Andrea Loi and Roberto Mossa, *Rigidity properties of holomorphic isometries into homogeneous kähler manifolds* in *Proceedings of the American Mathematical* Society, Vol. 152 (July 2024), Issue 7, Pages 3051 - 30621

The publisher's version is available at:

http://dx.doi.org/10.1090/proc/16754

RIGIDITY PROPERTIES OF HOLOMORPHIC ISOMETRIES INTO HOMOGENEOUS KÄHLER MANIFOLDS

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Abstract. We prove two rigidity results on holomorphic isometries into homogeneous Kähler manifolds. The first shows that a Kähler-Ricci soliton induced by the homogeneous metric of the Kähler product of a special generalized flag manifold (i.e. a flag of classical type or integral type) with a bounded homogeneous domain is trivial, i.e. Kähler-Einstein. In the second one we prove that: (i) a flat space is not relative to the Kähler product of a special generalized flag manifold with a homogeneous bounded domain, (ii) a special generalized flag manifold is not relative to the Kähler product of a flat space with a homogeneous bounded domain and (iii) a homogeneous bounded domain is not relative to the Kähler product of a flat space with a special generalized flag manifold. Our theorems strongly extend the results in [5], [6], [13], [14] and [24].

Contents

1. Introduction 1
2. A technical result on holomorphic Nash algebraic functions 5
3. The proofs of Theorem 1.1 and Theorem 1.2 8
References 12

1. Introduction

It is an interesting and classical problem to study rigidity properties of holomorphic isometries (i.e. Kähler immersions) from a Kähler manifold (M, g) into another Kähler manifold (S, g_S), namely those holomorphic maps $\varphi : M \to S$ such that $\varphi^* g_S = g$. The word *rigidity* has acquired several different meanings in the

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2000 *Mathematics Subject Classification.* 53C55, 32Q15, 53C24, 53C42 .

Key words and phrases. Kähler metrics, , Kähler-Einstein metrics; Kähler-Ricci solitons; homogeneous Kähler manifold; relatives Kähler manifolds; Rigidity properties.

The authors are supported by INdAM and GNSAGA - Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni, by GOACT - Funded by Fondazione di Sardegna and partially funded by PNRR e.INS Ecosystem of Innovation for Next Generation Sardinia (CUP F53C22000430001, codice MUR ECS00000038).

mathematical literature. One can ideally subdivide the known rigidity phenomena into three main directions, closely related to each other. The first direction tends to analyze when a local holomorphic isometry $\varphi : U \to S$ extends to M and its unicity. For example Calabi in his celebrated work [3] (see also [19] and [28]) proves that a holomorphic isometry of a real analytic Kähler manifold (M, g) into a finite or infinite dimensional complex space form (S, g_S) is unique up to rigid motions (see also [9] where it is shown that this is generically true for arbitrary real analytic Kähler manifolds) and the global extendability of a local holomorphic isometry when M is simply-connected. An interesting result in this direction is due to Huang and Yuan in [10] (see also [11]) where they study some algebro-geometric conditions for the extendibility of local holomorphic isometries between product of Hermitian symmetric spaces of compact type. Regarding the unicity of holomorphic isometries between bounded symmetric domains the reader is referred to [4, 21, 22, 25, 26, 27, 29].

The second direction deals with the following general question: *if the Kähler metric g_S on S is homogeneous, what can be said on the Kähler metric g on M if one assumes that it is* canonical *in a broad sense, e.g. it is Kähler-Einstein (KE) or more generally it has constant scalar curvature or it is extremal or it is a Kähler-Ricci soliton (KRS)?*

The case of holomorphic isometries of a Kähler-Einstein manifold (M, g) into a finite or infinite dimensional complex space form has attracted the interest of many mathematicians by becoming a subject on its own right (the reader is referred to Chapter 4 in [19] for an update material on the subject). For the case of holomorphic isometries of a complex manifold equipped with an extremal metric into finite or infinite dimensional complex space forms the reader is referred to [17].

Regarding KRS one has the following (see also [18]):

Theorem A ([13] and [14, (i) of Theorem 1.1]) *If (g, X) is a KRS on a complex manifold M (g is a Kähler metric and X is the solitonic vector field) and (M, g) admits a holomorphic isometry into either a finite dimensional (definite or indefinite) complex space form or a homogeneous bounded domain, then the soliton is trivial, i.e. g is KE.*

Recall that a homogeneous bounded domain is a complex bounded domain $\Omega \subset \mathbb{C}^n$ equipped with a homogeneous Kähler metric g_Ω. Notice that bounded symmetric domains with the Bergman metric are a very special examples of homogeneous bounded domains and that it could exist many homogeneous Kähler metrics on a bounded domain different from the Bergman metric (see [8] for details).

The third direction studies the obstruction of the existence of holomorphic isometries of a Kähler manifold (M, g) (of positive dimension) into two given Kähler manifolds (S, g_S) and $(S', g_{S'})$. If two such holomorphic isometries exist, (S, g_S) and $(S', g_{S'})$ are said to be *relatives* ([24] and [7]). For update results on relatives Kähler manifolds the reader is referred the survey paper [28] (see also the recent preprint [12]). Recently A. Loi and R. Mossa extend the result in [5] for bounded symmetric domains to arbitrary homogeneous domain, by proving the following:

Theorem B ([14, (ii) of Theorem 1.1]) *A homogeneous bounded domain and the (definite or indefinite) complex Euclidean space are not relatives.*

In this paper we address the problem of extending Theorem A and Theorem B to arbitrary homogeneous Kähler manifolds, i.e. those Kähler manifolds which are acted upon transitively by their biholomorphic isometry group.

Besides the homogeneous bounded domains, important examples of homogeneous Kähler manifolds are the flat ones and the generalized flag manifolds, the later being compact and simply-connected Kähler homogeneous manifolds. In this paper we restrict to the class of *special generalized flag manifolds* defined as follows.

Definition. *A generalized flag manifold $(\mathcal{C}, g_{\mathcal{C}})$ is said to be* special *if it satisfies one of the following conditions.*

(i) *$\mathcal{C}$ is of classical type;*
(ii) *$\mathcal{C}$ is of integral type, namely there exists a positive real number $\lambda \in \mathbb{R}^+$ such that $\lambda\omega_{\mathcal{C}}$ is integral (i.e. $\lambda\omega_{\mathcal{C}} \in H^2(\mathcal{C}, \mathbb{Z})$), where $\omega_{\mathcal{C}}$ denotes the Kähler form associated to $g_{\mathcal{C}}$.*

Recall that a generalized flag manifold $\mathcal{C} = G/K$ is of *classical type* if G is a semisimple compact Lie group of classical type (see, e.g. [16]).

