



FRANCESCO PAOLI
GAVIN ST. JOHN 

Editorial Introduction

Abstract. This is the Editorial Introduction to “S.I.: Strong and Weak Kleene Logics”.

1. Kleene Logics: An Overview

There is no denying that Stephen Cole Kleene (1909–1994) was one of the milestone figures in XXth century logic. In light of his decisive contributions to recursion theory, the lambda calculus, the study of incompleteness phenomena, and the foundations of intuitionistic mathematics, his importance for the development of mathematical logic can hardly be overestimated. Kleene, however, was not only a mathematician of genius whose deep theorems exerted an indisputable impact in all the fields where he happened to work. He was also a superb writer, with an uncommon skill for presenting heavy-duty mathematical topics in a clear and readable manner. His *Introduction to Metamathematics* [46] stands out, even by present-day standards, as a textbook of mathematical logic which manages to give a crystal-clear presentation of the foundational issues in mathematics without skirting their most intricate aspects or lowering the bar of rigour.

It is in § 64 of this book that Kleene introduces a distinction between two senses of the propositional connectives when we are reasoning about partial recursive predicates. If we compute the value of one such predicate P as applied to the argument x , the computation may terminate and return a classical value (0, “false”, or 1, “true”) if P is defined for x , or else return no value at all. In the latter case, it may be convenient to say that Px has value $\frac{1}{2}$ (“undefined”). In such a 3-valued setting, how are we to calculate the values of sentential compounds built by means of the connectives of negation, disjunction and conjunction?

Kleene intends to respect two minimal constraints:

Special Issue: Strong and weak Kleene logics
Edited by **Gavin St. John and Francesco Paoli.**

Received July 16, 2024

- The table of each connective should be exactly the classical one, when all the arguments have classical values.
- The table of each connective should be *regular*, meaning that whenever a column (row) contains a classical value in the row (column) of $\frac{1}{2}$, that column (row) should contain *that* classical value everywhere.

It is readily seen that the following tables, called by Kleene *strong* for this reason, maximise the number of classical entries among those that comply with the above criteria:

\neg		\wedge	0	$\frac{1}{2}$	1	\vee	0	$\frac{1}{2}$	1
0	1	0	0	0	0	0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
1	0	1	0	$\frac{1}{2}$	1	1	1	1	1

At the opposite end of the spectrum, we have the following *weak* tables, where the undefined value $\frac{1}{2}$ occurs as often as possible:

\neg		\wedge	0	$\frac{1}{2}$	1	\vee	0	$\frac{1}{2}$	1
0	1	0	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	1	0	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	1

Although Kleene calls these tables “logics”, this label is improper by contemporary terminological standards. In order to obtain logics from these algebras (which we call **SK** and **WK**, respectively), indeed, we have to turn them into logical matrices, via an appropriate selection of *designated values*. Two reasonable choices can be made: either we designate the true value only, or we designate both the true and the undefined value. As a result, we obtain four possible combinations:

- K_3 , known as *strong Kleene logic*, is based on the strong Kleene tables and has 1 as a single designated value;
- LP (the *Logic of Paradox*) is also based on the strong tables, but both 1 and $\frac{1}{2}$ are designated values;
- B_3 , or *Paracomplete Weak Kleene Logic*, is based on the weak tables and has 1 as a single designated value;
- finally, *Paraconsistent Weak Kleene Logic* PWK adopts the weak tables and designates both 1 and $\frac{1}{2}$.

The naturality of these logics is attested by the fact that they, or slight variants thereof, have been rediscovered several times in the history of modern logic. LP, for one, was originally a suggestion by Asenjo [4], but was independently introduced, and later investigated proof-theoretically and semantically, by Graham Priest, who made it the cornerstone of his dialethic approach to the truth-theoretic and set-theoretic paradoxes [59–61]. B_3 and PWK, on the other hand, are linguistic fragments of logics respectively introduced by Bochvar in 1938 [14] for dealing with the paradoxes in set theory, and by Halldén in 1949 [42] as a tool for reasoning in the presence of nonsensical sentences (see also Prior’s [63]).

