applications in models involving nonlinearities such as heat radiation, heat transfer, acoustics, elasticity, and electromagnetic problems; see [1,2]. Some of these models are mathematically represented in terms of boundary value problems having nonlinear boundary conditions which can be reformulated in terms of (1); see [3] and Section 6. In applicative contexts, the kernels k_1 and k_2 of (1) are smooth and/or weakly singular. For instance, in the interior Neumann problem for Laplace's equation the kernel k_1 is smooth whereas k_2 is a combination of a smooth function and a logarithmic kernel, i.e.

$$k_2(x,y)=\rho(x,y)+\psi(x)\log|x-y|, \tag{2}$$

with ρ and ψ smooth functions.

* Corresponding author.
E-mail addresses: fermo@unica.it (L. Fermo), annalucia.laguardia@unibas.it (A.L. Laguardia), concetta.laurita@unibas.it (C. Laurita), mariagrazia.russo@unibas.it (M.G. Russo).

https://doi.org/10.1016/j.amc.2024.129094
Received 24 July 2024; Received in revised form 20 September 2024; Accepted 27 September 2024

Available online 9 October 2024

Motivated by these applications, in this paper we treat (1) when the kernels are continuous functions and/or weakly singular at the bisector as, for example, $|x-y|^{\nu}$, $\nu > -1$ and $\log|x-y|$. Without losing the generality, we first consider the case when k_1 and k_2 are both smooth and the case when k_1 is smooth and k_2 is weakly singular. However, our approach can be also applied in other "mixed" situations, as, for instance, (2).

Let us note that if $k_1(x,y) \equiv 0$, then equation (1) is the classical nonlinear Hammerstein equation

$$f(y) - \int_{-1}^{1} k_2(x,y) h(x, f(x)) dx = g(y), \qquad y \in [-1,1],$$

which is one of the most frequently investigated nonlinear integral equations, since it occurs in applications in numerous areas. Several problems written in terms of ordinary and partial differential equations can be transformed into equations of Hammerstein type through Green's function [4–6]. An example is the following differential problem of the second order which describes the forced oscillations of finite amplitude of a pendulum [7]

$$\begin{cases} F_{yy}(y) + a^2 \sin F(y) = G(y), & y \in [0,1] \\ F(0) = F(1) = 0, \end{cases} \tag{3}$$

where F denotes the amplitude of oscillation, the constant $a \neq 0$ depends on the length of the pendulum and on the gravity, and the driving force G is periodic and odd. Problem (3) is equivalent to this nonlinear integral equation

$$F(y) + \int_{0}^{1} k(x,y) \left[G(x) - a^2 \sin(F(x)) \right] dx = 0,$$

with $k(x,y)$ the triangular function defined as

$$k(x,y) = \begin{cases} x(1-y), & x \in [0,y] \\ y(1-x), & x \in [y,1] \end{cases}.$$

A further example is the well-known Chandrasekhar H-equation

$$H(y) - cH(y) \int_{0}^{1} \frac{xs(x)}{y+x} H(x) dx = 1, \qquad c \in \mathbb{C},$$

where s is a given function and H is the unknown. It models various physical problems such as the radioactive transfer and the kinetic of gases and, setting $f(y) = [H(y)]^{-1}$, it can be written as an Hammerstein equation [4] in the unknown $f(y)$.
Other contexts of application are the network theory, optimal control systems and automation [8,9].

Many interesting papers on the approximation of the solution of Hammerstein equations have appeared in the last few years. The survey [4] provides a complete overview of methods that can be also applied to other kind of nonlinear integral equations. Detailed examples of existing methods are collocation methods [10,11], degenerate kernel methods [12], discrete Legendre spectral methods [13] also for weakly singular equations [14,15], and the more recent numerical techniques based on spline quasi-interpolation [16,17] and Gaussian spline rules [18].

In this paper, first we determine the functional spaces which the solution of the equation belongs to, and study the mapping properties of the involved integral operators by using suitable approximation tools. Then, we propose Nyström type methods based on the polynomial approximation. Although this approach has been widely applied to linear Fredholm integral equations of the second kind (see, for instance, [19–22]), this is the first time that it is developed for nonlinear second-kind equations, according to our knowledge. The Nyström method is based on a discretization of the integral operators which involves the Gauss-Legendre rule if the kernel is smooth or a suitable product rule, based on the Legendre nodes, if the kernel is weakly singular. Following [23], we prove the stability of the method in spaces equipped with the uniform norm and we provide new estimates of the error, deduced also thanks to the recent results given in [24]. Specifically, under suitable assumptions on the known functions, we prove that the rate of convergence of the method is comparable with the error of best polynomial approximation in the functional spaces where the solution lives.

We conclude the paper by applying the proposed method to the numerical solution of the interior Neumann problem for the Laplace equation having nonlinear boundary conditions, following an approach which has been already shown in [25,3].

The paper is structured as follows. Section 2 details the function spaces in which equation (1) is considered and provides some basic results concerning the error of best polynomial approximation. Section 3 focuses on the mapping properties of the involved integral operators and on the solvability of equation (1). Section 4 and Section 5 deal with the Nyström methods we propose when the kernels are smooth or weakly singular, respectively. In both situations, a theoretical study is provided together with some numerical experiments that show the performance of the method. Section 6 concerns an application of the described numerical approach to the Laplace equation with nonlinear Neumann boundary conditions. Section 7 contains the proofs of our results.

2. Function spaces and best polynomial approximation

Let us denote by $C \equiv C([-1,1])$ the Banach space of continuous functions on $[-1,1]$ with the uniform norm

$$\|f\|_C = \|f\|_\infty = \max_{x\in[-1,1]} |f(x)|,$$

and let us introduce the Sobolev–type space of order $1 \le r \in \mathbb{N}$

$$W^r = \left\{ f \in C : f^{(r-1)} \in AC(-1,1),\ \|f^{(r)}\varphi^r\|_\infty < \infty \right\},$$

where $\varphi(x) = \sqrt{1-x^2}$ and $AC(-1,1)$ denotes the set of all absolutely continuous functions on $(-1,1)$. We equip W^r with the norm

$$\|f\|_{W^r} = \|f\|_\infty + \|f^{(r)}\varphi^r\|_\infty.$$

The space $(W^r, \|\cdot\|_{W^r})$ is a Banach space.

For our aims, we also need to define the Sobolev space $\boldsymbol{W}^r(\mathcal{D})$ for bivariate functions $f : \mathcal{D} \to \mathbb{R}$, with $\mathcal{D}$ an open subset of $\mathbb{R}^2$. It is the set of all functions f in $\mathcal{D}$ such that for every 2-tuple of nonnegative integers $\ell = (\ell_1, \ell_2)$, with $|\ell| = \sum_{i=1}^{2} \ell_i \le r$, the mixed partial derivatives $D^\ell f = \frac{\partial^{\ell_1+\ell_2} f}{\partial x_1^{\ell_1} \partial x_2^{\ell_2}}$ exist and $\|D^\ell f\|_\infty < \infty$. We endow this space with the norm

$$\|f\|_{\boldsymbol{W}^r(\mathcal{D})} = \|f\|_\infty + \sum_{1\le|\ell|\le r} \|D^\ell f\|_\infty.$$

For functions of "intermediate" smoothness, we define the Zygmund space Z^λ, with $\lambda \in \mathbb{R}^+$, as follows

$$Z^\lambda = \left\{ f \in C \ : \ \sup_{t>0} \frac{\Omega_\varphi^k(f,t)}{t^\lambda} < \infty, \quad k \ge 1, k > \lambda \right\},$$

where the main part of the *φ-modulus of smoothness* $\Omega_\varphi^k(f,t)$ is defined as [26, p. 90]

$$\Omega_\varphi^k(f,t) = \sup_{0<\tau\le t} \max_{x\in I_{k\tau}} |\Delta_{\tau\varphi}^k f(x)|, \quad I_{k\tau} = [-1+(2k\tau)^2, 1-(2k\tau)^2], \tag{4}$$

with

$$\Delta_{\tau\varphi}^k f(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f\left(x + \frac{\tau\varphi(x)}{2}(k-2i) \right).$$

The space Z^λ is endowed with the norm

$$\|f\|_{Z^\lambda} = \|f\|_\infty + \sup_{t>0} \frac{\Omega_\varphi^k(f,t)}{t^\lambda},$$

and also $(Z^\lambda, \|\cdot\|_{Z_\lambda})$ is a Banach space.

From now on we will denote by C a generic positive constant that can be different in different formulas. Moreover will we write $C = C(a,b,...)$ to say that C is dependent on the parameters $a, b, ...$ and $C \ne C(a,b,...)$ to say that C is independent of them.

Denoting by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m, for functions $f \in C$, let us now define the error of best polynomial approximation as

$$E_m(f) = \inf_{P_m\in\mathbb{P}_m} \|f - P_m\|_\infty.$$

It is well known that (see, for instance, [26] and the references therein)

$$f \in C \iff \lim_{m\to\infty} E_m(f) = 0,$$

Moreover, it is known that the behavior of the best approximation error is strictly related to the smoothness of the function f. Indeed in order to estimate $E_m(f)$, we can use, for instance, the following weak-Jackson inequality [27]

$$E_m(f) \le C \int_0^{\frac{1}{m}} \frac{\Omega_\varphi^k(f,t)}{t} dt, \quad \forall f \in C, \quad C \ne C(m,f). \tag{5}$$

On the other hand, it is also well known the so called Favard inequality, which says that

$$E_m(f) \le \frac{C}{m^r} \|f\|_{W^r}, \qquad \forall f \in W^r, \tag{6}$$

where $C \neq C(m, f)$. A stronger result is that both the Sobolev and Zygmund spaces introduced before, can be characterized in terms of the best polynomial approximation error. Indeed, from [27, Th. 4.2.1, p. 40] and [28, Co.2.2, p. 224] it immediately follows that

$$f \in W^r \iff E_m(f) = \mathcal{O}\left(\frac{1}{m^r}\right), \qquad \forall r \in \mathbb{N}, \tag{7}$$

while by [27, Th.8.2.1., p. 94] and (5) it follows

$$f \in Z^\lambda \iff E_m(f) = \mathcal{O}\left(\frac{1}{m^\lambda}\right), \qquad \forall \lambda \in \mathbb{R}^+ \setminus \mathbb{N}. \tag{8}$$

We conclude the section with a very recent result [24] which provides a characterization of the error of the best polynomial approximation of composite functions and that will be useful in the sequel.

