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Emergence of chaotic dynamics in the Goodwin model with disequilibrium in the goods market

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A B S T R A C T

This study investigates the conditions for the emergence of chaotic dynamics in the Goodwin economy studied in Sordi and Vercelli (2014), where the economic fluctuations occur in the presence of a class struggle in the labor force while assuming disequilibrium in the produced goods market. Applying the Shilnikov theorem, we derive a parametric configuration leading to a chaotic region that sets the economy on an undesired indeterminate equilibrium path. We also apply a standard stabilizing algorithm to determine a solution for ending the chaos. Implications of this study are noteworthy, as may produce a more powerful instrument to detect the emergence of unregular (chaotic) cycles and the possible path-dependence of equilibrium trajectories from the initial endowments of an economy, which remain instead hidden when the standard Hopf bifurcation theorem is uniquely applied.

1. Introduction

A still-growing debate that dates back to the early 1990s continues in the literature regarding the reliability of economic forecasts in the presence of irregular (or chaotic) dynamics in highly nonlinear systems, aimed at formalizing the increasing complexity of contemporary economies (see Krugman, 1991; Matsuyama, 1991). The risk of erroneous decision making based on economic agents' incorrect expectations may result in the emergence of multiple equilibria, with indeterminate patterns for achieving the optimal solution. Subsequently, this could trap the economy in a low-growth steady state, characterized by high unemployment rates and a lack of capital accumulation. In such circumstances, understanding the adjustment process for selecting a high-growth equilibrium trajectory (i.e., the economically optimal path among the infinite number of possible equilibria) is crucial for policymakers. However, determining the causes that limit some economies to permanently lag behind others (thus preventing a significant economic take-off) remains a controversial topic. This debate has often revolved around the different roles of historical conditions and expectations in shaping the potential for multiple equilibrium paths (see Fukao and Benabou, 1993). The key challenge is to determine under which circumstances the initial conditions are decisive and when expectations take on a more dominant role. The obvious follow-up is to understand which equilibrium will prevail by asking will past events (history) create the preconditions for one steady state over another, or will economic agents' future expectations result in a self-fulfilling outcome? In summary, one scenario may involve an economy starting with a large stock of physical capital that eventually solidifies its acquired comparative advantage, which allows the current economic state to determine its future. Alternatively, firms' willingness to invest may hinge on the expectation that other firms will simultaneously accumulate capital, meaning that the adoption of certain investment strategies will depend on other competing agents' practices. Therefore, the key challenge is fostering convergent expectations around an optimally targeted level of investment. In reality, we may find evidence of circumstances in which the initial conditions are decisive and others where expectations have a critical influence. The calibration of the fundamental model parameters will provide an answer to the emerging scenario. For instance, if the chosen discount rate is sufficiently large, or the coefficient of risk aversion is small, history will outweigh expectations. In this case, individuals can be expected to heavily discount the future or adjust the path of consumption slowly, thus reducing the significance of future actions by other agents and eliminating the possibility of self-fulfilling prophecies (see Acemoglu and Jackson, 2015). A clear implication of this history-dependence (or path-dependence) is that any decision strategy, once undertaken, cannot be reversed and the economy faces a "locked-in" evolutionary equilibrium pattern. Conversely, when expectations matter more, factor reallocation is driven by expectations. Essentially, in the absence of adjustment costs, history is irrelevant, and any equilibrium can be reached through convergent expectations.

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However, the slower the economy adjusts, the more likely history will matter.

The standard instruments of local analysis techniques are insufficient for addressing this issue; instead, global bifurcation theorems are necessary to verify the presence of unique or multiple equilibrium paths outside the neighborhood of the steady state. This is even more crucial in systems that exhibit chaotic dynamics, where sensitivity to initial conditions and measurement errors that are intrinsic to random deterministic processes may lead to incorrect expectations (see Bullard and Butler, 1993; Kelsey, 1988). Understanding the range of model parameters that govern this evolution can help steer wandering equilibrium trajectories toward a stabilized state. Although highly nonlinear models can help explain the rationale behind business cycles that are not exactly repetitive, they may also produce aperiodic equilibrium paths. Depending on initial conditions, these patterns may be influenced by exogenous random shocks and become trapped in a chaotic attractor. In such cases, it is essential to coordinate the agents' expectations to achieve a perfect foresight equilibrium trajectory that is free from chaotic fluctuations. In conclusion, when considering models with chaotic dynamics, perfect foresight is not an appropriate way to model rational expectations. Therefore, only when the chaotic cycle is regularized can agents learn to predict the process toward a long run equilibrium. Subsequently, control methods are essential for stabilizing unstable equilibrium trajectories and mitigating chaotic fluctuations (e.g., Alexeeva et al., 2023).

To formally address the above problem, it might be useful to consider the growth cycle model proposed by Goodwin (1967), which has been recognized by the literature on macro-dynamics as a seminal framework for explaining the emergence of economic fluctuations that may characterize the mutual interaction of labor force and firm owners in the determination of the intended long run equilibrium. The model's structure is based on the Marxian inspiration that a decreased wage rate, while enhancing output growth and employment, has the further consequence of increasing claims for wage renewals.[1] However, decreased profitability because of higher wages may eventually depress economic growth, by pushing workers back into the unemployment pool, in a perpetual and unavoidable oscillating pattern. Basically, this has been predominantly proved in the existing bulk of literature by demonstrating the possible emergence of a standard or generalized Hopf bifurcation in some of the proposed variants of the standard framework aimed to formalize the distributive conflict of the Goodwinian economy.[2] Of higher interest are analyses regarding the emergence of chaotic dynamics in the original version of the model as already shown in Goodwin (1990) and its revisitation by Di Matteo and Sordi (2015), or in the presence of international trade (e.g., Ishiyama and Saiki, 2005), or as an outcome of time lags introduced into the dynamic equations of the model (e.g., Yoshida and Asada, 2007; Davila-Fernandez and Sordi, 2019; Sportelli and De Cesare, 2022). Interestingly, Jakimowicz (2010) numerically investigated the possible emergence of different chaotic attractors in the Goodwin (1951) nonlinear accelerator model with lagged investments.

In particular, as recently shown by Sordi and Vercelli (2014), the dynamic structure implied by Goodwin (1967) becomes even more interesting when complemented by the presence of disequilibrium in the goods market, where future investment planning is a crucial factor for accelerating capital accumulation, depending on the role of the expectations that drive produced output adjustment.[3] The resulting endogenous mechanism, again based on the standard Hopf bifurcation theorem, may eventually result in a possible economic crisis, when the misalignment of the expectations rate from its equilibrium level may produce a shortage of planned investments. As a consequence, we might face a situation where the lower the accumulation of future capital the lower the expected returns of realized output, which may also result in a lack of liquidity attributed to claims for wage increase, thus finally generating a variety of persistent dynamic fluctuations.[4] However, the extended model proposed in this study has additional interesting potential insights since it contains *in nuce* the characteristics of producing a chaotic attractor, which permits us to associate the emerging fluctuations to the random unpredictability of the equilibrium trajectories that behave chaotically around the long-term steady state. We argue that a theoretical analysis of such complex dynamics in continuous time is worth pursuing for a deeper understanding of the behavior of the global economy being studied, which is characterized by expectations as a crucial factor for the emergence of chaotic dynamics within a model featuring disequilibrium in the goods market.[5]

Therefore, our aim is to determine the conditions for the emergence of a limit cycle in a three-dimensional system, giving rise to periodic oscillations that result in persistent oscillations under small perturbations until a chaos frontier is reached. This particular bifurcation degeneracy originates in the neighborhood of a so-called Shilnikov singularity, where the equilibrium trajectories might be constrained to perpetually oscillate off the steady state, moving chaotically around a possible attracting bounded region, without ever attaining the intended steady state. As a consequence, any policy action that is intended to control and stabilize the emerging fluctuations becomes ineffective if the exact parametric configuration that produces the chaotic behavior is not determined in advance.

The remainder of this paper develops as follows. In Section 2, we present the Goodwin model with disequilibrium in the goods market as derived in Sordi and Vercelli (2014) and examine the long run properties of the implied three-dimensional vector field. In Section 3, we apply the Shilnikov theorem to determine the region of the parameter space in which the equilibrium undergoes a saddle-focus bifurcation, providing an example to show the appearance of the implied chaotic scenario. In Section 4, we present an algorithm for conducting chaos stabilization. The conclusion section reassesses the main findings of the paper and policy implications. The Appendix provides all calculations and necessary proofs.

2. Modified Goodwin model

In this section, we outline the three-dimensional model with

[1] Monetary, fiscal, or other real factors such as technical change, have generally been considered to be additional forces that can help to explain the presence of the conflict between workers and capitalists in current-day economic systems (e.g., Shah and Desai, 1981; Di Matteo, 1984; van der Ploeg, 1987; Sportelli, 1995; Foley, 2003, Manfredi and Fanti, 2004; Matsumoto, 2009; Makovinyiova and Zimka, 2011).

[2] See Barrales-Ruiz et al. (2021), Cajas Guijarro (2024), Bailly et al. (2024) for a deep review of both theoretical and empirical validations of the studies that have investigated the cycles implied by the distributive conflict in the labor market in the standard Goodwin model and its variants proposed by the macroeconomics scholars in recent years.

