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Two sides of the same coin: the $\mathcal{F}$-statistic and the 5-vector method

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Abstract
This work explores the relation between two data-analysis methods used in the search for continuous gravitational waves in LIGO-Virgo-KAGRA data: the $\mathcal{F}$-statistic and the 5-vector method. We show that the 5-vector method can be derived from a maximum likelihood framework similar to the $\mathcal{F}$-statistic. Our analysis demonstrates that the two methods are statistically equivalent, providing the same detection probability for a given false alarm rate. We extend this comparison to multiple detectors, highlighting differences from the standard approach that simply combines 5-vectors from each detector. In our maximum likelihood approach, each 5-vector is weighted by the observation time and sensitivity of its respective detector, resulting in efficient estimators and analytical distributions for the detection statistic. Additionally, we present the

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analytical computation of sensitivity for different searches, expressed in terms of the minimum detectable amplitude.

Keywords: gravitational wave data analysis, continuous gravitational wave, maximum likelihood statistics, neutron stars

1. Introduction

Spinning neutron stars with a non-axisymmetric mass distribution are promising targets for the emission of continuous gravitational wave (CW) radiation in the LIGO-Virgo-KAGRA [1–3] (LVK) observational frequency band.

At the source, the CW signal is quasi-monochromatic with the gravitational wave (GW) frequency $f_{\rm gw}$ that is proportional to the source rotation frequency according to the considered emission model [4]. At the detector, the CW signal has a phase modulation mainly due to the Doppler effect produced by the relative motion between the source and the Earth. Due to the Earth sidereal motion and the response of the detector to the coming signal, there is also an amplitude and phase modulation that splits the signals in five frequencies $f_{\rm gw}$,$f_{\rm gw} \pm f_\oplus$,$f_{\rm gw} \pm 2f_\oplus$, where $f_\oplus$ is the Earth sidereal frequency [5].

So far, there is no significant evidence of a CW signal in LVK data (for reference, see [6]). Latest results from the most sensitive analysis targeting known pulsars—the so-called *targeted searches*—set interesting upper limits [7, 8] that are now approaching theoretical constraints [4] on the pulsar ellipticity, i.e. the physical parameter that quantifies the mass distribution asymmetry with respect to the rotation axis.

In the context of the search for CWs in LVK data, two of the most sensitive pipelines developed in the last decades are the $\mathcal{F}$-statistic [9] and the 5-vector method [5]. Both methods are frequentist pipelines, generally defined as matched filtering techniques applied in the time-domain and in the frequency-domain, respectively. Although this definition is intuitive, it is theoretically not accurate since the matched filtering theory requires the exact knowledge for the expected signal. Indeed, even though the CW signal has a clear signature at the detector, the exact shape of the signal depends on two unknown parameters that fix the GW polarizations.

The $\mathcal{F}$-statistic is inferred [10] maximizing the likelihood ratio on the unknown parameters leaving the dependence on the source parameters that can be accurately known, as in targeted searches, or generally fixed from an optimized grid. Originally, the 5-vector method [5] has not been inferred from a maximum likelihood principle. Based on a complex-number formalism, the expected signal can be written in terms of two—plus and cross—polarization amplitudes. The 5-vector method defines two matched filters in the frequency domain for both the plus and cross polarization at the five frequencies where the signal is expected. The associated detection statistic corresponds to the linear combination of the squared modulus of these two matched filters [5]. The multidetector extension in [11] combines the 5-vectors from each detector defining the so-called 5n-vector, where n is the number of considered detectors.

In this paper, we show that the 5-vector method statistic can be inferred from a maximum likelihood approach with a re-definition of ad hoc coefficients. The noise and signal distributions of the inferred statistic are equivalent to the $\mathcal{F}$-statistic distributions showing that the two methods are statistically equivalent. We generalize the procedure to n detectors showing important differences with respect to the standard 5n-vector definition in [11].

The paper is organized as follows. In section 2, we provide an overview of maximum likelihood statistics introducing the $\mathcal{F}$-statistic. In section 3, we review the formalism and the state-of-art of the 5-vector method. In section 4, we infer the 5-vector statistic from a maximum likelihood approach generalizing to the multidetector case. In section 5, we describe some applications of the maximum likelihood statistics including the theoretical estimation of the minimum detectable signal amplitude for different searches.

2. Maximum likelihood approach

Assuming a signal $h(t)$ and additive noise $n(t)$, the detector output $x(t)$ can be written as:

$$x(t) = n(t) + h(t)\ . \tag{1}$$

The likelihood ratio [10] is defined as the probability to have a signal in the analyzed data divided by the probability to have just noise. It can be shown that for Gaussian noise:

$$\ln\Lambda \equiv \ln\frac{P(x|h)}{P(x|h=0)} = (x|h) - \frac{1}{2}(h|h)\,, \tag{2}$$

where the scalar product is defined for a small frequency band as:

$$(a|b) = 2\int_f^{f+\delta f}\frac{\tilde{a}(f')\,\tilde{b}(f')}{S_h(f')}\mathrm{d}f' \cong \frac{2}{S_h}\int_0^{T_{\mathrm{obs}}} a(t)\,b^*(t)\,\mathrm{d}t, \tag{3}$$

with S_h being the single-sided power spectral density, assumed constant in the narrow frequency band. The likelihood ratio in equation (2) depends on unknown parameters and, for composite hypothesis, the Neymann–Pearson lemma [12], i.e. the maximized likelihood ratio as the optimal statistics, can not be applied. In a frequentist framework, a simple and common approach to construct a detection statistic is to consider the likelihood ratio maximized over the unknown parameters.

As shown in [13], statistics inferred from the estimation of the maximum likelihood (hereafter MLE statistics) are not "optimal" in the Neyman–Pearson sense since Bayesian methods can be more powerful considering priors consistent with the parameter distribution.