Theorem 1. *Let $h : \mathcal{D} \to \mathbb{R}$, with $\mathcal{D}$ open subset of $\mathbb{R}^2$ and $\sigma : (-1,1) \to \mathbb{R}^2$ such that $Im(\sigma) \subseteq \mathcal{D}$. Assume that $h \in \mathbf{W}^r(\mathcal{D})$ and $\sigma(x) = (x, f(x))$ with $f \in W^r$, then*

$$E_m(h \circ \sigma) \leq C \left(\frac{2}{m}\right)^r B_r \, \|h\|_{\mathbf{W}^r(\mathcal{D})} \, \|f\|_{W^r}^s,$$

where $C = C(r)$ is a positive constant independent of h and f, B_r is the r-th Bell number, and the exponents s are defined as follows

$$s = \begin{cases} 0, & \text{if} \quad \|f\|_{W^r} \leq 1 \\ r, & \text{if} \quad \|f\|_{W^r} > 1 \end{cases}. \tag{9}$$

3. The solvability of equation (1)

The aim of this section is, firstly, to investigate the mapping properties of the operators involved in equation (1) and, then, to study its solvability in suitable subspaces of C.

We start with the case that both $k_1(x,y)$ and $k_2(x,y)$ are smooth functions.

Let us introduce the Fredholm operators

$$(K^i f)(y) = \int_{-1}^{1} k_i(x,y) f(x) dx, \qquad y \in [-1,1], \qquad i = 1,2, \tag{10}$$

and the so-called Nemytskii operator

$$(Hf)(x) = h(x, f(x)), \qquad x \in [-1,1]. \tag{11}$$

Then, equation (1) can be written as

$$(I - \mathcal{K}) f = g, \qquad \mathcal{K} = K^1 + K^2 H \tag{12}$$

where I is the identity operator.

It is well-known that if we assume the following hypothesis

[K1] The kernels $k_i(x,y)$, $i = 1,2$, are such that

$$\sup_{y \in [-1,1]} \int_{-1}^{1} |k_i(x,y)| dx < \infty,$$

$$\lim_{y \to \tilde{y}} \int_{-1}^{1} |k_i(x,y) - k_i(x,\tilde{y})| dx = 0, \quad \tilde{y} \in [-1,1];$$

then the linear Fredholm operators $K^i : C \to C$, $i = 1,2$, are compact, from which we can also deduce that they are completely continuous; see, for instance, [29].

Moreover, if we assume that

[H1] The function $h : [-1,1] \times \mathbb{R} \to \mathbb{R}$ is continuous;

[H2] The partial derivative $h_v(x,v) = \dfrac{\partial h(x,v)}{\partial v}$ exists and is continuous for $x \in [-1,1]$ and $v \in \mathbb{R}$;

then the Nemytskii operator $H : C \to C$ is well defined, bounded, and continuous because of the hypothesis **[H1]** (see, for instance, [30]) and is continuously Fréchet differentiable on the space C thanks to **[H2]** (see, for instance, [11, Lemma 4]).

Its Fréchet derivative, at $f \in C$, is given by the multiplicative linear operator defined as

$$[(H'f)\phi](x) = h_v(x, f(x))\phi(x), \qquad \forall \phi \in C, \quad x \in [-1,1].$$

For our aims, it is also useful to introduce the linear operator

$$(Gf)(y) = (K^2 f)(y) + g(y), \qquad y \in [-1,1].$$

Let us note that if the further condition

[G1] The right-hand side g is continuous in $[-1,1]$;

is fulfilled, the operator G inherits the same properties as K^2. Consequently, the composite operator $GH : C \to C$ is completely continuous.

Let us now rewrite equation (1) (or equivalently (12)) as the following fixed point problem

$$f(y) = (\mathcal{G}f)(y), \qquad (\mathcal{G}f)(y) = (K^1 f)(y) + (GHf)(y) = (\mathcal{K}f)(y) + g(y). \tag{13}$$

Under the assumptions **[K1]**, **[H1]**, **[H2]**, and **[G1]** the operator $\mathcal{G}$ is continuously Fréchet differentiable on the space C. For each $f \in C$, its derivative is given by

$$[(f)\phi](y) = [(K^1 + (GH)'f)\phi](y) = (K^1\phi)(y) + [G(H'f)\phi](y), \qquad \forall \phi \in C, \quad y \in [-1,1],$$

that is

$$[(\mathcal{G}'f)\phi](y) = \int_{-1}^{1} k_1(x,y)\phi(x)dx + \int_{-1}^{1} k_2(x,y)h_v(x,f(x))\phi(x)dx. \tag{14}$$

Moreover, if

[H3] The partial derivative $h_{vv}(x,v) = \dfrac{\partial^2 h(x,v)}{\partial^2 v}$ exists and is continuous for $x \in [-1,1]$ and $v \in \mathbb{R}$;

then the operator $\mathcal{G}$ also admits the second Fréchet derivative at $f \in C$, defined as [23]

$$[(\mathcal{G}''f)(\phi_1,\phi_2)](y) = \int_{-1}^{1} k_2(x,y)h_{vv}(x,f(x))\phi_1(x)\phi_2(x)dx, \quad \phi_1,\phi_2 \in C. \tag{15}$$

Remark 1. Let us observe that, if $h \in \mathbf{W}^r(\mathcal{D})$ where $\mathcal{D}$ is an open subset of $\mathbb{R}^2$ and $r \geq 2$, then the assumptions **[H1]**, **[H2]**, and **[H3]** are satisfied.

The analysis of the operator $\mathcal{G}$ is fundamental to state the existence of solutions of (13). Let us recall that a solution f^* of (13) is *geometrically isolated* if there exists a ball

$$B(f^*,\delta) = \{f \in C : \|f - f^*\|_\infty \leq \delta\},$$

for some $\delta > 0$ that does not contain any solution of (13) other than f^*.

Let us also remind that the *index* of a geometrically isolated solution f^* is the common value of the *rotation* of the vector field $I - K^1 + GH$ over all sufficiently small spheres centered at f^*; see, for instance, [31, p. 100] and [11, Section 2].

Next theorem, which can be deduced by the more general result [31, Theorem 21.6, p. 108], establishes the existence and the uniqueness of a geometrically isolated solution f^* of (13) with nonzero index.

Theorem 2. *Assume that the operator $\mathcal{G} : C \to C$ defined in (13) is completely continuous. Let f^* be such that $I - \mathcal{G}'f^*$ is invertible, where $\mathcal{G}'f^*$ is the Fréchet derivative of $\mathcal{G}f$ at the point f^*. Then f^* is a fixed point of $\mathcal{G}$. Moreover, assume that 1 is not an eigenvalue of $\mathcal{G}'f^*$. Then f^* is the unique nonzero index geometrically isolated solution of equation (13) in C.*

About the smoothness of the solution we can state the following result.

Theorem 3. *Let $\mathcal{K} = K^1 + K^2 H$ where K^1, K^2, and H are defined in (10) and (11), respectively. Under the assumptions of Theorem 2 if*

$$\sup_{x\in[-1,1]} \|k_i(x,\cdot)\|_{W^r} < \infty, \qquad i = 1,2, \tag{16}$$

then $\mathcal{K}f \in W^r$ for all $f \in C$. Consequently, if also $g \in W^r$ the solution f^* of (13) belongs to W^r.

Now, we consider the case when equation (1) presents a nonlinear operator having a symmetric weakly singular kernel at the bisector, i.e.

$$f(y) - \int_{-1}^{1} k_1(x,y) f(x) dx - \int_{-1}^{1} k_2(x,y) h(x, f(x)) dx = g(y), \qquad y \in [-1,1], \tag{17}$$

where

$$k_2(x,y) = \psi(x) k^*(|x-y|)$$

with ψ a smooth function on $[-1,1]$. Typical examples of the kernel $k^*(|x-y|)$ are

$$k^*(|x-y|) = |x-y|^\mu, \quad \mu > -1, \qquad k^*(|x-y|) = \log|x-y|. \tag{18}$$

About the solvability of this equation we have the following result.

Theorem 4. *Assume that* ***[H1]****,* ***[H2]****, and* ***[G1]*** *holds true. If*

[H4]

$$\max_{y\in[-1,1]} \left(\int_{-1}^{1} |k_1(x,y)| dx + \max_{x\in[-1,1]} \|h_v(x,\cdot)\|_\infty \int_{-1}^{1} |\psi(x) k^*(|x-y|)| dx \right) < 1$$

then equation (17) has a unique solution $f^* \in C$.

Assumption **[H4]** is quite restrictive. Nevertheless in the case (18) it is possible to show that **[K1]** is still satisfied (see [29]) and therefore Theorem 2 is still true for this special, but frequent, case.

Moreover in this special case, we can also state a result about the smoothness of the solution of the equation (17).

In what follows we adopt the notation

$$(\widetilde{K}^\mu f)(y) = \begin{cases} \displaystyle\int_{-1}^{1} \psi(x) |x-y|^\mu f(x) dx, & \mu > -1, \quad \mu \neq 0, \\ \displaystyle\int_{-1}^{1} \psi(x) \log|x-y| f(x) dx, & \mu = 0 \end{cases}. \tag{19}$$

Theorem 5. *Let $\mathcal{K} = K^1 + \widetilde{K}^\mu H$, $\mu > -1$, where K^1, H, and $\widetilde{K}^\mu$ are defined in (10), (11), and (19), respectively. Assume that equation (17) has a unique solution $f^* \in C$. If* ***[H1]*** *holds, $\psi \in C$, and*

$$\sup_{x\in[-1,1]} \|k_1(x,\cdot)\|_{Z^\lambda} < \infty, \quad \lambda > 0, \tag{20}$$

then, for all $f \in C$, $\mathcal{K} f \in Z^s$ where $s = \min\{\lambda, \mu+1\}$, when $\mu \neq 0$, while $s = \min\{\lambda, 1-\epsilon\}$, with $\epsilon > 0$ arbitrarily small, when $\mu = 0$. Consequently, if also $g \in Z^s$ the solution f^* of the equation $(I - \mathcal{K})f = g$ belongs to Z^s.

Remark 2. In the literature some estimates of the smoothness of the solution of Hammerstein integral equations with a weakly singular integral of the type (19) are known (see, for instance, [32] and [33]). Here, for a more general equation, we give minimal smoothness assumptions on the known functions, in order to determine the space which the solution of the equation belongs to.

4. The Nyström method for the case of continuous kernels

We introduce, now, a numerical method of Nyström type based on the Gauss-Legendre rule

$$\int_{-1}^{1} f(x) dx = \sum_{k=1}^{m} \lambda_k f(x_k) + e_m(f), \tag{21}$$

where λ_k is the kth Christoffel number, x_k is the kth zero of the orthonormal Legendre polynomial $p_m(x)$ of degree m, and $e_m(f)$ is the remainder term. Let us recall that [26, Theorem 5.1.6]

$$|e_m(f)| \le 4\, E_{2m-1}(f), \qquad \forall f \in C. \tag{22}$$

First, let us approximate the Fredholm integral operators (10) by defining the discrete operator K_m^i as follows

$$(K_m^i f)(y) = \sum_{k=1}^{m} \lambda_k k_i(x_k, y) f(x_k), \qquad i = 1, 2. \tag{23}$$

Now, consider the equation

$$(I - \mathcal{K}_m) f_m = g, \qquad \mathcal{K}_m = K_m^1 + K_m^2 H \tag{24}$$

i.e.

$$f_m = \mathcal{G}_m f_m, \qquad \text{with} \quad \mathcal{G}_m f = \mathcal{K}_m f + g \tag{25}$$

where f_m is an unknown function, approximating the solution of equation (13).

