Qualitative behavior of solutions of a chemotaxis system with flux limitation and nonlinear signal production

M.Marras [1], **Y. Chiyo** [2]

Abstract

In this paper we consider radially symmetric solutions of the following parabolic-elliptic cross-diffusion system

$$\begin{cases} u_t = \Delta u - \nabla(uf(|\nabla v|^2)\nabla v), \\ 0 = \Delta v - \mu(t) + g(u), \quad \mu(t) = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega g(u(\cdot,t))dx \\ u(x,0) = u_0(x), \end{cases}$$

in $\Omega \times (0,\infty)$, with Ω a ball in $\mathbb{R}^N$, $N \geq 1$ under homogeneous Neumann boundary conditions, $g(u)$ a regular function with the prototype $g(u) = u^k$, $u \geq 0$, $k > 0$. The function $f(\xi) = k_f(1+\xi)^{-\alpha}$, $k_f > 0$, describes gradient-dependent limitation of cross diffusion fluxes. Under suitable conditions on the data, we prove that the solution is global in time. If $N \geq 3$, under conditions on f, g and initial data, we prove that if the solution $u(x,t)$ blows up in L^∞-norm at finite time T_{max} then for some $p > 1$ it blows up also in L^p-norm. Moreover a lower bound of blow-up time is derived.

AMS Subject Classification Primary: 35B44; Secondary: 35Q92, 92C17.

Key Words: boundedness; finite-time blow-up; chemotaxis.

[1]Dipartimento di Matematica e Informatica, Universitá di Cagliari, via Ospedale 72, 09124 Cagliari (Italy), mmarras@unica.it

[2]Department of Mathematics, Tokyo University of Science 1-3, Kagurazaka, Shinjuku-ku, Tokyo 162-8601, (Japan), ycnewssz@gmail.com

1 Introduction

In this paper we consider the chemotaxis system with flux limitation and nonlinear signal production,

$$
(1.1) \qquad \begin{cases} u_t = \Delta u - \nabla(uf(|\nabla v|^2)\nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \mu(t) + g(u), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ \int_\Omega v(x,t)dx = 0, & t > 0, \end{cases}
$$

in $\Omega := B_R(0) \subset \mathbb{R}^N$ with $R > 0$, $N \geq 3$,

$$
(1.2) \qquad f \in C^2([0,\infty)),
$$

$\mu(t) = \frac{1}{|\Omega|}\int_\Omega g(u(x,t))dx > 0,$

$$
(1.3) \qquad g \in \cup_{\theta\in(0,1)} C^\theta_{loc}([0,\infty)) \cap C^1((0,\infty)),\ \text{nonnegative and nondecreasing.}
$$

On the boundary we assume no flux conditions. The initial data u_0 is nonnegative, radially symmetric function nonincreasing with respect to $|x|$ and

$$
(1.4) \qquad u_0 \in \cup_{\theta\in(0,1)} C^\theta(\bar{\Omega}).
$$

When $g(u) = u$, and the drift term depends on the chemical flux ∇v in the sense that $f(|\nabla v|^2) = \chi$, for a positive given constant χ, and $\mu(t)$ replaced by $v(x,t)$, the chemotaxis model (1.1) is just the classical Keller-Segel system. Keller-Segel systems ha been extensively studied in the last years: in particular the existence of global bounded solutions and the detection of some solutions blowing up in either finite or infinite time, in a great number of literature (see [1], [7], [9], [11], [19] [20], and the references therein).
In recent years, a large number of variants of the classical chemotaxis model have been introduced as a non linear diffusion terms (for instance $div(h(u)\nabla u)$, $div(|\nabla u|^{p-2}\nabla u)$, Δu^m (see [1], [11], [17] and the references therein), as well as nonlinear dependence of the chemical flux $(div(u^m\nabla v),\ div(uf(|\nabla v|)\nabla v))$.

In [2] and [3], Bellomo and Winkler analyzed the case where the diffusion term Δu is replaced by $div\Big(\frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\Big)$ and the chemotactic term is in the form $\chi\nabla\cdot\Big(\frac{u\nabla v}{\sqrt{1+|\nabla v|^2}}\Big)$. In [2] the authors obtain global existence of bounded classical solutions for arbitrary positive radial initial data $u_0 \in C^3(\Omega)$ when either $N \geq 2$ and $\chi < 1$ or $N = 1$ and $\int_\Omega u_0 < \frac{1}{\sqrt{(\chi^2-1)_+}}$.
In [3], the authors complement the above result on global existence by showing that in both cases $N \geq 2$ and $N = 1$ the previous conditions are optimal in the sense that if appropriate reverse inequalities hold, then finite-time blow-up may occur.
Chiyoda et al. (see [6]) generalized the blow-up results in [3] to the case u^p and u^q in the diffusion term and the chemotactic term respectively, obtaining the existence of blow-up solutions under some condition for χ and u_0 when $1 \leq p \leq q$.
Negreanu and Tello in [14] considered the system (1.1) with flux limitation $f(|\nabla v|) = \chi|\nabla v|^{p-2}$ and $v(x,t)$ instead of $\mu(t)$, and p satisfies

$$p \in (1,\infty),\ if\ N = 1,\ and\ p \in \big(1, \frac{N}{N-1}\big),\ if\ N \geq 2,$$

to derive uniform bounds in L^∞ and the existence of global in time solutions. Moreover, the steady states of the one-dimensional case are also considered and infinitely many non-constant solutions appear for $p \in (1,2)$ for any χ positive and a prescribed positive mass.
The complementary case for $p \in (\frac{N}{N-1}, 2)$ was studied in [18] where the author, for χ large enough, derived conditions on the initial data to obtain blow up phenomena at finite time for any regular solution of the problem.
In [16] Sastre-Gomez and Tello studied the existence of uniformly bounded solutions of chemotaxis system with flux limitation $f(|\nabla|v) = \chi|\nabla v|^{p-2}$ and with a logistic type growth term $\mu u(1-u)$, $\mu > 0$, for $p < \frac{3}{2}$ and any $N \geq 2$.
In [4] Boccardo and Tello studied the stationary case in presence of non linear flux limitation and a non negative source term $f(x) \in L^m(\Omega)$, $m > \max\{1, \frac{N}{2}\}$ and with subcritical signal production u^θ, θ positive and fulfilling $1 < p < \frac{N\theta}{N\theta-1}$, $1 < N\theta$ or $\max\{N,p\} < \frac{\theta+1}{\theta}$.

