



# Lifespan of solutions for a class of fourth order parabolic equations involving the Hessian

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## ABSTRACT

This paper deals with blow-up solutions of a class of initial-boundary value problems for a fourth order parabolic equation involving the Hessian. A lower bound for the lifespan of such solutions is derived.

## 1. Introduction

This paper deals with the following initial-boundary value problem for a fourth order parabolic equation involving the Hessian

$$\begin{cases} u_t + \Delta^2 u = |\det(D^2 u)|^{\frac{1}{2}} + f(u), & x \in \Omega, \quad t > 0, \\ u = 0, \quad \frac{\partial u}{\partial n} = 0 & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (1)$$

with  $\Omega$  a bounded domain in  $\mathbb{R}^2$  with smooth enough boundary  $\partial\Omega$ ,  $\frac{\partial}{\partial n}$  is the outward normal derivative on the boundary  $\partial\Omega$ ,  $f \in L^2(\Omega)$  is a non negative function,  $u_0(x)$ , is a non negative initial data defined in  $\Omega$ . In our context we assume  $f(u) = u|u|^{p-1}$ ,  $p > 1$ . In the literature there is a great interest to this type of equations for the physical applications as the epitaxial growth of a thin film [1–4]. A general model is

$$u_t + A_1 \Delta^2 u - A_2 \Delta u - A_3 \operatorname{div}(|\nabla u|^2 \nabla u) = g(x, t, u), \quad (2)$$

where  $u$  is the height of the surface of the film,  $A_1 \Delta^2 u$  denotes the capillary-driven surface diffusion,  $A_2 \Delta u$  describes the diffusion due to the evaporation-condensation,  $A_3 \operatorname{div}(|\nabla u|^2 \nabla u)$  represents the upward hopping of atoms and  $g$  corresponds to the mean deposition flux. Different variations of (2) have been studied in the past, deriving global existence as well as blow-up of the solutions. In [5] King et al. for the weak solutions to

$$\begin{cases} u_t + \Delta^2 u - \nabla \cdot (f(\nabla u)) = g(x, t), & \text{in } \Omega \times (0, T), \Omega \subset \mathbb{R}^N, \\ \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0, & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & \text{in } \Omega, \end{cases} \quad (3)$$

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proved existence, uniqueness and regularity in an appropriate functional space. Philippin and Vernier-Piro in [6] for the following problem with time dependent coefficients

$$\begin{cases} u_t + k_1(t)\Delta^2 u - k_2(t)\Delta u = k_3(t)u|u|^{p-1}, & x \in \Omega, t \in (0, t^*), \\ u = \frac{\partial u}{\partial n} = 0, \text{ or } u = \Delta u = 0, & x \in \partial\Omega, t \in (0, t^*), \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{4}$$

with  $k_i(t)$  positive derivable functions,  $\Omega \subset \mathbb{R}^N, N = 2, 3$ , estimated the blow-up time  $t^*$  under conditions on initial data and the coefficients  $k_i(t)$ , obtaining a lower bound for  $t^*$ ; moreover, a criterion ensuring that  $u$  cannot exist for all time was given and an upper bound for  $t^*$  attained.

For the following particular case of (2)

$$\begin{cases} u_t + \Delta^2 u + \mu \Delta u + \lambda \Delta(|\nabla u|^2) = f(x), & x \in \Omega \subset \mathbb{R}^N, t > 0, \\ \frac{\partial u}{\partial n} = \frac{\Delta u}{\partial n} = 0, & x \in \partial\Omega, t > 0 \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{5}$$