The next theorem contains some useful properties of the operators $\mathcal{K}_m$ essential for obtaining the stability and convergence of the method. Here and in the following all the involved operators will be considered as maps of C into itself.

Theorem 6. *Let $\mathcal{K}$ and $\mathcal{K}_m$ be the operators defined in (12) and (24), respectively, and let us assume that* [**K1**] *and* [**H1**] *are satisfied. Then, the sequence $\{\mathcal{K}_m\}_m$ is collectively compact and pointwise convergent to $\mathcal{K}$.*

Remark 3. Note that if, in addition to [**K1**] and [**H1**], the assumption [**G1**] is satisfied, from Theorem 6 we can deduce that the sequence $\{\mathcal{G}_m\}_m$ is collectively compact and pointwise convergent to $\mathcal{G}$.

Now, in order to compute the unknown solution f_m of equation (24), which has the explicit form

$$f_m(y) - \sum_{k=1}^{m} \lambda_k k_1(x_k, y) f_m(x_k) - \sum_{k=1}^{m} \lambda_k k_2(x_k, y) h(x_k, f_m(x_k)) = g(y),$$

let us collocate it at the points x_i for $i = 1, \dots, m$. In this way, we obtain the following nonlinear system of m equations in the m unknowns $a_i = f_m(x_i)$, $i = 1, \dots, m$,

$$\sum_{k=1}^{m} [\delta_{i,k} - \lambda_k k_1(x_k, x_i)] a_k - \sum_{k=1}^{m} \lambda_k k_2(x_k, x_i) h(x_k, a_k) = g(x_i), \quad i = 1, \dots, m, \tag{26}$$

where $\delta_{i,k}$ is the Kronecker symbol.

The solution $(a_1^*, \dots, a_m^*)$ of system (26), allows us to construct the Nyström interpolant

$$f_m(y) = \sum_{k=1}^{m} \lambda_k [k_1(x_k, y) a_k^* + k_2(x_k, y) h(x_k, a_k^*)] + g(y). \tag{27}$$

Theorem 7. *Assume* ***[K1], [H1], [G1],*** *and* ***[H2]*** *and* ***[H3]*** *in the ball $B(f^*, \delta)$, where f^* is a fixed point of the operator $\mathcal{G}$ defined in (13). Moreover, assume that 1 is not an eigenvalue of $\mathcal{G}' f^*$ where $\mathcal{G}' f^*$ is given in (14). Then, for m sufficiently large, say $m \geq m_0$, the operator $\mathcal{G}_m$ in (25) has a unique fixed point f_m in $B(f^*, \epsilon)$ with $0 < \epsilon \leq \delta$.*

Theorem 8. *Assume that the hypotheses of Theorem 7 are fulfilled. Let f^* be the unique fixed point of $\mathcal{G}$ in $B(f^*, \delta)$, for some $\delta > 0$. Assume that 1 is not an eigenvalue of $\mathcal{G}' f^*$ and, in addition, that for some $r \geq 1$,*

$$\sup_{x \in [-1,1]} \|k_i(x, \cdot)\|_{W^r} < \infty, \qquad \sup_{y \in [-1,1]} \|k_i(\cdot, y)\|_{W^r} < \infty, \qquad i = 1, 2 \tag{28}$$

$$g \in W^r, \qquad h \in \mathbf{W}^r(\mathcal{D}), \tag{29}$$

where $\mathcal{D}$ is an open subset of $\mathbb{R}^2$.

Then, for m sufficiently large (say $m \geq m_0$), denoted by f_m the unique fixed point of $\mathcal{G}_m$ in $B(f^*, \epsilon)$, with $0 < \epsilon \leq \delta$, we have

$$\|f^* - f_m\|_\infty = \mathcal{O}\left(\frac{1}{m^r}\right). \tag{30}$$

4.1. Numerical tests

In this section, we consider some numerical tests to confirm the effectiveness of our method.

In all the experiments, first we solve system (26) by using the classical Newton method or the Matlab routine `fsolve`. Then, we compute the Nyström interpolant f_m given in (27), and the relative discrete errors on a grid of 10^2 equidistant nodes y_i, $i = 1, \dots, 100$, in $[-1, 1]$, i.e.

Table 1
From left to right the numerical results for Example 1 and Example 2.

m	$\mathcal{E}_m$	iter
4	4.88e-08	6
8	4.90e-16	6

m	$\mathcal{E}_m$	iter
4	4.87e-03	5
8	2.32e-07	5
16	2.22e-16	5

Table 2
From left to right the numerical results for Example 3 and Example 4.

m	$\mathcal{E}_m$	iter	EOC_m
8	2.35e-04	4	3.85e+00
16	2.15e-05	4	3.45e+00
32	1.98e-06	4	3.44e+00
64	1.79e-07	4	3.47e+00
128	1.59e-08	4	3.49e+00
256	1.30e-09	4	3.61e+00

m	$\mathcal{E}_m$	iter	EOC_m
8	9.45e-04	18	7.96e+00
16	4.77e-05	65	4.31e+00
32	2.26e-06	21	4.40e+00
64	1.03e-07	20	4.45e+00
128	4.64e-09	20	4.48e+00
256	1.98e-10	20	4.55e+00

$$\mathcal{E}_m = \frac{\|f^* - f_m\|}{\|f^*\|}, \tag{31}$$

where $\|f\| = \max_{i=1,\dots,10^2} |f(y_i)|$ and f^* is the exact solution.

The first two examples, in which the exact solution is known, aim to make a comparison with other methods available in the literature. In the other two, the solution f^* is not known, and then we consider as exact the Nyström interpolant f_{512}. Moreover, in these specific tests, the known functions are not so smooth that we can show the efficacy of the method also in these cases.

In addition, we compute the estimated order of convergence

$$EOC_m = \frac{\log(\mathcal{E}_m/\mathcal{E}_{2m})}{\log 2}. \tag{32}$$

Example 1. Consider the Hammerstein equation

$$f(y) - \int_{-1}^{1} e^{y-2x}(f(x))^3 dx = e^{y-1}(e - e^2 + 1), \quad |y| \le 1,$$

whose exact solution is $f(x) = e^x$. Such equation has been considered in [18,34]. In [18] the machine precision is achieved by solving a nonlinear system of order 64, whereas in [34] the better convergence error is 10^{-10}. Table 1 shows that our method has a faster convergence. In fact, the machine precision is reached by solving a nonlinear system of $m = 8$ equations in 6 iterations. This is certainly due to the smoothness of the known functions which are analytic in $[-1, 1]$.

Example 2. Let us apply our method to the Hammerstein equation [35]

$$f(y) - \int_{-1}^{1} y\cos\left(\frac{\pi}{2}x\right)e^{f(x)}dx = \sin\left(\frac{\pi}{2}y\right) - \frac{4y}{\pi}\sinh(1) \quad |y| \le 1,$$

where the right-hand side term is fixed so that the exact solution is $f(x) = \sin\left(\frac{\pi}{2}y\right)$. Also in this case, the kernel and the right-hand side are analytic functions and then we expect a fast convergence. Looking at the errors (see Table 1) we can note that our method reaches the machine precision with $m = 16$. We remark that the best convergence error of the method presented in [35] is equal to 10^{-8}.

Example 3. In this test we consider the complete equation of the form (1)

$$f(y) - \int_{-1}^{1} y\cos x dx - \int_{-1}^{1} \frac{e^{x+y}\cos(x+1)}{x^2+5}\frac{dx}{1+(f(x))^2} = |y|^{\frac{5}{2}}, \quad |y| \le 1.$$

where kernels are both smooth and the right hand side term $g \in W_2$. Then, according to (30), the expected theoretical order of convergence is $\mathcal{O}(m^{-2})$. The numerical results reported in Table 2 (on the left) show a better convergence.

Example 4. Consider an equation in which the kernel of the nonlinear operator satisfies conditions (28) with $r = 3$

$$f(y) - \int_{-1}^{1} (x+y) f(x) dx - \int_{-1}^{1} |xy|^{\frac{7}{2}} (f(x))^3 dx = e^y + \log(3+y), \quad |y| \le 1.$$

Table 2 (on the right) shows, also in this case, that the numerical errors are better than the expected one, since the theoretical error is of the order $\mathcal{O}(m^{-3})$.

5. The Nyström method for the case of weakly singular kernels

Let us introduce a product rule which allows us to approximate integrals having kernels with weak singularity at the bisector of the following type

$$\mathcal{I}(f, y) = \int_{-1}^{1} \psi(x) k^*(|x-y|) f(x) dx.$$

The quadrature formula is given by

$$\mathcal{I}(f, y) = \sum_{k=1}^{m} \left[\lambda_k \sum_{i=0}^{m-1} p_i(x_k) M_i(y) \right] f(x_k) + r_m(f, y) = \mathcal{I}_m(f, y) + r_m(f, y), \tag{33}$$

where λ_k and x_k are the kth Christoffel number and the kth zero of the Legendre polynomial p_m, respectively, $M_i(y)$, $i = 0, 1, \dots, m-1$, are the so-called modified moments defined as

$$M_i(y) = \int_{-1}^{1} p_i(x) \psi(x) k^*(|x-y|)\, dx, \qquad i = 0, \dots, m-1, \tag{34}$$

and $r_m(f, y)$ is the quadrature error. From now on, we will denote the weights of rule (33) by

$$c_k(y) = \lambda_k \sum_{i=0}^{m-1} p_i(x_k) M_i(y), \qquad k = 1, \dots, m.$$

Next theorem provides the assumptions assuring the stability of the quadrature formula (33) and an estimate for the remainder term (see [26, Theorem 5.1.11] and [36]).

Theorem 9. *Let us assume that the kernel k^* and the function ψ satisfies*

$$\sup_{|y|\le 1} \int_{-1}^{1} \frac{|\psi(x) k^*(|x-y|)|}{\sqrt{\varphi(x)}} dx < \infty,$$

$$\sup_{|y|\le 1} \int_{-1}^{1} \psi(x) k^*(|x-y|) \left(1 + \log^+ \psi(x) k^*(|x-y|)\right) dx < \infty,$$

where $\log^+ \psi(x) k^*(|x-y|) = \log \max\{1, \psi(x) k^*(|x-y|)\}$. *Then, for each $f \in C$ one has*

$$\sup_m \sup_{|y|\le 1} |\mathcal{I}_m(f, y)| \le C \|f\|_\infty,$$

and

$$\sup_{|y|\le 1} |r_m(f, y)| \le C E_{m-1}(f), \tag{35}$$

with $C \ne C(m, f)$.