All the generalized flag manifolds with second Betti number equals to one are examples of generalized flag manifold of integral type. In particular all the Hermitian symmetric spaces of compact type with any homogeneous Kähler metric are generalized flag manifolds of integral type. Other examples of generalized flag manifolds of integral type are the KE ones since in this case the homogeneous Kähler form is integral being the first Chern class of the anticanonical bundle. Finally, one can construct examples of generalized flag manifolds of classical type but not of integral type by taking the Kähler product $(\mathcal{C} \times \mathcal{C}, \sqrt{2}g_{\mathcal{C}} \times g_{\mathcal{C}})$, where $(\mathcal{C}, g_{\mathcal{C}})$ is a generalized flag manifold of both classical and integral type.

The following theorems (and their corollaries) represent our main results.

Theorem 1.1. *Let (M, g) be a KRS. If (M, g) admits a holomorphic isometry into the Kähler product $\mathcal{C} \times \Omega$ of a special generalized flag manifold $\mathcal{C}$ and a homogeneous bounded domain Ω, then g is KE.*

By combining this theorem with Theorem A we immediately gets:

Corollary 1. *Let (M, g) be a Kähler manifold which admits a holomorphic isometry into either a (definite or indefinite) flat space $\mathcal{E}$, a special generalized flag manifold $\mathcal{C}$ or a homogeneous bounded domain Ω. Then any KRS on (M, g) is trivial.*

Remark 1. Theorem 1.1 cannot be extended to the Kähler product of a flat space with either a special generalized flag manifold or a homogeneous bounded domain. Indeed, one can easily exhibit a non-trivial KRS on $\mathbb{C} \times \mathbb{C}P^1$ (where $\mathbb{C}\mathrm{P}^1$ is the complex one-dimensional complex projective space with the Fubini-Study metric g_{FS}) and on $\mathbb{C} \times \mathbb{C}H^1$ (where $\mathbb{C}H^1$ is the complex one-dimensional hyperbolic space with the hyperbolic metric g_{hyp}). Indeed, one can easily verify that the vector field X on $\mathbb{C} \times \mathbb{C}P^1$ (resp. on $\mathbb{C} \times \mathbb{C}\mathrm{H}^1$) defined by $X(z, q) := -2\left(z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}\right)$ (resp. $X(z, q) := 2\left(z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}\right)$ is a solitonic vector field but the Kähler metric $g_0 \oplus g_{FS}$ (resp. on $g_0 \oplus g_{hyp}$) is not KE.

Remark 2. Notice that if $(\mathcal{C}, g_{\mathcal{C}})$ is a generalized flag manifold of integral type with $\lambda\omega_{\mathcal{C}}$ integral, then $(\mathcal{C}, \lambda g_{\mathcal{C}})$ admits a holomorphic isometry into a complex projective space $(\mathbb{C}P^N, g_{FS})$ (see. e.g. the proof of Theorem 1 in [20]). Thus, in order to prove Corollary 1 for a generalized flag manifold of integral type one can assume that this flag manifold is $(\mathbb{C}P^N, g_{FS})$ and then the triviality of the KRS in Corollary 1 can be deduced by Theorem A above.

Theorem 1.2. *Let $\mathcal{E}$, $\mathcal{C}$ and Ω be respectively a flat manifold, a special generalized flag manifold and a homogeneous bounded domain. Then the following facts hold true.*

- (i) *$\mathcal{E}$ is not relative to the Kähler product $\mathcal{C} \times \Omega$;*
- (ii) *$\mathcal{C}$ is not relative to the Kähler product $\mathcal{E} \times \Omega$;*
- (iii) *Ω is not relative to the Kähler product $\mathcal{E} \times \mathcal{C}$.*

It is worth pointing out that it could exist three Kähler manifolds (M_1, g_1), (M_2, g_2) and (M_3, g_3) such that (M_1, g_1) is not relative to either (M_2, g_2) or (M_3, g_3) but it is relative to $(M_2 \times M_3, g_2 \oplus g_3)$. Therefore in Theorem 1.2 one cannot restrict to a single factor of the Kähler product involved in order to achieve the conclusion. For example the one-dimensional complex projective spaces $(\mathbb{C}\mathrm{P}^1, g_{FS})$ and $(\mathbb{C}\mathrm{P}^1, 2g_{FS})$ where g_{FS} is the Fubini Study metric, are not relatives. Indeed, assume by contradiction that there exist an open set $U \subset \mathbb{C}$ and two holomorphic isometries $\varphi_1 : U \to \mathbb{C}P^1$ and $\varphi_2 : U \to \mathbb{C}P^1$ such that $\varphi_1^* g_{FS} = \varphi_2^*(2g_{FS})$. By restricting U if necessary one can assume that this maps are injective and hence $\varphi = \varphi_1 \circ \varphi_2^{-1} : \varphi(U) \to \mathbb{C}\mathrm{P}^1$ would be a holomorphic map such that $\varphi^* g_{FS} = 2g_{FS}$, in contrast with [3, Theorem 13]. Nevertheless, $(\mathbb{C}P^1 \times \mathbb{C}P^1, g_{FS} \oplus g_{FS})$ and $(\mathbb{C}\mathrm{P}^1, 2g_{FS}))$ are relatives (the holomorphic map $\varphi : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1, q \mapsto (q, q)$ satisfies $\varphi^*(g_{FS} \oplus g_{FS}) = 2g_{FS}$).

As a very special case of Theorem 1.2 one gets the following appealing result which is a strong extension of the results in [13], [14].