Winkler in [7] observed that, in the context of the growth of thin surfaces when exposed to molecular beam epitaxy, the model incorporates the linear effects of surface diffusion and the so-called ‘‘Schwochel barriers’’ reflected by the biharmonic term and the second order backward term  $-\mu \Delta u$ , while the term  $\Delta(|\nabla u|^2)$  accounts for the presence of an energy barrier that particles need to cross before they diffuse (see all the references in [7]). Winkler proved that if  $\Omega$  is convex,  $N \leq 3$ ,  $u_0$  and  $f$  belong to  $L^\infty$  with non negative integral on  $\Omega$ , then (5) possesses at least one global weak solution  $u$  with  $\int_\Omega u \geq 0$  for a.e.  $t > 0$ . For the problem (1), in the case  $|\det(D^2u)|^{\frac{1}{2}}$  and  $f(u)$  replaced by  $\det(D^2u)$  and  $\lambda f(x, t)$  respectively, Escudero et al. in [1], observing that the epitaxial growth process is dictated by the competition between the determinant for the Hessian matrix and the bilaplacian term, proved that a unique solution exists (Theorem 3.2), while the solution blows up in finite time in  $W_0^{2,2}(\Omega)$ -norm and  $W_0^{1,4}(\Omega)$ -norm for  $\Omega \subset \mathbb{R}^2$ , if the initial energy  $J(u_0) \leq d$ , where  $d > 0$  is the mountain pass level and with the  $W_0^{2,2}(\Omega)$ -norm of the initial data  $u_0$  strictly bounded by  $3 \int_\Omega u_{0,x} u_{0,y} u_{0,xy} dx$ ; moreover, in [2, Theorem 2.4], the authors obtained a blow up criterion; while Xu and Zhou in [8], for the case of  $\lambda = 0$ , solved an open question proposed in [1], i.e. blow-up in  $L^p(\Omega)$ - norm, proving that the solutions blow up in  $L^2(\Omega)$ -norm if  $J(u_0) \leq 0$  (see also [3,4]). In [1] (see Theorem 3.2), the problem admits a unique solution in  $X_{\bar{T}} := C(0, \bar{T}; W_0^{2,2}(\Omega)) \cap L^2(0, \bar{T}; W^{4,2}(\Omega)) \cap W^{1,2}(0, \bar{T}; L^2(\Omega))$  provided one of the following set of conditions holds:

- (i)  $u_0 \in W_0^{2,2}(\Omega)$ ,  $f \in L^2(0, \bar{T}; L^2(\Omega))$ , and  $\bar{T} > 0$  is sufficiently small;
- (ii)  $\bar{T} \in (0, \infty]$ ,  $f \in L^2(0, \bar{T}; L^2(\Omega))$ , and  $\|\Delta u_0\|_2^2$ , and  $|\lambda|$  are sufficiently small. Moreover, if  $[0, t^*)$  denotes the maximal interval of continuation of  $u$  and if  $t^* < \infty$ , then  $\|\Delta u\|_2^2 \rightarrow \infty$  as  $t \rightarrow t^*$ , ( $\|\cdot\|_2^2$  is the usual norm in  $L^2(\Omega)$ ). More recently Zhou in [3] solved an open question proposed in [1], with  $f = 0$ , i.e. blow-up in  $L^p(\Omega)$ -norm, by proving that if  $I(u_0) < \frac{\lambda_1}{6} \|u_0\|_2$ , ( $I(u) = \int_\Omega u_x u_y u_{xy}$ ,  $\lambda_1$  the least eigenvalue of the biharmonic operator), the solution blows up in finite time  $t^*$  in  $L^2(\Omega)$ -norm, estimating  $t^*$  from above. Following the idea in [1,3], we can observe that the solutions of (1) may blow-up in finite time. Our aim is to derive a lower bound of the lifespan of the solutions so that a safe interval of existence of the solutions can be derived. We point out that in [1] the authors consider the case of the source term depending only on  $x, t$ , while here we have the presence of a source term depending on the solution  $u$  of superlinear type. We remark that in the cited results only an upper bound of the blow-up time was obtained, but in practical situations it may be more interesting to derive some lower bounds for the lifespan. We prove the following theorem:

**Theorem 1.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$ , let  $u(x, t)$  be a blowing up solution in  $W_0^{2,2}(\Omega)$  of the initial–boundary value problem (1) with initial data  $u_0 \in W_0^{2,2}(\Omega)$ . Assume  $f(u) = u|u|^{p-1}$ ,  $p > 1$ . If  $\|\Delta u_0\|_2^2 > 0$ , then there exist two positive constants  $\alpha$  and  $\beta$  (independent of  $u$ ) such that the solution is bounded in  $W_0^{2,2}(\Omega)$ -norm (and in  $L^2(\Omega)$ -norm) in the interval  $[0, T]$  with*

$$T \leq \tilde{T} := \frac{1}{(p-1)\alpha} \log\left(1 + \frac{\alpha}{\beta} (\|\Delta u_0\|_2^2)^{1-p}\right) < t^*, \tag{6}$$

with  $t^*$  the blow-up time.

**Proof of Theorem 1.1**

We assume that  $u(x, t)$  is a blowing up solution of (1) in  $W_0^{2,2}(\Omega)$ -norm ( $\lim_{t \rightarrow t^*} \|\Delta u\|_2^2 = +\infty$ ). By constructing a differential inequality for  $\|\Delta u\|_2^2$ , we estimate a lower  $\tilde{T}$  of  $t^*$ . Then, if  $T \leq \tilde{T}$ ,  $[0, T]$  is a safe interval of existence of the solution  $u(x, t)$ . Then as a consequence,  $u(x, t)$  is bounded also in  $L^2(\Omega)$ -norm, since  $\|u\|_2^2 \leq \frac{1}{\lambda_1} \|\Delta u\|_2^2$ , by the first eigenvalue problem for the bilaplacian [9] with  $u = \frac{\partial u}{\partial n} = 0$  on the boundary. Testing the equation in (1) with  $\Delta^2 u$  we have

$$\int_\Omega (u_t \Delta^2 u + (\Delta^2 u)^2) dx = \int_\Omega |\det(D^2u)|^{\frac{1}{2}} \Delta^2 u dx + \int_\Omega f(u) \Delta^2 u dx. \tag{7}$$

Since

$$\int_\Omega (u_t \Delta^2 u) dx = \frac{1}{2} \frac{d}{dt} \|\Delta u\|_2^2, \tag{8}$$

then replacing (8) into (7), the following identity holds

$$\frac{1}{2} \frac{d}{dt} \|\Delta u\|_2^2 + \|\Delta^2 u\|_2^2 = \int_{\Omega} |\det(D^2 u)|^{\frac{1}{2}} \Delta^2 u \, dx + \int_{\Omega} f(u) \Delta^2 u \, dx. \tag{9}$$

By Schwarz inequality, for all  $t \in [0, T]$ , with  $T < t^*$ ,

$$\begin{aligned} \int_{\Omega} |\det(D^2 u)|^{\frac{1}{2}} \Delta^2 u \, dx &\leq \frac{1}{2} \left[ \mu \int_{\Omega} |\Delta^2 u|^2 \, dx + \frac{1}{\mu} \int_{\Omega} |\det(D^2 u)| \, dx \right] \\ &\leq \frac{\mu}{2} \|\Delta^2 u\|_2^2 + \frac{\tilde{C}}{2\mu} \|\Delta u\|_2^2, \quad \mu > 0, \end{aligned} \tag{10}$$

where we have used the following estimate

$$\int_{\Omega} |\det(D^2 u)| \, dx \leq C \int_{\Omega} |D^2 u|^2 \, dx \leq \tilde{C} \int_{\Omega} |\Delta u|^2 \, dx.$$

Moreover,

$$\int_{\Omega} f(u) \, \Delta^2 u \, dx \leq \|f(u)\|_2 \cdot \|\Delta^2 u\|_2 \leq \frac{\epsilon}{2} \|\Delta^2 u\|_2^2 + \frac{1}{2\epsilon} \|f(u)\|_2^2, \quad \epsilon > 0. \tag{11}$$

Back to the first equation, inserting (10) and (11) in (9)