2.1. $\mathcal{F}$-statistic

The likelihood can be expressed more clearly by rewriting the expected signal as a linear combination of four basis terms [10] each with amplitude λ^a :

$$h(t) = \sum_{a=1}^{4}\lambda^a h_a(t)\ . \tag{4}$$

Each term $h_a(t)$ corresponds to a particular combination of the phase evolution $\Phi(t)$ and of the sidereal modulation, which depends on the antenna pattern that defines the response of the detector to the GW signal.

The $\mathcal{F}$-statistic is obtained maximizing the likelihood function with respect to the four amplitude parameters that depend on the signal amplitude h_0, the inclination angle ι of the neutron star rotational axis with respect to the line of sight, the wave polarization ψ and the initial phase ϕ, while keeping the dependence on the source parameters (sky position and rotational parameters). Then, the $\mathcal{F}$-statistic, defined to be twice the logarithm of the maximized likelihood ratio, is:

$$\mathcal{F} = \sum_{a,b=1}^{4}\left(\Gamma^{-1}\right)^{a,b}(x|h_a)\,(x|h_b) \tag{5}$$

where the matrix Γ is the Fisher matrix [14], defined as:

$$\Gamma^{a,b} \equiv \left(\frac{\partial h(t)}{\partial \lambda_a} \middle| \frac{\partial h(t)}{\partial \lambda_b} \right) = (h_a | h_b) , \tag{6}$$

and the maximized values for λ_a are:

$$\overline{\lambda}_a = \sum_{b=1}^{4} \left(\Gamma^{-1}\right)^{a,b} (x|h_b) . \tag{7}$$

Assuming stationary Gaussian noise, it can be shown [9] that the $\mathcal{F}$-statistic satisfies a χ^2 distribution with 4 degrees of freedom and, in the presence of a signal, it has a non-centrality parameter equal to the squared optimal signal-to-noise ratio (SNR) [15]: $\rho^2 \equiv (h(t)|h(t))$.

3. The 5-vector method

The difference between the $\mathcal{F}$-statistic and the 5-vector method arises from the formalism used for the expected signal $h(t)$. In the 5-vector context[7], the signal is written as the real part of:

$$h(t) = H_0 \mathbf{A} \cdot \mathbf{W} e^{j(\omega_{\mathrm{gw}} t + \phi)} \equiv H_0 \left(H_+ \mathbf{A}^+ + H_\times \mathbf{A}^\times\right) \cdot \mathbf{W} e^{j(\omega_{\mathrm{gw}} t + \phi)} \tag{8}$$

with $\omega_{\mathrm{gw}} = 2\pi f_{\mathrm{gw}}$. In bold, we refer to an array of five complex components.

The vector $\mathbf{W}$ in equation (8) is $\mathbf{W} = e^{jk\Theta}$ with $k = \{0, \pm 1, \pm 2\}$, and Θ is the local sidereal angle [5][8]. The product $\mathbf{A}^{+/\times} \cdot \mathbf{W}$ is defined as $\mathbf{A}^{+/\times} \cdot \mathbf{W} = \sum_{i=1}^{5} A_i^{+/\times} W_i^*$.

The amplitude H_0 is linked to the classical amplitude h_0 (defined in [10]) by:

$$H_0 = h_0 \sqrt{\frac{1 + 6\cos^2 \iota + \cos^4 \iota}{4}} , \tag{9}$$

while $H_{+/\times}$ are the polarization functions:

$$H_+ = \frac{\cos(2\psi) - j\eta \sin(2\psi)}{\sqrt{1+\eta^2}} \qquad H_\times = \frac{\sin(2\psi) + j\eta \cos(2\psi)}{\sqrt{1+\eta^2}} , \tag{10}$$

that depend on the polarization angle ψ and η:

$$\eta = -\frac{2\cos \iota}{1 + \cos^2 \iota} . \tag{11}$$

The extended expressions of the five components of the so-called 5-vector template $\mathbf{A}^{+/\times}$, which include the detector response to the GW signal, can be found in [5].

The signal in equation (8) is composed of two templates that depend on two overall polarization amplitudes, $H_0 H_{+/\times} e^{j\phi}$. The unknown polarization amplitudes are estimated through two matched filters [5] in the frequency domain[9]:

$$\hat{H}_{+/\times} = \frac{\mathbf{X} \cdot \mathbf{A}^{+/\times}}{|\mathbf{A}^{+/\times}|^2} , \tag{12}$$

[7] Assuming that we have heterodyned data, corrected for the Doppler and spin-down modulation (see [16] for more details).

[8] As in [5], we will indicate with $\mathbf{W}^* \equiv e^{-jk\Omega_\oplus t} = \mathbf{W} e^{jk(\alpha - \beta)}$ the *5-vector generator* where α is the source right ascension and β the detector longitude.

[9] The absolute value is defined in terms of the 5-vector product $|\mathbf{A}^+|^2 \equiv \mathbf{A}^+ \cdot \mathbf{A}^+$.

where

$$\mathbf{X} = \int_{T_{\text{obs}}} x(t)\, \mathrm{e}^{-j\mathbf{k}\Omega_{\oplus}t} \mathrm{e}^{-j\omega_{gw}t} \mathrm{d}t \qquad \text{with} \qquad \mathbf{k} = (0, \pm 1, \pm 2), \tag{13}$$

is the data 5-vector computed from the detector data, $x(t)$. In this work, we infer the two filters from a maximum likelihood approach[10].

The detection statistic is defined from the estimators (i.e. the matched filters) of the plus and cross polarization amplitudes:

$$S = |\mathbf{A}^{+}|^4 |\hat{H}_{+}|^2 + |\mathbf{A}^{\times}|^4 |\hat{H}_{\times}|^2. \tag{14}$$

In [5] (see figure 1), it is shown that this definition improves the ROC (Receiver Operating Characteristic) curve with respect to considering equal coefficients or just squared coefficients $|\mathbf{A}^{+/\times}|^2$ if the two polarization carry different 'weights'.

To generalize the statistic to the multidetector case, the estimators in equation (12) are defined using the 5n-vector [18], i.e. the combination of the 5-vector from each of the n considered detectors: $\mathbf{X} = [\mathbf{X}_1, \ldots, \mathbf{X}_n]$ and $\mathbf{A}^{+/\times} = [\mathbf{A}_1^{+/\times}, \ldots, \mathbf{A}_n^{+/\times}]$.