Corollary 2. *Any two among a flat space $\mathcal{E}$, a special generalized flag manifold $\mathcal{C}$ or a homogeneous bounded domain Ω are not relatives.*

Since each compact homogeneous KE manifold is the Kähler product of a flat complex torus and generalized flag KE manifold (see, e.g. ([2])) Theorem 1.2 gives:

Corollary 3. *A homogeneous bounded domain is not relative to a compact KE homogeneous manifold.*

The proofs of Theorem 1.1 and of (i) of Theorem 1.2 are obtained by some results on KRS proved in [13] and [14] and on the structure of the Calabi diastasis function of special generalized flag manifolds (Lemma 3.1) and of homogeneous bounded domains (Lemma 3.2). On the other hand the proof of (ii) and (iii) of Theorem 1.2 and in particular the fact that a special generalized flag manifold is not relative to a homogeneous bounded domain, are based on Theorem 2.2 below, a new and technical result, dealing with transcendental properties of holomorphic Nash algebraic functions.

The paper contains two more sections. Section 2 is dedicated to the proof of Theorem 2.2 while Section 3 to the proofs of Theorems 1.1 and 1.2.

The authors would like to thank Yuan Yuan for all interesting discussions and comments on Nash algebraic holomorphic functions.

2. A TECHNICAL RESULT ON HOLOMORPHIC NASH ALGEBRAIC FUNCTIONS

Let $\mathcal{N}^m$ be the set of real analytic functions $\xi : V \subset \mathbb{C}^m \to \mathbb{R}$ defined in a neighbourhood $V \subset \mathbb{C}^m$ of the origin, such that its real analytic extension $\tilde{\xi}(z, w)$ around $(0, 0) \in V \times \mathrm{Conj}\, V$ is a holomorphic Nash algebraic function. We define

$$\mathcal{F} = \{\xi\,(f_1, \dots, f_m) \mid \xi \in \mathcal{N}^m,\ f_j \in \mathcal{O}_0,\ f_j(0) = 0,\ j = 1, \dots, m,\ m > 0\},$$

where $\mathcal{O}_0$ denotes the germ of holomorphic functions around $0 \in \mathbb{C}$. We set

$$\widetilde{\mathcal{F}} = \{\xi\,(f_1, \dots, f_m) \in \mathcal{F} \mid \xi \text{ is of diastasis-type},\ m > 0\}. \tag{1}$$

We say (see also [13]) that a real analytic function defined on a neighborhood U of a point p of a complex manifold M is of *diastasis-type* if in one (and hence any) coordinate system $\{z_1, \dots, z_n\}$ centered at p its expansion in z and $\bar{z}$ does not contains non constant purely holomorphic or anti-holomorphic terms (i.e. of the form z^j or $\bar{z}^j$ with $j > 0$).

Before proving Theorem 2.2 we need a lemma.

Lemma 2.1. *Let $f_0, f_1, \dots, f_s$ be non zero Nash algebraic holomorphic functions, such that*

$$f_0^{c_0} f_1^{c_1} \dots f_s^{c_s} = 1, \qquad c_0, \dots, c_s \in \mathbb{R}. \tag{2}$$

If $\{c_0, \dots, c_s\}$ are linearly independent over $\mathbb{Q}$, then each f_j, $j = 0, \dots, s$ is a constant function.

Proof. Assume for example that f_0 is not constant. Let $z_0 \in \mathbb{C} \cup \{\infty\}$ such that $\lim_{z \to z_0} f_0(z) = \infty$. Without loss of generality we can assume $z_0 \in \mathbb{C}$ otherwise we apply the following argument to the Nash algebraic holomorphic function $\frac{1}{f_0}$. Moreover, up to a translation, we can also assume that $z_0 = 0$.

By the Puiseux expansion we can write $f_\ell(z) = (1 + o(1)) d_\ell z^{k_\ell / N_\ell}$, where $d_\ell \in \mathbb{C} \setminus \{0\}$, $\frac{k_\ell}{N_\ell} \in \mathbb{Q}_{<0}$, $\ell = 1, \dots, s$. From (2) we have that $\frac{k_0 c_0}{N_0} + \dots + \frac{k_s c_s}{N_s} = 0$. Since $\lim_{z \to z_0} f_0(z) = \infty$, we see that k_0 and at least one of $k_1, \dots, k_s$ are not zero in contrast with the linearly independence of $\{c_0, \dots, c_s\}$ over $\mathbb{Q}$. □

The following theorem is the technical result needed in the proof of (ii) and (iii) of Theorem 1.2. It is a generalization of [6, Theorem 2.1 (iii)] (even for $s = 1$).

Theorem 2.2. *Let $\psi_\ell = \xi_\ell\left(f_{\ell,1}, \dots, f_{\ell,m_\ell}\right) \in \widetilde{\mathcal{F}}$, $\ell = 0, 1, \dots, s$, be such that*

$$\psi_1^{c_1} \dots \psi_s^{c_s} = \psi_0, \quad \psi_0(0) \neq 0, \quad c_1, \dots, c_s \in \mathbb{R}. \tag{3}$$

If $\{c_1, \dots, c_s, 1\}$ are linearly independent over $\mathbb{Q}$, then $\psi_1, \dots, \psi_s$ are constant functions.

Proof. Let us write

$$\psi_k = \xi_k\left(f_1^{(k)}, \dots, f_{m_k}^{(k)}\right) \in \mathcal{F}, \ k = 0, \dots, s. \tag{4}$$

We rename the functions involved

$$(\varphi_1, \dots, \varphi_t) = \left(f_1^{(0)}, \dots, f_{m_0}^{(0)}, \dots, f_1^{(s)}, \dots, f_{m_s}^{(s)}\right) \tag{5}$$

and set

$$S = \{\varphi_1, \dots, \varphi_t\}.$$

Let D be an open neighborhood of the origin of $\mathbb{C}$ on which each φ_j, $j = 1, \dots, t$, is defined. Consider the field $\mathfrak{R}$ of rational function on D and its field extension $\mathfrak{F} = \mathfrak{R}(S)$, namely, the smallest subfield of the field of the meromorphic functions on D, containing rational functions and the elements of S. Let l be the transcendence degree of the field extension $\mathfrak{F}/\mathfrak{R}$. If $l = 0$, then each element in

S is holomorphic Nash algebraic, hence $\psi_1, \ldots, \psi_s$ would be constant functions by Lemma 2.1. Assume then that $l > 0$. Without loss of generality we can assume that $\mathcal{G} = \{\varphi_1, \ldots, \varphi_l\} \subset S$ is a maximal algebraic independent subset over $\mathfrak{R}$. Then there exist minimal polynomials $P_j\left(z, X, Y\right)$, $X = (X_1, \ldots, X_l)$, such that

$$P_j\left(z, \Phi(z), \varphi_j(z)\right) \equiv 0, \ \forall j = 1, \ldots, t,$$

where $\Phi(z) = (\varphi_1(z), \ldots, \varphi_l(z))$.