In [22], Winkler consider the following parabolic-elliptic system

$$
(1.5)\qquad \begin{cases} u_t = \Delta u - \nabla(uf(|\nabla v|^2)\nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \mu + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}
$$

with Ω a ball in $\mathbb{R}^N$, $N \geq 3$, $\mu(t) = \frac{1}{|\Omega|}\int_\Omega u dx > 0$, $\int_\Omega v dx = 0$, $f \in C^2([0,\infty))$.
The author proves that if $f(|\nabla v|^2) \geq k_f(1 + |\nabla v|^2)^{-\alpha}$ with $0 < \alpha < \frac{N-2}{2(N-1)}$, $k_f > 0$ then throughout a considerably large set of radially symmetric initial data, the blow-up phenomenon, with respect to the L^∞ norm of u, occurs in finite time.
This result is complemented by a second statement which ensures that in general (not necessarily symmetric settings), if either $N = 1$ and $\alpha \in \mathbb{R}$ is arbitrary, or $N \geq 2$ and $\alpha > \frac{N-2}{2(N-1)}$, $f(|\nabla v|^2) \leq k_f(1 + |\nabla v|^2)^{-\alpha}$, then for arbitrary nonnegative and continuous initial data, a global bounded classical solution exists.
In [12] the authors proved that if the solution (u, v) of problem (1.5), with $f(|\nabla v|^2) = k_f(1 + |\nabla v|^2)^{-\alpha}$, blows up in L^∞-norm at finite time T_{max} then for some $p > 1$ it blows up also in L^p-norm and derive a lower bound of T_{max}.

In [13] the authors considered the problem (1.5) with $f(|\nabla v|^2) = k_f(1 + |\nabla v|^2)^{-\alpha}$, $0 < \alpha < \frac{N-2}{2(N-1)}$ and a source term of logistic type: $g(u) = \lambda u - \mu u^k$, $\lambda,\ \mu > 0$, $k > 1$

$$
(1.6)\qquad \begin{cases} u_t = \Delta u - \nabla(uf(|\nabla v|^2)\nabla v) + g(u), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \mu + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}
$$

with Ω a ball in $\mathbb{R}^N$, $N \geq 3$, $\mu = \frac{1}{|\Omega|}\int_\Omega u dx > 0$, $\int_\Omega v dx = 0$, $f \in C^2([0,\infty))$.
Under smallness conditions on α and k the authors prove that the solution $u(x,t)$ blows up in L^∞-norm at finite time T_{max} and for some $p > 1$ it blows

up also in L^p-norm. In addition a lower bound of blow-up time is derived. Finally, under largeness conditions on α or k, they prove that the solution is global and bounded in time.

For the case of nonlinear signal production we refer the paper [21] where Winkler considered the problem

$$(1.7)\qquad \begin{cases} u_t = \Delta u - \chi\nabla(u\nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \mu(t) + g(u), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

in a ball $\Omega \subset \mathbb{R}^N$, with $\mu(t) = \frac{1}{|\Omega|}\int_\Omega g(u(x,t))dx$ and $g \in \cup_{\theta\in(0,1)} C^\theta_{loc}([0,\infty)) \cap C^1((0,\infty))$ nonnegative and nondecreasing. If $g(u) \geq \mathsf{k}u^k$ for all $u \geq 1$ and some $k > \frac{2}{N}$, $\mathsf{k} > 0$, the author proved that the solution (u,v) of (1.7) blows up in finite time T_{max}. On the contrary if $g(u) \leq Ku^k$, for all $u \geq 1$ with some $K > 0$ and $k < \frac{2}{N}$, then $T_{max} = \infty$ and $\|u(\cdot,t)\|_{L^\infty} \geq C$, with some $C > 0$.

In this paper we are interested to analyze the behavior in time of the solution of the parabolic-elliptic system (1.1) in presence of flux limitation term and non linear signal production in the form

$$(1.8)\qquad f(|\nabla v|^2) = \frac{k_f}{(1+|\nabla v|^2)^\alpha}, \quad \alpha > 0.$$

The first of our main result asserts global existence and boundedness of the solution (Section 3).

Theorem 1.1 (Global existence and boundedness). *Let* $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geq 3$, $R > 0$ *and let* f *and* g *satisfy* (1.2), (1.8) *and* (1.3) *respectively. Assume*

$$\alpha > \frac{N-2}{2(N-1)}$$

and

$$(1.9)\qquad g(u) \leq K_1 u^k, \quad u \geq 0,$$

with some $K_1 > 0$ *and*

$$k < \frac{2}{N}. \tag{1.10}$$

Then for all radially symmetric nonnegative initial data $u_0 \in C^0(\bar{\Omega})$*, system* (1.1) *possesses a unique global classical solution* (u, v) *in* $\Omega \times (0, \infty)$*, which is bounded in the sense that*

$$\sup_{t \in (0,\infty)} \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty. \tag{1.11}$$

Remark 1.1. *We observe that if* $\alpha = 0$ *we obtain the problem* (1.7)*, studied in [21], where if* $k < \frac{2}{N}$ *the author proved that the solution is global and bounded. Thus, in some cases when* $\alpha < \frac{N-2}{2(N-1)}$*, and for suitably small powers* k*, a possibly alternative to prevent blow-up can be obtained.*

The purpose in Section 4 is to prove that the solutions of (1.1) blow up at finite time in L^p-norm, for some $p > 1$, if they blow up in L^∞-norm.

Theorem 1.2 (Finite-time blow-up in L^p-norm)**.** *Let* $\Omega = B_R(0) \subset \mathbb{R}^N$*,* $N \geq 3$ *and* $R > 0$*and let* f *satisfy* (1.2), (1.8) *and* (1.3) *respectively. Assume*

$$0 < \alpha < \frac{N-2}{2(N-1)},$$

and

$$g(u) \leq K_2 u^k, \quad u \geq 0, \tag{1.12}$$

with some $K_2 > 0$ *and*

$$k > \frac{2}{N}.$$

If a classical solution (u, v) *of* (1.1) *for* $t \in (0, T_{max})$*, blows up at finite time* T_{max} *in* L^∞*-norm, then for all* $p > \frac{N}{2}k$*,*

$$\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^p(\Omega)} = \infty.$$

2 Preliminaries

In this section we state a few helpful results, most of which have already been proven elsewhere. In some cases, for the convenience of the reader, we state the conditions differently from the original source, in such a way that the lemmas become more easily applicable in the present form.

