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Euler's "Tentamen": historical and mathematical aspects on the consonance theory

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Abstract. The *Tentamen novae theoriae musicae* is a treatise in which Euler elaborated a new music theory using mathematics. The aim of this paper is to explain his theoretical system to justify the pleasure of listening to music and to analyze differences and similarities with other consonance theories.

Keywords: Euler · Tentamen · consonance theory

1 Introduction

Since the XVII century, important mathematicians as Euler (1707-1783) and d'Alembert (1717-1783) worked on music theories. This is obviously not the result of a "historical chance". On the contrary, it represents prolongation of a tradition in which mathematics helps to describe music from an acoustical and theoretical point of view. Probably the first ancient example is the Pythagorean scale, defined by Pythagoreans starting from the first four natural numbers and the observation of the sounds produced by the division of the string of a monochord, determining the ratios of the consonant intervals of octave, fifth and fourth. Didymus and Ptolemy developed the same idea to describe intervals as mathematical ratios defining a new musical scale, and Zarlino (1517-1590) resumed it in *Le istitutioni harmoniche* [22]. They did not know, but this idea is totally agree with harmonic series.

After composers as Bach (1685-1750), Handel (1685-1759), Rameau (1683-1764), Haydn (1732-1809), Mozart (1756-1791), music had a profound change by abandoning the medieval counterpoint in favor of a new harmony. This change had to be explained, the western music needed a new theory, and it inspired scientist with a passion for music. Several scientists dedicated workd on it: Descartes (1596-1650) with *Compendium musicae* [9], Mersenne (1588-1648) with *Harmonie universelle* [17] and Leibniz [16] in several letters.

Euler wrote severel works on music [12], whose first and best known is *Tentamen novae theoriae musicae ex certissismis harmoniae principiis dilucide expositae* [11], written in Latin in 1731 and published in 1739. The *Tentamen* *novae theoriae musicae* is a treatise in which Euler elaborated a new music theory using mathematics. In recent years, interest on this work by Euler has been increasing [2] [8] [3] [13]. Probably this interest also arises from the important developments of the Mathematical Music Theory of the last thirty years linked to the Tonnetz and his generalizations [20] [7] [10] [1] [4] [6]. The Tonnetz is a 2-dimensional simplicial complex which tiles the Euclidean plane with triangles representing major and minor triads. This structure today is interesting because is a model for the neo-Riemannian operations P, L and R, from which generalizations have been introduced in Mathematical Music Theory [15] [5]. A graph similar to the Tonnetz appeared in a the chapter of *Tentamen* in order to represent some intervals of the just intonation in a scheme.

In this paper, we will focus on Euler's theoretical system to justify the pleasure of listening to music. More precisely, we will analyze differences and similarities with other consonance theories. We will start, in section 2, with a historical-musical context in which Euler lived and his possible musical knowledges. In sections 3 and 4 we summarize and explained the mathematical ideas on sound and on the pleasure of consonance, described in the first four chapters of the *Tentamen*. Finally, in the last section we will analyze his consonance theory by comparing it with other theories.

2 Some historical aspects on Euler's musical interests and the birth of the "Tentamen"

In agreement with De Piero [8], we note that Euler's interests on music began very early.

The *Dissertatio physica de sono*, written in 1727 to compete for the chair of Physics in Basel, is a demonstration of this. In this dissertation the physical foundations of music are provided; these concepts were taken up and expanded in the first chapter of *Tentamen*. The *Dissertatio* did not allow Euler to obtain the chair in Basel, but showed the mathematician's interest in the subject and laid the foundations for the next and more important essay, already completed in 1731, and which originally bore the name of *Tractatus de musica*. De Piero points out that the news of the writing of this essay is present in a letter that Euler sent to his teacher Johann Bernoulli on May 25, 1731 from St Petersburg, where he moved in 1727. In this letter the mathematician showed that he had designed the entire work and that he wrote most of it, moreover he has not yet proposed a title, but speaks indefinitely of *Systema Musicus*. But, in the reply sent on 11 August 1731, his teacher introduced for the first time the term *Tractatus Musices*, a term that led Euler to encode the first title in *Tractatus de Musica*, as we learn from the letter sent to Bernoulli on December 20, 1738. According to De Piero, we believe the title was established before 1738, but we have the first evidence of it only on that date. However, the work was published only in 1739 in St Petersburg, printed by the Academy of Sciences with the new title *Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucidae expositae*.