Moreover, by the definition of minimal polynomial

$$\frac{\partial P_j\left(z, X, Y\right)}{\partial Y}\left(z, \Phi(z), \varphi_j(z)\right) \not\equiv 0, \ \forall j = 1, \ldots, t.$$

on D. Thus, by the algebraic version of the existence and uniqueness part of the implicit function theorem, there exist a connected open subset $U \subset D$ with $0 \in \overline{U}$ and Nash algebraic functions $\hat{\varphi}_j(z, X)$, defined in a neighborhood $\hat{U}$ of $\{(z, \Phi(z)) \mid z \in U\} \subset \mathbb{C}^n \times \mathbb{C}^l$, such that

$$\varphi_j(z) = \hat{\varphi}_j\left(z, \Phi(z)\right), \ \forall j = 1, \ldots, t.$$

for any $z \in U$.

Let us denote

$$\left(\hat{f}_1^{(0)}(z, X), \ldots, \hat{f}_{m_0}^{(0)}(z, X), \ldots, \hat{f}_1^{(s)}(z, X), \ldots, \hat{f}_{m_s}^{(s)}(z, X)\right) = (\hat{\varphi}_1(z, X), \ldots, \hat{\varphi}_t(z, X)),$$

(notice that $\hat{f}_i^{(k)}(z, \Phi(z)) = f_i^{(k)}(z)$ for all $k = 0, \ldots, s$ and $i = 1, \ldots, m_k$). We define

$$\hat{F}_k(z, X) := \left(\hat{f}_1^{(k)}(z, X), \ldots, \hat{f}_{m_k}^{(k)}(z, X)\right), \ k = 0, \ldots, s.$$

Consider the function

$$\Psi(z, X, w) := \tilde{\xi}_0\left(\hat{F}_0(z, X), F_0(w)\right)\tilde{\xi}_1\left(\hat{F}_1(z, X), F_1(w)\right)^{-c_1} \cdots \left(\tilde{\xi}_s\hat{F}_s(z, X), F_s(w)\right)^{-c_s},$$

where $\tilde{\xi}_j$ is the real analytic extension of ξ_j in a neighbourhood of $(0, 0) \in \mathbb{C}^{m_j} \times \operatorname{Conj}\mathbb{C}^{m_j}$ and $F_k(w) = \left(f_1^{(k)}(w), \ldots, f_{m_k}^{(k)}(w)\right)$. By shrinking U if necessary we can assume that $\Psi(z, X, w)$ is defined on $\hat{U} \times U$.

We claim that $\Psi(z, X, w)$ is identically equal to one on this set. Recalling that $\psi_k(z, w) = \tilde{\xi}_k\left(F_k(z), F_k(w)\right)$ is of diastasis-type, we see that $\tilde{\xi}_k(\hat{F}_k(z, X), F_k(0)) = \psi_k(0)$, $k = 0, \ldots, s$. Since $0 \in \overline{U}$, it follows by (3) that $\Psi(z, X, 0) \equiv 1$. Hence, in order to prove the claim, it is enough to show that the logarithmic differentiation

$\partial_w^{\log}\Psi := \partial_w \Psi/\Psi$ with respect w, namely

$$(\partial_w^{\log}\Psi)(z,X,w) = \frac{\partial_w \tilde{\xi}_0\left(\hat{F}_0(z,X), F_0(w)\right)}{\tilde{\xi}_0\left(\hat{F}_0(z,X), F_0(w)\right)} - c_1 \frac{\partial_w \tilde{\xi}_1\left(\hat{F}_1(z,X), F_1(w)\right)}{\tilde{\xi}_1\left(\hat{F}_1(z,X), F_1(w)\right)} - \cdots - c_s \frac{\partial_w \tilde{\xi}_s\left(\hat{F}_s(z,X), F_s(w)\right)}{\tilde{\xi}_s\left(\hat{F}_s(z,X), F_s(w)\right)}$$

vanishes for all $w \in U$. Assume, by contradiction, that there exists $w_0 \in U$ such that $(\partial_w^{\log}\Psi)(z,X,w_0) \not\equiv 0$. Since $(\partial_w^{\log}\Psi)(z,X,w_0)$ is Nash algebraic in (z,X) there exists a holomorphic polynomial $P(z,X,t) = A_d(z,X)t^d + \cdots + A_0(z,X)$ with $A_0(z,X) \not\equiv 0$ such that $P(z,X,(\partial_w^{\log}\Psi)(z,X,w_0)) = 0$. Since, by (3) and (4) we have $\Psi(z,\Phi(z),w) \equiv 1$, we get $(\partial_w^{\log}\Psi)(z,\Phi(z),w) \equiv 0$. Thus $A_0(z,\Phi(z)) \equiv 0$ which contradicts the fact that $\varphi_1(z),\dots,\varphi_l(z)$ are algebraic independent over $\mathfrak{R}$. Hence $(\partial_w^{\log}\Psi)(z,X,w) \equiv 0$ and the claim is proved.

Therefore

$$\tilde{\xi}_0\left(\hat{F}_0(z,X), F_0(w)\right) = \left(\tilde{\xi}_1\left(\hat{F}_1(z,X), F_1(w)\right)\right)^{c_1} \cdots \left(\tilde{\xi}_s\left(\hat{F}_s(z,X), F_s(w)\right)\right)^{c_s},$$

for every $(z,X,w) \in \hat{U} \times U$. By fixing $w \in U$ and applying Lemma 2.1 we deduce that $\psi_1,\dots,\psi_s$ are constant functions. The proof of the theorem is complete. □

3. THE PROOFS OF THEOREM 1.1 AND THEOREM 1.2

Given a real analytic Kähler metric g on a complex manifold M one can introduce in a neighborhood of a point $p \in M$, a very special Kähler potential D_p^g for the metric g, the celebrated *Calabi's diastasis function* (see [3] or to [19] for details). Among all the potentials the diastasis is characterized by the fact that in every coordinate system $\{z_1,\dots,z_n\}$ centered at p

$$D_p^g(z,\bar{z}) = \sum_{|j|,|k|\geq 0} a_{jk} z^j \bar{z}^k,$$

with $a_{j0} = a_{0j} = 0$ for all multi-indices j. Clearly the diastasis D_p^g is a function of diastasis-type as defined above.