By using standard arguments well-established in the context of parabolic-elliptic Keller-Segel type systems, we state the local existence and extensibility of solutions to (1.1) (see e.g. [8], [21] and [22]):

Lemma 2.1. *Let $N \geq 1$, and assume that $\Omega = B_R(0) \subset \mathbb{R}^N$ for some $R > 0$, f, g and u_0 satisfy respectively* (1.2), (1.3) (1.4). *Then there exist $T_{max} \in (0, \infty]$ and a unique pair (u, v) of functions*

$$u \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})),$$
$$v \in \cap_{q>N} L^\infty_{loc}([0, T_{max}); W^{1,q}(\Omega)) \cap C^{2,0}(\bar{\Omega} \times (0, T_{max})),$$

which solves (1.1) *in the classical sense in $\Omega \times (0, T_{max})$. Moreover, we have $u > 0$ in $\Omega \times (0, T_{max})$, and both $u(\cdot, t)$ and $v(\cdot, t)$ are radially symmetric with respect to $x = 0$ for all $t \geq 0$. Finally,*

$$\text{if} \quad T_{max} < \infty, \quad \text{then} \quad \limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.1}$$

We next give some properties of the Neumann heat semigroup which will be used later. For the proof, see [5, Lemma 2.1] and [19, Lemma 1.3].

Lemma 2.2. *Let $(e^{t\Delta})_{t \geq 0}$ be the Neumann heat semigroup in Ω, and let $\mu_1 > 0$ denote the first non zero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then there exist $k_1, k_2 > 0$ which depend only on Ω and have the following properties:*

(i) if $1 \leq q \leq \mathrm{p} \leq \infty$, then

$$\|e^{t\Delta} z\|_{L^{\mathrm{p}}(\Omega)} \leq k_1 \big(1 + t^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{\mathrm{p}})}\big) e^{-\mu_1 t} \|z\|_{L^q(\Omega)}, \quad \forall\, t > 0 \tag{2.2}$$

holds for all $z \in L^q(\Omega)$ satisfying $\int_\Omega z = 0$.

(ii) If $1 < q \leq \mathrm{p} \leq \infty$, then

$$\|e^{t\Delta} \nabla \cdot \boldsymbol{z}\|_{L^{\mathrm{p}}(\Omega)} \leq k_2 \big(1 + t^{-\frac{1}{2} - \frac{N}{2}(\frac{1}{q} - \frac{1}{\mathrm{p}})}\big) e^{-\mu_1 t} \|\boldsymbol{z}\|_{L^q(\Omega)}, \quad \forall\, t > 0 \tag{2.3}$$

is valid for any $\boldsymbol{z} \in C^1(\bar{\Omega})$ which satisfy $\boldsymbol{z} \cdot \nu = 0$ on $\partial\Omega$.

We observe that since constants are invariant under $e^{t\Delta}$ we can use (2.2) writing $\bar{z} = \frac{1}{|\Omega|}\int_\Omega z\,dx$ so that we have $\int_\Omega (z-\bar{z})\,dx = 0$ (see [19]) .

In Section 3 we will use the Gagliardo–Nirenberg inequality in the following form.

Lemma 2.3. *Let Ω be a bounded and smooth domain of $\mathbb{R}^N$ with $N \geq 1$. Let $\mathsf{r} \geq 1$, $1 \leq \mathsf{q} < \mathsf{p} \leq \infty$, $\mathsf{s} > 0$. Then there exists a constant $C_{\mathrm{GN}} > 0$ such that*

$$\|f\|^\gamma_{L^{\mathsf{p}}(\Omega)} \leq C_{\mathrm{GN}}\Big(\|\nabla f\|^{\gamma a}_{L^{\mathsf{r}}(\Omega)}\|f\|^{\gamma(1-a)}_{L^{\mathsf{q}}(\Omega)} + \|f\|^\gamma_{L^{\mathsf{s}}(\Omega)}\Big) \tag{2.4}$$

for all $f \in L^{\mathsf{q}}(\Omega)$ with $\nabla f \in (L^{\mathsf{r}}(\Omega))^N$ and $a := \frac{\frac{1}{\mathsf{q}}-\frac{1}{\mathsf{p}}}{\frac{1}{\mathsf{q}}+\frac{1}{N}-\frac{1}{\mathsf{r}}} \in (0,1)$, $\gamma > 0$.

See [15] for more details.

3 Boundedness

The purpose of this section is to prove the boundedness result of Theorem 1.1.
To this end we derive an L^p-estimate for u with some $p > 1$.

Lemma 3.1. *Assume that the conditions of Theorem1.1 hold. Then for some $p > 1$ there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \leq C$$

for all $t \in (0, T_{\max})$.

Proof. We consider some $p > 1$. We note that, since $k < \frac{2}{N}$ we have $p > 1 > \frac{N}{2}k$. By straightforward calculations, we have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx &= \int_\Omega u^{p-1}\Delta u dx - \int_\Omega u^{p-1}\nabla\cdot\big(uf(|\nabla v|^2)\nabla v\big)dx \\ &=: I_1(t) + I_2(t).\end{aligned} \tag{3.1}$$

We first deal with the term I_1. Integrating by parts and using the Neumann boundary condition of (1.1) imply

$$\begin{aligned}I_1(t) &= -(p-1)\int_\Omega u^{p-2}|\nabla u|^2 dx \\ &= -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx = -\frac{4(p-1)}{p^2}\big\|\nabla u^{\frac{p}{2}}\big\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.2}$$