Actually, the idea of writing an essay on music dates back to the period spent in Basel, this can be seen from Euler's first notebook, which the Swedish mathematician Gustav Enestroem dated to 1726 [5], therefore one year before the publication of the *Dissertatio*. From the notes in the notebook the work, in its first formulation, should have been titled *Musices Theoreticae Systema*, and should have been divided into three sections: *De compositiones solius discantus*, *De compositione integrorum concertorum* , *De compositione certarum specierum*.

The real reason why Euler gave up on this project, completing another completely different one, is not known. According to Ferdinand Rudio [6] the transfer of the mathematician to St Petersburg and other kind of works would have forced him to postpone his musical projects. However, this does not convince De Piero, because in 1731 Euler already spoke to Bernoulli about his projects, claiming that he had already conceived the overall plan of the work and that he had already drawn up most of it. De Piero hypothesizes that at first Euler started from *Dissertatio physica de sono* and, once the conceptual foundations of a physical nature had been laid, he was so passionate about hypothesizing the drafting of an essay expressly dedicated to composition. However, he also hypothesizes that Euler may have known that Johann Mattheson was dedicating himself to a similar essay in the same years [7]. However, the problem remains open.

Although Euler wrote about music theory, there are no particular testimonies of meetings or discussions with musicians. But, according to De Piero [8], during the period at the Frederick the Great's court he probably met musicians and composers such as Carl Philipp Emanuel Bach[8], Johann Joachim Quantz[9] or the brothers Carl Heinrich e Johann Gottlieb Graun[10]. Moreover, in 1747 Johann Sebastian Bach[11] went to the Frederick the Great's court, who asked him to compose music on the basis of a theme composed by himself, which will then be collected in the famous *Musical offer*. Also in St Petersburg many musicians as Baldassare Galuppi [12] stayed during the reign of Catherine II. According to De Piero, although there is no direct evidence, a mathematician as interested in music as Euler and employed in the same court may hardly remain outside of all this. The only evidence of disquisitions with musicians of the time are

[5] G. Ernstroem, *Bericht an die Eulerkommission der Schweizerschen naturforschenden Gesellshaft iber die Eulerchen Manuskripte der Petersburg Akademie*, in "Jahresbericht der Deutchen Mathematiker-Vereinigung, 22 ", 1913, p.197.

[6] See [8] p. 7.

[7] This is the essay *Grosse General-Bass-Schuule*, published in 1731 in Hamburg and also cited in *Tentamen*.

[8] Carl Philipp Emanuel Bach (1714-1788), composer, p.45 [21].

[9] Johann Joachim Quantz (1697-1773), composer and music theorist, p. 715 [21].

[10] Carl Heinreich Graun (1701-1759), Johann Gottlieb Graun (1702 o 1703-1771), composers, p. 360 [21].

[11] Johann Sebastian Bach (1685-1750), composer, pa. 46 [21]

[12] Baldassarre Galuppi (1706-1785), known as Buranello, Master of the Ducal Chapel of San Marco in Venice, stayed there from 1756 to 1768, p. 333 [21].

correspondence exchanges. To date, four letters are known: two between Euler and Giuseppe Tartini [13] and two more between the Swiss mathematician and Jean-Philippe Rameau [14].