In [7], it is defined a normalized statistic $\tilde{S}$ for the multidetector analysis:

$$\tilde{S} = \frac{|\mathbf{A}^{+}|^4}{\sum_{i=1}^{n} \sigma_i^2 T_i |\mathbf{A}_i^{+}|^2} |\hat{H}_{+}|^2 + \frac{|\mathbf{A}^{\times}|^4}{\sum_{k=1}^{n} \sigma_k^2 T_k |\mathbf{A}_k^{\times}|^2} |\hat{H}_{\times}|^2 \tag{15}$$

where σ_i^2 and T_i are the variance of the time-domain data distribution in a frequency band around f_{gw} (usually few tenths of Hz wide) and the observation time in the ith detector, respectively. In the case of Gaussian noise, the distribution of the statistic $\tilde{S}$ in equation (15) is a Gamma distribution $\Gamma(x;2,1)$ with shape 2 and scale parameter 1.

If a signal is present, the distribution of $\tilde{S}$ is proportional to a 4D χ^2 distribution with non-centrality parameter λ:

$$\tilde{S} \sim 2 \cdot \chi^2 (2x; 4, \lambda), \tag{16}$$

$$\lambda = 2H_0^2 \left(\frac{|\mathbf{A}^{+}|^4 |H_{+}|^2}{\sum_{i=1}^{n} \sigma_i^2 T_i |\mathbf{A}_i^{+}|^2} + \frac{|\mathbf{A}^{\times}|^4 |H_{\times}|^2}{\sum_{k=1}^{n} \sigma_k^2 T_k |\mathbf{A}_k^{\times}|^2} \right). \tag{17}$$

The extended derivation for the distributions of $\tilde{S}$ can be found in [19].

4. Maximum likelihood approach to the 5-vector

In this section, we infer the 5-vector method from a maximum likelihood approach. First, we consider the single detector case and then, we generalize to the multidetector scenario considering different observation time and sensitivity for each detector.

4.1. Single detector

Following [10], we can express the likelihood as

$$\ln \Lambda = Re \left\{ (x|h_{+}) - \frac{1}{2}(h_{+}|h_{+}) + (x|h_{\times}) - \frac{1}{2}(h_{\times}|h_{\times}) \right\} \tag{18}$$

[10] The matched filter theory is generally linked to maximum likelihood estimation. For more details, see [17] section 7.4.2.

where $Re\{\}$ indicates the real part, and

$$h_{+/\times} = H_0 H_{+/\times} e^{j\phi} \mathbf{A}^{+/\times} \cdot \mathbf{W} e^{j\omega_{gw} t}. \tag{19}$$

Indeed, due to the orthogonality of the GW polarization bases, it follows that $(h_+|h_\times) = (h_\times|h_+) = 0$.

The polarization amplitudes $H_0 H_{+/\times} e^{j\phi}$ correspond to the λ^α in equation (4) with the difference that the 5-vector formalism considers the linear combination of only two terms due to their complex nature. Let us indicate with

$$\lambda_+ \equiv H_0 H_+ e^{j\phi} \qquad \text{and} \qquad \lambda_\times \equiv H_0 H_\times e^{j\phi}. \tag{20}$$

Evaluating the scalar product defined in equation (3)

$$(x|h_+) = \frac{2T_0}{S_h} \left(\frac{1}{T_0} \int_0^{T_0} x(t) \mathbf{W}^* e^{-j\omega_{gw} t} dt \right) \lambda_+^* \left(\mathbf{A}^+\right)^* = \frac{2T_0}{S_h} \lambda_+^* \mathbf{X} \cdot \mathbf{A}^+, \tag{21}$$

we can introduce the data 5-vector $\mathbf{X}$[11], i.e. the Fourier components of the data at the expected five frequencies:

$$\mathbf{X} = \frac{1}{T_0} \int_0^{T_0} x(t) \mathbf{W}^* e^{-j\omega_{gw} t} dt, \tag{22}$$

and also the product between 5-vectors $\mathbf{X} \cdot \mathbf{A}^+ = \sum_{i=1}^5 X_i (A_i^+)^*$, in analogy with [5]. The second term in the likelihood is[12]:

$$(h_+|h_+) = \frac{2}{S_h} |\lambda_+|^2 |\mathbf{A}^+|^2 \left(\int_0^{T_0} dt \right) = \frac{2T_0}{S_h} |\lambda_+|^2 |\mathbf{A}^+|^2. \tag{23}$$

It follows that:

$$\ln \Lambda = \frac{T_0}{S_h} \sum_p^{+,\times} \left[\lambda_p^* \mathbf{X} \cdot \mathbf{A}^p + \lambda_p \left(\mathbf{X} \cdot \mathbf{A}^p\right)^* - |\lambda_p|^2 |\mathbf{A}^p|^2 \right]. \tag{24}$$

The derivatives with respect to the complex polarization amplitudes are:

$$\frac{\partial \ln \Lambda}{\partial \lambda_{+/\times}^*} \propto \left(\mathbf{X} \cdot \mathbf{A}^{+/\times} \right) - \lambda_{+/\times} |\mathbf{A}^{+/\times}|^2 \tag{25}$$

that are set to zero if:

$$\left(\lambda_{+/\times}\right)_{\text{MAX}} \equiv \hat{H}_{+/\times} = \frac{\mathbf{X} \cdot \mathbf{A}^{+/\times}}{|\mathbf{A}^{+/\times}|^2}. \tag{26}$$

The maximized likelihood ratio with these estimators results in:

$$(\ln \Lambda)_{\text{MAX}} = \frac{T_0}{S_h} \left(|\mathbf{A}^+|^2 |\hat{H}_+|^2 + |\mathbf{A}^\times|^2 |\hat{H}_\times|^2 \right) \propto |\mathbf{A}^+|^2 |\hat{H}_+|^2 + |\mathbf{A}^\times|^2 |\hat{H}_\times|^2. \tag{27}$$

The common factor T_0/S_h is irrelevant since it does not influence the ROC curves that are invariant to any monotonic transformation [20]. The result is in agreement with [5], where it is stated: *'If we take the two coefficients proportional to the square of the absolute values of the signal 5-vectors (i.e.* $\mathbf{A}^{+/\times}$*), we have the well-known F-statistics, which is an equalization*

[11] Note that the 5-vector definition has here an additional factor T_0^{-1} with respect to [5] that is important for the multidetector extension.