We next estimate the term I_2. Again by an integration by parts and the Neumann boundary condition of (1.1), we derive

$$
\begin{aligned}
I_2(t) &= (p-1)\int_\Omega u^{p-1} f(|\nabla v|^2)\nabla u\cdot\nabla v dx\\
&= \frac{p-1}{p}\int_\Omega f(|\nabla v|^2)\nabla u^p\cdot\nabla v dx\\
&= \frac{p-1}{p}\int_\Omega \nabla\cdot\big(u^p f(|\nabla v|^2)\nabla v\big)dx - \frac{p-1}{p}\int_\Omega u^p\nabla\cdot\big(f(|\nabla v|^2)\nabla v\big)dx\\
&= -\frac{p-1}{p}\int_\Omega u^p f(|\nabla v|^2)\Delta v dx - \frac{p-1}{p}\int_\Omega u^p f'(|\nabla v|^2)\nabla v\cdot\nabla\big(|\nabla v|^2\big)dx
\end{aligned}
$$

Here, by using $f(|\nabla v|^2) = k_f\frac{1}{(1+|\nabla v|^2)^\alpha}$ and the second equation in (1.1), it follows that

$$
\begin{aligned}
I_2(t) = &-\mu(t)k_f\frac{p-1}{p}\int_\Omega \frac{u^p}{(1+|\nabla v|^2)^\alpha}dx\\
&+ k_f\frac{p-1}{p}\int_\Omega\frac{u^p g(u)}{(1+|\nabla v|^2)^\alpha}dx\\
&+ \alpha k_f\frac{p-1}{p}\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2)}{(1+|\nabla v|^2)^{\alpha+1}}dx.
\end{aligned}
\tag{3.3}
$$

Neglecting in (3.3) the negative term $\mu(t)k_f\frac{p-1}{p}\int_\Omega\frac{u^p}{(1+|\nabla v|^2)^\alpha}dx$ and using $\frac{1}{(1+|\nabla v|^2)^\alpha}\le 1$ and $g(u)\le K_1u^k$, we arrive at

$$
I_2(t)\le k_fK_1\frac{p-1}{p}\int_\Omega u^{p+k}dx + \alpha k_f\frac{p-1}{p}\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2)}{(1+|\nabla v|^2)^{\alpha+1}}dx. \tag{3.4}
$$

Now, to estimate the first term in (3.4), we use the Gagliardo–Nirenberg inequality (2.4) with $\mathsf{p}=\gamma=\frac{2(p+k)}{p}$, $\mathsf{q}=\mathsf{r}=\mathsf{s}=2$. So it follows that

$$
\begin{aligned}
\int_\Omega u^{p+k}dx &= \|u^{\frac{p}{2}}\|^{\frac{2(p+k)}{p}}_{L^{\frac{2(p+k)}{p}}(\Omega)}\\
&\le C_{GN}\left(\big\|\nabla u^{\frac{p}{2}}\big\|^{\frac{2(p+k)}{p}\theta_1}_{L^2(\Omega)}\big\|u^{\frac{p}{2}}\big\|^{\frac{2(p+k)}{p}(1-\theta_1)}_{L^2(\Omega)} + \big\|u^{\frac{p}{2}}\big\|^{\frac{2(p+k)}{p}}_{L^2(\Omega)}\right),
\end{aligned}
\tag{3.5}
$$

where $\theta_1 := \frac{\frac12-\frac{p}{2(p+k)}}{\frac12+\frac1N-\frac12} = \frac{Nk}{2(p+k)}\in(0,1)$ since $k<\frac{2}{N}$ and $p>1>\frac{N}{2}k$.
Noting from $p>\frac{Nk}{2}$ that $\frac{p+k}{p}\theta_1 = \frac{Nk}{2p}<1$ and applying the Young inequality

to the first term in the right-hand side of (3.5) yield

$$\begin{aligned}
&\left\|\nabla u^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\frac{2(p+k)}{p}\theta_1}\left\|u^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\frac{2(p+k)}{p}(1-\theta_1)}\\
&\quad\le \frac{Nk}{2p}\varepsilon_1\left\|\nabla u^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2+\frac{2p-Nk}{2p}\frac{1}{\varepsilon_1{}^{\frac{Nk}{2p-Nk}}}\left\|u^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{2\frac{2(p+k)-Nk}{2p-Nk}}
\end{aligned}$$

for all $\varepsilon_1>0$, which along with (3.5) implies

$$\begin{aligned}
\int_\Omega u^{p+k}dx &\le C_{GN}\frac{Nk}{2p}\varepsilon_1\left\|\nabla u^{\frac{p}{2}}\right\|_{L^2(\Omega)}^2\\
&\quad + C_{GN}\frac{2p-Nk}{2p}\frac{1}{\varepsilon_1{}^{\frac{Nk}{2p-Nk}}}\left\|u^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{2\frac{2(p+k)-Nk}{2p-Nk}}+C_{GN}\left\|u^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{\frac{2(p+k)}{p}}.
\end{aligned}\tag{3.6}$$

We next estimate the second term in the right-hand side of (3.4). To this end we consider the second equation of (1.1) in radial variables. We have

$$\left(r^{N-1}v_r\right)_r=r^{N-1}\mu(t)-r^{N-1}g(u).\tag{3.7}$$

Integrating (3.7) from 0 to r, we obtain

$$\begin{aligned}
r^{N-1}v_r&=\mu(t)\int_0^r\rho^{N-1}d\rho-\int_0^r\rho^{N-1}g(u(\rho,t))d\rho\\
&=\frac{r^N}{N}\mu(t)-\int_0^r\rho^{N-1}g(u(\rho,t))d\rho.
\end{aligned}\tag{3.8}$$

Differentiating (3.8), we have

$$(N-1)r^{N-2}v_r+r^{N-1}v_{rr}=r^{N-1}\mu(t)-r^{N-1}g(u),$$

which in conjunction with (3.7) yields

$$\begin{aligned}
v_{rr}&=\mu(t)-g(u)-\frac{N-1}{r}v_r\\
&=\mu(t)-g(u)-\frac{N-1}{r^N}\Big[\frac{r^N}{N}\mu(t)-\int_0^r\rho^{N-1}g(u)d\rho\Big]\\
&=\frac{1}{N}\mu(t)-g(u)+\frac{N-1}{r^N}\int_0^r\rho^{N-1}g(u)d\rho.
\end{aligned}\tag{3.9}$$