Reading the *Tentamen*, however, it is evident that Euler knew very well the musical theories of that time and of the past. In fact, the conceptual structure of the work starts from the Pythagorean principles of harmony, influenced by the writings of Marin Mersenne, René Descartes and Gottfried Wilhelm von Leibniz. The latter, as librarian and historian of the court of Hannover, provided consultancy for the staging of the court shows, consequently maintained daily relationships with musicians and singers and this allowed him to develop very original and interesting ideas about music. These ideas were not collected in a specific text, but can be traced in letters sent to mathematicians and theorists. Euler quoted Leibniz [15] as evidence of that any piece of music is composed of the exponents of only the numbers $2, 3$ and 5 since only the numerical ratios based on such numbers or on the respective multiples produce listening pleasure, while too complex numerical relationships cannot be perceived as pleasure.

Although he is never mentioned in the *Tentamen*, Descartes is the author who most left his mark froma methodological point of view. There are several points in common between Euler's essay and that of the French mathematician, *Compendium Musicae*, published in Utrecht in 1650. In fact, the Swiss mathematician took up the concept according to which the causes of *delectare et movere affectus* are to be found in the relationship of duration or time between sounds and in the relationship between high and low pitches [16], and according to both the perception of pleasure by the hearing is governed by simple arithmetic ratios, because they are more easily understood.

3 Sound and hearing

The first principle from which Euler starts, mainly presented in the first chapter of the *Tentamen*, is about physical nature and concerns the science of sounds [17]. On the other hand this aspect was not addressed by Descartes and, according to De Piero, this constitutes the most original contribution provided by Euler. For the Swiss mathematician, sounds are vibrations of the air perceived as multiple by the ears[18]. Given two sounds, we understand the relation between them by

[13] Giuseppe Tartini (1692-1770), composer, violinist and music theorist, p. 880 [21].

[14] Jean-Philippe Rameau (1683-1764), composer and music theorist, p. 725 [21].

[15] L. Euler, *Tentamen*, chap. X, par.19.

[16] R. Descartes, emph Compendium Musicae, Utrecht, 1650, ed. mod. *Abregé de musique*, Édition nouvelle, in traduction, presentation and notes by Fr. de Buzon, Paris, Presses Universitaires de France, 1987. Cf. p. 55: " Media ad finem, vel soni affectiones duae sunt praecipue: nempe huius differentiae, in ratione durationis vel temporis, et in ratione intensionis circa acutum aut grave ".

[17] We observe that throughout the work, Euler speaks of *sound* but never of *sound waves*.

[18] This is clearly the frequency of the sound wave.

the ratio of the number of vibrations carried out for one, with the number of vibrations carried out for the other in the same time. For example, if there were 3 vibrations for the first, while for the second there would be 2 of them, we know their relation and consequently their order, by observing the ratio of the numbers 3 and 2 which is 3:2. Then, we observe that Euler obtained the same ratio introduced in Pythagorean scale and in just intonation for describing the interval of fifth.

The second principle, introduced in the preface and exposed above all in the second chapter, invests in the causes of pleasure, questioning why a person likes or dislikes music. In the preface to the Tentamen, Euler argues that consonances do not depend on human habits. In support of this, he quotes Pythagoras, who identified the cause of the pleasure produced by the consonances in the mathematical relationships of the intervals, although he did not understand how these relationships are perceived by hearing. Euler notes that European music is not appreciated by barbarians [19] and vice versa. The Basel mathematician attributes this to the complexity of European musical composition, made up of various melodic lines that intertwine with each other and which are not easily distinguishable and noticeable by few trained ears. Again, he observes that in many countries it is commonly believed that the octave, fifths, fourths, thirds and sixths are consonant intervals, while tritons, sevenths, seconds and all the others that can be constituted are dissonant. Euler's aim is to investigate the causes of this judgment which he considers universal. Euler therefore proposes a classification of the degrees of pleasure of consonances based on mathematical relationships.