Now, let us describe the Nyström method for solving equation (17) which we also write as

$$(I - \mathcal{K}) f = g, \qquad \mathcal{K} = K^1 + K^2 H$$

where H is given in (11) and

$$(K^1 f)(y) = \int_{-1}^{1} k_1(x,y) f(x) dx, \qquad (K^2 f)(y) = \int_{-1}^{1} \psi(x) k^*(|x-y|) f(x) dx.$$

For this purpose, we introduce the operators

$$(K_m^1 f)(y) = \sum_{k=1}^{m} \lambda_k k_1(x_k, y) f(x_k), \qquad (K_m^* f)(y) = \sum_{k=1}^{m} c_k(y) f(x_k), \tag{36}$$

and consider the equation

$$(I - \mathcal{K}_m) f_m = g, \qquad \mathcal{K}_m = K_m^1 + K_m^* H,$$

i.e.

$$f_m = \mathcal{G}_m f_m, \quad \text{with} \quad \mathcal{G}_m f_m = (\mathcal{K}_m f_m) + g, \tag{37}$$

where f_m is an unknown function.

At this point, in order to compute the solution f_m of equation (37) which has the explicit form

$$f_m(y) - \sum_{k=1}^{m} \lambda_k k_1(x_k, y) f_m(x_k) - \sum_{k=1}^{m} c_k(y) h(x_k, f_m(x_k)) = g(y),$$

we collocate it at the points $x_i, i = 1, ..., m$, obtaining the nonlinear system

$$\sum_{k=1}^{m} [\delta_{ik} - \lambda_k k_1(x_k, x_i)] a_k - \sum_{k=1}^{m} c_k(x_i) h(x_k, a_k) = g(x_i), \quad i = 1, \dots, m, \tag{38}$$

in the m unknowns $a_i = f_m(x_i)$.
The solution $(a_1^*, \dots, a_m^*)$ allows us to construct the Nyström interpolant as follows

$$f_m(y) = \sum_{k=1}^{m} \lambda_k k_1(x_k, y) a_k^* + \sum_{k=1}^{m} c_k(y) h(x_k, a_k^*) + g(y).$$

The difficulty in applying this procedure is the construction of the entries of the matrix of the nonlinear system (38) and, in particular, the computation of the constants $c_k(y)$, $k = 1, \dots, m$, for fixed y. Indeed, by their definition, the crucial point is the computation of the modified moments (34), that can be carried out only for some special kernels k^*. Fortunately, for kernels of type (18) the modified moments can be exactly computed by using the well known recurrence relations for Legendre polynomials [37]. Moreover, as we have already underlined, in this case Theorem 2 still holds true, and therefore Theorem 7 follows also for equation (37), that means (37) is unisolvent.

Concerning the convergence, in the special case (18) we have the following result.

Theorem 10. *Assume the hypotheses of Theorem 7 and Theorem 9 are true. Let f^* be the unique fixed point of $\mathcal{G} = K^1 + \widetilde{K}^\mu H + g$ in $B(f^*, \delta)$, for some $\delta > 0$. Assume that 1 is not an eigenvalue of $\mathcal{G}' f^*$ and in addition for some $\lambda > 0$, $-1 < \mu \le 0$,*

$$\sup_{x \in [-1,1]} \|k_1(x, \cdot)\|_{Z^\lambda} < \infty, \qquad \sup_{y \in [-1,1]} \|k_1(\cdot, y)\|_{Z^\lambda} < \infty, \tag{39}$$

$$g \in Z^s, \quad \text{where } s = \begin{cases} \min\{\lambda, 1+\mu\}, & \mu \neq 0, \\ \min\{\lambda, 1-\epsilon\}, & \mu = 0, \ \epsilon > 0 \text{ arbitrarily small.} \end{cases}$$

Then, for m sufficiently large (say $m \ge m_0$), denoted by f_m the unique fixed point of $\mathcal{G}_m$ in $B(f^*, \epsilon)$, with $0 < \epsilon \le \delta$, we have

$$\|f^* - f_m\|_\infty = \mathcal{O}\left(\frac{1}{m^s}\right).$$

Remark 4. The assumptions of Theorem 10 assure that $f^* \in Z^s$, $0 < s < 1$. If $\mu > 0$ and $\lambda > 1$, then s could be greater than 1. In this case, we can apply Theorem 8 recalling that $f^* \in W^{[s]}$.

5.1. Numerical experiments

In this subsection we give three numerical experiments to show the performance of the method. As in the regular case, we consider the relative discrete errors as in (31). To solve the numerical system (38), we used the classical Newton method or the Matlab function `fsolve`.

Table 3
The numerical results for Example 7.

m	$\mathcal{E}_m$	iter	EOC_m
8	2.93e-03	6	2.01e+00
16	7.81e-04	6	1.91e+00
32	2.03e-04	6	1.94e+00
64	5.16e-05	6	1.98e+00
128	1.25e-05	6	2.05e+00
256	2.51e-06	6	2.31e+00

Example 5. Consider the equation proposed in [15]

$$f(y) - \int_0^1 |x-y|^{-\frac{1}{2}} (f(x))^2 dx = g(y), \quad y \in [0,1],$$

where $g(y) = [y(1-y)]^{\frac{1}{2}}] + \frac{16}{15} y^{\frac{5}{2}} + 2y^2(1-y)^{\frac{1}{2}} + \frac{4}{3} y(1-y)^{\frac{3}{2}} + \frac{2}{5}(1-y)^{\frac{5}{2}} - \frac{4}{3} y^{\frac{3}{2}} - 2y(1-y)^{\frac{1}{2}} - \frac{2}{3}(1-y)^{\frac{3}{2}}$ and the exact solution is $f(x) = [x(1-x)]^{\frac{1}{2}}$. The best convergence error in [15] is 10^{-4} whereas our method produces an error $\mathcal{E}_4 = 1.97e-14$. The numerical results definitely overcome the theoretical expectation but this is due to the fact that the function $h(x, f(x) = x(1-x)$ is a polynomial and the product rule is exact.

Example 6. Let us test the method to the following equation already considered in [15],

$$f(y) - \int_0^1 \log|x-y| \sin(\pi f(x)) dx = 1, \quad y \in [0,1],$$

where the exact solution is $f(x) = 1$. Transformed the equation into $[-1,1]$, it becomes

$$f\left(\frac{y+1}{2}\right) - \frac{1}{2}\int_{-1}^1 \log|y-x| \sin\left(\pi f\left(\frac{x+1}{2}\right)\right) dx - \frac{\log 2}{2}\int_{-1}^1 \sin\left(\pi f\left(\frac{x+1}{2}\right)\right) dx = 1, \qquad |y| \le 1,$$

The best convergence error in [15] is 10^{-4} whereas in our case we have $\mathcal{E}_4 = 6.66e-16$.

Example 7. Consider the equation

$$f(y) - \int_{-1}^1 x^2 y f(x) dx - \int_{-1}^1 |x-y|^{-1/2} \frac{1}{1+f(x)^2} dx = \sqrt{y+1}, \quad |y| \le 1.$$

In this case, we do not known the exact solution. For increasing values of m, Table 3 reports the relative errors exhibiting a better performance than the expected one according to the theoretical estimate, which is $\mathcal{O}(m^{-\frac{1}{2}})$, as also confirmed by the estimated order of convergence reported in the last column.

6. An application to Boundary Integral Equations

In this section, we show an application of the Nyström method described in Section 5 for the numerical solution of a nonlinear boundary integral equation (BIE) arising from the reformulation of a nonlinear boundary value problem (BVP) for Laplace's equation.

Let us consider the interior Neumann problem over a bounded simply connected planar domain $D \subset \mathbb{R}^2$ with smooth boundary Γ. It consists in finding a function $u \in C^2(D) \cap C^1(\overline{D})$ that satisfies

$$\begin{cases} \Delta u(P) = 0, & P \in D, \\ \dfrac{\partial u(P)}{\partial n_P} = -\bar{h}(P, u(P)) + \bar{g}(P), & P \in \Gamma, \end{cases} \tag{40}$$

where n_P denotes the exterior unit normal to Γ at the point P, while the function $\bar{h}(P, v)$ defined in $\Gamma \times \mathbb{R}$ is nonlinear in v and is assumed sufficiently smooth.

It is known that (see, for instance, [25]) the harmonic function u satisfying (40) is the solution of the following nonlinear BIE of the second kind

$$u(P)-\frac{1}{\pi}\int_{\Gamma}u(Q)\frac{\partial}{\partial n_Q}\left[\log|P-Q|\right]d\sigma(Q)-\frac{1}{\pi}\int_{\Gamma}\bar{h}(Q,u(Q))\log|P-Q|d\sigma(Q)$$
$$=\frac{1}{\pi}\int_{\Gamma}\bar{g}(Q)\log|P-Q|d\sigma(Q),\qquad P\in\Gamma \tag{41}$$

which can be deduced from Green's representation formula for u

$$u(P)=\frac{1}{2\pi}\int_{\Gamma}u(Q)\frac{\partial}{\partial n_Q}\left[\log|P-Q|\right]d\sigma(Q)-\frac{1}{2\pi}\int_{\Gamma}\frac{\partial u(Q)}{\partial n_Q}\log|P-Q|d\sigma(Q),\qquad P\in D, \tag{42}$$

taking into account the boundary condition in (40).
Once equation (41) has been solved, one can use the known function u on Γ along with its known normal derivative on the boundary given in (40) in order to compute the unknown solution u on the domain D using formula (42).

Hence, we are interested in the numerical solution of (41). In order to transform the BIE (41) into an equivalent 1D integral equation on the interval $[-1,1]$, firstly we introduce a parametric representation of the curve Γ

$$\boldsymbol{\gamma}(x)=(\xi(x),\eta(x))\in\Gamma,\qquad x\in[-1,1]. \tag{43}$$

We assume that γ traverses Γ in a counter-clockwise direction (i.e. it is such that the domain D is on the left of Γ) and $\xi,\eta\in C^2[-1,1]$, with

$$|\gamma'(x)|\neq 0,\qquad \forall x\in[-1,1].$$

Moreover, in order to achieve higher orders of convergence for the numerical method and, consequently, more accurate approximations of the solution, we adopt some already known regularization strategies (see, for instance, [38–42]) considering a smoothing transformation $\phi_q(x)$, such that

$$\phi_q(x)=\begin{cases}-1+(x+1)^q, & x\in[-1,-1+\epsilon],\\ 1-(1-x)^q, & x\in[1-\epsilon,1],\end{cases} \tag{44}$$

for some small $\epsilon>0$ and some smoothing exponent $q\geq 1$. Note that in the case $q=1$ we have $\phi_1(x)=x$, which means that no smoothing transformation is applied.