Writing in radial variables the second term of (3.4) and using (3.9), we have

$$
\begin{aligned}
(3.10)\qquad & \int_\Omega u^p \frac{\nabla v\cdot\nabla(|\nabla v|^2)}{(1+|\nabla v|^2)^{\alpha+1}}dx \\
&= \omega_N \int_0^R u^p \frac{N v_r (v_r^2)_r}{(1+v_r^2)^{\alpha+1}} r^{N-1}dr \\
&= 2N\omega_N \int_0^R u^p \frac{v_r^2 v_{rr}}{(1+v_r^2)^{\alpha+1}} r^{N-1}dr \\
&= 2\mu(t)\omega_N \int_0^R u^p \frac{v_r^2}{(1+v_r^2)^{\alpha+1}} r^{N-1}dr \\
&= -2N\omega_N \int_0^R u^p \frac{v_r^2 g(u)}{(1+v_r^2)^{\alpha+1}} r^{N-1}dr \\
&\quad + 2N(N-1)\omega_N \int_0^R u^p \frac{v_r^2}{(1+v_r^2)^{\alpha+1}} \frac{1}{r}\Big(\int_0^r \rho^{N-1} g(u)d\rho\Big)dr \\
&\le 2\mu(t)\omega_N \int_0^R u^p r^{N-1}dr \\
&\quad + 2N(N-1)K_1\omega_N \int_0^R u^p \frac{1}{r}\Big(\int_0^r \rho^{N-1}u^k d\rho\Big)dr \\
&:= I_{21} + I_{22},
\end{aligned}
$$

where we neglected the negative term $-2N\omega_N \int_0^R u^p \frac{v_r^2 g(u)}{(1+v_r^2)^{\alpha+1}} r^{N-1}dr$ and used the fact that $\frac{v_r^2}{(1+v_r^2)^{\alpha+1}} \le 1$ and hypothesis (1.9) $g(u)\le K_1 u^k$. To estimate the term I_{21} we use $\mu(t) = \frac{1}{|\Omega|}\int_\Omega g(u)dx \le \frac{K_1}{|\Omega|}\int_\Omega u^k dx$ and the Young inequality, we obtain

$$
\begin{aligned}
(3.11)\qquad I_{21} &= 2\mu(t)\omega_N \int_0^R u^p r^{N-1}dr \\
&\le 2K_1 \frac{\omega_N}{|\Omega|}\left(\int_\Omega u^k(x,t)dx\right)\left(\int_\Omega u^p(x,t)dx\right) \\
&\le 2K_1 \frac{\omega_N}{|\Omega|}\Big(\int_\Omega u^{p+k}dx\Big)^{\frac{k}{p+k}} |\Omega|^{\frac{p}{p+k}} \cdot \Big(\int_\Omega u^{p+k}dx\Big)^{\frac{p}{p+k}} |\Omega|^{\frac{k}{p+k}} \\
&= 2K_1\omega_N \int_\Omega u^{p+k}dx.
\end{aligned}
$$

Also, by using the Young inequality, we can estimate I_{22} as

(3.12)
$$
\begin{aligned}
I_{22} &= 2N(N-1)K_1\omega_N \int_0^R u^p \frac{1}{r}\Big(\int_0^r \rho^{N-1}u^k d\rho\Big)dr \\
&\le 2N(N-1)K_1\omega_N \int_0^R u^p \frac{1}{r}\Big(\int_0^r \rho^{N-1}d\rho\Big)^{\frac{p}{p+k}}\Big(\int_0^r u^{p+k}\rho^{N-1}d\rho\Big)^{\frac{k}{p+k}}dr \\
&= 2N(N-1)K_1\omega_N \int_0^R u^p \frac{1}{r}\Big(\frac{r^N}{N}\Big)^{\frac{p}{p+k}}\Big(\int_0^r u^{p+k}\rho^{N-1}d\rho\Big)^{\frac{k}{p+k}}dr \\
&= 2N^{\frac{k}{p+k}}(N-1)K_1\omega_N{}^{\frac{p}{p+k}} \int_0^R u^p r^{\frac{Np}{p+k}-1}\Big(\omega_N\int_0^r u^{p+k}\rho^{N-1}d\rho\Big)^{\frac{k}{p+k}}dr \\
&\le 2N^{\frac{k}{p+k}}(N-1)K_1\omega_N{}^{\frac{p}{p+k}} \Big(\int_0^R u^p r^{\frac{Np}{p+k}-1}dr\Big)\Big(\int_\Omega u^{p+k}dx\Big)^{\frac{k}{p+k}}
\end{aligned}
$$

Here, to estimate the term $\int_0^R u^p r^{\frac{Np}{p+k}-1}dr$ we choose $\varepsilon > 0$ such that the Hölder inequality holds:

$$
\begin{aligned}
\int_0^R u^p r^{\frac{Np}{p+k}-1}dr &\le \Big(\int_0^R u^{p+k+\varepsilon}r^{N-1}dr\Big)^{\frac{p}{p+k+\varepsilon}}\Big(\int_0^R r^\delta dr\Big)^{\frac{k+\varepsilon}{p+k+\varepsilon}} \\
&= \omega_N^{-\frac{p}{p+k+\varepsilon}}\Big(\int_\Omega u^{p+k+\varepsilon}dx\Big)^{\frac{p}{p+k+\varepsilon}}\Big(\frac{R^{\delta+1}}{\delta+1}\Big)^{\frac{k+\varepsilon}{p+k+\varepsilon}}
\end{aligned}
\tag{3.13}
$$

with $\delta = \delta(\varepsilon) := \big[\frac{p(p+k+N\varepsilon)}{(p+k)(p+k+\varepsilon)} - 1\big]\frac{p+k+\varepsilon}{k+\varepsilon} > -1$, for all $\varepsilon > 0$.
Applying the inequality (3.13) to (3.12), we obtain

$$
I_{22} \le c_1\Big(\int_\Omega u^{p+k+\varepsilon}dx\Big)^{\frac{p}{p+k+\varepsilon}}\Big(\int_\Omega u^{p+k}dx\Big)^{\frac{k}{p+k}},
\tag{3.14}
$$

where $c_1 = c_1(\varepsilon, \omega_N, N, K_1) := 2N^{\frac{k}{p+k}}(N-1)K_1\omega_N^{\frac{p\varepsilon}{(p+k)(p+k+\varepsilon)}}\Big(\frac{R^{\delta+1}}{\delta+1}\Big)^{\frac{k+\varepsilon}{p+k+\varepsilon}}$.
Substituting (3.11) and (3.14) in (3.10) derives

$$
\begin{aligned}
&\int_\Omega u^p \frac{\nabla v\cdot\nabla(|\nabla v|^2)}{(1+|\nabla v|^2)^{\alpha+1}}dx \\
&\quad\le 2\omega_N\int_\Omega u^{p+k}dx + c_1\Big(\int_\Omega u^{p+k+\varepsilon}dx\Big)^{\frac{p}{p+k+\varepsilon}}\Big(\int_\Omega u^{p+k}dx\Big)^{\frac{k}{p+k}}.
\end{aligned}
\tag{3.15}
$$