4 Pleasure and consonance

Music is formed by sounds played together that Euler defines consonance. We observe that it does not distinguish between consonance and dissonance, as historically it has always been used and as it is still usual today, every compound sound is consonant and may or may not be liked. According to Euler, pleasure consists in the exact perceptibility of sounds and their relationships: two or more sounds like when one perceives the relationship that the numbers of the vibrations emitted have between them, vice versa they are dislike when no order is felt. We perceive pleasure, if from that structure we understand how all the parts intertwine with each other, and how their actions all converge. For Euler where there is order, there is perfection, and that the rule or law of the order corresponds to the goal which marks the perfection. The order of sounds primarily consists on two types: according to the pitch and according to the duration.

As already mentioned, Euler defines sound as successive beats produced in the air in a certain order, so a sound is distinctly perceived if all the beats from the hearing organs are heard and their order is recognized. In general, perception of order can occur in two ways:

[19] De Piero hypothesizes that what Euler defines *barbarians* are populations living outside Europe.

1. when the rule is known;
2. when the rule is not known, but can be deduced by looking the structure.

In music the order is perceived according to this latter way, in fact it is through listening that the order that sounds have among themselves is understood. Since the perception of perfection produces pleasure, according to Euler a composition pleases if the order of the sounds is perceived. It may happen that some people perceive this order that others do not hear, which is why the same music can like some and not others.

4.1 Study of chords with two sounds

Given two sounds, their relationship is perceived through the ratios that the number of hits, emitted at the same time, have between them. From this, Euler defines the degrees of pleasure.

The first and simplest degree of pleasure is the unison, represented by the numerical ratio $1:1$. On the other hand, two sounds having a double ratio, therefore $1:2$, are part of the second degree of pleasure. The sounds expressed with the numerical ratio $1:3$ and the ratio $1:4$ belong to the third degree. In fact, the first is expressed by small numbers, so it is easily perceptible, the second would apparently seem a more complex ratio, but it is obtained simply by dividing the ratio $1:2$ by 2, therefore it is not very difficult to distinguish from the latter. For this reason Euler believes that both relationships are part of the third degree of pleasure. Similarly, the ratios $1:8$, $1:16$ belong to the fourth and fifth degree of pleasure, respectively. More generally, the ratio $1:2^n, n \in \mathbb{N}$, corresponds to the degree of pleasure $n+1$.

If the ratio contains divisors different by 1 and 2, the degree of pleasure is less. But the degree of pleasure is also estimated by looking at the magnitude of the numbers: the ratio $1:5$ is simpler than $1:7$, although the latter is no simpler than $1:8$. For the ratios $1:p$, where p is a prime number, is easy to determine the degree of pleasure. In fact since $1:2$ belongs to the second degree and $1:3$ to the third one, then $1:5$ belongs to the fifth and $1:7$ to the seventh. More generally, if p is prime, the ratio $1:p$ belongs to the degree of pleasure p.

It follows that if the ratio $1:p$ refers to the degree m, the ratio $1:2p$ belongs to the degree $m+1$. In fact, multiplying the number p by 2, the perception of the ratio requires the perception of $1:p$ and the division by 2, an operation that increases the degree of pleasure by one unit. Similarly $1:4p$ belongs to the degree $m+2$. More generally, $1:(2^n p)$ belongs to the degree $m+n$. Similarly, the degree of pleasure of the relationship $1:(pq)$, with p and q prime numbers, is equal to $p+q-1$, since $1:(pq)$ is composed by $1:p$ and $1:q$. This also applies to $1:(PQ)$ with any positive integer P and Q. Similarly, the ratio $1:(pqr)$, with p,q,r prime, being constituted by $1:(pq)$ and $1:r$ whose degrees of pleasure are $p+q-1$ and r respectively, will have degree of pleasure $p+q+r-2$. Iterating the reasoning, the degree of the relationship $1:(pqrs)$ will be $p+q+r+s-3$, and so on.