[12] The relation is valid if T_0 is a multiple of the sidereal day.

of the response at the two modes'. The same result can be also found in [21] but expressed in the $\mathcal{F}$-statistic formalism.

The noise and signal distributions of $2(\ln\Lambda)_{\mathrm{MAX}}$ correspond to the $\mathcal{F}$-statistic distributions showing that the two methods are statistically equivalent, i.e. for a fixed false alarm probability they provide the same detection probability. The extended proof of this equivalence is provided in the next section, describing the multidetector extension.

The definition of the data 5-vector in equation (22) with the factor $1/T_0$ entails the constant factor T_0/S_h in equation (27). Considering the "normalized" definition in equation (15) with $n=1$, the constant factor is $1/(T_0 S_h)$; the difference arises due to the different definition of the data 5-vector. The weighted definition is important for the multidetector extension, as shown in the next section.

4.2. Multidetector

Let us consider n detectors assuming stationary Gaussian noise with different variance and observation time for each detector. The new form of the likelihood depends on the expression of the scalar product in equation (3) that is (assuming uncorrelated noise):

$$(\mathbf{a}|\mathbf{b}) \cong 2\sum_{i=1}^{n}\int_0^{T_i}\frac{a_i(t)\,b_i^*(t)}{S_i}\mathrm{d}t. \tag{28}$$

The scalar products in the multidetector case are:

$$\left(\mathbf{x}|\mathbf{h}_{+/\times}\right) = 2\lambda_{+/\times}^*\sum_{i=1}^{n}\frac{T_i}{S_i}\left(\mathbf{X}_i\cdot\mathbf{A}_i^{+/\times}\right) \tag{29}$$

where $\mathbf{X}_i$ is the data 5-vector for the ith detector, as defined in equation (22). Since

$$\left(\mathbf{h}_{+/\times}|\mathbf{h}_{+/\times}\right) = 2|\lambda_{+/\times}|^2\sum_{i=1}^{n}\frac{T_i|\mathbf{A}_i^{+/\times}|^2}{S_i}, \tag{30}$$

the likelihood is maximized for:

$$\left(\lambda_{+/\times}\right)_{\mathrm{MAX}} \equiv \hat{H}_{+/\times} = \left(\sum_{i=1}^{n}\frac{T_i\mathbf{X}_i\cdot\mathbf{A}_i^{+/\times}}{S_i}\right)\left(\sum_{k=1}^{n}\frac{T_k|\mathbf{A}_k^{+/\times}|^2}{S_k}\right)^{-1}. \tag{31}$$

This is quite different from the standard definition of the estimators through the 5n-vector in [18] generalizing the single detector case:

$$\hat{H}_{+/\times} = \frac{\mathbf{X}\cdot\mathbf{A}^{+/\times}}{|\mathbf{A}^{+/\times}|^2} = \left(\sum_{i=1}^{n}\mathbf{X}_i\cdot\mathbf{A}_i^{+/\times}\right)\left(\sum_{k=1}^{n}|\mathbf{A}_k^{+/\times}|^2\right)^{-1}, \tag{32}$$

where $\mathbf{X} = [\mathbf{X}_1,\ldots,\mathbf{X}_n]$ and $\mathbf{A}^{+/\times} = [\mathbf{A}_1^{+/\times},\ldots,\mathbf{A}_n^{+/\times}]$. The two estimators definition in equations (31) and (32) coincide if each detector has the same observation time and sensitivity.

From the maximization of the likelihood, it follows that:

$$(\ln\Lambda)_{\mathrm{MAX}} = \sum_{p=+,\times}\left(\sum_{k=1}^{n}\frac{T_k|\mathbf{A}_k^p|^2}{S_k}\right)^{-1}\left[\left(\sum_{i=1}^{n}\frac{T_i\mathbf{X}_i\cdot\mathbf{A}_i^p}{S_i}\right)\left(\sum_{r=1}^{n}\frac{T_r\mathbf{X}_r\cdot\mathbf{A}_r^p}{S_r}\right)^*\right]. \tag{33}$$

From equation (33), if we define the weighted data 5n-vectors as

$$\tilde{\mathbf{X}} = \left[\sqrt{\frac{T_1}{S_1}}\mathbf{X}_1,\ldots,\sqrt{\frac{T_n}{S_n}}\mathbf{X}_n\right] \tag{34}$$

and the weighted template 5n-vectors

$$\tilde{\mathbf{A}}^{+/\times} = \left[\sqrt{\frac{T_1}{S_1}}\mathbf{A}_1^{+/\times}, \ldots, \sqrt{\frac{T_n}{S_n}}\mathbf{A}_n^{+/\times}\right], \tag{35}$$

we can re-write the maximum likelihood statistics:

$$(\ln\Lambda)_{\mathrm{MAX}} = |\tilde{\mathbf{A}}^{+}|^2|\tilde{H}_{+}|^2 + |\tilde{\mathbf{A}}^{\times}|^2|\tilde{H}_{\times}|^2 \tag{36}$$

where

$$\tilde{H}_{+/\times} = \frac{\tilde{\mathbf{X}}\cdot\tilde{\mathbf{A}}^{+/\times}}{|\tilde{\mathbf{A}}^{+/\times}|^2} \equiv \left(\sum_{i=1}^{n}\tilde{\mathbf{X}}_i\cdot\tilde{\mathbf{A}}_i^{+/\times}\right)\left(\sum_{k=1}^{n}|\tilde{\mathbf{A}}_k^{+/\times}|^2\right)^{-1}. \tag{37}$$

The statistic equation (36) has the same form of equation (27) inferred for the single-detector case, but with weighted 5n-vectors.