Using (3.15) in (3.4) and applying the Young and Hölder inequalities, we see that

$$
\begin{aligned}
I_2(t) &\le k_f \frac{p-1}{p}(K_1 + 2\omega_N \alpha) \int_\Omega u^{p+k} dx \\
&\quad + c_1 \alpha k_f \frac{p-1}{p} \Big(\int_\Omega u^{p+k+\varepsilon} dx \Big)^{\frac{p}{p+k+\varepsilon}} \Big(\int_\Omega u^{p+k} dx \Big)^{\frac{k}{p+k}} \\
&\le k_f \frac{p-1}{p}(K_1 + 2\omega_N \alpha) \int_\Omega u^{p+k} dx \\
&\quad + c_1 \alpha k_f \frac{p-1}{p} \frac{p}{p+k} \Big(\int_\Omega u^{p+k+\varepsilon} dx \Big)^{\frac{p+k}{p+k+\varepsilon}} \\
&\quad + c_1 \alpha k_f \frac{p-1}{p} \frac{k}{p+k} \int_\Omega u^{p+k} dx \\
&= k_f \frac{p-1}{p} \Big(K_1 + 2\omega_N \alpha + c_1 \frac{k}{p+k} \Big) \int_\Omega u^{p+k} dx \\
&\quad + c_1 \alpha k_f \frac{p-1}{p+k} \Big(\int_\Omega u^{p+k+\varepsilon} dx \Big)^{\frac{p+k}{p+k+\varepsilon}} \\
&\le c_2 \Big(\int_\Omega u^{p+k+\varepsilon} dx \Big)^{\frac{p+k}{p+k+\varepsilon}},
\end{aligned}
\tag{3.16}
$$

where $c_2 := |\Omega|^{\frac{\varepsilon}{p+k+\varepsilon}} k_f \frac{p-1}{p}(K_1 + 2\omega_N\alpha + c_1 \frac{k}{p+k}) + c_1 \alpha k_f \frac{p-1}{p+k}$. Combining (3.2) and (3.16) with (3.1), we arrive to

$$
\frac{1}{p}\frac{d}{dt} \int_\Omega u^p dx \le -\frac{4(p-1)}{p^2} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + c_2 \Big(\int_\Omega u^{p+k+\varepsilon} dx \Big)^{\frac{p+k}{p+k+\varepsilon}}. \tag{3.17}
$$

In the last term of (3.17) we use the Gagliardo–Nirenberg inequality (2.4) with $\gamma = 2\frac{p+k}{p}$, $\mathsf{p} = 2\frac{p+k+\varepsilon}{p}$, $\mathsf{q} = \mathsf{s} = \frac{2}{p}$ and $\mathsf{r} = 2$, and the mass conservation law $\|u(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}$ for all $t \in (0, T_{\max})$, we obtain

$$
\begin{aligned}
\Big(\int_\Omega u^{p+k+\varepsilon} dx \Big)^{\frac{p+k}{p+k+\varepsilon}} &= \big\| u^{\frac{p}{2}} \big\|^{2\frac{p+k}{p}}_{L^{2\frac{p+k+\varepsilon}{p}}(\Omega)} \\
&\le C_{GN} \left(\|\nabla u^{\frac{p}{2}}\|^{2\frac{p+k}{p}\theta_2}_{L^2(\Omega)} \|u^{\frac{p}{2}}\|^{2\frac{p+k}{p}(1-\theta_2)}_{L^{\frac{2}{p}}(\Omega)} + \|u^{\frac{p}{2}}\|^{2\frac{p+k}{p}}_{L^{\frac{2}{p}}(\Omega)} \right) \\
&\le C_{GN} \left(\|\nabla u^{\frac{p}{2}}\|^{2\frac{p+k}{p}\theta_2}_{L^2(\Omega)} \|u\|^{(p+k)(1-\theta_2)}_{L^1(\Omega)} + \|u\|^{p+k}_{L^1(\Omega)} \right) \\
&\le c_3 \left(\|\nabla u^{\frac{p}{2}}\|^{2\frac{p+k}{p}\theta_2}_{L^2(\Omega)} + 1 \right)
\end{aligned}
\tag{3.18}
$$

with some $c_3 > 0$, where $\theta_2 := \frac{\frac{p}{2} - \frac{p}{2(p+k+\varepsilon)}}{\frac{p}{2} + \frac{1}{N} - \frac{1}{2}} \in (0,1)$. Moreover, for $0 < \epsilon < \frac{(p+k)(2+Nk)}{N+k-2}$ we obtain

$$\frac{p+k}{p}\theta_2 = \frac{\frac{p}{2} + \frac{k}{2} - \frac{p+k}{2(p+k+\varepsilon)}}{\frac{p}{2} + \frac{1}{N} - \frac{1}{2}} < 1.$$

Hence, applying the Young inequality to the right-hand side of (3.18) implies

$$\left(\int_\Omega u^{p+k+\varepsilon} dx\right)^{\frac{p+k}{p+k+\varepsilon}} \leq \frac{2(p-1)}{p^2 c_2} \|\nabla u^{\frac{p}{2}}\|^2_{L^2(\Omega)} + c_4$$

with some $c_4 > 0$, which together with (3.17) yields

$$\frac{d}{dt}\int_\Omega u^p dx + \frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx \leq p c_2 c_4. \tag{3.19}$$