So, if p is prime, the degree of pleasure of $1 : p^2$ is $2p - 1$, and for $1 : p^3$ is $3p - 2$. More generally $1 : p^n$ belongs to the degree of pleasure $np - n + 1$. Therefore, since $1 : q^m$ belongs to the degree $mq - m + 1$, the ratio $1 : (p^n q^m)$ will belong to the degree

$$np + mq - n - m + 1 \tag{1}$$

So for any number P, in order to determine the degree of pleasure of a ratio $1 : P$ we have to represent the ratio in simple factors and the degree will be obtained by subtracting from their sum the number of factors subtracted by one. To clarify, Euler also shows the following example.

Example 1 *We look for the degree of the ratio* $1 : 72$.

First we factorize 72, then $72 = 2^3 \cdot 3^2$. *Using the 1, we have* $3 \cdot 2 + 2 \cdot 3 + 2 - 3 + 1 = 8$.

Therefore the degree of pleasure of the ratio $1 : 72$ *is* 8.

In [12] it is generalized it as the following general formula that Euler does not write, . Given two sounds such that their ratio is $\frac{1}{P}$, we factorize P as

$$P = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_n^{\alpha_n}.$$

Then, the degree $d(P)$ of pleasure of $\frac{1}{P}$ is

$$d(P) = \sum_{i=1}^{n} (\alpha_i p_i - \alpha_i) + 1 \tag{2}$$

4.2 Study of chords with more than two sounds

At this point, Euler continues his theory by examining relationships between more than two numbers, that is, he analyzes the degree of pleasure obtained by more than two sounds. Given two prime numbers p, q, in the ratio of three numbers such as $1 : p : q$ we also perceive $1 : p$ and $1 : q$, the perception here is equal to that of $1 : (pq)$. Similarly, the ratio of four numbers $1 : p : q : r$, where p, q, r are always prime numbers, belongs to the same degree as $1 : (pqr)$. Euler proposes another ex:

Example 2 *Suppose we have 4 sounds expressed by the following numbers:* $1 : 2 : 3 : 5$. *The degree of pleasure of these sounds is the same as those expressed by the ratio* $1 : 30$, *therefore:* $2 + 3 + 5 - 3 + 1 = 8$.

Therefore they belong to the eighth degree of pleasure.

The Swiss mathematician observes that these prime numbers must be all unequal. In fact, $1 : p : p$ is perceived exactly as $1 : p$. Similarly, to perceive the relationship $1 : (pr) : (qr) : (ps)$, only the ratios $1 : p, 1 : q, 1 : r, 1 : s$ are

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Gr. II.	2:5.	Gr. IIX.	3:7.	3:64.	1:160.
1:2.	1:18.	1:14.	1:25.	1:256.	5:32.
Gr. III.	2:9.	2:7.	1:28.	Gr. X.	1:162.
1:3.	1:24.	1:30.	4:7.	1:42.	2:81.
1:4.	3:8.	2:15.	1:45.	3:14.	1:216.
Gr. IV.	1:32.	3:10.	5:9.	6:7.	8:27.
1:6.	Gr. VII.	5:6.	1:60.	1:50.	1:288.
2:3.	1:7.	1:40.	3:20.	2:25.	9:32.
1:8.	1:15.	5:8.	4:15.	1:56.	1:384.
Gr. V.	3:5.	1:54.	5:12.	7:8.	3:128.
1:5.	1:20.	2:27.	1:80.	1:90.	1:512.
1:9.	4:5.	1:72.	5:16.	2:45.	
1:12.	1:27.	8:9	1:81.	5:18.	
3:4.	1:36.	1:96.	1:108.	9:10.	
1:16.	4:9.	3:32.	4:27.	1:120.	
Gr. VI.	1:48.	1:128.	1:144.	3:40.	
1:10.	3:16.	Gr. IX.	9:16.	5:24.	
	1:64.	1:21.	1:192.	8:15.	

Fig. 1: The first ten degree of pleasure in *Tentamen*, p. 61.

needed, it is not necessary to count twice $1:p$ and $1:r$. Therefore, the degree of pleasure is the same as the ratio $1:pqrs$.

At this point Euler determine the universal rule in order to know the degree of pleasure in perceiving the ratio of several numbers proposed at the same time: the least common multiple is determined and the 1 is used. Thus, Euler shows in a table to which degrees correspond all the least common multiples.