[3] A recent study by Setterfield and Wheaton (2024) developed an interesting investigation of the growth cycles generated by the Goodwin model framework in the presence of endogenous variation in the "animal spirits" of investing decision makers, acting in the presence of market uncertainty and therefore forming erroneous short-term expectations for the appropriate adjustments to be made to correct the emerging disequilibrium problems.

[4] Interesting analyses of the effects produced by market disequilibrium on the dynamics related to capacity utilization, employment rates, income distribution, and the adjustments of prices and quantities of produced goods can be found in Skott (1989), Barbosa-Filho and Taylor (2006), Flaschel (2009, 2015), Sasaki (2013), von Armin and Barrales (2015), Araujo et al. (2019), Rada et al. (2023), among others.

[5] Grassetti et al. (2020) applied an interesting discretization procedure to the Goodwin (1967) model that preserves the original dynamic properties of the continuous model in terms of equilibria and their stability and bifurcation characteristics.

disequilibrium in the goods market presented in Sordi and Vercelli (2014), which is a variant of the Goodwin (1967) framework economy that is defined by the labor share, the employment rate, and the capital-output ratio as an endogenous variable, employed to illustrate the emerging imbalances in the goods market. The original mathematical notation is maintained in the present analysis. Therefore, we keep the variable q for the level of output; k for the stock of physical capital; $\sigma = k/q$ for the capital-output ratio. In the labor market: n is the labor force, l measures the level of employment and w the real wage rate. Additionally, $a = q/l$ stands for labor productivity. Hence, $u = wl/q = w/a$ is the labor share and $v = l/n$ is the employment rate. Finally, as standard in the literature, both labor force and labor productivity are assumed to grow at positive constant rates, $\frac{\dot{a}}{a} = \alpha > 0$ and $\frac{\dot{n}}{n} = \beta > 0$, being dotted variables as their time derivatives. In what follows, we provide a detailed sketch of the model derivation.

The first starting assumption relates to the level of planned investments, whose evolution is based on the accelerator mechanism according to the following equation:

$$I = k^d - k \tag{1}$$

where I expresses the level of investment as the gap between the actual stock of capital, k, and its target (i.e., desired) level, k^d. The latter can be further expressed as $k^d = \overline{\sigma} q^{exp}$, which proportionally relates k^d to the (constant) desired target of the capital-output ratio ($\overline{\sigma}$) and the expected output (q^{exp}). Expectations are assumed to be of an extrapolative type, and are obtained as follows:

$$q^{exp} = q + \tau\dot{q}, \ \ \tau \geq 0 \tag{2}$$

where the agents' output forecast is based on the observed current level (q) and past values, maintaining the assumption that the observed patterns will continue to hold, so that variation ($\dot{q}$) will enter the determination of future output at the rate $\tau \geq 0$, which represents the coefficient of expectations. Basically, when $\tau = 0$, expectations are static and current values are assumed to persist; instead, when $\tau < 0$, expectations are simply conservative, due to the fact that the expected values always lag behind the effective current ones. More interestingly for our scope is the case when $\tau > 0$, because the expected values become an extrapolation of the evolution of current values. As pointed out by Enthoven and Arrow (1956), the coefficient of expectations assumes the crucial role of a destabilizing force in the quest for market stability, as long as incorrect expectations may result in an output solution that is going to be unstable, and thus modified in the long run, when expectations fail to be verified if not correctly (i.e., rationally) anticipated.

Another crucial assumption of the model is that the level of planned investments determined according to Eq. (1) does not match the amount of savings held by the capitalist (firm owning) sector, and will result in a disequilibrium in the goods market via possible undesired excesses in the demand or supply sides of the economy. Therefore, it appears necessary to introduce the following dynamic equation:

$$\dot{q} = g_n q + \eta[I - (1-u)q] \tag{3}$$

which is a simple representation of an adjustment process of the output variation toward the equilibrium exogenous growth rate $(g_n = \alpha + \beta)$, given the imbalance of current investments, I, from total savings, $(1 - u)q$, which impacts output growth at a rate of $\eta > 0$. This may contribute to expanding the current level of output as long as investments exceed savings. In contrast, if total savings displace investments, less production occurs and a negative growth in output is expected, which is commonly known as the paradox of thrift. In an open economy, parameter η would determine the proportion of output variation due to a positive or negative imbalance between investments and savings, which is also equal to the imbalance in the current account in national accounting (i.e., the difference between exports and imports). However, since Sordi and Vercelli (2014) consider a closed economy model, it reflects the existence of excess demand or supply in the goods market, thus making the capital-output ratio endogenous. This highlights also the interaction between distributive cycles and financial instability, when investments ought to be financed through the creation of new debt.

To close the model, Sordi and Vercelli (2014) considered the following Phillips curve equation:

$$\widehat{w} = F(v, \widehat{v}) \tag{4}$$

where the growth rate of the real wage rate ($\widehat{w} = \dot{w}/w$), representing the workers' bargaining power in the labor market, is positively related both to the employment rate (v) and its growth rate ($\widehat{v} = \dot{v}/v$). This indicates that the speed at which the labor force enters the occupation sector also contributes as a push factor in the request for higher wages.[6] Without any loss of generality, the $F(\cdot)$ function is assumed to be of the following additive form:

$$F(v, \widehat{v}) = f(v) + \delta\widehat{v} \tag{5}$$

where $f(v) = -\gamma + \frac{\rho}{(1-v)^2}$ is chosen in Sordi and Vercelli (2014) to guarantee that the employment rate is constrained within the unit circle of the non-negative orthant, and is such that $f'(v) > 0$, given $\gamma > 0$ and $\rho > 0$ as the two positive scale parameters.[7] Additionally, $\delta > 0$ measures the positive effect resulting in the wage growth rate due to an increase in the employment rate. It can also be interpreted as a measure of a hysteresis effect resulting from the bargaining process between firms and working force that incorporates a procyclical evolution of real salaries (see, Sportelli, 1995; Sportelli and De Cesare, 2022).

We can now derive the main equations of the model. First, a log-derivative of the expression $u = w/a$ obtains the following:

$$\frac{\dot{u}}{u} = \frac{\dot{w}}{w} - \alpha = F(v, \widehat{v}) - \alpha = -\gamma + \frac{\rho}{(1-v)^2} + \delta\widehat{v} - \alpha \tag{6}$$

which determines the growth rate equation of the labor share.

Second, by log-differentiating the expressions $v = l/n$ and $l = q/a$, we derive the following:

$$\frac{\dot{v}}{v} = \frac{\dot{l}}{l} - \frac{\dot{n}}{n} = \frac{\dot{q}}{q} - \frac{\dot{a}}{a} - \frac{\dot{n}}{n} = \frac{\dot{q}}{q} - g_n \tag{7}$$

Now, recalling Eqs. (1) and (2), and given that $I = \overline{\sigma} q^{exp} - k$ and $k = \sigma q$, with a bit of algebra we can re-express Eq. (3) in the following form:

$$\dot{q} = g_n q + \eta[\overline{\sigma}(q + \tau\dot{q}) - \sigma q - (1-u)q] \tag{8}$$

or in growth-rate terms, as follows:

$$\frac{\dot{q}}{q} = \frac{g_n + \eta[\overline{\sigma} - \sigma - (1-u)]}{(1 - \eta\overline{\sigma}\tau)} \tag{9}$$

which can be substituted into Eq. (7).

Finally, given that $\dot{k} = I$, and using the expression in Eq. (2), we can derive the following:

$$\frac{\dot{k}}{k} = \frac{\overline{\sigma}(q + \tau\dot{q})}{k} - 1 \tag{10}$$

[6] A theoretical justification for a positive relationship between real wages and employment level (and the speed of its growth) can be easily found in the literature of non-competitive labor markets. When workers and firm owners engage in bargaining on due wages, the unemployment rate represents a measure of the outside option of being employed in an alternative job. In our context, this means also that an increase in employment rate would result in a higher outside option, and a subsequently higher bargained wage for the working-class members.

[7] A fully detailed derivation of the properties of function $f(v)$ to produce equilibrium trajectories totally restricted within the unit box is provided in Desai et al. (2006).

which defines the growth rate of physical capital.

Then, taking a log-differentiation of the expression $k = \sigma q$ we can easily obtain the following:

$$\frac{\dot{\sigma}}{\sigma} = \frac{\dot{k}}{k} - g \tag{11}$$

where $g = \dot{q}/q$. Hence, by substituting Eq. (10) into Eq. (11), and collecting similar terms, we finally obtain the following:

$$\frac{\dot{\sigma}}{\sigma} = \frac{\overline{\sigma}(1+\tau g)}{\sigma} - 1 \ - g \tag{12}$$

which defines the growth rate value of the endogenized capital-output ratio entering the full model dynamics.