Again by the Gagliardo–Nirenberg inequality (2.4) with $\gamma = \frac{2}{p}$, $\mathsf{p} = \mathsf{r} = 2$ and $\mathsf{q} = \mathsf{s} = \frac{2}{p}$, and the mass conservation law, we observe

$$\begin{aligned}
\|u\|_{L^p(\Omega)} &= \|u^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^2(\Omega)} \\
&\leq C_{GN}\left(\|\nabla u^{\frac{p}{2}}\|^{\frac{2}{p}\theta_3}_{L^2(\Omega)} \|u^{\frac{p}{2}}\|^{\frac{2}{p}(1-\theta_3)}_{L^{\frac{2}{p}}(\Omega)} + \|u^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^{\frac{2}{p}}(\Omega)}\right) \\
&\leq c_5\left(\|\nabla u^{\frac{p}{2}}\|^{\frac{2}{p}\theta_3}_{L^2(\Omega)} + 1\right)
\end{aligned}$$

with some $c_5 > 0$, where $\theta_3 := \frac{\frac{p}{2} - \frac{1}{2}}{\frac{p}{2} + \frac{1}{N} - \frac{1}{2}} \in (0,1)$, which implies

$$\|\nabla u^{\frac{p}{2}}\|^2_{L^2(\Omega)} \geq c_5 \|u\|^{\frac{p}{\theta_3}}_{L^p(\Omega)} - 1 \tag{3.20}$$

with some $c_6 > 0$. A combination of (3.19) and (3.20) yields

$$\frac{d}{dt}\int_\Omega u^p dx + c_7\left(\int_\Omega u^p dx\right)^{\frac{p}{\theta_3}} \leq c_8$$

with some $c_7, c_8 > 0$. Upon an ODE comparison argument this inequality warrants that

$$\int_\Omega u^p dx \leq \max\left\{\left(\frac{c_8}{c_7}\right)^{\frac{\theta_3}{p}}, \int_\Omega u_0^p dx\right\},$$

which proves the conclusion. □

We are now in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. *Since, for all $t \in (0,\infty)$ we have $\|u(\cdot,t)\|_{L^1(\Omega)} = \|u_0(x)\|_{L^1(\Omega)} < \infty$, then $\|u^k(\cdot,t)\|_{L^{\frac{1}{k}}(\Omega)} < \infty$ for all $p = \frac{1}{k} > \frac{N}{2}$, thus $\nabla v(\cdot,t) \in W^{1,p^*}(\Omega)$ with $p^* = \frac{Np}{N-p}$ and $t \in (0,\infty)$. From Sobolev embedding Theorem we obtain $W^{1,p^*}(\Omega) \subset L^{p^*}(\Omega)$ and from the first equation we obtain $\|u(\cdot,t)\|_{L^p(\Omega)} < \infty$ for all $p > \frac{N}{2}$. From which we can conclude $\|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$.*

4 Blow up in L^p-norm. Proof of theorem 1.2

In this section we prove that if the solution (u,v) of system (1.1) blows up in $L^\infty(\Omega)$-norm in finite time T_{max} then for some $p > 1$ it blows up also in $L^p(\Omega)$-norm. This result allows us to determine a lower estimate T of the blow up time T_{max}, such that we can obtain a safe time interval of existence of the solution (u,v). To this end, we first prove the following Lemma.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded and smooth domain. Let (u,v) be a classical solution of system (1.1), let*

$$f(|\nabla v|^2) = k_f \frac{1}{(1+|\nabla v|^2)^\alpha}, \quad 0 < \alpha < \frac{N-2}{2(N-1)}, \tag{4.1}$$

if for some $p > \frac{N}{2}k$, $k > \frac{2}{N}$, there exists $C > 0$ such that

$$\|u(x,t)\|_{L^p(\Omega)} \leq C, \quad t \in (0,T_{max}),$$

then for some $\bar{C} > 0$ we have

$$\|u(x,t)\|_{L^\infty(\Omega)} \leq \bar{C}. \tag{4.2}$$

Proof. We set $t_0 := \max\{0, t-1\}$ and using the representation formula for u we have

$$\begin{aligned} u(\cdot,t) &= e^{(t-t_0)\Delta}u(\cdot,t) - k_f \int_{t_0}^t e^{(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\frac{\nabla v(\cdot,s)}{(1+|\nabla v(\cdot,t)|^2)^\alpha}\Big)\,ds \\ &= u_1(\cdot,t) + u_2(\cdot,t) \end{aligned}$$

and

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_1(\cdot,t)\|_{L^\infty(\Omega)} + \|u_2(\cdot,t)\|_{L^\infty(\Omega)}. \tag{4.3}$$

We recall that if $1 \le q \le \mathrm{p} \le \infty$, then

$$\|e^{t\Delta} z\|_{L^{\mathrm{p}}(\Omega)} \le k_1\big(1 + t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{\mathrm{p}})}\big)e^{-\mu_1 t}\|z\|_{L^q(\Omega)}, \quad \forall\, t > 0,$$

so that, with $\mathrm{p} = \infty$ and $q = 1$ we obtain

$$\|u_1(\cdot,t)\|_{L^\infty(\Omega)} = \|e^{(t-t_0)\Delta}u(\cdot,t)\|_{L^\infty(\Omega)} \le k_1\big(1 + (t-t_0)^{-\frac{N}{2}}\big)e^{-\mu_1 t}.\|u\|_{L^1(\Omega)}$$

If $t \le 1$ then $t_0 = 0$ and from the maximum principle $\|u_1(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}$.
If $t > 1$ then $t - t_0 = 1$ and we obtain

$$\begin{aligned} \|u_1(\cdot,t)\|_{L^\infty(\Omega)} &= k_1\big(1 + (t-t_0)^{-\frac{N}{2}}\big)e^{-\mu_1(t-t_0)}\|u\|_{L^1(\Omega)} \\ &\le 2k_1\|u_0\|_{L^1(\Omega)} = 2k_1\bar{m}|\Omega|, \end{aligned} \tag{4.4}$$

with $\bar{m} := \frac{1}{|\Omega|}\int_\Omega u(x,t)dx = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx$.