To infer the noise distribution of $(\ln\Lambda)_{\mathrm{MAX}}$ assuming stationary and Gaussian noise, we start from:

$$\tilde{H}_{+/\times} \sim \mathrm{Gauss}\left(x;0,\tilde{\sigma}^2_{+/\times}\right) \qquad |\tilde{H}_{+/\times}|^2 \sim \mathrm{Exp}\left(x;\tilde{\sigma}^2_{+/\times}\right) \tag{38}$$

with

$$\tilde{\sigma}^2_{+/\times} = \sum_{i=1}^{n}\frac{T_i\,(S_i)^{-1}\,S_i T_i^{-1}|\tilde{\mathbf{A}}_i^{+/\times}|^2}{|\tilde{\mathbf{A}}^{+/\times}|^4} = \frac{1}{|\tilde{\mathbf{A}}^{+/\times}|^2}. \tag{39}$$

The factor $T_j(S_j)^{-1}$ comes from the weight of the data 5n-vectors in equation (34), while $S_jT_j^{-1}$ from the definition of the 5-vector in equation (22).

The coefficients in equation (36) are exactly $1/\tilde{\sigma}^2_{+/\times}$ that normalize the distribution of $(\ln\Lambda)_{\mathrm{MAX}}$ to the $\Gamma(x;2,1)$ (i.e. to the distribution of the sum of two exponential random variable with rate parameter 1). It follows that the noise distribution of $2(\ln\Lambda)_{\mathrm{MAX}}$ is

$$2\,(\ln\Lambda)_{\mathrm{MAX}} \sim \frac{1}{2}\Gamma\left(\frac{x}{2};2,1\right) \equiv \chi^2\,(x;4) \tag{40}$$

a central χ^2 distribution with 4 degrees of freedom corresponding to the noise distribution of the $\mathcal{F}$-statistic.

Since the signal distributions of the estimators in equation (31) are:

$$\hat{H}_{+/\times} \sim \mathrm{Gauss}\left(x;\frac{2H_0H_{+/\times}}{\tilde{\sigma}^2_{+/\times}},\tilde{\sigma}^2_{+/\times}\right), \tag{41}$$

the signal distribution of $(\ln\Lambda)_{\mathrm{MAX}}$ is proportional to a non-central χ^2 distribution,

$$(\ln\Lambda)_{\mathrm{MAX}} \sim 2\cdot\chi^2\left(2x;4,\rho^2\right), \tag{42}$$

where the parameter ρ^2 is:

$$\rho^2 \equiv (\mathbf{h}_+|\mathbf{h}_+) + (\mathbf{h}_\times|\mathbf{h}_\times) = H_0^2\sum_{p}^{+/\times}\left(\frac{2|H_p|^2}{\tilde{\sigma}_p^2}\right), \tag{43}$$

i.e. the optimal SNR. It follows that the signal distribution of $2(\ln\Lambda)_{\mathrm{MAX}}$ is

$$2\,(\ln\Lambda)_{\mathrm{MAX}} \sim \chi^2\left(x;4,\rho^2\right). \tag{44}$$

As shown for the noise distribution, the signal distribution of $2(\ln\Lambda)_{\mathrm{MAX}}$ correspond to the signal distribution of the $\mathcal{F}$-statistic with the same non-centrality parameter by definition, proving the statistical equivalence of the two statistics.

The relation between ρ^2 and λ defined in equation (17), and consequently between $(\ln\Lambda)_{\mathrm{MAX}}$ and $\tilde{S}$, is described in section 5.1.

4.2.1. Toy case: n co-located detectors. For n co-located detectors, the signal 5-vectors coincide:

$$|\mathbf{A}_k^{+/\times}|^2 = |\mathbf{A}_0^{+/\times}|^2\,, \forall k = 1,..,n. \tag{45}$$

Assuming the same observation time T_0 for each detector, it follows that:

$$\tilde{\sigma}^2_{+/\times} = \left(\sum_{k=1}^{n} \frac{T_k|\mathbf{A}_k^{+/\times}|^2}{S_k}\right)^{-1} = \frac{1}{T_0|\mathbf{A}_0^{+/\times}|^2}\left(\sum_{k=1}^{n}\frac{1}{S_k}\right)^{-1} = \frac{1}{T_0|\mathbf{A}_0^{+/\times}|^2}\frac{\mathcal{H}}{n}\,, \tag{46}$$

where $\mathcal{H}$ is the harmonic mean of the S_k. It follows that n co-located detectors correspond to a single detector with sensitivity equal to the harmonic mean divided by n. The relation

$$\min\{S_1,\ldots,S_n\} \leqslant \mathcal{H} \leqslant n\min\{S_1,\ldots,S_n\} \tag{47}$$

entails that:

$$\frac{\min\{S_1,\ldots,S_n\}}{n} \leqslant \frac{\mathcal{H}}{n} \leqslant \min\{S_1,\ldots,S_n\}\,. \tag{48}$$

For n co-located detectors with the same observation time, we always have an improvement in the detection sensitivity, differently from what is found for the classic definition of the 5n-vector.

4.3. Real data

In real analysis, we cannot use the theoretical expressions in [5] for $\mathbf{A}^{+/\times}$ as the matched filters would not take into account the presence of gaps in the data and the effect of all pre-processing operations.

In [5], it is proposed to define the signal 5-vector simulating in the time domain the signal $+/\times$ components with the theoretical expressions, but applying all the cleaning procedures and removing the time windows corresponding to gaps in the data. The obtained time series $s^{+/\times}(t)$ are used to define the signal 5-vectors

$$\mathbf{A}^{+/\times} = \frac{1}{T_0}\int_0^{T_0} s^{+/\times}(t)\,\mathbf{W}^*\mathrm{e}^{-j\omega_{\mathrm{gw}}t}\mathrm{d}t\,, \tag{49}$$

i.e. computing the Fourier transform[13] at the expected five frequencies as in the case of the data 5-vector. The results shown so far are not influenced by this definition using real data since $\mathbf{A}^{+/\times}$ have a constant value fixing the source and detector position.