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta u\|_2^2 &\leq -\|\Delta^2 u\|_2^2 + \frac{\mu}{2} \|\Delta^2 u\|_2^2 + \frac{\tilde{C}}{2\mu} \|\Delta u\|_2^2 + \frac{\epsilon}{2} \|\Delta^2 u\|_2^2 \\ &+ \frac{1}{2\epsilon} \|f(u)\|_2^2 = \left(-1 + \frac{\mu}{2} + \frac{\epsilon}{2}\right) \|\Delta^2 u\|_2^2 + \frac{\tilde{C}}{2\mu} \|\Delta u\|_2^2 + \frac{1}{2\epsilon} \|f(u)\|_2^2. \end{aligned}$$

Choosing  $\mu = \epsilon = 1$  we have  $-1 + \frac{\mu}{2} + \frac{\epsilon}{2} = 0$ . Now, for  $f(u) = u |u|^{p-1}$ , from  $\|w\|_q \leq S_q \|\Delta w\|_2$ , valid for  $w \in W_0^{2,2}(\Omega)$ ,  $q > 1$ ,  $S_q$  the best Sobolev constant, we have with  $q = 2p$

$$\int_{\Omega} u^{2p} \, dx \leq S_{2p}^{2p} \left( \int_{\Omega} (\Delta u)^2 \, dx \right)^p.$$

It follows

$$\frac{d}{dt} \|\Delta u\|_2^2 \leq \tilde{C} \|\Delta u\|_2^2 + S_{2p}^{2p} (\|\Delta u\|_2^2)^p. \tag{12}$$

With  $\chi(t) := \|\Delta u(\cdot, t)\|_2^2$ ,  $\alpha := \tilde{C}$ ,  $\beta := S_{2p}^{2p}$ , then (12) can be rewritten as

$$\chi' \leq \alpha \chi + \beta \chi^p. \tag{13}$$

We note that since  $\lim_{t \rightarrow t^*} \|\Delta u(\cdot, t)\|_2^2 = \lim_{t \rightarrow t^*} \chi(t) = +\infty$ , there exists a time  $t_1 \in (0, t^*)$  such that  $\chi(t) \geq \chi(0) > 0$  for all  $t \in [t_1, t^*)$  and  $\chi(t_1) = \chi(0)$ .

Let  $\phi(t) = \chi^{1-p}(t)$ , for  $t \in [t_1, t^*)$ . Then from the inequality (13) we find

$$\phi'(t) + \alpha(p-1)\phi(t) \geq -\beta(p-1), \quad t \in [t_1, t^*). \tag{14}$$

Integrating (14) from  $t_1$  to  $t$  we have

$$\begin{aligned} \int_0^t \left( \phi(\tau) e^{\alpha(p-1)\tau} \right)' \, d\tau &> \int_{t_1}^t \left( \phi(\tau) e^{\alpha(p-1)\tau} \right)' \, d\tau \\ &\geq -\beta(p-1) \int_{t_1}^t e^{\alpha(p-1)\tau} \, d\tau > -\beta(p-1) \int_0^t e^{\alpha(p-1)\tau} \, d\tau \end{aligned}$$

and recalling that  $\phi(t) \rightarrow 0$  as  $t \rightarrow t^*$ , we easily get

$$t^* > \frac{1}{(p-1)\alpha} \log\left(1 + \frac{\alpha}{\beta} (\|\Delta u_0\|_2^2)^{1-p}\right) := \tilde{T}. \tag{15}$$

(15) implies that  $\Delta u$  is bounded in  $L^2(\Omega)$ -norm in  $[0, T]$ ,  $T \leq \tilde{T}$ .

As a consequence the solution exists bounded in  $L^2(\Omega)$ -norm in the interval  $[0, T]$ , since  $\|u\|_2^2 \leq \frac{1}{\lambda_1} \|\Delta u\|_2^2$ .

### Data availability

No data was used for the research described in the article.

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