In conclusion, the Eqs. (6), (7), and (12) are thus used to represent the three-dimensional system of differential equations in the variables (u, v, σ), that describe the Sordi and Vercelli (2014) Goodwin-like model with disequilibrium in the goods market:

$$\begin{aligned}
\dot{v} &= \frac{\eta}{1-\eta\overline{\sigma}\tau}(\overline{\sigma}\tau g_n + \overline{\sigma} - \sigma - 1 + u)v \\
\dot{u} &= \left[f(v) + \frac{\delta\eta}{1-\eta\overline{\sigma}\tau}(\overline{\sigma}\tau g_n + \overline{\sigma} - \sigma - 1 + u) - \alpha\right]u \\
\dot{\sigma} &= \frac{1}{1-\eta\overline{\sigma}\tau}\{(\overline{\sigma}-\sigma)(1-\eta\overline{\sigma}\tau) + (\overline{\sigma}\tau - \sigma)[g_n + \eta(\overline{\sigma} - \sigma - 1 + u)]\}
\end{aligned} \tag{S}$$

At the equilibrium, $\dot{v} = \dot{u} = \dot{\sigma} = 0$, we obtain the following from (*S*):

$$v^* = 1 - \frac{\sqrt{(\alpha+\gamma)\rho}}{\alpha+\gamma} \tag{13.a}$$

$$u^* = 1 - g_n\sigma^* \tag{13.b}$$

$$\sigma^* = \frac{(1+\tau g_n)\overline{\sigma}}{1+g_n} \tag{13.c}$$

The triplet $P^* \equiv (v^*,\ u^*, \sigma^*)$ represents the unique steady state of the economy described by system (*S*), which can be approximated around the stationary equilibrium as follows:

$$\begin{pmatrix} \dot{v}\dot{u}\dot{\sigma} \end{pmatrix} = \mathscr{J}\begin{pmatrix} v-v^* \\ u-u^* \\ \sigma-\sigma^* \end{pmatrix} \tag{14}$$

where

$$\mathscr{J} = \begin{pmatrix} 0 & \frac{\eta v^*}{1-\eta\overline{\sigma}\tau} & -\frac{\eta v^*}{1-\eta\overline{\sigma}\tau} \\ \frac{2\rho u^*}{(1-v^*)^3} & \frac{\delta\eta u^*}{1-\eta\overline{\sigma}\tau} & -\frac{\delta\eta u^*}{1-\eta\overline{\sigma}\tau} \\ 0 & -\frac{\eta\overline{\sigma}(1-\tau)}{(1-\eta\overline{\sigma}\tau)(1+g_n)} & -(1+g_n) + \frac{\eta\overline{\sigma}(1-\tau)}{(1-\eta\overline{\sigma}\tau)(1+g_n)} \end{pmatrix} \tag{15}$$

is the Jacobian matrix associated with (*S*), at the steady state $(v^*,\ u^*, \sigma^*)$.

The eigenvalues of $\mathscr{J}$ are the solutions of the following characteristic equation:

$$\det(\lambda\mathbf{I} - \mathscr{J}) = \lambda^3 - \mathrm{Tr}(\mathscr{J})\lambda^2 + \mathrm{B}(\mathscr{J})\lambda - \mathrm{Det}(\mathscr{J}) \tag{16}$$

given the identity matrix, **I**, and

$$\mathrm{Tr}(\mathscr{J}) = \frac{\delta\eta u^*}{1-\eta\overline{\sigma}\tau} - (1+g_n) + \frac{\eta\overline{\sigma}(1-\tau)}{(1-\eta\overline{\sigma}\tau)(1+g_n)} \tag{17.1}$$

$$\mathrm{B}(\mathscr{J}) = -\frac{\eta u^*}{1-\eta\overline{\sigma}\tau}\left[\delta(1+g_n) + \frac{2\rho v^*}{(1-v^*)^3}\right] \tag{17.2}$$

$$\mathrm{Det}(\mathscr{J}) = \frac{2\rho(1+g_n)\eta u^* v^*}{(1-\eta\overline{\sigma}\tau)(1-v^*)^2} \tag{17.3}$$

where $\mathrm{Tr}(\mathscr{J})$, $\mathrm{Det}(\mathscr{J})$ and $\mathrm{B}(\mathscr{J})$ are the trace, the determinant, and the sum of principal minors of order 2 of $\mathscr{J}$, respectively.

Lemma 1. *Consider the following condition:*

$$\tau < \frac{1}{\eta\overline{\sigma}} < 1 \tag{18}$$

then the equilibrium, P^*, may exhibit global indeterminacy.

Proof.. By applying the Routh-Hurwitz criterion, Sordi and Vercelli (2014) demonstrated that, when expectations are moderately extrapolative (i.e., $\frac{1}{\eta\overline{\sigma}} < \tau < 1$), the number of negative roots associated with (16) is always one, given that $\mathrm{Tr}(\mathscr{J}) < 0$, $\mathrm{Det}(\mathscr{J}) < 0$, $\mathrm{B}(\mathscr{J}) > 0$, and $[-\mathrm{B}(\mathscr{J}) + \mathrm{Det}(\mathscr{J})/\mathrm{Tr}(\mathscr{J})] > 0$, which is sufficient to say that the equilibrium is locally asymptotically stable. Instead, if condition (18) holds, then $\mathrm{B}(\mathscr{J}) < 0$ and $\mathrm{Det}(\mathscr{J}) > 0$, whereas $\mathrm{Tr}(\mathscr{J})$ has no definite sign, which allows to consider the possibility of non-uniqueness and indeterminacy of the equilibrium trajectories around P^*. ■

The novelty of our contribution is to determine the exact parametric restrictions necessary for the emergence of a Shilnikov (1965) chaotic attractor in system (*S*). We construct some examples using the same set of parameters employed in Sordi and Vercelli (2014) to validate the robustness of our results. The remainder of the paper is devoted to this end.

3. Emergence of Shilnikov chaos

Assuming the result obtained by Bella et al. (2017) on the application of the Shilnikov theorem to a three-dimensional system of differential equations.

Definition 1. *Consider the following dynamic system:*

$$\frac{dx}{dt} = f(x,\mu),\ x \in \mathbb{R}^3,\ \mu \in \mathbb{R}^1,$$

with f sufficiently smooth. Assume that f has a hyperbolic saddle-focus equilibrium point, $\widetilde{x} = 0$, at the bifurcation parameter $\widetilde{\mu} = 0$, implying that the eigenvalues of the Jacobian matrix, $\mathscr{J} = Df$, are in the form φ and $\psi \pm \omega i$, where φ, ψ, and ω are real constants with $\varphi\psi < 0$, and with a saddle quantity $s = |\varphi| - |\psi| \neq 0$. Let

$$\Phi \equiv \mathrm{B}(\mathscr{J}) + \mathrm{Tr}(\mathscr{J})^2 = 0 \tag{19}$$

be the bifurcating condition that separates the region of parameters where the system exhibits a saddle-focus dynamics with $s > 0$, from the region where the equilibrium is saddle path stable with $s < 0$. Then, if we can determine that, when crossing the boundary Φ, a region associated with a saddle-focus dynamics with positive saddle quantity exists, then a sequence of Smale-horseshoes along the homoclinic orbit connecting the equilibrium to itself emerges, giving rise to a Shilnikov chaotic attractor.

Applying condition (19) to system (*S*) allows us to characterize a

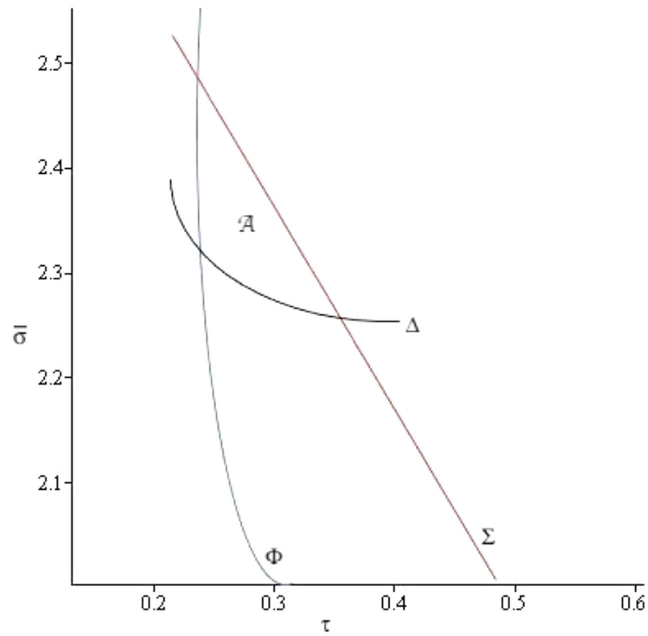

Fig. 1. The Shilnikov parametric set.

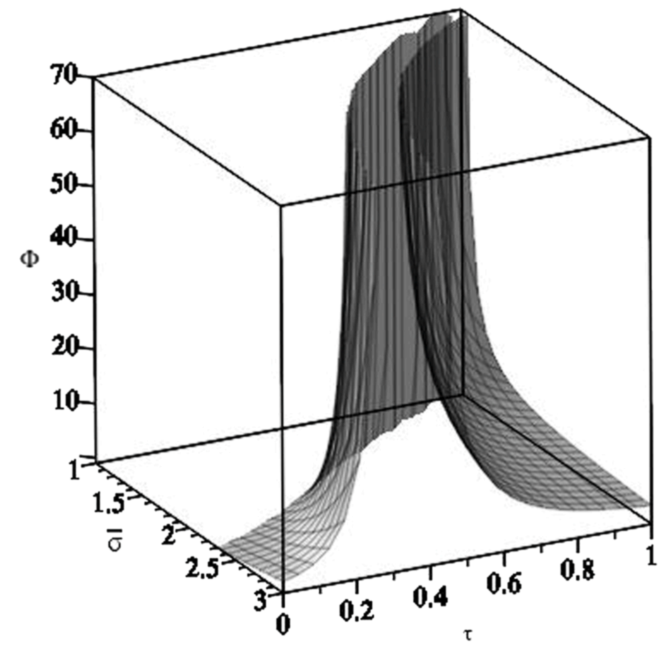

Fig. 2. The Shilnikov bifurcation curve.

bifurcating surface in terms of some critical parameters. For convenience, we decide to keep the pair of parameters $(\tau, \overline{\sigma})$ free, setting all the others as in Sordi and Vercelli (2014). This choice is motivated by the fact that both are crucial in the determination of global indeterminacy in the model, as suggested by the relevance of condition (18) in Lemma 1. Moreover, since we are dealing with a Goodwin economy in the presence of disequilibrium in the production of goods, this choice will permit us to characterize the conditions for the emergence of chaotic solutions depending on the rate measuring the degree of expectations due to current output variation, τ, and desired capital-output ratio, $\overline{\sigma}$.[8]

Denote the set of the parameters $\Omega \equiv \{\eta, \alpha, \beta, g_n, \gamma, \rho, \delta\}$, and assume $\overline{\Omega} \in \Omega$ as in Sordi and Vercelli (2014). Then, Eq. (19) reduces to a nonlinear function of the remaining parameters, $\Phi \equiv \Phi(\tau, \overline{\sigma}) = 0$, which is represented by the blue curve in Fig. 1.