The following two lemmata are crucial in both the proofs of Theorem 1.1 and Theorem 1.2.

Lemma 3.1. *Let $\mathcal{C}$ be a generalized flag manifold of special type and $g_{\mathcal{C}}$ its homogeneous metric. Then around any point $p \in \mathcal{C}$ there exists a system of coordinates, centered in the origin, such that the diastasis $D_0^{g_{\mathcal{C}}}$ satisfies*

$$e^{D_0^{g_{\mathcal{C}}}} \in \widetilde{F}^{c_1} \cdots \widetilde{F}^{c_s}, \qquad c_1,\dots,c_s \in \mathbb{R}^+. \tag{6}$$

Proof. Assume that $\mathcal{C}$ is of classical type. Given $p \in \mathcal{C}$ by [15, Theorem 3.5], there exists a system of Bochner coordinates $(z_1, \dots, z_n)$ centered at p and dense in $\mathcal{C}$ such that the diastasis associated of $g_{\mathcal{C}}$ in a neighborhood of p reads as

$$D_0^{g_{\mathcal{C}}}(z) = \sum_{j=1}^{s} c_j \log\left(h_j(z)\right) \tag{7}$$

where $c_1, \dots, c_s \in \mathbb{R}^+$ and $h_1, \dots, h_s$ are real analytic functions (see [15, Theorem 3.5] for their explicit expression), the Kähler potential (7) has been firstly constructed in [1, Proposition 8.2] then in [15, Theorem 3.5] it has been showed that it is a diastasis function. In [15, pag. 9] (see also [16]) it is proven that the functions $h_1, \dots, h_s$ are polynomials. Thus (6) holds true and by [1] we see that $g_{\mathcal{C}}$ is integer if and only if $c_1, \dots, c_s \in \mathbb{Z}^+$. Moreover, for any choice of $c_1, \dots, c_s \in \mathbb{R}^+$ in (7) we obtain the diastasis function of a homogeneous metric on $\mathcal{C}$. Assume now that $\mathcal{C}$ is of integral type, so that there exists a Kähler immersion $F : \mathcal{C} \to \mathbb{C}P^N$ such that $g_{\mathcal{C}} = \lambda F^* g_{FS}$, for some $\lambda \in \mathbb{R}^+$ (cfr. Remark 2). Fix a system of coordinates $z_1, \dots, z_n$ for $\mathcal{C}$ around p. From the hereditary property of the diastasis function, we get

$$D_0^{g_{\mathcal{C}}}(z) = \lambda \log\left(1 + \|F(z)\|^2\right),$$

where $\|F(z)\|^2 = |f_1(z)|^2 + \cdots + |f_N(z)|^2 \in \widetilde{F}$ and $f_1, \dots, f_N$ are the component of F written in the affine coordinates of $\mathbb{C}P^N$ around $f(0)$. Clearly (6) is satisfied with $s = 1$ and $c_1 = \lambda$ and $\omega_{\mathcal{C}}$ is integral if and only if $c_1 \in \mathbb{Z}^+$. □

Lemma 3.2. ([14, Proof of Theorem 1.1]) *Let (Ω, g_Ω) be a homogeneous bounded domain. Consider its realization as a Siegel domain of $\mathbb{C}^n$, then the diastasis function for the metric g_Ω at given point $v \in \Omega$ is given by:*

$$D_v^{g_\Omega}(u) = \sum_{k=1}^{r} \gamma_k \log\left(\frac{F_k(u, \overline{u})\, F_k(v, \overline{v})}{F_k(u, \overline{v})\, F_k(v, \overline{u})}\right), \tag{8}$$

where γ_k are positive real numbers and F_k are non constant rational holomorphic functions, $k = 1, \dots, r$. In particular

$$e^{D_0^{g_\Omega}(u)} \in \widetilde{F}^{\gamma_1} \cdots \widetilde{F}^{\gamma_r}. \tag{9}$$

We are now in the position to prove Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Let (g, X) be a KRS on M and let $\varphi : M \to \mathcal{C} \times \Omega$ be a holomorphic isometric immersion, i.e. $\varphi^*(g_{\mathcal{C}} \oplus g_\Omega) = g$, where $g_{\mathcal{C}}$ and g_Ω are the homogeneous Kähler metrics on $\mathcal{C}$ and Ω respectively. Choose complex coordinates $\{z_1, \dots, z_n\}$ for M centered at $p \in M$ where Calabi's diastasis function D_p^g is defined. By the hereditary property of the diastasis function, we have that

$$D_p^g(z) = D_{\varphi_{\mathcal{C}}(p)}^{g_{\mathcal{C}}}(\varphi_{\mathcal{C}}(z)) + D_{\varphi_\Omega(p)}^{g_\Omega}(\varphi_\Omega(z)), \tag{10}$$

on a neighbourhood of p and where $\varphi := (\varphi_{\mathcal{C}}, \varphi_{\Omega})$. From Lemma 3.1 and 3.2 we see that

$$e^{D^g_p(z)} \in \widetilde{F}^{c_1} \dots \widetilde{F}^{c_s} \widetilde{F}^{\gamma_1} \dots \widetilde{F}^{\gamma_r}.$$

Thus, by [14, Proposition 4.1], the metric g is forced to be KE. □

Proof of (i) of Theorem 1.2. Assume by contradiction that $\mathcal{E}$ and $\mathcal{C} \times \Omega$ are relatives. Thus there exist a neighbourhood $U \subset \mathbb{C}$ of the origin $0 \in \mathbb{C}$ and two holomorphic immersions $\varphi_{\mathcal{E}} : U \to \mathcal{E}$ and $\varphi : U \to \mathcal{C} \times \Omega$ such that

$$\varphi_{\mathcal{E}}^* g_0 = \varphi^* \left(g_{\mathcal{C}} \oplus g_{\Omega} \right), \tag{11}$$

where $g_{\mathcal{C}}$ and g_{Ω} are the homogeneous Kähler metrics on $\mathcal{C}$ and Ω respectively and $g_{\mathcal{E}}$ is the flat metric on $\mathcal{E}$. Since we are working locally we can assume that $\mathcal{E}$ is the complex Euclidean space $\mathbb{C}^m$, $g_{\mathcal{E}}$ is the flat metric g_0 on $\mathbb{C}^m$ and that $\varphi_{\mathcal{E}}(0) = 0 \in \mathbb{C}^m$, $m = \dim \mathcal{E}$.