Then, by introducing in (41) the change of variables $x=\bar{\gamma}(x)$ and $y=\bar{\gamma}(y)$ with

$$\bar{\gamma}(x)=\gamma(\phi_q(x))=(\xi(\phi_q(x)),\eta(\phi_q(x)))=:(\bar{\xi}(x),\bar{\eta}(x))\qquad x\in[-1,1]. \tag{45}$$

we can rewrite the BIE as follows

$$u(\bar{\gamma}(y))-\int_{-1}^{1}k_1(x,y)u(\bar{\gamma}(x))dx-\frac{1}{\pi}\int_{-1}^{1}\bar{h}(\bar{\gamma}(x),u(\bar{\gamma}(x)))|\bar{\gamma}'(x)|\log|\bar{\gamma}(y)-\bar{\gamma}(x)|dx$$
$$=-\frac{1}{\pi}\int_{-1}^{1}\bar{g}(\bar{\gamma}(x))|\bar{\gamma}'(x)|\log|\bar{\gamma}(y)-\bar{\gamma}(x)|dx, \tag{46}$$

where

$$k_1(x,y)=\frac{1}{\pi}\begin{cases}\dfrac{\bar{\eta}'(x)[\bar{\xi}(x)-\bar{\xi}(y)]-\bar{\xi}'(x)[\bar{\eta}(x)-\bar{\eta}(y)]}{[\bar{\xi}(x)-\bar{\xi}(y)]^2+[\bar{\eta}(x)-\bar{\eta}(y)]^2}, & x\neq y,\\[2ex] \dfrac{1}{\pi}\dfrac{\bar{\xi}'(x)\bar{\eta}''(x)-\bar{\eta}'(x)\bar{\xi}''(x)}{2[\bar{\xi}'(x)^2+\bar{\eta}'(x)^2]}, & x=y.\end{cases}$$

Now, first we split the logarithmic kernel $\log|\bar{\gamma}(y)-\bar{\gamma}(x)|$ as follows

$$\log|\bar{\gamma}(y)-\bar{\gamma}(x)|=\log\frac{|\bar{\gamma}(y)-\bar{\gamma}(x)|}{|x-y|}+\log|x-y|. \tag{47}$$

Then, following a numerical trick in [43] (see, also, [44]) in order to avoid numerical cancellation, when $|x-y|<eps$ (eps denotes the machine precision) we use the approximation

$$\log\frac{|\bar{\gamma}(y)-\bar{\gamma}(x)|}{|x-y|}\simeq\log|\bar{\gamma}'(x)|.$$

Now, setting $f(x)=u(\bar{\gamma}(x))$, $h(x,f(x))=\bar{h}(\bar{\gamma}(x),u(\bar{\gamma}((x)),$

$$\rho(x,y)=\begin{cases}\frac{1}{\pi}|\bar{\gamma}'(x)|\log|\bar{\gamma}'(x)| & |x-y|<eps,\\ \frac{1}{\pi}|\bar{\gamma}'(x)|\log\frac{|\bar{\gamma}(y)-\bar{\gamma}(x)|}{|x-y|}, & \text{otherwise,}\end{cases}$$

$\psi(x)=\frac{1}{\pi}|\bar{\gamma}'(x)|$, $k_2(x,y)=\rho(x,y)+\psi(x)\log|x-y|$, and, finally,

$$g(y)=-\frac{1}{\pi}\int_{-1}^{1}\bar{g}(\bar{\gamma}(x))|\bar{\gamma}'(x)|\log|\bar{\gamma}(y)-\bar{\gamma}(x)|dx, \tag{48}$$

the integral equation (46) takes the form

$$f(y)=(\mathcal{G}f)(y), \qquad \text{with} \quad (\mathcal{G}f)(y)=(K_1f)(y)+(K_2Hf)(y)+g(y) \tag{49}$$

with the operators K_i, $i=1,2$, and H defined as in (10) and (11), respectively.

In order to approximate the solution of (49), we apply a Nyström type method which is a combination of the methods described in sections 4 and 5. More precisely, it consists in solving the following approximating equation

$$f_m(y)=(\mathcal{G}_mf_m)(y) \tag{50}$$

with the operator $\mathcal{G}_m$ defined as

$$(\mathcal{G}_mf)(y)=(K_m^1f)(y)+(K_m^2Hf)(y)+g(y),$$

where K_m^1 is the operator given in (23), while K_m^2 is the operator defined as follows

$$(K_m^2f)(y)=\sum_{k=1}^{m}\left[\lambda_k\rho(x_k,y)+c_k(y)\psi(x_k)\right]f(x_k).$$

The collocation of equation (50) at the Legendre zeros leads to the nonlinear system

$$f_m(x_k)=(\mathcal{G}_mf_m)(x_k), \qquad k=1,\dots,m, \tag{51}$$

whose solutions are $f_m(x_i)$, $i=1,\dots,m$.

Once computed, these values can be used in order to construct an approximation of the harmonic function u, solution of the boundary value problem (40), at any point P of the interior domain D. First, using the parameterization (45) in (42), for any $P\equiv(x_P,y_P)\in D$ we represent the potential $u(P)$ as

$$\begin{aligned}u(x_P,y_P)&=\frac{1}{2}\int_{-1}^{1}\frac{\bar{\eta}'(x)[\bar{\xi}(x)-x_P]-\bar{\xi}'(x)[\bar{\eta}(x)-y_P]}{[\bar{\xi}(x)-x_P]^2+[\bar{\eta}(x)-y_P]^2}f(x)dx\\ &+\frac{1}{2}\int_{-1}^{1}[h(x,f(x))-\bar{g}(\bar{\gamma}(x))]|\bar{\gamma}'(x)|\log\left|(x_P,y_P)-\bar{\gamma}(x)\right|dx,\end{aligned} \tag{52}$$

where f is the solution of the integral equation (49). Then, we approximate the potential u in (52), by the following function

$$\begin{aligned}u_m(x_P,y_P)&=\frac{1}{2}\sum_{k=1}^{m}\lambda_k\frac{\bar{\eta}'(x_k)[\bar{\xi}(x_k)-x_P]-\bar{\xi}'(x_k)[\bar{\eta}(x_k)-y_P]}{[\bar{\xi}(x_k)-x_P]^2+[\bar{\eta}(x_k)-y_P]^2}f_m(x_k)\\ &+\frac{1}{2}\sum_{k=1}^{m}\lambda_k\left[h(x,f_m(x_k))-\bar{g}(\bar{\gamma}(x_k))\right]|\bar{\gamma}'(x_k)|\log\left|(x_P,y_P)-\left(\bar{\xi}(x_k),\bar{\eta}(x_k)\right)\right|,\end{aligned} \tag{53}$$

where the solutions $f_m(x_i)$, $i=1,\dots,m$, of the solved nonlinear system are employed.

We observe that in the computation of the right-hand sides of such system the function $g(y)$ given in (48) needs to be evaluated at the collocation points. When we are not able to compute analytically the integral, proceeding as in [44], we approximate it taking into account (47) and using a proper combination of the Gauss-Legendre formula and a product quadrature rule with a large number of knots.

6.1. Numerical experiments

In this subsection, we are going to show some numerical examples in which the method described in sections 4, 5 has been applied for approximating the solution of the interior Neumann problem (40) in some planar domain D with smooth boundary Γ.

Table 4
Errors for the potential u in Example 8.

	$q=1$		$q=2$	
m	$\|u-u_m\|_\Gamma$	$\|u-u_m\|_D$	$\|u-u_m\|_\Gamma$	$\|u-u_m\|_D$
8	6.93e-02	2.71e-01	4.59e-01	3.58e-01
16	2.36e-03	5.94e-02	1.14e-02	1.42e-01
32	3.94e-04	4.98e-03	6.60e-05	1.48e-02
64	1.01e-04	3.65e-05	5.92e-07	1.21e-03
128	2.56e-05	3.89e-08	3.76e-08	2.59e-05
256	6.44e-06	2.45e-09	3.19e-09	2.84e-09
512	1.61e-06	1.53e-10	1.98e-09	1.71e-14

Table 5
Errors for the potential u in Example 9.

	$q=1$		$q=2$	
m	$\|u-u_m\|_\Gamma$	$\|u-u_m\|_D$	$\|u-u_m\|_\Gamma$	$\|u-u_m\|_D$
8	5.27e-01	6.28e-01	6.71e-01	8.71e-01
16	5.65e-02	9.57e-02	9.83e-02	1.95e-01
32	1.23e-03	6.01e-03	1.84e-03	1.52e-02
64	7.58e-04	2.23e-04	3.38e-05	1.20e-03
128	3.16e-04	1.00e-06	6.53e-07	2.60e-05
256	9.89e-05	4.54e-08	4.21e-08	5.54e-09
512	2.70e-05	3.06e-09	1.53e-09	3.18e-10

The reported error $\|u-u_m\|_\Gamma$ is the maximum error at the node points on Γ while the error $\|u-u_m\|_D$ represents the maximum error at 600 points sampled randomly in the interior domain D. Moreover, in our tests, whenever necessary, we have used as smoothing transformation $\phi(x)$ (see (44)) the following one adopted in [39,40]

$$\phi_q(x) = \frac{2\int_{-1}^{x} \left(1-t^2\right)^{q-1} dt}{\int_{-1}^{1} \left(1-t^2\right)^{q-1} dt} - 1, \qquad x \in [-1,1], \quad q \geq 1.$$

Example 8. We consider the problem (40) defined on the planar region D bounded by the ellipse Γ of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

for given values of (a,b). We choose the function $\bar{h}$ as follows (see [25])

$$\bar{h}(P,v) = v + \sin v,$$

and the function $\bar{g}$ such that the exact solution of (40) is the harmonic function

$$u(x,y) = e^x \cos y. \tag{54}$$

Table 4 contains the numerical results obtained by applying the numerical method (51), (53) in the case $(a,b) = (1,2)$.

Example 9. In this second example, again taken from [25], we solve the interior Neumann problem (40) defined on the same domain D considered in the previous example. Here, we assume the boundary condition defined by the nonlinear function

$$\bar{h}(P,v) = |v|v^3$$

and the function $\bar{g}$ chosen such that the exact solution u of (40) is the one given in (54).
The obtained numerical results are shown in Table 5.

Example 10. We consider the amoeba-like domain D bounded by the curve Γ having the following parametric representation

$$\gamma(x) = R(\pi(x+1))e^{\mathrm{i}\pi(x+1)}, \qquad x \in [-1,1]$$

with $R(x) = e^{\cos x}\cos^2 2x + e^{\sin x}\sin^2 2x$ (see Fig. 1).