To demonstrate that the equilibrium P^* of system (S) is a saddle focus we need to prove that the associated Jacobian matrix, $\mathscr{J}$, has a pair of roots with non-zero imaginary part. Solving the characteristic equation in (16) with the Cardano's formula for cubic equations provides the following solutions:

$$\lambda_1 = \frac{\mathrm{Tr}(\mathscr{J})}{3} + (\zeta + \xi)$$

$$\lambda_{2,3} = \frac{\mathrm{Tr}(\mathscr{J})}{3} - \frac{(\zeta + \xi)}{2} \pm \sqrt{3}\ \frac{(\zeta - \xi)}{2} i$$

with $\zeta = \sqrt[3]{-\frac{\theta}{2} + \sqrt{\Delta}}$ and $\xi = \sqrt[3]{-\frac{\theta}{2} - \sqrt{\Delta}}$, where $\Delta = \left(\frac{\pi}{3}\right)^3 + \left(\frac{\theta}{2}\right)^2$ is the discriminant. Furthermore, $\pi = \frac{3\mathrm{B}(\mathscr{J}) - \mathrm{Tr}(\mathscr{J})^2}{3}$, $\theta = -\,\mathrm{Det}(\mathscr{J}) -\ 2\frac{\mathrm{Tr}(\mathscr{J})^3}{27} + \frac{\mathrm{Tr}(\mathscr{J})\mathrm{B}(\mathscr{J})}{3}$, and $i = \sqrt{-1}$ is the imaginary unit. Hence, a saddle-focus emerges only if $\Delta > 0$, which is represented above the black curve in Fig. 1 in the bi-dimensional space $(\tau, \overline{\sigma})$.

Additionally, as demonstrated in Bella et al. (2017), the saddle quantity is positive, $s > 0$, when $\Sigma \equiv \frac{2}{3}\mathrm{Tr}(\mathscr{J}) + \frac{(\zeta+\xi)}{2} < 0$, and is satisfied below the red curve in Fig. 1.

Finally, since (19) implies that $\Phi = 0$ when $\Delta = s = 0$, then we can infer that the equilibrium P^* undergoes a saddle-focus dynamics, with $\Delta > 0$ and $s > 0$, which represents the onset for the emergence of a chaotic attractor, above the blue surface reported in Fig. 1, where $\Phi > 0$.

The results from Fig. 1 suggest some interesting economic interpretations concerning the restrictions obtained on the chosen parameters, and link the expectations rate, τ, to the capital-output ratio, $\overline{\sigma}$. In particular, the parametric area of our interest for the emergence of the Shilnikov attractor is identified in the narrow combination of the pair of parameters $(\tau, \overline{\sigma})$, called A, in between of the three curves.

In this scenario, it appears that the moderative expectation rate $(\tau < 1)$, that Sordi and Vercelli (2014) found necessary to guarantee local uniqueness and equilibrium determinacy might instead lead to a chaotic regime that pushes the periodic oscillations surrounding the equilibrium onto the bounded attracting region outside of the steady state if it is associated to particular capital-output ratio levels in region A, $\overline{\sigma} \in (2.26, 2.48)$. Hence, unpredictability concerning policy actions to restore long-term stability becomes a problem that policymakers must confront to avoid continuous and irregular episodes of oscillatory economic activity.

We illustrate a robustness check of our result in Fig. 2, presenting the bifurcation curve, $\Phi \equiv \Phi(\tau, \overline{\sigma})$, in the full three-dimensional space.

It is clear that if we choose the parameters within the area A defined in Fig. 1, we must expect a positive value in condition (19), $\Phi > 0$, to satisfy the Shilnikov bifurcation requirement. In particular, if we set the pair $(\tau, \overline{\sigma}) = (0.35, 2.4)$, then $\Phi = 63, 49178772$, which proves our statement.

To ease the numerical computation necessary to draw the resulting chaotic attractor, we first need to put system (S) in the following convenient normal form:[9]

[8] Since Φ is both positive and increasingly positive when η is close to one, and the pair $(\tau, \overline{\sigma})$ is chosen within the area A implied by Fig. 1, then the requirements for chaotic dynamics implied by the Shilnikov theorem can hold even if we set $\eta = 1$, as in Sordi and Vercelli (2014), which represents a 1:1 impact of total savings on output growth. This allows us to ease the graphical representation by restricting Eq. (19) to the manageable three-dimensional functional form $\Phi(\tau, \overline{\sigma})$.

[9] Notably, x and y become auxiliary variables for the rate of employment, v, and of the labor income share, u, respectively, whereas z is a simple linear transformation of the original capital-output ratio, σ. Therefore, any cyclical oscillation of the auxiliary variables (x, y, z) are thus topologically equivalent to the economic fluctuations in the vector field in the original variables (v, u, σ).

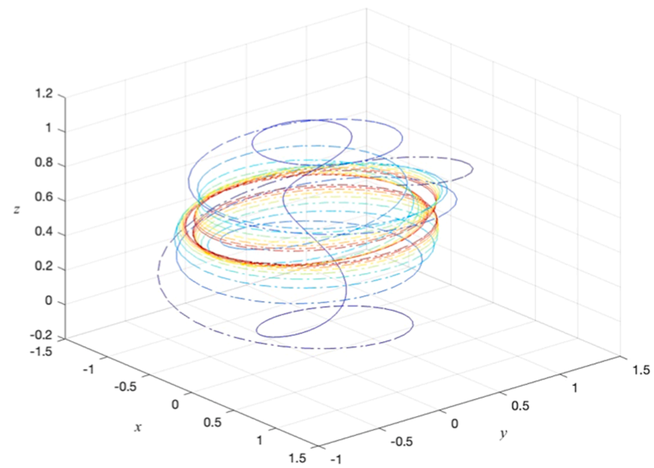

Fig. 3. The Shilnikov chaotic attractor.

$$\begin{pmatrix} \dot{x}\dot{y}\dot{z} \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ dx^2 + hx^3 \end{pmatrix} \quad (20)$$

where (x, y, z) is a new set of coordinates arising from the near-identity transformation, $\varepsilon_1 = \text{Det}(\mathscr{J})$, $\varepsilon_2 = -\text{B}(\mathscr{J})$, $\varepsilon_3 = \text{Tr}(\mathscr{J})$, and where d and h are combinations of the coefficients of the nonlinear terms, as demonstrated in Gamero et al. (1999).[10]

Example 1. *Consider the set of parameters* $\overline{\Omega} \equiv (1,\ 0.035, 0.03, 0.065, 0.04, 0.0002, 0.15) \in \Omega$*. Assume* $(\tau, \overline{\sigma}) = (0.35, 2.4)$*, within region* A *denoted in Fig. 1. Obtaining that* $\omega = 10.50282683$ *and* $(\varphi, \psi) = (0.8079226560, -0.9118945944)$*, and* $s = 9.69490417 > 0$*, with* $\Phi = 63.49178772 > 0$*, the requirements in Definition 1 for the Shilnikov theorem are satisfied. Given that* $\varepsilon_1 = 15.58924890, \varepsilon_2 = 15.48665271$ *and* $\varepsilon_3 = 8.886981514$ *and* $(d, h) = (1.75, 0.1096947638)$*, the evolution of the chaotic attractor is represented in Fig. 3.*

The dynamics of the economy along the spiral structure of the chaotic attractor resembles periods of irregular oscillatory activity as far as the phase dynamics starts to move away from the saddle-focus point. It is now convenient to represent the evolution of the original variable of the model, and check for the presence of meaningful results in our simulations, by using the conversion via the transformation matrix explained in Appendix C. This is clarified in Fig. 4, which presents the time profile of the original variables of the model along the Shilnikov chaotic scenario. The economic implication behind this result allows us to conclude that, given the initial value of our labor-market variables (u, v), in presence of a chaotic attractor, a *continuum* of initial values of the goods market variable (σ) exists giving rise to a pattern of unpredictable admissible equilibria, which implies therefore global indeterminacy of the equilibrium.