Write $\varphi := (\varphi_{\mathcal{C}}, \varphi_{\Omega})$ with $\varphi_{\mathcal{C}} : U \to \mathcal{C}$ and $\varphi_{\Omega} : U \to \Omega$. Again from the hereditary property of the diastasis function we get

$$D^{g_0}_{\varphi_{\mathcal{E}}(0)}(\varphi_{\mathcal{E}}(z)) = D^{g_{\mathcal{C}}}_{\varphi_{\mathcal{C}}(0)}(\varphi_{\mathcal{C}}(z)) + D^{g_{\Omega}}_{\varphi_{\Omega}(0)}(\varphi_{\Omega}(z))$$

From Lemma 3.1 and 3.2 we see that

$$e^{D^{g_0}_{\varphi_{\mathcal{E}}(0)}(z)} \in \widetilde{F}^{c_1} \dots \widetilde{F}^{c_s} \widetilde{F}^{\gamma_1} \dots \widetilde{F}^{\gamma_r}.$$

Thus, by the transcendental properties of holomorphic Nash algebraic functions proved in [14, Theorem 2.1] we deduce that $\varphi_{\mathcal{E}}$ is constant, in contrast with the hypothesis that $\varphi_{\mathcal{E}}$ is an immersion. The proof of (i) is completed. □

Proof of (ii) and (iii) of Theorem 1.2. We start by proving (ii) and (iii) in the special case $\mathcal{E}$ is zero dimensional, namely we prove that $\mathcal{C}$ and Ω are not relatives. As we have already pointed out at in the introduction its proof strongly relies on Theorem 2.2.

Let $U \subset \mathbb{C}$ be a neighbourhood of the origin and assume by contradiction that there exists two holomorphic immersions $\varphi_{\mathcal{C}} : U \to \mathcal{C}$ and $\varphi_{\Omega} : U \to \Omega$ such that

$$\varphi_{\mathcal{C}}^* g_{\mathcal{C}} = \varphi_{\Omega}^* g_{\Omega}, \tag{12}$$

with $\varphi_{\mathcal{C}} : U \to \mathcal{C}$ and $\varphi_{\Omega} : U \to \Omega$. So that in a neighbourhood of $0 \in \mathbb{C}$ we have

$$D^{g_{\mathcal{C}}}_{\varphi_{\mathcal{C}}(0)}(\varphi_{\mathcal{C}}(z)) = D^{g_{\Omega}}_{\varphi_{\Omega}(0)}(\varphi_{\Omega}(z)).$$

Let us pass to the coordinates centered at $\varphi_{\mathcal{C}}(0)$ for $\mathcal{C}$ given by Lemma 3.1 and to the realization of Ω as a Siegel domain. From Lemma 3.1 and Lemma 3.2 combined with (7) and (8) we see that in a neighbourhood of $0 \in \mathbb{C}$ there exist $\psi_1, \dots, \psi_s$,

$\phi_1, \dots, \phi_r \in \widetilde{F}$ such that

$$D^{g_{\mathcal{C}}}_{\varphi_{\mathcal{C}}(0)}(\varphi_{\mathcal{C}}(z)) = \log\left(\psi_1^{c_1} \cdots \psi_s^{c_s}\right) \tag{13}$$

and

$$D^{g_\Omega}_{\varphi_\Omega(0)}(\varphi_\Omega(z)) = \log\left(\phi_1^{\gamma_1}, \dots, \phi_r^{\gamma_r}\right). \tag{14}$$

Up to indexes order, we can assume that there exist $1 \leq s' \leq s$, such that $\{c_1, \dots, c_{s'}\} \subset \{c_1, \dots, c_s\}$ are maximal subsets of linearly independent real numbers over $\mathbb{Q}$. Thus for suitable coefficients $a_{jk} \in \mathbb{Q}$, we have

$$c_j = \sum_{k=1}^{s'} a_{jk} c_k, \quad j = s'+1, \dots, s. \tag{15}$$

Set

$$\tilde{\psi}_k := \psi_k \psi_{s'+1}^{a_{s'+1,k}} \cdot \ldots \cdot \psi_s^{a_{sk}}, \quad k = 1, \dots s'. \tag{16}$$

Therefore we have

$$1 = e^{D^{g_{\mathcal{C}}}_{\varphi_{\mathcal{C}}(0)}(\varphi_{\mathcal{C}}(z)) - D^{g_\Omega}_{\varphi_\Omega(0)}(\varphi_\Omega(z))} = \tilde{\psi}_1^{c_1} \cdot \ldots \cdot \tilde{\psi}_{s'}^{c_{s'}} \cdot \phi_1^{-\gamma_1} \cdot \ldots \cdot \phi_r^{-\gamma_r}. \tag{17}$$

Let us complete $\{c_1, \dots, c_{s'}\}$ to a maximal subset

$$\{c_1, \dots, c_{s'}, \gamma_1, \dots, \gamma_{r'}\} \subset \{c_1, \dots, c_{s'}, \gamma_1, \dots, \gamma_r\}, \qquad 0 \leq r' \leq r,$$

of linearly independent real numbers over $\mathbb{Q}$ (with $r' = 0$ we mean that $\{c_1, \dots, c_{s'}\}$ is already maximal). So that

$$\gamma_j = \sum_{k=1}^{s'} b_{jk} c_k + \sum_{k=1}^{r'} t_{jk} \gamma_k, \qquad j = r'+1, \dots, r$$

we can rewrite (17) as follows

$$\begin{aligned} 1 = &\left(\tilde{\psi}_1 \phi_{r'+1}^{-b_{r'+1,1}} \cdots \phi_r^{-b_{r1}}\right)^{c_1} \cdots \left(\tilde{\psi}_{s'} \phi_{r'+1}^{-b_{r'+1,s'}} \cdots \phi_r^{-b_{rs'}}\right)^{c_{s'}} \\ &\cdot \left(\phi_1 \phi_{r'+1}^{t_{r'+1,1}} \cdots \phi_r^{t_{r1}}\right)^{-\gamma_1} \cdots \left(\phi_{r'} \phi_{r'+1}^{t_{r'+1,r'}} \cdots \phi_r^{t_{rr'}}\right)^{-\gamma_{r'}} \end{aligned}$$