We assume as exact solution of the BVP (40), with the nonlinear function $\bar{h}(P,v) = v^3$, the function

$$u(x,y) = \sin(x)\cosh(y)$$

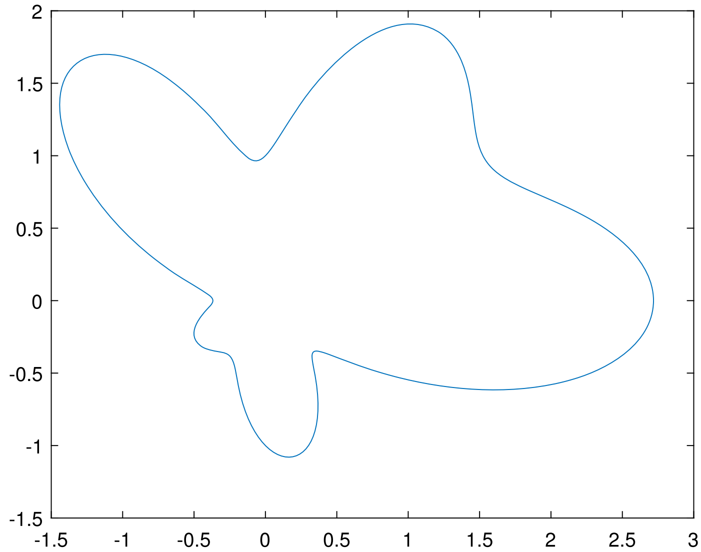

Fig. 1. The boundary Γ of the domain D in Example 10.

Table 6
Errors for the potential u in Example 10.

	$q = 1$	
m	$\|u - u_m\|_\Gamma$	$\|u - u_m\|_D$
16	4.23e-01	4.72e-01
32	1.99e-01	1.09e-01
64	5.68e-03	8.61e-03
128	3.25e-05	1.38e-04
256	1.94e-07	6.71e-08
512	4.86e-08	1.42e-12

and determine the corresponding function $\bar{g}$ at the right-hand side of the Neumann boundary condition. The errors occurred in the computation of the solution at the nodes of curve Γ and at some random points in the interior domains are reported in Table 6.

We observe that we have obtained accurate numerical results already without using any smoothing transformation ($q = 1$), even better that those obtained for $q > 1$, when one needs larger values of m in order to achieve higher accuracy.

7. Proofs

In this section, we collect the proofs of all our main results. From now on, for the sake of simplicity, when we handle with the norm of an operator T, we will omit the subscript $C \to C$ in the norm, that is we set $\|T\| = \|T\|_{C\to C}$.

Proof of Theorem 3. For any fixed $x \in [-1, 1]$, let $P_m(x, y)$ and $Q_m(x, y)$ be the polynomial of best approximation with respect to the variable y of $k_1(x, y)$ and $k_2(x, y)$, respectively, i.e.

$$E_m(k_1(x,\cdot)) = \|k_1(x,\cdot) - P_m(x,\cdot)\|_\infty, \quad E_m(k_2(x,\cdot)) = \|k_2(x,\cdot) - Q_m(x,\cdot)\|_\infty,$$

and introduce the operator $\tilde{\mathcal{K}}_m$ defined as follows

$$(\tilde{\mathcal{K}}_m f)(y) = \int_{-1}^{1} P_m(x,y) f(x)dx + \int_{-1}^{1} Q_m(x,y) h(x, f(x))dx.$$

Denoted by $\mathcal{K} = K^1 + K^2 H$ where K^i, $i = 1, 2$, are given in (10) and H is defined in (11), we have

$$|(\mathcal{K}f)(y) - (\tilde{\mathcal{K}}_m f)(y)| \le \int_{-1}^{1} |k_1(x,y) - P_m(x,y)|\,|f(x)|dx + \int_{-1}^{1} |k_2(x,y) - Q_m(x,y)|\,|h(x,f(x))|dx$$

$$\leq \sup_{|x|\leq 1} |k_1(x,y) - P_m(x,y)| \int_{-1}^{1} |f(x)|dx + \sup_{|x|\leq 1} |k_2(x,y) - Q_m(x,y)| \int_{-1}^{1} |h(x,f(x))|dx.$$

Then, since $\tilde{\mathcal{K}}_m f$ is a polynomial of degree at most m, for any $f \in C$, it follows

$$E_m(\mathcal{K}f) \leq 2\|f\|_\infty \sup_{|x|\leq 1} E_m(k_1(x,\cdot)) + \sup_{|x|\leq 1} E_m(k_2(x,\cdot)) \int_{-1}^{1} |h(x,f(x))|dx.$$

Therefore, by the assumptions (16) on the kernels k_i, $i = 1,2$, and on the function h, and by applying (7), we deduce that $\mathcal{K}f \in W^r$. Hence, if $g \in W^r$, the solution of equation (12) $f = \mathcal{K}f + g \in W^r$. □

Proof of Theorem 4. For simplicity set

$$\mathcal{G}f = K^1 f + K^2 Hf + g,$$

with K^1 and K^2 defined as in (10), and with $k_2(x,y) = \psi(x)k^*(|x-y|)$, while H is defined in (11). Then the solvability of equation (17) is equivalent to the existence of a fixed point of the operator $\mathcal{G}$ in C.

Then, for $f_1, f_2 \in C$, we get

$$\|\mathcal{G}f_1 - \mathcal{G}f_2\|_\infty \leq \max_{y\in[-1,1]} \left[\int_{-1}^{1} |k_1(x,y)||f_1(x) - f_2(x)|\, dx + \int_{-1}^{1} |\psi(x)k^*(|x-y|)||h(x,f_1(x)) - h(x,f_2(x))|\, dx \right]$$

$$\leq \|f_1 - f_2\|_\infty \max_{y\in[-1,1]} \left(\int_{-1}^{1} |k_1(x,y)|\, dx + \max_{x\in[-1,1]} \|h_v(x,\cdot)\|_\infty \int_{-1}^{1} |\psi(x)|\, |k^*(|x-y|)|\, dx \right)$$

Therefore, under the assumption **[H4]**, we deduce that $\mathcal{G}$ is a contraction mapping on C and consequently it has a unique fixed point. □

Proof of Theorem 5. We want to estimate $E_m(\mathcal{K}f)$ in order to apply (8) and deduce the class of the function $\mathcal{K}f$.

First of all we underline that

$$E_m(\mathcal{K}f) \leq E_m(K^1 f) + E_m(\widetilde{K}^\mu Hf).$$

About $E_m(K^1 f)$, proceeding, for instance, as done in the proof of Theorem 3 we have

$$E_m(K^1 f) \leq \mathcal{C} \sup_{|x|\leq 1} E_m(k_1(x,\cdot)), \quad \mathcal{C} \neq \mathcal{C}(m)$$

and hence, under the assumptions (20), we get

$$E_m(K^1 f) \leq \frac{\mathcal{C}}{m^\lambda}, \quad \mathcal{C} \neq \mathcal{C}(m).$$

Therefore from (8) we deduce $K^1 f \in Z^\lambda$.

Consider now $E_m(\widetilde{K}^\mu Hf)$. Using inequality (5), we deduce

$$E_m(\widetilde{K}^\mu Hf) \leq \mathcal{C} \int_0^{\frac{1}{m}} \frac{\Omega_\varphi^k(\widetilde{K}^\mu Hf, t)}{t}\, dt, \quad \mathcal{C} \neq \mathcal{C}(m,f), \quad k \geq 1. \tag{55}$$

Thus, we have to estimate the main part of the modulus of continuity of $\widetilde{K}^\mu Hf$. We can proceed following step by step the proof of Lemma 4.1 in [45]. Therefore, since we are assuming that **[H1]** holds true and $\psi \in C$, we get

$$\Omega_\varphi^k(\widetilde{K}^\mu Hf, t) \leq \mathcal{C}\|h(\cdot,f)\|_\infty \|\psi\|_\infty \begin{cases} t^{1+\mu}, & \mu \neq 0,\ k > 1+\mu, \\ t\log t^{-1}, & \mu = 0,\ k \geq 1, \end{cases}$$

and, consequently, by (55) we have

$$E_m(\widetilde{K}^\mu Hf) \leq \mathcal{C} \begin{cases} \dfrac{1}{m^{1+\mu}}, & \mu \neq 0,\ k > 1+\mu, \\ \frac{\log m}{m}, & \mu = 0,\ k \geq 1. \end{cases}$$

From these estimates and (8) we get that $\widetilde{K}^\mu Hf \in Z^r$, where $r = 1+\mu$ when $\mu \neq 0$, while $r = 1-\epsilon$, with ϵ sufficiently small, when $\mu = 0$, and the theorem follows. □

Proof of Theorem 6. First, note that the well-known Gauss-Legendre formula (21) is convergent. Therefore, the sequences $\{K_m^i\}_m$ $i=1,2$ given in (23) are collectively compact and pointwise convergent to the integral operators K^i $i=1,2$ defined in (10); see for instance [46, Theorem 12.8]. Consequently, the assertion follows taking into account that the operator $\mathcal{K}_m$ is the sum of the operator K_m^1 and the composition of K_m^2 with the operator H in (11) (see, for instance, [31, p.74]). □

Proof of Theorem 7. The existence of a fixed point $f_m \in B(f^*, \epsilon)$ for the operator $\mathcal{G}_m$, for m sufficiently large, say $m \geq m_0$, follows by [23, Theorem 3]. In fact, the first four hypotheses of Theorem 3 in [23] are guaranteed by Theorem 6 and the use of our assumptions. In addition, the fixed point f^* has nonzero index and is isolated, i.e. $f^* \in B(f^*, \epsilon_0)$ with $0 < \epsilon_0 \leq \delta$. This can be deduced by [31, Theorem 21.6, p.108] or [47, p.136], taking into account that g verifies [**G1**], h satisfies [**H1**] and [**H2**], and 1 is not an eigenvalue of $\mathcal{G}' f^*$.

Let us now prove the uniqueness by showing that

$$\|f - \mathcal{G}_m(f)\|_\infty > 0, \qquad \forall f \in B(f^*, \epsilon), \tag{56}$$

i.e. f cannot be a solution of (25).

First, let us note that for sufficiently large m, the operators $(I - \mathcal{G}_m' f)^{-1}$, $f \in B(f^*, \epsilon)$, $0 < \epsilon \leq \epsilon_0$, exist and are uniformly bounded w.r.t. m, i.e. there exists a constant $C \neq C(m)$ such that

$$\|(I - \mathcal{G}_m' f)^{-1}\| < 2C, \qquad f \in B(f^*, \epsilon), \qquad m \geq m_0, \tag{57}$$

where, for $\phi \in C$,

$$[(\mathcal{G}_m' f)\phi](y) = \sum_{k=1}^m \lambda_k k_1(x_k, y)\phi(x_k) + \sum_{k=1}^m \lambda_k k_2(x_k, y) h_v(x_k, f(x_k))\phi(x_k), \quad y \in [-1, 1].$$

This can be obtained by proceeding as in the proof of [23, Theorem 4] by virtue of [**H1**] [**H2**] and [**H3**].