5. Application

In this section, we briefly summarize some useful applications and properties of the MLE 5-vector. First, considering the Fisher matrix formalism, we show that the estimators in

[13] Given the finite duration of the Fourier transform, part of the energy of the Fourier components will be spread into lateral bands decreasing the power of the signal in the peaks.

equation (31) are statistically efficient. Then, we provide a theoretical estimation of the minimum detectable amplitude for different searches. In appendix, we also describe a solid framework for the construction of a grid in the overall parameter space.

5.1. Fisher matrix

The $\mathcal{F}$-statistic is usually written in terms of the Fisher matrix as shown in section 2.1. In the 5-vector formalism, the elements of the Fisher matrix are (with $h(t)$ defined in equation (8)):

$$\Gamma^{++} = \left(\frac{\partial h(t)}{\partial \lambda_+}\middle|\frac{\partial h(t)}{\partial \lambda_+}\right) = 2\sum_{k=1}^{n}\frac{T_k|\mathbf{A}_k^+|^2}{S_k}, \tag{50}$$

$$\Gamma^{\times\times} = \left(\frac{\partial h(t)}{\partial \lambda_\times}\middle|\frac{\partial h(t)}{\partial \lambda_\times}\right) = 2\sum_{k=1}^{n}\frac{T_k|\mathbf{A}_k^\times|^2}{S_k}, \tag{51}$$

$$\Gamma^{+\times} = \left(\frac{\partial h(t)}{\partial \lambda_+}\middle|\frac{\partial h(t)}{\partial \lambda_\times}\right) = 2\sum_{k=1}^{n}\frac{T_k\mathbf{A}_k^+\cdot\left(\mathbf{A}_k^\times\right)^*}{S_k} = 0 = \left(\Gamma^{\times +}\right)^*. \tag{52}$$

The $\Gamma^{+\times}$ (and hence $\Gamma^{\times+}$) goes to zero since $\mathbf{A}_k^+\cdot(\mathbf{A}_k^\times)^* = 0, \forall k$ from the theoretical expressions in [5] due to the orthogonality of the GW polarizations.

The diagonal elements are the inverse of the coefficients in equation (33) since in the Fisher formalism, the MLE statistic is expressed as:

$$(\ln\Lambda)_{\text{MAX}} = \frac{1}{2}\sum_{p}^{+/\times}\left(\Gamma^{-1}\right)^{pp}\left(\mathbf{x}\middle|\frac{\partial h(t)}{\partial \lambda_p}\right)\left(\mathbf{x}\middle|\frac{\partial h(t)}{\partial \lambda_p}\right)^*. \tag{53}$$

The maximum likelihood estimators can be also written in terms of the Fisher matrix:

$$\hat{H}_+ \equiv \frac{1}{2}\left(\Gamma^{-1}\right)^{++}\left(\mathbf{x}\middle|\frac{\partial h(t)}{\partial \lambda_+}\right), \qquad \hat{H}_\times \equiv \frac{1}{2}\left(\Gamma^{-1}\right)^{\times\times}\left(\mathbf{x}\middle|\frac{\partial h(t)}{\partial \lambda_\times}\right), \tag{54}$$

and the inverse of the Fisher matrix element is a lower bound (the so-called Cramér–Rao bound) on the variance of any unbiased estimator:

$$2\left(\Gamma^{-1}\right)^{++/\times\times} \equiv \tilde{\sigma}^2_{+/\times} \tag{55}$$

with $\tilde{\sigma}^2_{+/\times}$ defined in equation (39). Indeed, the Fisher matrix is often called the covariance matrix, since the maximum likelihood estimators distribution tends to a jointly Gaussian distribution with covariance matrix equal to Γ^{-1}.

The variances $\sigma^2_{+/\times}$ for the estimators defined in equation (32) are:

$$\sigma^2_{+/\times} = \sum_{i=1}^{n}\frac{S_i\cdot T_i\cdot|\mathbf{A}_i^{+/\times}|^2}{|\mathbf{A}^{+/\times}|^4}, \tag{56}$$

and the Cramér–Rao bound entails that $\sigma^2_{+/\times} \geqslant \tilde{\sigma}^2_{+/\times}$, i.e. the variance of the estimators in equation (31) is smaller by definition than the variance of the standard estimators in equation (32).

In addition, since $\sigma^2_{+/\times} \geqslant \tilde{\sigma}^2_{+/\times}$, ρ^2 defined in equation (43) is the largest non-centrality parameter that can be defined fixing the source parameters (for example, $\rho^2 \geqslant \lambda$ in equation (17)). It follows that the new definition of the 5n-vector in equation (22) maximize the optimal SNR as in the $\mathcal{F}$-statistic.

The Cramér–Rao bound implies that the ROC curve of $(\ln\Lambda)_{\text{MAX}}$ is always superior or equal to that of $\tilde{S}$. This is because $(\ln\Lambda)_{\text{MAX}}$ and $\tilde{S}$ share the same noise and signal distributions, but with different non-centrality parameters. Since $\rho^2 \geqslant \lambda$, this consistently leads to better ROC curves.

The difference between equations (36) and (15) is also in the estimators in equation (31). The parameter estimation improves (i.e. the distribution in equation (38) has smaller variance) using the estimators that maximize the likelihood.

5.2. Sensitivity estimation

The sensitivity of a specific search is generally defined as the minimum detectable amplitude:

$$h_{\min} \approx C\sqrt{\frac{S_h(f)}{T}} \tag{57}$$

where $S_h(f)$ is the power spectral density at frequency f, and T the effective observation time. The efficiency factor C depends on the considered search. For example, for a targeted search $C \approx 11$, while for a narrowband search [11] considering a grid in the frequency-frequency derivative space, C is a function of the number N of points explored in the parameter space (see for example, figure 5 in [11]). In this section, we show that the factor C can be computed analytically from the theoretical distributions in the assumption of stationary Gaussian noise avoiding Monte–Carlo injections studies.