By applying Theorem 2.2, we conclude that

$$\tilde{\psi}_k = A_k \phi_{r'+1}^{b_{r'+1,k}} \cdot \ldots \cdot \phi_r^{b_{rk}}, \quad A_k \in \mathbb{R}, \quad k = 1, \dots, s'. \tag{18}$$

and

$$\phi_k = B_k \phi_{r'+1}^{-t_{r'+1,k}} \cdot \ldots \cdot \phi_r^{-t_{rk}}, \quad B_k \in \mathbb{R}, \quad k = 1, \dots, r'. \tag{19}$$

Let us choose $\tilde{c}_1, \dots, \tilde{c}_{s'} \in \mathbb{Z}^+$ and $\tilde{\gamma}_1, \dots, \tilde{\gamma}_{r'} \in \mathbb{R}^+$ such that

$$\tilde{\gamma}_j = \sum_{k=1}^{s'} b_{jk} \tilde{c}_k + \sum_{k=1}^{r'} t_{jk} \tilde{\gamma}_k > 0, \qquad j = r'+1, \dots, r \tag{20}$$

and

$$\tilde{c}_j := a_{j1} \tilde{c}_1 + \cdots + a_{js'} \tilde{c}_{s'} \in \mathbb{Z}^+, \quad j = s'+1, \dots, s, \tag{21}$$

where the a_{jk} are the coefficients appearing in (15). Then (18), (19) and (20) yield

$$\begin{aligned}
&\tilde{\psi}_1^{\tilde{c}_1} \cdot \ldots \cdot \tilde{\psi}_{s'}^{\tilde{c}_{s'}} \cdot \phi_1^{-\tilde{\gamma}_1} \cdot \ldots \cdot \phi_{r'}^{-\tilde{\gamma}_{r'}} \\
&= \left(A_1 \phi_{r'+1}^{b_{r'+1,1}} \cdot \ldots \cdot \phi_r^{b_{r1}}\right)^{\tilde{c}_1} \cdot \ldots \cdot \left(A_{s'} \phi_{r'+1}^{b_{r'+1,s'}} \cdot \ldots \cdot \phi_r^{b_{rs'}}\right)^{\tilde{c}_{s'}} \\
&\cdot \left(B_1 \phi_{r'+1}^{-t_{r'+1,1}} \cdot \ldots \cdot \phi_r^{-t_{r1}}\right)^{-\tilde{\gamma}_1} \cdot \ldots \cdot \left(B_{r'} \phi_{r'+1}^{-t_{r'+1,r'}} \cdot \ldots \cdot \phi_r^{-t_{rr'}}\right)^{-\tilde{\gamma}_{r'}} \\
&= C \phi_{r'+1}^{\tilde{\gamma}_{r'+1}} \cdot \ldots \cdot \phi_r^{\tilde{\gamma}_r},
\end{aligned} \tag{22}$$

for some $C \in \mathbb{R}$. Using (16) and (21), we get

$$\begin{aligned}
\tilde{\psi}_1^{\tilde{c}_1} \cdot \ldots \cdot \tilde{\psi}_{s'}^{\tilde{c}_{s'}} &= \left(\psi_1 \psi_{s'+1}^{a_{s'+1\,1}} \cdot \ldots \cdot \psi_s^{a_{s1}}\right)^{\tilde{c}_1} \cdot \ldots \cdot \left(\psi_{s'} \psi_{s'+1}^{a_{s'+1\,s'}} \cdot \ldots \cdot \psi_s^{a_{ss'}}\right)^{\tilde{c}_{s'}} \\
&= \psi_1^{\tilde{c}_1} \cdot \ldots \cdot \psi_{s'}^{\tilde{c}_{s'}} \cdot \psi_{s'+1}^{a_{s'+1\,1}\tilde{c}_1 + \cdots + a_{s'+1\,s'}\tilde{c}_{s'}} \cdot \ldots \cdot \psi_s^{a_{s1}\tilde{c}_1 + \cdots + a_{ss'}\tilde{c}_{s'}} \\
&= \psi_1^{\tilde{c}_1} \cdot \ldots \cdot \psi_s^{\tilde{c}_s}.
\end{aligned}$$

The previous equation together with (22) yield

$$\partial\overline{\partial} \log\left(\psi_1^{\tilde{c}_1} \cdot \ldots \cdot \psi_s^{\tilde{c}_s}\right) = \partial\overline{\partial} \log\left(\tilde{\psi}_1^{\tilde{c}_1} \cdot \ldots \cdot \tilde{\psi}_{s'}^{\tilde{c}_{s'}}\right) = \partial\overline{\partial} \log\left(\phi_1^{\tilde{\gamma}_1} \cdot \ldots \cdot \phi_r^{\tilde{\gamma}_r}\right).$$

From this equation we deduce that $\mathcal{C}$ equipped with the homogeneous metric $\tilde{g}_{\mathcal{C}}$ whose diastasis is obtained by replacing in (13) the constants $c_1, \ldots, c_s$ with the positive integers $\tilde{c}_1, \ldots, \tilde{c}_s$ and Ω equipped with the metric $\tilde{g}_\Omega$ whose diastasis is obtained by replacing in (14) $\gamma_1, \ldots, \gamma_r$ with $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_r$, are relatives. Since the coefficients $\tilde{c}_1, \ldots, \tilde{c}_s$ are positive integers it follows that the metric $\tilde{g}_{\mathcal{C}}$ is projectively induced (cfr. Remark 2 above) contradicting [23] where it is shown that a homogeneous bounded domain and a projective manifold cannot be relatives.