Now, for $f \in B(f^*, \epsilon)$ we have

$$f - \mathcal{G}_m(f) = [I - \mathcal{G}_m' f_m](f - f_m) - [\mathcal{G}_m(f) - \mathcal{G}_m(f_m) - \mathcal{G}_m' f_m (f - f_m)],$$

and then, using the reverse triangle inequality,

$$\|f - \mathcal{G}_m(f)\|_\infty \geq \|I - \mathcal{G}_m' f_m\| \|f - f_m\|_\infty - \|\mathcal{G}_m(f) - \mathcal{G}_m(f_m) - \mathcal{G}_m' f_m (f - f_m)\|_\infty. \tag{58}$$

Therefore, by (57), since $f_m \in B(f^*, \epsilon)$ we have

$$\|I - \mathcal{G}_m' f_m\| \geq \frac{1}{2C}. \tag{59}$$

In addition, from [**H3**] we can deduce that $\mathcal{G}_m$ admits the second Frechet derivative given by

$$[(\mathcal{G}_m'' f)(\phi_1, \phi_2)](y) = \sum_{k=1}^m \lambda_k k_2(x_k, y) h_{vv}(x_k, f(x_k))\phi_1(x_k)\phi_2(x_k), \qquad \phi_1, \phi_2 \in C,$$

and we have

$$\max\{\|\mathcal{G}'' f\|, \|\mathcal{G}_m'' f\|\} \leq C_1, \qquad C_1 = C_1(f^*, \delta), \quad C_1 \neq C_1(m), \quad f \in B(f^*, \delta),$$

with $\|\cdot\|$ denoting the norm of a bilinear form from $C \times C \to C$, and $\mathcal{G}'' f$ given in (15).

Standard arguments [48, Chapter 17, p. 500] lead to

$$-\|\mathcal{G}_m(f) - \mathcal{G}_m(f_m) - \mathcal{G}_m' f_m (f - f_m)\|_\infty \geq -\frac{C_1}{2}\|f - f_m\|_\infty^2 \geq -\epsilon C_1 \|f - f_m\|_\infty.$$

Therefore, by applying (59) and the above inequality in (58), by fixing $\epsilon < (2CC_1)^{-1}$, we have

$$\|f - \mathcal{G}_m(f)\|_\infty \geq \left[\frac{1}{2C} - \epsilon C_1\right] \|f - f_m\|_\infty, \qquad m > m_0, \qquad f \in B(f^*, \epsilon),$$

namely (56). □

Proof of Theorem 8. First let us prove that

$$\|f^* - f_m\|_\infty \leq C \|(\mathcal{G} - \mathcal{G}_m) f^*\|_\infty, \tag{60}$$

where $C \neq C(m)$. To this end, let us proceed as in the proof of Theorem 4 in [23]. Then, by (13) and (25) we write

$$f^* - f_m = \mathcal{G} f^* - \mathcal{G}_m f_m.$$

Therefore,

$$(I-\mathcal{G}'_m f^*)(f^*-f_m)=(\mathcal{G}-\mathcal{G}_m)f^*-[\mathcal{G}_m(f_m-f^*)-\mathcal{G}'_m f^*(f_m-f^*)],$$

or equivalently

$$f^*-f_m=(I-\mathcal{G}'_m f^*)^{-1}\{(\mathcal{G}-\mathcal{G}_m)f^*-[\mathcal{G}_m(f_m-f^*)-\mathcal{G}'_m f^*(f_m-f^*)]\},$$

from which it follows

$$\|f^*-f_m\|_\infty\le\|(I-\mathcal{G}'_m f^*)^{-1}\|\{\|(\mathcal{G}-\mathcal{G}_m)f^*\|_\infty+\|\mathcal{G}_m(f_m-f^*)-\mathcal{G}'_m f^*(f_m-f^*)\|_\infty\}.$$

By applying

$$\|(I-\mathcal{G}'_m f^*)^{-1}\|<\mathcal{C},\qquad m\ge m_0$$

and [48, Chapter 17, p. 500]

$$\|\mathcal{G}_m(f_m-f^*)-\mathcal{G}'_m f^*(f_m-f^*)\|_\infty\le\frac{\mathcal{C}_1}{2}\|f^*-f_m\|_\infty^2,$$

we have

$$\|f^*-f_m\|_\infty\le\frac{\mathcal{C}}{1-\frac{\mathcal{C}\mathcal{C}_1}{2}\|f^*-f_m\|_\infty}\|(\mathcal{G}-\mathcal{G}_m)f^*\|_\infty.$$

Taking $\epsilon<(2\mathcal{C}\mathcal{C}_1)^{-1}$, since $f_m\in B(f^*,\epsilon)$, we get

$$\|f^*-f_m\|_\infty\le\frac{\mathcal{C}}{1-\frac{\mathcal{C}\mathcal{C}_1}{2}\epsilon}\|(\mathcal{G}-\mathcal{G}_m)f^*\|_\infty \tag{61}$$

from which we deduce (60) being the denominator less than 1/2.

Now, by using the definition of $\mathcal{G}$ and $\mathcal{G}_m$, we have

$$(\mathcal{G}-\mathcal{G}_m)f^*=(K^1-K^1_m)f^*+(K^2-K^2_m)Hf^*.$$

By (61) and by applying (22), we deduce

$$\begin{aligned}\|f^*-f_m\|_\infty&\le\mathcal{C}\left[\|(K^1-K^1_m)f^*\|_\infty+\|(K^2-K^2_m)Hf^*\|_\infty\right]\\&=\mathcal{C}\left[\sup_{|y|\le1}|e_m(k_1(\cdot,y)f^*)|+\sup_{|y|\le1}|e_m(k_2(\cdot,y)h(\cdot,f^*))|\right]\\&\le\mathcal{C}\left[\sup_{|y|\le1}E_{2m-1}(k_1(\cdot,y)f^*)+\sup_{|y|\le1}E_{2m-1}(k_2(\cdot,y)h(\cdot,f^*))\right].\end{aligned}$$

Therefore, by exploiting the following estimate

$$E_{2m}(f_1f_2)\le\|f_1\|_\infty E_m(f_2)+2\|f_2\|_\infty E_m(f_1),\quad\forall f_1,f_2\in C,$$

we have

$$\begin{aligned}\|f^*-f_m\|_\infty\le\mathcal{C}\Bigg[&\|f^*\|_\infty\sup_{|y|\le1}E_{m-1}(k_1(\cdot,y))+2\sup_{|y|\le1}\|k_1(\cdot,y)\|_\infty E_{m-1}(f^*)\\&+\|h(\cdot,f^*)\|_\infty\sup_{|y|\le1}E_{m-1}(k_2(\cdot,y))+2\sup_{|y|\le1}\|k_2(\cdot,y)\|_\infty E_{m-1}(h(\cdot,f^*))\Bigg].\end{aligned}$$

Now, by the assumptions (28) on the kernels k_i and taking (6) into account, we get

$$\begin{aligned}\|f^*-f_m\|_\infty\le\mathcal{C}&\sup_{|y|\le1}\|k_1(\cdot,y)\|_{W^r}\left[\frac{\|f^*\|_\infty}{m^r}+2E_{m-1}(f^*)\right]\\&+\sup_{|y|\le1}\|k_2(\cdot,y)\|_{W^r}\left[\frac{1}{m^r}\|h(\cdot,f^*)\|_\infty+2E_{m-1}(h(\cdot,f^*))\right].\end{aligned}$$

Moreover, note that by the hypothesis (29) on g and by virtue of Theorem 3, we can deduce that $f^*\in W^r$. Therefore, we can apply Theorem 1. Then, we obtain

$$\|f^*-f_m\|_\infty\le\frac{\mathcal{C}}{m^r}\left(\sup_{|y|\le1}\|k_1(\cdot,y)\|_{W^r}\|f^*\|_{W^r}+\sup_{|y|\le1}\|k_2(\cdot,y)\|_{W^r}\left[\|h(\cdot,f^*)\|_\infty+2^rB_r\|h\|_{\mathbf{W}^r(D)}\|f^*\|^s_{W^r}\right]\right),$$

from which the assertion. □

Proof of Theorem 10. The proof can be led following step by step the proof of Theorem 8, just by substituting operator K^2 and K_m^2 with K^μ defined in (19) and K_m^* defined in (36), respectively, and arriving to the following inequality, also using (35),

$$\begin{aligned} \|f^* - f_m\|_\infty &\le C \left[\|(K^1 - K_m^1)f^*\|_\infty + \|(\widetilde{K}^\mu - K_m^*)f^*\|_\infty \right] \\ &= C \left[\sup_{|y|\le 1} |e_m(k_1(\cdot,y)f^*)| + \sup_{|y|\le 1} |r_m(h(\cdot,f^*),y)| \right] \\ &\le C \left[\sup_{|y|\le 1} E_{2m-1}(k_1(\cdot,y)f^*) + \sup_{|y|\le 1} E_{m-1}(h(\cdot,f^*)) \right] \end{aligned} \tag{62}$$

where $C \neq C(m,f)$.

For the first term in the brackets, under the assumptions (39), using (5), and the same arguments in the proof of Theorem 8, we get

$$\sup_{|y|\le 1} E_{2m-1}(k_1(\cdot,y)f^*) \le C \frac{\|f^*\|_{Z^s}}{m^s}. \tag{63}$$

Concerning the second term, by (5) we have

$$E_{m-1}(h(\cdot,f^*)) \le C \int_0^{\frac{1}{m}} \frac{\Omega_\varphi^k(h(\cdot,f^*),t)}{t} dt, \tag{64}$$

and then, it is crucial to estimate the modulus of smoothness $\Omega_\varphi^k(h(\cdot,f^*),t)$. Under the assumptions we made, we can consider $k=1$. By the definition (4), we have

$$\begin{aligned} |\Delta_{\tau\varphi} h(x,f^*(x))| &= \left| h\left(x+\tau\frac{\varphi(x)}{2}, f^*\left(x+\tau\frac{\varphi(x)}{2}\right)\right) - h\left(x-\tau\frac{\varphi(x)}{2}, f^*\left(x-\tau\frac{\varphi(x)}{2}\right)\right)\right| \\ &\le \left| h\left(x+\tau\frac{\varphi(x)}{2}, f^*\left(x+\tau\frac{\varphi(x)}{2}\right)\right) - h\left(x-\tau\frac{\varphi(x)}{2}, f^*\left(x+\tau\frac{\varphi(x)}{2}\right)\right)\right| \\ &+ \left| h\left(x-\tau\frac{\varphi(x)}{2}, f^*\left(x+\tau\frac{\varphi(x)}{2}\right)\right) - h\left(x-\tau\frac{\varphi(x)}{2}, f^*\left(x-\tau\frac{\varphi(x)}{2}\right)\right)\right| \\ &= T_1(x) + T_2(x). \end{aligned} \tag{65}$$