There is a vast literature on all four such logics, teeming with both proof-theoretic studies and semantical enquiries. Moreover, all the strong and weak Kleene logics have been applied, although with different degrees of intensity and with an uneven distribution in time, to several problems in philosophy, computer science, linguistics, and artificial intelligence. Let us survey some of these developments in more detail.

1.1. Strong Kleene Logics

The Strong Kleene logics K_3 and LP drew the logicians’ attention quicker than B_3 or PWK. In part, this is probably due to the fact that they can be formulated as extensions of Belnap and Dunn’s logic [13, 51], which attracted a considerable amount of interest, from day one, among philosophers and computer scientists alike. Further advantages of K_3 and LP over their weak cousins are that their conjunctions and disjunctions have genuine lattice properties, and that there is no such thing as a “contaminating value” in need of a convincing intuitive explanation. Also, philosophical interpretations and applications of these logics were more readily available. Let us only mention that K_3 is a bedrock logic for Kripke’s influential theory of truth [48] and other related proposals in the same area [36, 49]. Here, the value $\frac{1}{2}$ receives a convenient interpretation as a *truth value gap* (“neither true nor false”) that can be assigned to various self-referential statements containing the truth predicate. The dual approach, where $\frac{1}{2}$ is viewed as a *truth value glut* (“both true and false”), has been adopted by Graham Priest in his dialethic account of the paradoxes in truth theory, set theory, and the theory of vagueness (to name only a few examples of application). Crucially, the glutty value counts as designated, whence such theories have LP as a base logic instead of K_3 . Let us also notice that K_3 has been widely used in artificial intelligence as a model of partial information [1] and nonmonotonic reasoning [73].

In other respects, neither K_3 nor LP are especially well-behaved. It is easy to see that K_3 is theoremless, while LP lacks a conditional that obeys Modus Ponens. For these reasons, the existing Hilbert systems for K_3 and LP [2, 39, 64] share some features with natural deduction calculi, in that they contain a large stock of inference rules (calculi for K_3 are purely inferential, indeed). Proper natural deduction calculi for K_3 have been given by Urquhart [74], Priest [61], Tamminga [72] and Petrukhin [54]. In the same papers by Priest and Petrukhin one also finds natural deduction calculi for LP (see also [47]). While the above efforts are not too dissimilar in their approaches, the sequent calculi for strong Kleene logics follow either one of two very different strategies. In some calculi [39, 65] the consequence relation of the logic at issue matches (in the terminology of [5]) the *internal* relation of the calculus itself—where the internal relation of a calculus \mathcal{C} is the relation obtaining between a set of formulas Γ and a formula φ iff there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \Rightarrow \varphi$ is a provable sequent of \mathcal{C} . In other calculi [58, 66], the appropriate match is with the *external* relation of \mathcal{C} —the relation obtaining between a set of formulas Γ and a formula φ iff the sequent $\Rightarrow \varphi$ is derivable in \mathcal{C} from $\{\Rightarrow \psi : \psi \in \Gamma\}$. The latter approach seems more flexible, at least in so far as it yields a modular sequent formulation for several logics in the vicinity of K_3 and LP [58]. In particular, the calculus for K_3 (for LP) is a version of the calculus for CL without the structural rule of Identity (of Cut) and with the addition of elimination rules for the connectives.

Neither K_3 nor LP are even protoalgebraic. Thus, both logics stop short of the bottom rung of the (traditional) Leibniz hierarchy in abstract algebraic logic [40]. Despite that, they are somewhat tractable from an algebraic viewpoint [2, 53, 64, 68]. Remarkably, the class of algebra reducts of reduced Suszko models of both K_3 and LP is the class of *Kleene lattices*, time-honoured structures first introduced by Kalman in the 1950s [45] and later studied by several authors (see e.g. [23, 25]). Interest in this class was also spurred by the observation that Kleene lattices arise as reducts of MV-algebras [26] and other notable algebras.

Before wrapping up this terse survey of strong Kleene logics, let us mention a recent development that has fueled a copious literature in the last few years. *Strict-Tolerant Logic* (ST) is a non-Tarskian logic defined on the strong Kleene matrices, where transitivity of consequence is not unrestrictedly valid [30, 31, 67]. It can be naturally formulated also as a first-order logic and used to underpin nontrivial naive theories of truth and vagueness. Interestingly, it validates the same inferences as CL, whence its proponents claim to have shown that a disquotational truth predicate or an unconstrained tolerance principle can coexist with a classical logical basis.