Our results are confirmed by computing the Lyapunov exponent associated to the spiraling structure, which is given by $\ell = 0.00233866243431977$, whose positiveness suggests a motion of equilibrium trajectories that start to diverge from the stable arm leading to the saddle-focus equilibrium, which indicates the presence of instability in the transitional dynamics until they produce an erratic and chaotic behavior.[11] The economic implications of this result are noteworthy, indicating that for any small change in the initial conditions of the original variables the model can produce a relevant change in the equilibrium dynamics along the basin of the emerging chaotic attractor.

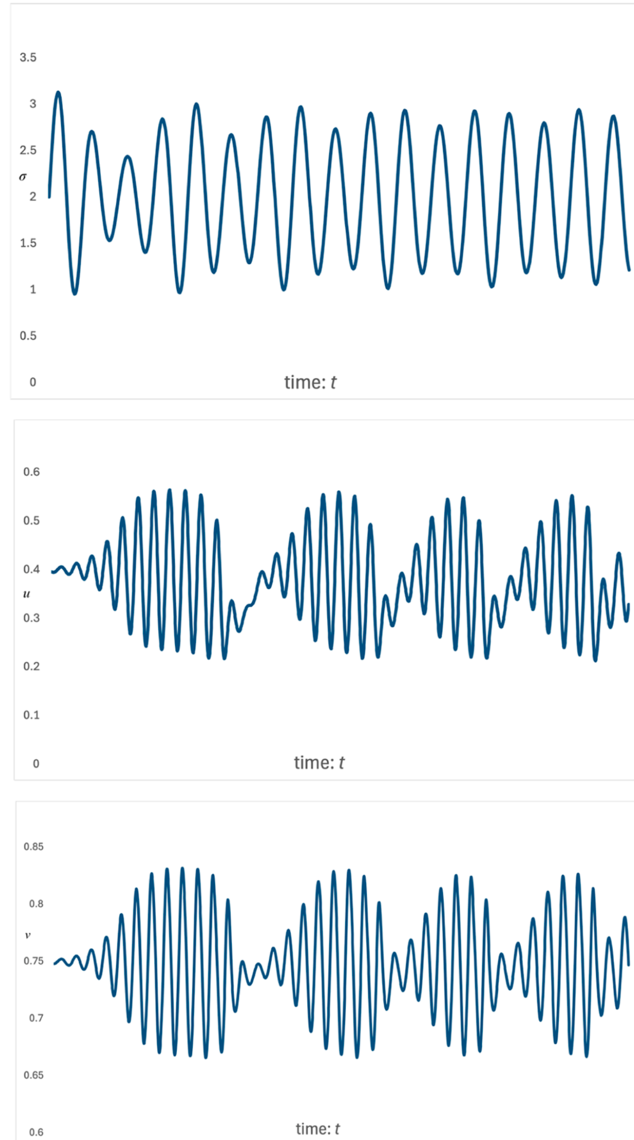

Fig. 4. The time profile of the original variables.

To provide an example, this behavior is outlined in Fig. 5 for the level of the capital-output ratio level, σ.

The simulation proves that with small changes in the initial condition, $\sigma(0)$, the equilibrium trajectories start to wander in a confined area, thus confirming that the resulting chaotic attractor is robust to small variations in the chosen bifurcation parameter, τ. The experiment also reveals that, for any small change in the value of the expectation rate, τ, within the chaotic interval implied by Fig. 1, solution trajectories starting at different initial values of σ show that the adjustment of output in the goods market begins to exhibit an oscillating behavior moving toward the attracting region surrounding the equilibrium level $\sigma^* =$

[10] The use of normal forms is a crucial feature for numerical simulations in standard computer algebra, since they are less costly in terms of memory requirements in the related computation, which may be largely impracticable when using the original canonical forms. Another advantage, as demonstrated by Freire et al. (2002), is that the hypernormal form in Eq. (20) is topologically equivalent to (S). Therefore, the change of parameters in the transformed system is a local diffeomorphism, that preserves the dynamics of the original system in the new ambient space (see also Wiggins, 1991).

[11] We use a standard routine in R software to compute the Lyapunov exponent.

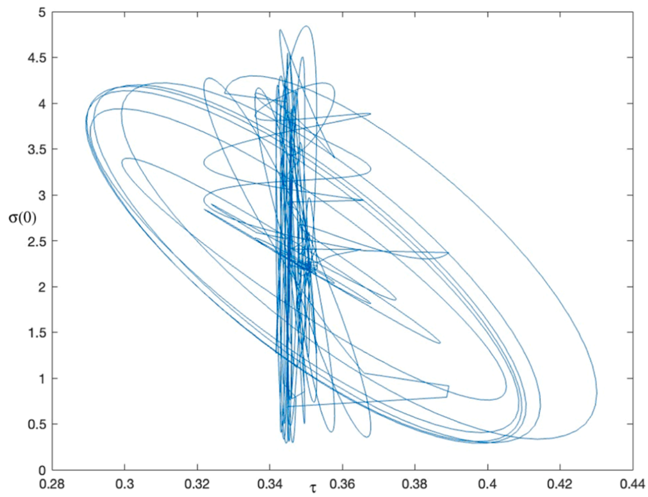

Fig. 5. Sensibility of the initial condition $\sigma(0)$ along the chaotic attractor.

2.304788732 implied by the set of parameters in Example 1.[12] This also confirms that economies that start with similar initial conditions may start to follow completely different patterns of unpredictable admissible equilibria at some point, which implies global indeterminacy. Therefore, any policy action devoted to achieving the intended steady state may fail in the presence of the economic uncertainty implied by the trapping chaotic attractor.

4. Chaos control and the OGY algorithm

This section applies the technique proposed by Ott et al. (1990), also known as OGY algorithm, which describes a correction mechanism to stabilize the emerging chaotic motion in a three-dimensional model of differential equations, thanks to the perturbation of a chosen parameter of the model (see, Bella and Mattana, 2020, for a full description and application of the procedure). Since expectations are a crucial aspect of the Goodwin economy with disequilibrium in the goods market outlined throughout this study, it is convenient to use in the algorithm the expectation rate, τ, as the appropriate control parameter that is necessary to change the sign of the eigenvalues and restore the stability of the steady state solution.

As technically described in Appendix D, our task is to identify the particular value of the expectation parameter (τ^s) that guarantees the existence of one negative real solution and two roots with positive real parts to the cubic characteristic equation in (16), which also implies full stability of the equilibrium point, P^*. Mathematically, this means that the discriminant in the Cardano's formula must be positive at $\tau = \tau^s$, $\Delta(\tau^s) > 0$, though associated with $\Phi < 0$. Therefore, as illustrated in Fig. 1, for any given value of $\overline{\sigma}$, the choice of τ must be located in the narrow area above the Δ-curve and below the Φ-curve, as confirmed by the following example.

Example 2. *Let us denote a set of parameters for the emergence of a chaotic attractor, at which system S has a saddle-focus equilibrium with positive saddle quantity and the Shilnikov theorem is satisfied as in Example 1. Let us select* $\tau \equiv \tau^s = 0.2$*, as derived by the algorithm presented in Appendix D to obtain the controlled system. Then* $\mathscr{J}$ *has one real eigenvalue and a pair of eigenvalues with positive real parts* ($\lambda_1 = -1.656124958$, $\lambda_2 = 0.52214489 + 1.551703134i$, $\lambda_3 = 0.52214489 - 1.551703134i$), *which implies that the equilibrium solution is asymptotically stable.*

[12] We conduct the simulation using R software and show the joint movements of the different levels of initial conditions of $\sigma(0)$ for different levels of the bifurcation parameter τ along the region of the chaotic attractor.

We can now assert the following:

Definition 2. *Consider a Goodwin economy with disequilibrium in the goods market and moderately extrapolative expectations that evolve along a chaotic attractor. If the expectation rate falls below unity and around 0.2, then the economy can stabilize the resulting irregular waves and may subsequently approach the intended steady-state solution without any cyclical behavior.*

Finally, Fig. 6 illustrates the time profile of the capital-output ratio, $\sigma(t)$, in the presence of a chaotic attractor (blue curve), obtained with $\tau = 0.35$ in Example 1, and the stabilized series (red curve) obtained with $\tau \equiv \tau^s = 0.2$ in Example 2.

The waves in Fig. 6 that are generated by the spiral attractor exhibit periods in which the expectation ratio lowers when approaching the equilibrium point through the stable arm of the saddle-focus dynamics (at $\tau - \tau^s \approx 0$). However, this also lowers the profitability of the produced goods, and firms may start to claim for less employed labor force. In contrast, bursts of oscillatory activity follow anew when the economy experiences a phase of growing expectations (at $\tau - \tau^s \neq 0$), with increasing productivity of capital invested in the goods market pushing the need for workers to sustain the production process, while driving back the dynamics of the economy on the divergent spiral branch of the saddle-focus. Therefore, it can be inferred that a policy action adopted to favor an upswing in future expectations in the market can eventually trap the economy in a chaotic attractor.

5. Conclusions and policy implications

The historical initial endowments of a country may dictate the long-term trajectory it follows, making the coexistence of countries with both low and high levels of per capita income possible. The equilibrium outcomes subsequently become highly sensitive to the economic fundamentals prevailing during the early stages of development. Consequently, multiple steady states may arise, undermining the potential for economies with different levels of the initial factors to converge at the same optimal equilibrium. Indeterminacy then becomes a possible scenario, and active policy measures may be needed to break such low-growth traps and move the economy toward the desired equilibrium. In contrast, even economies with similar or nearly identical initial conditions can experience multiple equilibrium paths. Failure in the coordination of agents' expectations may result in a low-growth outcome driven by self-fulfilling prophecies. Interventions to shape the formation of expectations will be necessary to escape this new trap. This is particularly relevant for the standard Goodwin model, where the formalization of the class struggle between capitalists and workers may generate a sequence of self-sustaining cycles of economic expansions and recessions, making the economy highly vulnerable to instability and crisis.