We now estimate $\|u_2(x,t)\|_{L^\infty(\Omega)}$.
By using (2.3) $\|e^{t\Delta}\nabla\cdot\mathbf{z}\,\|_{L^{\mathrm{p}}(\Omega)} \le k_2\big(1 + t^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{q}-\frac{1}{\mathrm{p}})}\big)e^{-\mu_1 t}\|\mathbf{z}\,\|_{L^q(\Omega)}, \quad \forall\, t > 0.$

Since $\frac{|\nabla v(\cdot,s)|}{(1+|\nabla v|^2)^\alpha} \le |\nabla v(\cdot,s)|^{1-2\alpha}$, we have

$$\begin{aligned} &\|u_2(\cdot,t)\|_{L^\infty(\Omega)} \\ &\le k_2 k_f \int_{t_0}^t \big(1 + (t-s)^{-\frac{1}{2}-\frac{N}{2q}}\big)e^{-\mu_1(t-s)}\Big\|u(\cdot,s)\frac{\nabla v(\cdot,s)}{(1+|\nabla v|^2)^\alpha}\Big\|_{L^q(\Omega)} ds \\ &\le k_2 k_f \int_{t_0}^t \big(1 + (t-s)^{-\frac{1}{2}-\frac{N}{2q}}\big)e^{-\mu_1(t-s)}\Big\|u(\cdot,s)|\nabla v(\cdot,s)|^{1-2\alpha}\Big\|_{L^q(\Omega)} ds. \end{aligned} \tag{4.5}$$

By Holder inequality we obtain

$$\|u(\cdot,s)|\nabla v(\cdot,s)|^{1-2\alpha}\|_{L^q(\Omega)} \le \|u(\cdot,s)\|_{L^{\frac{q}{2\alpha}}(\Omega)}\|\nabla v(\cdot,s)\|^{1-2\alpha}_{L^q(\Omega)} \tag{4.6}$$

Now, using (1.12), since $\|u\|_{L^p(\Omega)} \le C$ and $p > \frac{N}{2}k > k$ there exists $C_0 = C_0(q)$ such that

$$\|g(u)\|_{L^q(\Omega)} \le C_0 K_2 \|u\|^k_{L^p(\Omega)} \le C_0 K_2 C^k = C_1 \quad on\ (0, T_{max}),$$

for any $q > N > \max\{1, \frac{1}{k}\}$.
Thus, the elliptic regularity theory to the second equation in (1.1) tell us that $\nabla v \in W^{1,p}$ and

$$\|\nabla v\|^{1-2\alpha}_{L^q(\Omega)} \le C_2, \tag{4.7}$$

for some $C_2 > 0$. By using (4.7) in (4.6) we arrive to

$$\begin{aligned} &\|u(\cdot,s)|\nabla v(\cdot,s)|^{1-2\alpha}\|_{L^q(\Omega)} \\ &\le \|u(\cdot,s)\|_{L^{\frac{q}{2\alpha}}(\Omega)} \|\nabla v(\cdot,s)\|^{1-2\alpha}_{L^q(\Omega)} \le C_2 \|u(\cdot,s)\|_{L^{\frac{q}{2\alpha}}(\Omega)} \\ &\le C_3 \|u(\cdot,s)\|^\theta_{L^\infty(\Omega)} \|u(\cdot,s)\|^{1-\theta}_{L^1(\Omega)} = C_4 \bar{m}^{1-\theta} \|u(\cdot,s)\|^\theta_{L^\infty(\Omega)} \end{aligned} \tag{4.8}$$

with

$$\theta = 1 - \frac{2\alpha}{q} \ \in (0,1) \tag{4.9}$$

and $\bar{m} = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx$.

Inserting (4.8) in (4.5) we arrive to

(4.10)

$$\begin{aligned} &\|u_2(\cdot,t)\|_{L^\infty(\Omega)} \le C_4 \int_0^t [1 + (t-s)^{-\frac{1}{2}-\frac{N}{2q}} e^{-\mu_1(t-s)}] ds \sup_{t\in[0,T]} \|u(\cdot,t)\|^\theta_{L^\infty(\Omega)} \\ &\le B \sup_{t\in[0,T]} \|u(\cdot,t)\|^\theta_{L^\infty(\Omega)}, \end{aligned}$$

where $B := C_4[1 + \int_0^\infty (t-s)^{-\frac{1}{2}-\frac{N}{2q}} e^{-\mu_1(t-s)} ds] > 0$ is finite, because $\frac{1}{2} + \frac{N}{2q} < 1$ $(q > N)$.

The substitution of (4.4) and (4.10) in (4.3) leads to

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le A + B \sup_{t\in[0,T]} \|u(\cdot,t)\|^\theta_{L^\infty(\Omega)}, \tag{4.11}$$

with $A := 2k_1\bar{m}|\Omega|$. From inequality (4.11) we can write

$$\sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq A + B\Big(\sup_{t\in[0,T]} \|u(\cdot,t)\|_{L^\infty(\Omega)}\Big)^{\theta}, \tag{4.12}$$

with θ in (4.9).
The inequality (4.12) leads to (4.2). $\square$

Now we are ready to proof Theorem 1.2.

Proof of Theorem 1.2. If the solution (u, v) of system (1.1) blows up at finite time t^* in L^∞-norm, by contradiction we suppose that for some $p > \frac{N}{2}k$, the solution (u, v) is bounded $\forall t > 0$. From Lemma 4.1 if the solution of (1.1) is bounded in L^p-norm, it is bounded also in L^∞-norm. This reach a contradiction. Then (u, v) blows up in L^p-norm for some $p > \frac{N}{2}k$.

Acknowledgments

M. Marras is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Financial disclosure

M. Marras is partially supported by the research project *Analysis of PDEs in connection with real phenomena*, CUP F73C22001130007, funded by Fondazione di Sardegna annuality (2022); by the grant PRIN n. PRIN-2022ZXZTN2: *Nonlinear differential problems with applications to real phenomena* and by the grant INDAM-GNAMPA Project, CUP E53C22001930001.

Declaration

Compliance with Ethical Standards Disclosure of potential conflicts of interest:The authors declare that there are no conflicts of interest regarding the publication of the paper. Research involving Human Participants and/or Animals: Not applicable. Informed consent: Not applicable.

Author Contribution All authors contributed equally to this work.

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