On the other hand, several classical *metainferences*—including, as we have seen, transitivity—fail in ST. The extent to which Strict-Tolerant Logic deflects from CL, therefore, is an object of some controversy (see e.g. [6, 8, 34]). This debate originated two interesting sidelines: a literature on *metainferential logics*, investigating consequence relations between inferences, metainferences or even more complex objects [7, 33]; and a study of *mixed consequence relations*, many-valued logics whose designated values are different according as one is evaluating a premiss or a conclusion [24].

1.2. Weak Kleene Logics

At first sight, the *infectiousness* of the value $\frac{1}{2}$ in the weak Kleene tables—meaning that any sentential compound where some subformula is evaluated as $\frac{1}{2}$ also gets the value $\frac{1}{2}$ —may be puzzling. It has been variously suggested to read this value as “meaningless” [42], “paradoxical” [14], “non-significant” (a gloss that subsumes, but is more general than, the “meaningless” interpretation [35]), or “off-topic” [11]. On top of that, PWK poses the additional problem of justifying why $\frac{1}{2}$ is infectious (hence bad) and designated (hence good) at the same time. Needless to say, there has been and still is a lively debate on how to get out of this quandary [18, 28, 42, 50]. Such discussions have originated a number of interesting applications for weak Kleene logics, including the analysis of computation errors [35], of expert opinion pooling [38, 71], of vagueness [42], and of ignorance [16].

Two theorems, respectively due to Urquhart [74] and Ciuni and Carrara [27], have refined our understanding of weak Kleene logics and paved the way for unsuspected connections with other approaches and domains. For any set $\Gamma \cup \{\varphi\}$ of formulas in the language of such logics, we have that $\Gamma \vdash_{B_3} \varphi$ iff either (1) Γ is a classically inconsistent set or (2) $\Gamma \vdash_{CL} \varphi$ and the propositional variables in φ are included in the variables in Γ ; dually, we have that $\Gamma \vdash_{PWK} \varphi$ iff either (1) φ is a classical theorem or (2) $\Gamma \vdash_{CL} \varphi$ and there is a nonempty $\Delta \subseteq \Gamma$ such that the propositional variables in Δ are included in the variables in φ . In other words, the valid entailments of weak Kleene logics can be obtained by applying to the set of classically valid entailments an appropriate variable inclusion strainer, one that possibly admits of exceptions in correspondence of inconsistent sets of premisses or of tautologous conclusions. This may ring a bell to some universal algebraists. Recall that a variety of algebras is said to be *regular* if it only satisfies identities where the same variables occur on both sides. Algebras in some regular varieties admit a representation as *Plonka sums* over semilattice direct

systems of algebras of the same type (the *fibres* of the system), all belonging to a given—usually better known and more “manageable”—irregular subvariety of such. The Płonka sum construction, named after its creator Jerzy Płonka, is a very natural and flexible construction, which in nontrivial cases preserves all and only the regular identities satisfied in all the fibres [55–57, 69, 70]. This construction can be generalised from algebras to logical matrices (see e.g. [18]), permitting a uniform semantics for logics obeying variable inclusion patterns of the above kind. More precisely, for each logic L it is possible to define its *left variable inclusion companion* L^l and its *right variable inclusion companion* L^r , whose valid entailments are obtained from the L -valid entailments by applying appropriate variable inclusion strainers, from premisses to conclusions and from conclusions to premisses, respectively. Unsurprisingly, we have that $B_3 = CL^r$ and $PWK = CL^l$.¹ Logics of left and right variable inclusion can be given a semantics in terms of Płonka sums over different types of semilattice direct systems of matrices, where the behaviour of deductive filters is responsible for the leftward or rightward direction of the resulting variable inclusion pattern. In this more general framework (which relates to earlier taxonomies proposed e.g. in [29] within a multiple-conclusion setting), matrices with infectious elements are just a very special case of Płonka sums of matrices where one or more fibres have a trivial algebra reduct.