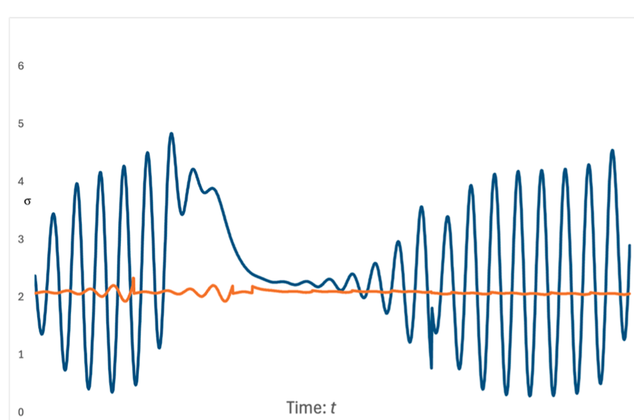

Fig. 6. Uncontrolled and controlled capital-output ratio.

Our scope in the present study has been twofold. We first proposed an application of the standard Shilnikov theorem to the Goodwin model with disequilibrium in the goods market described in Sordi and Vercelli (2014) and proved that a chaotic attractor might appear in a plausible set of the crucial model parameters for the expectations rate and the desired capital-output ratio. The standard normal form theory was used to characterize the simplest form that the original three-dimensional system of the cited economy might assume to enable the software computation of the chaotic trajectories in standard computer-based routines. Moreover, by using the appropriately derived transformation matrix, we were able to represent the time evolution of the original variables along the chaotic attractor. Additionally, we focused on the capital-output ratio to describe its sensitivity dependence to initial conditions when moving chaotically, with the expectations rate set at the implied Shilnikov bifurcation (i.e., chaotic) interval. As a second step of the investigation, we applied the commonly known OGY algorithm, which is a standard mathematical procedure used to characterize the parametric restrictions on the expectations rate, as the crucial policy instrument for controlling chaos and restoring the stability of a dynamic system, which is necessary for achieving the conditional saddle-path stability of the equilibrium.

The proposed analysis is quite a novelty in the literature related to the study of the growth cycle arising in the Goodwin (1967) framework, where Hopf bifucation analysis has predominantly been conducted. However, this can produce misleading results because it implies the existence of a predictable cycle via the computation of the standard first Lyapunov coefficient. This is inadequate in the presence of chaotic dynamics, since the equilibrium trajectories cycling around the long run equilibrium become totally unpredictable and thus produce undesired wavelike fluctuations of non-regular periodicity depending on the initial conditions of the variables describing the model.

Our results also represent a step forward in the literature that has investigated the possibility of chaos emergence in the Goodwin (1967) model, indicating the presence of a Rössler-type attracting set and time lags that characterize the production of new capital goods, and concluding that active policy interventions might tend to amplify the conflict inherent to the model and thus induce the related economic fluctuations. The existence of a chaos-attracting set has various policy implications since it can provide a better characterization of the fluctuations that may arise in an economy that follow unpredictable trajectories. This has an unavoidable impact on the reliability of forecasts, since erroneous decisions are based on incorrect expectations on the future evolution of equilibrium patterns. This could produce a resulting attracting trapping region, with its equilibrium dynamics characterized by a low-growth evolutionary pattern. Determining the set of the expectations rate that produces the odd chaotic dynamics and the set at which stability can be restored is crucial for an economy that wants to escape the emergence of a low-growth trapping region and allow the economic take-off and catch up with advanced economies, characterized by higher employment rates and higher capital-output ratios via correct adjustment process in the goods market driven by correct expectations rates.

The elegance of the Shilnikov bifurcation theorem is that it can also help to characterize the route to chaos dynamics, since the spiral attractor arises due to a rupture of the periodic orbit in the Andronov-Hopf setting, whose boundary forms an unstable manifold wandering off the cycle but spiraling in the vicinity until a homoclinic loop is formed. Mathematically, this depends on the structure of the pair of complex conjugate eigenvalues associated with the Jacobian matrix of the system that describe the emergence of a saddle focus necessary to the Shilnikov bifurcation theorem. When the real part of the pair vanishes then the eigenvalues result as purely complex, therefore producing a Hopf orbit.

The proposed theoretical investigation is also robust to real observed data and demonstrated the empirical plausibility of possible irregular chaotic fluctuations in contemporary political scenarios. Specifically, our simulations in Section 3 validated the possibility of a time evolution of the employment rate oscillating around 0.75, with a range between 0.65 and 0.85, which is more coherent with data in recent years, and below the value obtained in Sordi and Vercelli (2014) given by $v = 0.9484$, which is quite high for standard real economies. Additionally, the sensitivity of the initial capital-output ratio value to the expectations rate adheres to United States records which have been documented to oscillate below 4.0, as it appears in all reports issued by the Bureau of Economic Analysis of the U.S. Department of Commerce.

A possible extension of the present study could introduce heterogeneous agents to engage the idea of endogenizing the stochastic (idiosyncratic) process that drives the working-class agents' decision-making in the acceptance of randomly-distributed job opportunities (and related salaries) to investigate the possibility, both theoretically end empirically, of the emergence of chaos in the presence of such a nondeterministic framework. We leave this inquiry for future research.

CRediT authorship contribution statement

Giovanni Bella: Writing – review & editing, Writing – original draft, Software, Methodology, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.strueco.2025.01.005.

Appendix

A. Saddle focus eigenvalues

Let $\mathscr{J}$ denote the Jacobian matrix of system (S), evaluated at the equilibrium. The eigenvalues of $\mathscr{J}$ are the solutions of the characteristic equation:

$$det(\lambda\mathbf{I} - \mathscr{J}) = \lambda^3 - Tr(\mathscr{J})\lambda^2 + B(\mathscr{J})\lambda - Det(\mathscr{J}) \tag{A.1}$$

and are derived with the application of the Cardano's formula for a cubic equation, implying the following roots:

$$\lambda_1 = \frac{\mathrm{Tr}(\mathscr{J})}{3} + (\zeta + \xi) \tag{A.2}$$

$$\lambda_{2,3} = \frac{\mathrm{Tr}(\mathscr{J})}{3} - \frac{(\zeta + \xi)}{2} \pm \sqrt{3}\ \frac{(\zeta - \xi)}{2} i \tag{A.3}$$

with $\zeta = \sqrt[3]{-\frac{\theta}{2} + \sqrt{\Delta}}$ and $\xi = \sqrt[3]{-\frac{\theta}{2} - \sqrt{\Delta}}$, where $\Delta = \left(\frac{\pi}{3}\right)^3 + \left(\frac{\theta}{2}\right)^2$ is the discriminant. Furthermore, $\pi = \frac{3\mathrm{B}(\mathscr{J}) - \mathrm{Tr}(\mathscr{J})^2}{3}$, $\theta = -\ \mathrm{Det}(\mathscr{J}) - 2\frac{\mathrm{Tr}(\mathscr{J})^3}{27} + \frac{\mathrm{Tr}(\mathscr{J})\mathrm{B}(\mathscr{J})}{3}$, and $i = \sqrt{-1}$ is the imaginary unit. Hence, if $\Delta > 0$, (A.1) exhibits one real root (λ_1) and a pair of complex conjugate eigenvalues ($\lambda_{2,3}$), in the form $\lambda_1 = \varphi$ and $\lambda_{2,3} = \psi \pm \omega i$, which represents a saddle-focus equilibrium dynamics, necessary for the emergence of the chaotic attractor in the Shilnikov theorem. In addition, by applying the Routh-Hurwitz criterion to the characteristic Eq. (A.1), we can determine if the equilibrium is unique (and stable) or if the solution is indeterminate, depending on the sign change in the scheme:

$$-\ \mathrm{Tr}(\mathscr{J})\quad - B(\mathscr{J}) + \frac{\mathrm{Det}(\mathscr{J})}{\mathbf{Tr}(\mathscr{J})}\ \mathrm{Det}(\mathscr{J}) \tag{A.4}$$

In the case proposed in Lemma 1, if condition (18) holds, then we have that $\mathrm{B}(\mathscr{J}) < 0$ and $\mathrm{Det}(\mathscr{J}) > 0$, whereas $\mathrm{Tr}(\mathscr{J})$ has no definite sign. Hence, we may have that:

(i) if $\mathbf{Tr}(\mathscr{J}) > 0$, then the sequence in (A.4) is
$-\ +\ +\ +$
(ii) if $\mathbf{Tr}(\mathscr{J}) < 0$, then the sequence in (A.4) is
$-\ -\ \pm\ +$

In either case, the sequence implies one variation and two permanence of sign. This means that (A.1) has three roots, one of which is real and positive and the other two exhibit negative real parts. The resulting equilibrium is therefore always indeterminate.