The minimum detectable amplitude corresponds to the value of the amplitude that entails a desired value of the detection probability for a fixed false-alarm probability. As shown, the signal distribution is ruled by the non-centrality parameter ρ^2, which is proportional to the squared amplitude H_0^2. To estimate $h_{\min}$, we can invert the relation in equation (43) fixing the value of $\rho^{95\%,1\%}$ that entails a detection probability of 95% for a false alarm of 1%:

$$h_{\min} \approx 1.32\sqrt{\frac{\rho^{95\%,1\%}}{|\mathbf{A}^{+}|^2 + |\mathbf{A}^{\times}|^2}}\sqrt{\left(\sum_{i=1}^{n}\frac{T_i}{S_i}\right)^{-1}}, \tag{58}$$

taking the average over the polarization parameters ($-\pi/2 \leqslant \psi \leqslant \pi/2$, $-1 \leqslant \cos\iota \leqslant 1$ uniformly distributed). The factor 1.32 takes into account the conversion factor between H_0 and the standard amplitude h_0 [22]. Since the statistic distributions are fixed for any pulsars and any detectors, $\rho^{95\%,1\%} \approx 24$[14]. Assuming right ascension and declination uniformly distributed, and averaging the theoretical expressions (equation (17)–(18) in [5]) for $\mathbf{A}^{+/\times}$ entails $|\mathbf{A}^{+}|^2 + |\mathbf{A}^{\times}|^2 \approx 0.4$. It follows that:

$$C \approx 1.32\sqrt{\frac{\rho^{95\%,1\%}}{0.4}} \approx 10.3\,. \tag{59}$$

If we are exploring a parameter space using a template grid with a N points, we need to take into account the trial factors that decrease the search sensitivity and the false alarm, i.e we have to consider $\rho^{95\%,1\%/N}$ (see figure 1, for the relation with N). If we are considering a narrowband/directed search for a specific target, we can avoid the average over the sky position and consider the actual value of $|\mathbf{A}^{+}|^2 + |\mathbf{A}^{\times}|^2$.

[14] The value $\rho^{95\%,1\%} \approx 24$ is inferred fixing the value of the statistic that entails a false alarm of 1% from the noise distribution and then, using this fixed value to select the non-centrality parameter that entails a detection probability of 95% from the signal distribution.

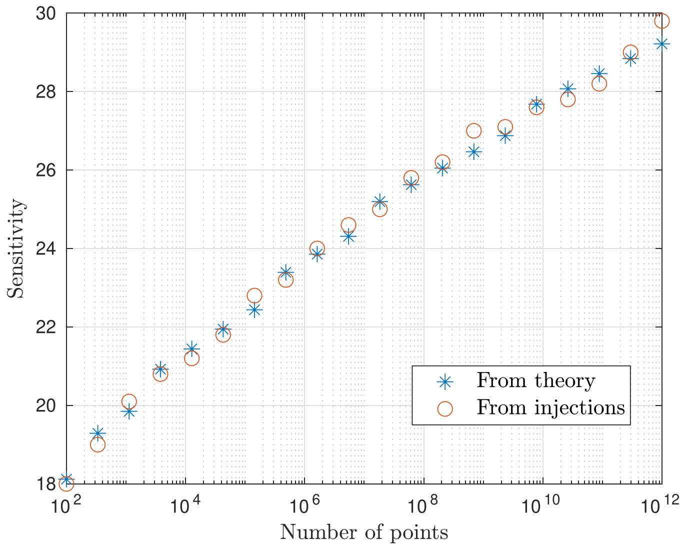

Figure 1. Sensitivity of a narrowband search using the 5n-vector method. The figure shows (blue stars) the factor C in equation (57) as a function of the number of points N explored in the parameter space from the theoretical computation in equation (60). The red circles correspond to the results in figure 5 of [11], inferred from a software injection campaign.

In the multidetector case, averaging over the sky positions, we have:

$$h_{\min} \approx 1.32\sqrt{\frac{\rho^{95\%,1\%/N}}{0.4}}\sqrt{\left(\sum_{i=1}^{n}\frac{T_i}{S_i}\right)^{-1}}. \tag{60}$$

In a semicoherent search [23], the observation time is generally divided in M chunks of length T_{coh} (with T_{coh} a multiple of the sidereal day) and individually analyzed. Then, the detection statistic is defined as the sum of the statistics computed for each chunk. This type of search allows to typically perform more robust analysis since there is no phase continuity between the segments. To asses the sensitivity, we need to consider the signal distribution of the statistics sum that is a 4M-D χ^2 distribution with non centrality parameter:

$$\rho_{\text{SC}} = 2H_0^2 T_{\text{coh}}\left(|H_+|^2|\mathbf{A}^+|^2 + |H_\times|^2|\mathbf{A}^\times|^2\right)\sum_{i=1}^{M}\frac{1}{S_i} \tag{61}$$

considering, for simplicity, the single detector case. It follows that:

$$h_{\min} \approx 1.32\sqrt{\frac{\rho_{\text{SC}}^{95\%,1\%}}{0.4}}\sqrt{\left(T_{\text{coh}}\sum_{i=1}^{M}\frac{1}{S_i}\right)^{-1}}, \tag{62}$$

noting that the $\rho_{\text{SC}}^{95\%,1\%}$ is fixed considering the distribution of the statistics sum, and S_i is the detector sensitivity in the ith data chunk.

5.3. O4 run and Virgo detector

As a toy case for the targeted search sensitivity estimation, we can approximate the current observing run O4 for the LIGO-Virgo detectors [24]. We assume equal sensitivity S_h and observation time T for the two LIGO detectors. Since Virgo joined the O4 run in the second part with lower sensitivity, we assume $A \cdot S_h$ and $T/2$ for Virgo.