Proof-theoretic approaches are also to be found in the literature. For Hilbert systems, see below. Like for strong Kleene logics, both “internal” and “external” sequent calculi have been proposed — in the former [20, 32] some rules (either operational or structural) are constrained by appropriate variable inclusion riders, while in the latter (see e.g. [52] or the paper by Da Ré et al. in this special issue) no such restrictions are present. Other contributions present Indrzejczak-style [44] bisequent calculi and single-conclusion [12, 54, 62] or multiple-conclusion [21] natural deduction calculi.

2. The Conference Trends in Logic XXIII

As per its website presentation, “Trends in Logic is the conference series of the journal *Studia Logica*, aimed at worldwide promotion of logic and *Studia Logica*. The series began in 2003, and has been held annually at different logic centres. Apart from Poland, Trends in Logic conferences were held in

¹The variable inclusion companions of intuitionistic logic, instead, are investigated in [10].

Denmark, China, Belgium, The Netherlands, Germany, Italy, USA, Georgia, Argentina, Brazil, Russia and Ukraine. The series has been instrumental in increasing the visibility of *Studia Logica* and elevating its international standing”.

Trends in Logic XXIII, “Strong and Weak Kleene Logic”, was hosted by the Department of Mathematics and Computer Science and by the Department of Pedagogy, Psychology, Philosophy of the University of Cagliari, on July 18–20, 2022, and held in a dual mode (in-person and online). Its programme included an opening address by Tomasz Jarmuzek (on behalf of the Editor in Chief of *Studia Logica*, Jacek Malinowski), invited talks by Graham Priest, Anna Romanowska, Peter Jipsen, Paul Egré, Tommaso Moraschini, Pablo Cobreros, Massimiliano Carrara, and Eduardo Barrio, as well as 28 contributed talks spanning a wide range of different themes: the proof-theoretic analysis of strong and weak Kleene logics, their applications to philosophical problems (vagueness, Carnap’s categoricity problem) or to formal epistemology (logics of ignorance, theories of epistemic commitment, theories of probability, theories of evidential and counterfactual conditionals), their relations with other many-valued logics, connexive logics, logics of non-falsity and exact truth, Strict-Tolerant Logic and other mixed consequence relations. Some talks also addressed the more general topics of left and right variable inclusion logics and Płonka sums, while other contributions were devoted to pseudo-Kleene and Kleene lattices and their expansions and variants.

It is fair to say that the conference achieved, at least as a preliminary step, its goal of fostering the collaboration and interaction among the different communities working in the area, as witnessed by the many talks that touched upon broad-spectrum cross-disciplinary topics. This volume includes some of the papers presented at the conference, representing many of the topics discussed during the sessions.

3. Outline of the Special Issue

Three crucial threads in the current research into Kleene logics and their attendant algebraic structures lie at the centre of the papers that have been collected in the present special issue.

1. *Proof theory of weak Kleene logics.* The proof-theoretical analysis of weak Kleene logics is comparatively less developed than that of their strong cousins—many problems that have long been solved in the latter

context, or have at least been addressed by a number of satisfactory proposals, are still the object of an active investigation in the weak Kleene realm. The paper “Finite Hilbert systems for Weak Kleene logics”, by Vitor Greati, Sérgio Marcelino, and Umberto Rivieccio, attacks one such problem. The paracomplete weak Kleene logic B_3 and its paraconsistent companion PWK have been given Hilbert-style axiomatisations in [15, 17–19]; such axiom systems, however, either have rules with linguistic constraints or contain infinitely many axiom schemata. The paper by Greati et al., via a detour through multiple-conclusion systems, introduces *finite* axiomatic calculi for both logics that are free from linguistic constraints. The paper “Non-reflexive nonsense: Proof theory of paracomplete Weak Kleene logic”, by Bruno Da Ré, Damian Szmuc, and María Inés Corbalán, takes its cue from the sequent calculus in [52], whose external consequence relation coincides with the consequence relation of PWK, and develop a similar approach for B_3 , where Identity, rather than Cut, is dropped. Observe that this is not a mere exercise in adjusting the framework from the paraconsistent to the paracomplete case, since a number of difficulties arise that are inherent to B_3 and call for a specific solution, given by the authors through a recourse to new methods. Finally, the paper “On Woodruff’s constructive nonsense logic”, by Jonas R.B. Arenhart and Hitoshi Omori, examines a constructive variant of Halldén’s logic of nonsense suggested by Woodruff [75], surveying its properties, discussing some of its philosophical shortcomings, and re-examining it against the backdrop of a two-dimensional semantics for many-valued logics developed by Hans Herzberger [43].