B. Saddle quantity

Assume the standard saddle quantity definition:

$$s = |\varphi\ | - |\psi| \tag{B.1}$$

and substitute the explicit values for φ and ψ obtained in (A.2) and (A.3), respectively. (B.1) can also written as:

$$s = \sqrt{\left(\frac{\mathrm{Tr}(\mathscr{J})}{3} + \zeta + \xi\right)^2} - \sqrt{\left(\frac{\mathrm{Tr}(\mathscr{J})}{3} - \frac{\zeta + \xi}{2}\right)^2}$$

which can be rationalized to:

$$s = \frac{\left(\frac{\mathrm{Tr}(\mathscr{J})}{3} + \zeta + \xi\right)^2 - \left(\frac{\mathrm{Tr}(\mathscr{J})}{3} - \frac{\zeta+\xi}{2}\right)^2}{\sqrt{\left(\frac{\mathrm{Tr}(\mathscr{J})}{3} + \zeta + \xi\right)^2} + \sqrt{\left(\frac{\mathrm{Tr}(\mathscr{J})}{3} - \frac{\zeta+\xi}{2}\right)^2}} = \frac{\frac{3}{2}(\zeta + \xi)\left(\frac{2\mathrm{Tr}(\mathscr{J})}{3} + \frac{\zeta+\xi}{2}\right)}{\sqrt{\left(\frac{\mathrm{Tr}(\mathscr{J})}{3} + \zeta + \xi\right)^2} + \sqrt{\left(\frac{\mathrm{Tr}(\mathscr{J})}{3} - \frac{\zeta+\xi}{2}\right)^2}}$$

Since the saddle-focus dynamics implied by the Shilnikov theorem requires a spiraling structure of the complex conjugate eigenvalues, and a convergence to the long run equilibrium through the stable arm of the leading (real) eigenvalue. it is therefore necessary that $\varphi = \frac{\mathrm{Tr}(\mathscr{J})}{3} + \zeta + \xi < 0$ and $\psi = \frac{\mathrm{Tr}(\mathscr{J})}{3} - \frac{\zeta+\xi}{2} > 0$, with $\varphi\psi < 0$. This also implies that $\zeta + \xi < 0$, because if we suppose that $\zeta + \xi > 0$, then we should have $\mathrm{Tr}(\mathscr{J}) < 0$ to have $\varphi < 0$. But this would determine $\psi < 0$, which contradicts the assumptions of the theorem. Hence, $s > 0$ needs:

$$\Sigma \equiv \frac{2\mathrm{Tr}(\mathscr{J})}{3} + \frac{\zeta + \xi}{2} < 0$$

which represents the locus for a positive saddle quantity associated to system (S).

C. The Hypernormal form computation

Consider system (S) in the following extended form:

$$\begin{pmatrix} \dot{v}\dot{u}\dot{\sigma} \end{pmatrix} = \mathscr{J}\begin{pmatrix} v \\ u \\ \sigma \end{pmatrix} + \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \tag{C.1}$$

where the g_i elements are the second order terms that result after a Taylor expansion of the original system. Since the Shilnikov theorem in Definition 1 requires that $\mathscr{J}$ exhibits a structure of one negative, φ, and a pair of complex conjugate eigenvalues, $\psi \pm \omega i$, the appropriate eigenbasis needs to solve the following system:

$$\begin{aligned}\mathscr{J}\mathbf{u} &= \psi\mathbf{u} - \omega\mathbf{v}\\ \mathscr{J}\mathbf{v} &= \omega\mathbf{u} + \psi\mathbf{v}\\ \mathscr{J}\mathbf{z} &= \varphi\mathbf{z}\end{aligned} \tag{C.2}$$

where:

$$\mathbf{u} = \begin{pmatrix} u_1\\ u_2\\ u_3\end{pmatrix} = \begin{pmatrix} \dfrac{v_3 j^*_{13} - \psi}{\omega}\\ \dfrac{v_3 j^*_{23}}{\omega}\\ \dfrac{v_3\left(j^*_{33} - \psi\right)}{\omega}\end{pmatrix}, \ \mathbf{v} = \begin{pmatrix} v_1\\ 0\\ v_3\end{pmatrix} = \begin{pmatrix} 1\\ 0\\ \varphi/j^*_{13}\end{pmatrix}, \ \mathbf{z} = \begin{pmatrix} z_1\\ z_2\\ z_3\end{pmatrix} = \begin{pmatrix} j^*_{23}j^*_{12} - \left(j^*_{22} - \varphi\right)j^*_{13}\\ \varphi j^*_{23}\\ -\varphi\left(j^*_{22} - \varphi\right)\end{pmatrix}$$

being $\mathscr{J} = \begin{pmatrix} j^*_{11} & j^*_{12} & j^*_{13}\\ j^*_{21} & j^*_{22} & j^*_{23}\\ j^*_{31} & j^*_{32} & j^*_{33}\end{pmatrix}$ **as in** (15).

The transformation matrix $T = [\mathrm{u}, \mathrm{v}, \mathrm{z}]$ permits to convert system (C.1) in the new coordinates:

$$\begin{pmatrix} \dot{w}_1\\ \dot{w}_2\\ \dot{w}_3\end{pmatrix} = \begin{pmatrix} \psi & -\omega & 0\\ \omega & \psi & 0\\ 0 & 0 & \varphi\end{pmatrix}\begin{pmatrix} w_1\\ w_2\\ w_3\end{pmatrix} + \begin{pmatrix} F_1\\ F_2\\ F_3\end{pmatrix} \tag{C.3}$$

where:

$$\begin{pmatrix} F_1\\ F_2\\ F_3\end{pmatrix} = \begin{pmatrix} \overline{F}_{1a}w_1w_2 + \overline{F}_{1b}w_1w_3 + \overline{F}_{1c}w_2w_3 + \overline{F}_{1d}w_1^2 + \overline{F}_{1e}w_2^2 + \overline{F}_{1f}w_3^2\\ \overline{F}_{2a}w_1w_2 + \overline{F}_{2b}w_1w_3 + \overline{F}_{2c}w_2w_3 + \overline{F}_{2d}w_1^2 + \overline{F}_{2e}w_2^2 + \overline{F}_{2f}w_3^2\\ \overline{F}_{3a}w_1w_2 + \overline{F}_{3b}w_1w_3 + \overline{F}_{3c}w_2w_3 + \overline{F}_{3d}w_1^2 + \overline{F}_{3e}w_2^2 + \overline{F}_{3f}w_3^2\end{pmatrix}$$

are the nonlinear terms in the new variables, and the $\overline{F}_{ij}$ coefficients are combinations of the original parameters of the model. Therefore,:

$$\begin{pmatrix} v\\ u\\ \sigma\end{pmatrix} = \mathbf{T}\begin{pmatrix} w_1\\ w_2\\ w_3\end{pmatrix} = \begin{bmatrix} u_1 & 1 & z_1\\ u_2 & 0 & z_2\\ u_3 & v_3 & z_3\end{bmatrix}\begin{pmatrix} w_1\\ w_2\\ w_3\end{pmatrix} = \begin{pmatrix} u_1w_1 + w_2 + z_1w_3\\ u_2w_1 + z_2w_3\\ u_3w_1 + v_3w_2 + z_3w_3\end{pmatrix}$$

is the coordinate change operation.

Finally, using the algorithm proposed by Gamero et al. (1999), system (C.3) can be transformed into a hypernormal form, that simplifies the nonlinear parts, with the following unfoldings:

$$\begin{pmatrix} \dot{x}\dot{y}\dot{z}\end{pmatrix} = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3\end{bmatrix}\begin{pmatrix} x\\ y\\ z\end{pmatrix} + \begin{pmatrix} 0\\ 0\\ dx^2 + hx^3\end{pmatrix} \tag{C.4}$$

where, as already defined in the text, $\varepsilon_1 = \mathbf{Det}(\mathscr{J})$, $\varepsilon_2 = -\mathbf{B}(\mathscr{J})$, $\varepsilon_3 = \mathbf{Tr}(\mathscr{J})$**, and where** $d = \overline{F}_{3d}$ **and** $h = \overline{F}_{1a}\overline{F}_{3d} - \overline{F}_{1d}\overline{F}_{3a} + \overline{F}_{2e}\overline{F}_{1d} - \overline{F}_{2d}\overline{F}_{3b} - \overline{F}_{3c}\overline{F}_{3d}$**.**

D. *The OGY algorithm*

The algorithm for proving the controllability of a given system requires that the nonlinear system be written in state-space notation. We first put the linear part of system (20) in the following form:

$$\dot{\mathbf{w}} = \mathscr{J}\mathbf{w} + \mathbf{MKw}, \tag{D.1}$$

where $\mathbf{w} = (w_1, w_2, w_3)^T$, while $\mathscr{J}$ is as in (15). Moreover, $\mathbf{M} = \left(\frac{\partial \dot{w}_1}{\partial \tau^s}, \frac{\partial \dot{w}_2}{\partial \tau^s}, \frac{\partial \dot{w}_3}{\partial \tau^s}\right)^T$, while $\mathbf{K} = (k_1, k_2, k_3)$ is a (1×3) vector. System (D.1) is then put into the following first-companion form:

$$\dot{\boldsymbol{\omega}} = (\mathbf{A} - \mathbf{BK})\boldsymbol{\omega}. \tag{D.2}$$

where the vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T$ results from the transformation $\mathbf{w} = \mathbf{P}\omega$, and $\boldsymbol{A} = \mathbf{P}^{-1}\mathscr{J}\mathbf{P}$ is given by:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3\end{bmatrix}, \tag{D.3}$$

where $\mathbf{B} = \mathbf{P}^{-1}\mathbf{M}$. In detail, the transformation matrix $\mathbf{P}$ must be chosen to satisfy the product $\mathbf{P} = \mathbf{NW}$, with

$$\mathbf{N} = \left[\mathbf{B}, \mathscr{J}\mathbf{B}, \mathscr{J}^2\mathbf{B}\right] \tag{D.4}$$

and

$$\mathbf{W} = \begin{bmatrix} \varepsilon_2 & \varepsilon_3 & 1 \\ \varepsilon_3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{D.5}$$