By using the derivability of h with respect to the first variable, we get

$$T_1(x) \le \tau \left| \frac{\partial h}{\partial x}\left(\xi_1, f^*\left(x+\frac{\tau}{2}\varphi(x)\right)\right)\right|, \qquad \xi_1 \in \left[x-\frac{\tau}{2}\varphi(x), x+\frac{\tau}{2}\varphi(x)\right] \tag{66}$$

and similarly, by the derivability of h with respect to the second variable, being f^* continuous, we write

$$T_2(x) \le \left| \frac{\partial h}{\partial y}\left(x-\frac{\tau}{2}\varphi(x), f^*(\xi_2)\right)\right| \left|\Delta_{\tau\varphi} f^*(x)\right|, \xi_2 \in \left[x-\frac{\tau}{2}\varphi(x), x+\frac{\tau}{2}\varphi(x)\right]. \tag{67}$$

Hence, by combining (66) and (67) in (65) and considering that we are assuming $h \in W^1$, we have

$$|\Delta_{\tau\varphi} h(x,f^*(x))| \le \|h\|_{W^1} \left[\tau + \left|\Delta_{\tau\varphi} f^*(x)\right|\right],$$

from which we can conclude that

$$\Omega_\varphi(h(\cdot,f^*),t) \le \|h\|_{W^1} \left[t + \Omega_\varphi(f^*,t)\right].$$

Therefore, since by Theorem 5, it is $f^* \in Z^s$, by (64) we get

$$E_{m-1}(h(\cdot,f^*)) \le C\|h\|_{W^1} \left[\frac{1}{m} + \frac{1}{m^s}\right] \le \frac{C}{m^s}\|h\|_{W^1}.$$

The assertion follows by combining the above result and (63) into (62). □

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors are member of the INdAM Research group GNCS and the TAA-UMI Research Group. This research has been accomplished within "Research ITalian network on Approximation" (RITA).

L. Fermo, C. Laurita, and M.G. Russo are partially supported by the PRIN 2022 PNRR project no. P20229RMLB financed by the European Union - NextGeneration EU and by the Italian Ministry of University and Research (MUR).

L. Fermo is also partially supported by the INdAM-GNCS project 2024 "Algebra lineare numerica per problemi di grandi dimensioni: aspetti teorici e applicazioni".

The authors have no other competing interests to declare that are relevant to the content of this article.

Finally the authors want to thank the anonymous referees for the accurate reading of the paper, the remarks and suggestions, which improved the quality of the paper.

References

[1] R. Bialecki, A.J. Nowak, Boundary value problems in heat conduction with nonlinear material and nonlinear boundary conditions, Appl. Math. Model. 5 (6) (1981) 417–421.
[2] M.A. Kelmanson, Solution of nonlinear elliptic equations with boundary singularities by an integral equation method, J. Comput. Phys. 56 (2) (1984) 244–258.
[3] K. Ruotsalainen, W. Wendland, On the boundary element method for some nonlinear boundary value problems, Numer. Math. 53 (1988) 299–314.
[4] K.E. Atkinson, A survey of numerical methods for solving nonlinear integral equations, J. Integral Equ. Appl. 4 (1992) 1–32.
[5] D. Pascali, S. Sburlan, Nonlinear Mappings of Monotone Type, Springer, 1978.
[6] A. Wazwaz, Linear and Nonlinear Integral Equations: Methods and Applications, 1st edition, Springer Publishing Company, Incorporated, 2011.
[7] C. Chidume, A. Adamu, L. Okereke, Iterative algorithms for solutions of Hammerstein equations in real Banach spaces, Fixed Point Theory Appl. 12 (2020).
[8] V. Dolezale, Monotone Operators and Its Applications in Automation and Network Theory, Studies in Automation and Control, Elsevier, New York, 1979.
[9] K. Narendra, P. Gallman, An iterative method for the identification of nonlinear systems using a Hammerstein model, IEEE Trans. Autom. Control 11 (3) (1966) 546–550.
[10] S. Kumar, The numerical solution of Hammerstein equations by a method based on polynomial collocation, J. Aust. Math. Soc. Ser. B 31 (3) (1990) 319–329.
[11] S. Kumar, I.H. Sloan, A new collocation-type method for Hammerstein integral equations, Math. Comput. 48 (1987) 585–593.
[12] H. Kaneko, Y. Xu, Degenerate kernel method for Hammerstein equations, Math. Comput. 56 (193) (1991) 141–148.
[13] P. Das, M.M. Sahani, G. Nelakanti, G. Long, Legendre spectral projection methods for Fredholm–Hammerstein integral equations, J. Sci. Comput. 68 (2016) 213–230.
[14] M. Mandal, K. Kant, G. Nelakanti, Discrete Legendre spectral methods for Hammerstein type weakly singular nonlinear Fredholm integral equations, Int. J. Comput. Math. 98 (11) (2021) 2251–2267.
[15] M. Mandal, G. Nelakanti, Superconvergence results for weakly singular Fredholm-Hammerstein integral equations, Numer. Funct. Anal. Optim. 40 (5) (2019) 548–570.
[16] D. Barrera, F. El Mokhtari, M.J. Ibáñez, D. Sbibih, Non-uniform quasi-interpolation for solving Hammerstein integral equations, Int. J. Comput. Math. 97 (1–2) (2020) 72–84.
[17] C. Dagnino, A. Dallefrate, S. Remogna, Spline quasi-interpolating projectors for the solution of nonlinear integral equations, J. Comput. Appl. Math. 354 (2019) 360–372.
[18] D. Barrera, M. Bartoň, I. Chiarella, S. Remogna, On numerical solution of Fredholm and Hammerstein integral equations via Nyström method and Gaussian quadrature rules for splines, Appl. Numer. Math. 174 (2022) 71–88.
[19] M. De Bonis, G. Mastroianni, Projection methods and condition numbers in uniform norm for Fredholm and Cauchy singular integral equations, SIAM J. Numer. Anal. 44 (4) (2006) 1351–1374.
[20] L. Fermo, M. Russo, Numerical methods for Fredholm integral equations with singular right-hand sides, Adv. Comput. Math. 33 (2010) 305–330.
[21] L. Fermo, C. Laurita, A Nyström method for a boundary integral equation related to the Dirichlet problem on domains with corners, Numer. Math. 130 (1) (2015) 35–71.
[22] C. Laurita, A new stable numerical method for Mellin integral equations in weighted spaces with uniform norm, Calcolo 57 (3) (2020).
[23] K.E. Atkinson, The numerical evaluation of fixed points for completely continuous operators, SIAM J. Numer. Anal. 10 (1973) 799–807.
[24] L. Fermo, C. Laurita, M. Russo, On the error of best polynomial approximation of composite functions, 2023, arXiv:2308.06043 [math.NA].
[25] K.E. Atkinson, G. Chandler, Boundary integral equation methods for solving Laplace's equation with nonlinear boundary conditions: the smooth boundary case, Math. Comput. 55 (192) (1990) 451–472.
[26] G. Mastroianni, G. Milovanovic, Interpolation Processes: Basic Theory and Applications, Springer Monographs in Mathematics, 2008.
[27] Z. Ditzian, V. Totik, Moduli of Smoothness, Springer-Verlag, New York, 1987.
[28] U. Luther, M. Russo, Boundedness of the Hilbert transformation in some weighted Besov type spaces, Integral Equ. Oper. Theory 36 (2) (2000) 220–240.
[29] K.E. Atkinson, The Numerical Solution of Integral Equations of the Second Kind, Cambridge University Press, Cambridge, 1997.
[30] C. Pötzsche, Urysohn and Hammerstein operators on Hölder spaces, Analysis 42 (4) (2022) 205–240.
[31] M.A. Krasnosel'skii, P. Zabreiko, Geometrical Methods of Nonlinear Analysis, Springer-Verlag, Berlin, 1984.
[32] H. Kaneko, R. Noren, Y. Xu, Regularity of the solution of Hammerstein equations with weakly singular kernel, Integral Equ. Oper. Theory 13 (5) (1990) 660–670.
[33] A. Pedas, G. Vainikko, Superconvergence of piecewise polynomial collocations for nonlinear weakly singular integral equations, J. Integral Equ. Appl. 9 (4) (1997) 379–406.
[34] A. Shahsavaran, F. Fotros, An effective and simple scheme for solving nonlinear Fredholm integral equations, Math. Model. Anal. 27 (2) (2022) 215–231.
[35] J. Rashidinia, H. Khosravian Arab, A. Parsa, Gauss-Legendre quadrature for solution of Hammerstein integral equations, Math. Sci. 5 (4) (2011) 345–354.
[36] D. Mezzanotte, D. Occorsio, M.G. Russo, Combining Nyström methods for a fast solution of Fredholm integral equations of the second kind, Mathematics 9 (21) (2021) 2652.
[37] G. Szegö, Orthogonal Polynomials, Amer. Math. Soc., vol. 23, Amer. Math. Soc. Colloq. Publ., Providence, 1975.
[38] J. Elschner, Y. Jeon, I.H. Sloan, E.P. Stephan, The collocation method for mixed boundary value problem on domains with curved polygonal boundaries, Numer. Math. 76 (1997) 355–381.
[39] G. Monegato, L. Scuderi, Global polynomial approximation for Symm's equations on polygons, Numer. Math. 86 (2000) 655–683.
[40] L. Scuderi, A Chebyshev polynomial collocation BIEM for mixed boundary value problems on nonsmooth boundaries, J. Integral Equ. Appl. 14 (2002) 179–221.
[41] L. Fermo, C. Laurita, A Nyström method for mixed boundary value problems in domains with corners, Appl. Numer. Math. 149 (2020) 65–82.
[42] C. Laurita, A numerical method for the solution of exterior Neumann problems for the Laplace equation in domains with corners, Appl. Numer. Math. 119 (2017) 248–270.

[43] G. Monegato, L. Scuderi, A polynomial collocation method for the numerical solution of weakly singular and singular integral equations on non-smooth boundaries, Int. J. Numer. Methods Eng. 58 (2003) 1985–2011.
[44] L. Fermo, C. Laurita, On the numerical solution of a boundary integral equation for the exterior Neumann problem on domains with corners, Appl. Numer. Math. 94 (2015) 179–200.
[45] G. Mastroianni, M. Russo, W. Themistoclakis, Numerical methods for Cauchy singular integral equations in spaces of weighted continuous functions, in: I. Gohberg, et al. (Eds.), Recent Advances in Operator Theory and Its Applications, Birkhäuser Basel, Basel, 2005, pp. 311–336.
[46] R. Kress, Linear Integral Equations, Applied Mathematical Sciences, vol. 82, Springer-Verlag, Berlin, 1989.
[47] M.A. Krasnosel'skii, Topological Methods in the Theory of Nonlinear Integral Equations, Macmillan, New York, 1964.
[48] L. Kantorovich, G. Akilov, Functional Analysis, Pergamon Press, Oxford, 1982.