We can now prove (ii) of Theorem 1.2 (the proof of (iii) is omitted since it follows the same pattern). Assume by contradiction that there exist two holomorphic maps $\varphi_{\mathcal{C}} : U \to \mathcal{C}$ and $\varphi = (\varphi_{\mathcal{E}}, \varphi_\Omega) : U \to \mathcal{E} \times \Omega$ such that

$$\varphi_{\mathcal{C}}^* g_{\mathcal{C}} = \varphi^*(g_{\mathcal{E}} \oplus g_\Omega),$$

where $U \subset \mathbb{C}$ is a neighbourhood of the origin. By arguing as in proof of part (i), we see that the previous equation yields

$$e^{D^{g_0}_{\varphi_{\mathcal{E}}(0)}(z)} \in \widetilde{F}^{c_1} \ldots \widetilde{F}^{c_s} \widetilde{F}^{-\gamma_1} \ldots \widetilde{F}^{-\gamma_r}.$$

Again by [14, Theorem 2.1] we see that $\varphi_{\mathcal{E}}$ is constant, and so $\varphi_{\mathcal{C}}^* g_{\mathcal{C}} = \varphi_\Omega^* g_\Omega$, showing that $\mathcal{C}$ and Ω are relatives, in contrast with the first part of the proof. □

References

[1] D. V. Alekseevsky, A. M. Perelomov, *Invariant Kahler-Einstein metrics on compact homogeneous spaces*, Funct. Anal. Appl., 20 (3) (1986) 171-182.

[2] A. L. Besse, *Einstein manifolds*, Reprint of the 1987 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2008.

[3] E. Calabi, *Isometric Imbedding of Complex Manifolds*, Ann. of Math. Vol. 58 No. 1, 1953.

[4] S. T. Chan, N. Mok, *Asymptotic total geodesy of local holomorphic curves exiting a bounded symmetric domain and applications to a uniformization problem for algebraic subsets*, J. Differential Geom. 120 (2022), no. 1, 1-49.

[5] X. Cheng, Y. Hao, *On the non-existence of common submanifolds of Kähler manifolds and complex space forms*, Ann. Global Anal. Geom. 60 (2021), no. 1, 167-180.

[6] X. Cheng, A. J. Di Scala, Y. Yuan, *Kähler submanifolds and the Umehara algebra*, Int J. of Math. Vol. 28, No. 4 (2017) 1750027, 13 pp.

[7] A. J. Di Scala, A. Loi, *Kähler manifolds and their relatives*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 9 (2010), no. 3, 495–501.

[8] A. J. Di Scala, H. Ishi, and A. Loi, *Kähler immersions of homogeneous Kähler manifolds into complex space forms*, Asian J. Math. 16 (2012), 479-487.

[9] M. L. Green, *Metric rigidity of holomorphic maps to Kähler manifolds*, J. Diff. Geom. 13 (1978), no. 2, 279–286.

[10] X. Huang, Y. Yuan, *Holomorphic isometry from a Kähler manifold into a product of complex projective manifolds*, Geom. Funct. Anal. 24 (3) (2014) 854-886.

[11] X. Huang, Y. Yuan, *Submanifolds of Hermitian symmetric spaces*, Analysis and geometry, 197-206, Springer Proc. Math. Stat., 127, Springer, Cham, 2015.

[12] A. Loi, R. Mossa, *On holomorphic isometries into blow-ups of* $\mathbb{C}^n$, arXiv:2208.03208

[13] A. Loi, R. Mossa, *Kähler immersions of Kähler-Ricci solitons into definite or indefinite complex space forms* Proc. Amer. Math. Soc. 149 (2021), no. 11, 4931-4941

[14] A. Loi, R. Mossa, *Holomorphic isometries into bounded homogeneous domains* to appear in Proc. Amer. Math. Soc., arXiv:2205.11297,

[15] A. Loi, R. Mossa, F. Zuddas, The log-term of the disc bundle over a homogeneous Hodge manifold, Ann. Global Anal. Geom. 51 (2017), no. 1, 35-51.

[16] A. Loi, R. Mossa, F. Zuddas, Bochner Coordinates on Flag Manifolds, Bull. Braz. Math. Soc., New Series 50 (2018), no 2, 497-514.

[17] A. Loi, F. Salis, F. Zuddas, *Extremal Kähler metrics induced by finite or infinite dimensional complex space forms*, J. Geom. Anal. 31 (2021), no. 8, 7842-7865.

[18] A. Loi, F. Salis, F. Zuddas, *Kähler-Ricci solitons induced by infinite dimensional complex space forms*, Pacific Journal of Mathematics 316 (2022), 183–205.

[19] A. Loi, M. Zedda, *Kähler Immersions of Kähler Manifolds into Complex Space Forms*, Lecture Notes of the Unione Matematica Italiana **23**, Springer, (2018).

[20] A. Loi, F. Zuddas, *On the Gromov width of homogeneous Kähler manifolds*, Differential Geom. Appl. 47 (2016), 130–132.

[21] N. Mok; S. C. Ng, *Germs of measure-preserving holomorphic maps from bounded symmetric domains to their Cartesian products*, J. Reine Angew. Math. 669 (2012), 47-73.

[22] N. Mok, *Some recent results on holomorphic isometries of the complex unit ball into bounded symmetric domains and related problems*, Geometric complex analysis, 269–290, Springer Proc. Math. Stat., 246, Springer, Singapore, 2018.

[23] R. Mossa, *A bounded homogeneous domain and a projective manifold are not relatives*, Riv. Mat. Univ. Parma 4 (2013), no. 1, 55-59.

[24] M. Umehara, *Kähler submanifolds of complex space forms*, Tokyo J. Math. 10 (1987), 203-214.

[25] H. Upmeier, K. Wang, G. Zhang, *Holomorphic isometries from the unit ball into symmetric domains*, Int. Math. Res. Not. IMRN 2019, no. 1, 55-89.

[26] M. Xiao, *Holomorphic isometric maps from the complex unit ball to reducible bounded symmetric domains*, J. Reine Angew. Math. 789 (2022), 187-209.

[27] M. Xiao, Y. Yuan, *Holomorphic maps from the complex unit ball to type IV classical domains*, J. Math. Pures Appl. (9) 133 (2020), 139-166.

[28] Y. Yuan, *Local holomorphic isometries, old and new results*, Proceedings of the Seventh International Congress of Chinese Mathematicians, Vol. II, 409-422, Somerville, MA: International Press, 2019.

[29] Y. Yuan, Y. Zhang, *Rigidity for local holomorphic isometric embeddings from B^n into $B^{N_1} \times \cdots \times B^{N_m}$ up to conformal factors*, J. Differential Geom. 90 (2012), no. 2, 329-349.

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