The factor A depends on the frequency and generally, it should be between $2 \leqslant A \leqslant 5$. Using the sensitivity estimation in equation (60), considering only the two LIGO detectors we have:

$$h_{\min}^{(2)} \approx C\sqrt{\frac{S_h}{T}}\frac{1}{\sqrt{2}}, \tag{63}$$

while including Virgo,

$$h_{\min}^{(3)} \approx C\sqrt{\frac{S_h}{T}}\sqrt{\frac{2A}{4A+1}}. \tag{64}$$

It follows that $h_{\min}^{(3)} \approx h_{\min}^{(2)}\sqrt{4A/(4A+1)}$; i.e there is an improvement between 6% for $A=2$ and 2.5% for $A=5$ including Virgo data with respect to considering only the two LIGO detectors for the O4 run.

6. Conclusion

In this paper, we derived the 5-vector method and its related detection statistic using a maximum likelihood approach. While the relation between the 5-vector statistic and the $\mathcal{F}$-statistic was intuitively recognized in [5] for a single detector, we extended this to multiple detectors, highlighting significant differences from the classic multidetector extension in [11]. The maximum likelihood approach provides different, statistically efficient estimators of the polarization amplitudes by accounting for the varying sensitivity and observation time of each detector. We demonstrated that the 5-vector statistic derived from the maximum likelihood is statistically equivalent to the $\mathcal{F}$-statistic, as they share the same distributions. The 5-vector entails a simpler formalism with two overall complex amplitudes and the matched filter between arrays of only five components.

The definition of the data 5-vector introduced in equation (22) from the maximum likelihood estimation adds a factor that is the inverse of the observation time with respect to the classic definition in [5]. For the multidetector case, the resulting 5n-vector definition in equation (34) combines the 5-vector from each detector with weights that are the square root of the ratio between the corresponding observation time and sensitivity. This definition allows to consider multiple detector also if the detectors sensitivity and/or observation time vary significantly between the detectors.

For future CW searches using the 5-vector formalism, we recommend using the 5n-vector from the maximum likelihood approach for multidetector analysis. This approach offers a multidetector extension that always improves the detection sensitivity, efficient estimators for the polarization amplitudes, analytical theoretical distributions for the detection statistic, and a solid framework for constructing a template grid. Additionally, we provided an analytical computation of the sensitivity for different searches, estimating the minimum detectable amplitude. These theoretical estimations will simplify and significantly reduce the computational cost of search sensitivity estimation, bypassing extensive Monte–Carlo analyses.

Recently in [25], the authors have used the likelihood in the $\mathcal{F}$-statistic formalism for a Bayesian parameter estimation assuming priors on the unknown parameters. Similarly, using the framework developed in this work, we can use the 5-vector likelihood for a 'Bayesian 5-vector method' for parameter estimation but also to estimate the Bayesian evidence. The comparison and relation with other methods (like the $\mathcal{B}$-statistic in [13] that outperforms the $\mathcal{F}$-statistic marginalizing the likelihood with physical priors) will be the subject of future work.

Data availability statement

No new data were created or analysed in this study.

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Appendix. Phase metric

If either the source parameters are not known with the required accuracy for a targeted search or when performing an *all-sky* search, a template grid is usually built in the search parameter space. In the last years, several works (see for example [26–29]) have described the implementation of a template grid using the $\mathcal{F}$-statistic. Generally, the grid is chosen minimizing the distance between any point in parameter space and the nearest grid point. Recently, [30] showed that standard template banks do not always maximize the expected number of detections and for high dimensional space parameters (above eight dimensions), random template banks outperform the best known lattices [31].

A first attempt to set a template grid using the 5-vector formalism is described in [21], where the authors refer to the statistic equation (15) with squared coefficients as '$\mathcal{F}$-statistic'. In this section, we generalize the results in [21] to the multidetector case using a slightly different approach and following [15].

The signal distribution of the MLE statistic in equation (36) is proportional to a 4D χ^2 distribution with non centrality parameter ρ^2. We can generalize the results to the multidetector case evaluating the non centrality parameter, expliciting equation (43):

$$\rho^2 = 2H_0^2\left(|H_+|^2\sum_{i=1}^{n}\frac{T_i|\mathbf{A}_i^+|^2}{S_i} + |H_\times|^2\sum_{i=1}^{n}\frac{T_i|\mathbf{A}_i^\times|^2}{S_i}\right). \tag{A.1}$$

This optimal (i.e. from the optimal filter) SNR does not depend on the initial phase, and ρ scales linearly with the overall amplitude and with the square root of the observation time, as expected.

Let us suppose that we are searching with a template set parameters $\vec{\theta}_t$ a CW signal with real parameters $\vec{\theta}$, where $\vec{\theta}_t = \vec{\theta} + \Delta\vec{\theta}$. The mismatch function is defined as:

$$\mu\left(\mathcal{A},\vec{\theta}_t;\vec{\theta}\right) = \frac{\rho^2\left(\mathcal{A},\vec{\theta}\right) - \rho^2\left(\mathcal{A},\vec{\theta}_t\right)}{\rho^2\left(\mathcal{A},\vec{\theta}\right)}, \tag{A.2}$$

and it can be written as (assuming summation on repeated indexes):

$$\mu\left(\mathcal{A},\vec{\theta}_t;\vec{\theta}\right) = g_{ij}\left(\mathcal{A},\vec{\theta}_t\right)\Delta\theta_i\Delta\theta_j + \mathcal{O}\left(\Delta\vec{\theta}^3\right) \tag{A.3}$$

where g_{ij} is the normalized projected Fisher matrix as defined in [15], depending on the unknown polarization amplitudes $\mathcal{A}$. A more practical mismatch measure can be constructed taking the mean of the eigenvalues of $g_{ij}(\mathcal{A}, \vec{\theta}_t)$ defining an averaged metric as in [32], $\overline{g}_{ij}(\vec{\theta}_t)$.

Considering long-duration observation times ($T_{\rm obs}$ of a few days), the metric $\overline{g}_{ij}(\boldsymbol{\theta}_t)$ can be approximated by the so-called 'phase metric' [15], which neglects amplitude modulation but retains detector-specific phase modulation. In recent decades, several studies have explored the definition and computation of the phase metric. All the results from these studies can be readily applied to the 5-vector, as the phase metric depends solely on the signal phase modulation.

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