2. *Kleene posets and lattices, expansions thereof, and Płonka sums.* None of the Kleene logics, strong or weak, are even protoalgebraic, at least in their standard signature. However, as we have remarked above, all of them are to some extent pliant to an algebraic investigation. The classes of algebras that take centre stage in such enquiries are Kleene posets and lattices [45] (for strong logics) and (generalised) involutive bisemilattices [18] (for weak logics). The latter are representable as Płonka sums of Boolean algebras and provide one of the best-behaved exemplifications of the Płonka construction. Some papers in this issue pick this more algebraic strand. In “Representability of Kleene posets and Kleene lattices”, Ivan Chajda, Helmut Länger and Jan Paseka show how to obtain Kleene lattices and posets by a refinement of the twist product construction, specifying conditions under which Kleene lattices or posets are representable in this form. The paper also investigates different types

of completions of Kleene posets. In “Distributive PBZ*-lattices”, Claudia Mureşan investigates paraorthomodular Brouwer-Zadeh lattices with the *-property (PBZ*-lattices), the abstract counterparts of structures of effects in a Hilbert space—where effects can be viewed as the mathematical representatives of unsharp quantum-theoretic properties [41]. She examines the distributive subvariety of PBZ*-lattices, whose members have Kleene lattices as reducts, and proves *inter alia* that it contains an infinite ascending chain of subvarieties—in sharp contrast with the variety of Kleene lattices, which only contains one nontrivial subvariety. In “De Morgan-Łonka sums”, Thomas Randriamahazaka identifies some varieties of algebras that arise e.g. in the semantics of Angell’s logic of analytic containment [3] and whose valid identities satisfy properties stronger than regularity. Such varieties are not docile to the traditional Łonka methods because the values of some operations are not computed according to the defining clauses of Łonka sums. Randriamahazaka generalises the Łonka construction to the effect that the indexing algebra is no longer a semilattice, but a semilattice with involution. Analogues of the main structure theorems for Łonka sums are given in this new context, and a purely algebraic completeness proof for Fine’s truthmaker semantics for Angell’s logic [37] is obtained as a corollary.

3. *Philosophical and mathematical applications of Kleene logics.* As already observed, Kleene logics were originally devised with an eye to computer science applications, and for all their seeming simplicity, there is no shortage of contexts (whether philosophical, mathematical, or practical) where they can be significantly utilised. A philosophical problem connected with strong Kleene logics is explored in S. Kaan Tabakci’s “Categoricity problem for LP and K_3 ”. Carnap [22] first noticed that there are non-Boolean models that satisfy the provable inferences of Classical Logic, and since then the issue as to whether, and how, we can attain categoricity for CL has been much debated. Tabakci provides three different restrictions on admissible models that deliver categoricity results for LP and K_3 . Victor Aranda’s “Propositional type theory of indeterminacy” defines a propositional type theory whose hierarchy of types contains partial functions, and where the undefined value has a strong Kleene interpretation. A semantics and a complete proof system are given for this theory. Finally, “A new game theoretic semantics (GTS-2) for Weak Kleene logics”, by Massimiliano Carrara, Filippo Mancini, Michele Pra Baldi, and Wei Zhu, presents and discusses a game theoretic semantics for B_3 and PWK, and shows how it generalises to a

broader family of logics of variable inclusion. This semantics improves on a preceding proposal by Baškent [9] from the point of view of its philosophical justification.

Funding Open access funding provided by Università degli Studi di Cagliari within the CRUI-CARE Agreement.

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F. Paoli, G. St. John

G. ST. JOHN
Gavinstjohn@gmail.com