Controllability requires matrix $\mathbf{N}$ to have full rank. Since, in our case, matrix $\mathbf{A}$ is non-degenerate, the controllability of system (20) by means of changes in τ^s is feasible and produces the following sign sequence: $\varepsilon_1 = \mathrm{Det}(\mathscr{J})\langle 0,\ \varepsilon_2 = -B(\mathscr{J})\rangle 0,\ \varepsilon_3 = \mathrm{Tr}(\mathscr{J}) < 0$ and $[-B(\mathscr{J}) + \mathrm{Det}(\mathscr{J})/\mathrm{Tr}(\mathscr{J})] > 0$, so that our characteristic Eq. (A.1) may exhibit three solutions: one real (and negative) eigenvalue and two eigenvalues with positive real parts, which assures that the solution of system (S) is fully stable.∎

Data availability

No data was used for the research described in the article.

References

Acemoglu, D., Jackson, M.O., 2015. History, expectations, and leadership in the evolution of social norms. Rev. Econ. Stud. 82 (2), 423–456.

Alexeeva, T., Diep, Q.B., Kuznetsov, N., Zelinka, I., 2023. Forecasting and stabilizing chaotic regimes in two macroeconomic models via artificial intelligence technologies and control methods. Chaos Solit. Fractals 170, 113377.

Araujo, R.A., Davila-Fernandez, M.J., Nunes Moreira, H., 2019. Some new insights on the empirics of Goodwin's growth-cycle model. Struct. Change Econ. Dyn. 51, 42–54.

Bailly, H., Mortier, F., Giraud, G., 2024. Empirical analysis of a Debt-augmented Goodwin model for the United States. Struct. Change Econ. Dyn. 70, 619–633.

Barbosa-Filho, N.H., Taylor, L., 2006. Distributive and demand cycles in the US economy - a structuralist Goodwin Model. Metroeconomica 57 (3), 389–411.

Barrales-Ruiz, J., Mendieta-Muñoz, I., Rada, C., Tavani, D., Von Arnim, R., 2021. The distributive cycle: evidence and current debates. J. Econ. Surv. 36 (2), 468–503.

Bella, G., Mattana, P., 2020. Chaos control in presence of financial bubbles. Econ. Lett. 193, 109314.

Bella, G., Mattana, P., Venturi, B., 2017. Shilnikov chaos in the Lucas model of endogenous growth. J. Econ. Theory 172, 451–477.

Bullard, J., Butler, A., 1993. Nonlinearity and chaos in economic models: implications for policy decisions. Econ. J. 103 (419), 849–867.

Cajas Guijarro, J., 2024. An extended Goodwin model with endogenous technical change and labor supply. Struct. Change Econ. Dyn. 70, 699–710.

Davila-Fernandez, M., Sordi, S., 2019. Path dependency, distributive cycles and export capacity in a BoPC growth model. Struct. Change Econ. Dyn. 50, 258–272.

Desai, M., Mosley, H., Pemberton, M., 2006. A clarification of the Goodwin model of the growth cycle. J. Econ. Dyn. Control 30 (2), 661–670.

Di Matteo, M., 1984. Alternative monetary policies in a classical growth cycle. In: Goodwin, R.M., Kruger, M., Vercelli, A. (Eds.), Nonlinear Models of Fluctuating Growth. Springer-Verlag, Berlin, pp. 14–24.

Di Matteo, M., Sordi, S., 2015. Goodwin in Siena: economist, social philosopher and artist. Camb. J. Econ. 39 (6), 1507–1527.

Enthoven, A.C., Arrow, K.J., 1956. A theorem on expectations and the stability of equilibrium. Econometrica 24 (3), 288–293.

Flaschel, P., 2009. The Macrodynamics of Capitalism: Elements for a Synthesis of Marx, Keynes, and Schumpeter. Springer.

Flaschel, P., 2015. Goodwin's MKS system: a baseline macro model. Camb. J. Econ. 39 (6), 1591–1605.

Foley, D.K., 2003. Endogenous technical change with externalities in a classical growth model. J. Econ. Behav. Organ. 52, 167–189.

Freire, E., Gamero, E., Rodriguez-Luis, A.J., Algaba, A., 2002. A note on the triple-zero linear degeneracy: normal forms, dynamical and bifurcation behaviors of an unfolding. Int. J. Bifurc. Chaos 12, 2799–2820.

Fukao, K., Benabou, R., 1993. History versus expectations: a comment. Q. J. Econ. 108 (2), 535–542.

Gamero, E., Freire, E., Rodríguez-Luis, A.J., Ponce, E., Algaba, A., 1999. Hypernormal form calculation for triple-zero degeneracies. Bull. Belg. Math. Soc. Simon Stevin 6 (3), 357–368.

Goodwin, R., 1990. The complex dynamics of innovation, output, and employment. Struct. Change Econ. Dyn. 1 (1), 119–131.

Goodwin, R.M., 1951. The nonlinear accelerator and the persistence of business cycles. Econometrica 19 (1), 1–16.

Goodwin, R.M., 1967. A growth cycle. In: Feinstein, C.H. (Ed.), Socialism, Capitalism and Economic Growth. Cambridge University Press, Cambridge, pp. 54–58.

Grassetti, F., Guzowska, M., Michetti, E., 2020. A dynamically consistent discretization method for Goodwin model. Chaos Solit. Fractals 130, 109420.

Ishiyama, K., Saiki, Y., 2005. Unstable periodic orbits embedded in a chaotic economic dynamics model. Appl. Econ. Lett. 12, 749–753.

Jakimowicz, A., 2010. Numerical explorations of R. M. Goodwin's business cycle model. Nonlinear Dyn. Psychol. Life Sci. 14 (1), 69–83.

Kelsey, D., 1988. The economics of chaos or the chaos of economics. Oxf. Econ. Pap. 40 (1), 1–31.

Krugman, P., 1991. Increasing returns and economic geography. J. Political Econ. 99, 483–499.

Makovínyiová, K., Zimka, R., 2011. On the bifurcation of a torus in a small open economy model. Tatra Mt. Math. Publ. 48 (1), 125–134.

Manfredi, P., Fanti, L., 2004. Cycles in dynamic economic modelling. Econ. Model. 21, 573–594.

Matsumoto, A., 2009. Note on Goodwin's 1951 nonlinear accelerator model with an investment delay. J. Econ. Dyn. Control 33, 832–842.

Matsuyama, K., 1991. Increasing returns, industrialization, and indeterminacy of equilibrium. Q. J. Econ. 106, 617–650.

Ott, E., Grebogi, C., Yorke, J.A., 1990. Controlling chaos. Phys. Rev. Lett. 64, 1196–1199.

Rada, C., Tavani, D., von Arnim, R., Zamparelli, L., 2023. Classical and Keynesian models of inequality and stagnation. J. Econ. Behav. Organ. 211, 442–461.

Sasaki, H., 2013. Cyclical growth in a Goodwin-Kalecki-Marx model. J. Econ. 108 (2), 145–171.

Setterfield, M., Wheaton, G., 2024. Animal spirits and the Goodwin pattern. SSRN Working Paper 482049. https://doi.org/10.2139/ssrn.4820492.

Shah, A., Desai, M., 1981. Growth cycles with induced technical change. Econ. J. 91, 1006–1010.

Shilnikov, L.P., 1965. A case of the existence of a denumerable set of periodic motions. Sov. Math. Dokl. 6, 163–166.

Skott, P., 1989. Effective demand, class struggle and cyclical growth. Int. Econ. Rev. 30 (1), 231–247 (Philadelphia).

Sordi, S., Vercelli, A., 2014. Unemployment, income distribution and debt-financed investment in a growth cycle model. J. Econ. Dyn. Control 48, 325–348.

Sportelli, M., De Cesare, L., 2022. A Goodwin type cyclical growth model with two-time delays. Struct. Change Econ. Dyn. 61, 95–102.

Sportelli, M.C., 1995. A Kolmogorov generalized predator-prey model of Goodwin's growth cycle. J. Econ. 61 (1), 35–64.

Van der Ploeg, F., 1987. Growth cycles, induced technical change, and perpetual conflict over the distribution if income. J. Macroecon. 9 (1), 1–12.

Von Arnim, R., Barrales, J., 2015. Demand-driven Goodwin cycles with Kaldorian and Kaleckian features. Rev. Keynes. Econ. 3 (3), 351–373.

Wiggins, S., 1991. Introduction to Applied Nonlinear Dynamical Systems and Chaos. Springer-Verlag, New York.

Yoshida, H., Asada, T., 2007. Dynamic analysis of policy lag in a Keynes-Goodwin model: stability, instability, cycles and chaos. J. Econ. Behav. Organ. 62 (3), 441–469.