Example 3 *Let the numbers* $72, 80, 100, 112$ *be consider. Their factorizations are respectively:* $2^3 \cdot 3^2, 2^4 \cdot 5, 2^2 \cdot 5^2, 2^4 \cdot 7$*. So, the least common multiple is* $2^4 \cdot 3^2 \cdot 5^2 \cdot 7 =$ *25200, and it belongs to the twenty-third degree.*

5 Comparison between *Tentamen* and other mathematical consonance theories

As already mentioned, studies on consonance and dissonance has been central to music theory since ancient Greece.

But what does it mean consonance and dissonance? Several music theorists have been investigating to try and answer this question, such as Hindemith [14] and Tenney [19]. We can observe that there are semantic problems on the terms consonance and dissonance: we do not have a strict and universal definition of them. During the history many definitions and different meanings are found. For instance, only focusing on western music tradition, we observe that major and minor thirds and sixths were considered dissonant in antiquity but, in 14th century, they were accepted as consonant. For Tenney [19], there are five different forms of the consonance-dissonance conception (CDC).

> "Before the rise of polyphonic practice they were used in an essentially melodic sense, to distinguish degrees of affinity, agreement, similarity, or relatedness between pitches sounding successively. During the first four centuries of the development of polyphony they were used to describe an aspect of the sonorous character of simultaneous dyads, relatively independent of any musical context in which they might occur. In the 14th century the CDC began to change (again) in conjunction with the newly developing rules of counterpoint, and a new system of interval-classification emerged which involved the perceptual clarity of the lower voice in a polyphonic texture) and of the text which it carried). In the early 18th century, 'consonance' and 'dissonance' came to be applied to individual tones in a chord, giving rise to a new interpretation of these terms which would eventually yield results in diametric opposition to all of the earlier forms of the CDC. Finally-in the mid-19th century-a conception f consonance and dissonance arose in which 'dissonance' was equated with "roughness," and this had implications quite different from those of earlier forms of the CDC."

Without a clear and precise definition of consonance it is difficult to developed a theory able to explain the nature of consonance and dissonance in musical perception, because it is not clear what this theory would have to explain. From this point of view, Euler's idea of not distinguishing consonance and dissonance but different levels of consonance may be interesting.

Moreover, consonance is perceived in different way in different cultures. For all these reasons, we may say that concept of consonance is not universal.

In addition to this, we know that in a musical piece the perception of consonance do not only depends from the individual chords, but also for the chord sequences. In other terms: the concept of consonance is both local and global. This problem was known also by Euler indeed, in chapter V, he studied also the succession of consonance. But he considered the preeminence of pitch over that of the duration, since that one is measured by the frequencies of vibration, Euler brings back the evaluation of the musical pleasure to the arithmetic measurement of the proportions related to the sounds. Therefore, he brings back musical science using the theory of proportions already used in the Ancient Greece with Pythagoreans and Ptolomy or by Zarlino. Despite the same starting point, there are differences stressing the originality of Euler. Initially, the theory of the basel mathematician exceeds by far the simple consideration of the ratios of the frequencies of two sounds. Then, contrary to his predecessors, who had also founded their theory on the proportions without any explanation, Euler introduces a philosophical argumentation, in which by the proportions one arrives at the musical pleasure, via the order and the perfection. As for the Leibniz source, one can undoubtedly recognize from it the influence in the distinction which Euler makes between the two modes of perception of the order.

Moreover, as we see in table 1, we note that the ratios representing intervals in just intonation are all included also in Euler's theory. This is an interesting point, because the ratios of just intonation are totally agree with the physical theory of natural harmonics and harmonic series. We note that Euler's table 1 does not include the ratios 15 : 16 and 32 : 45. This is why his table represents only the first ten degree of pleasure. It is not specified in the *Tentamen*, but given a ratio $P : Q$, where $P, Q \in \mathbb{N}$, $P = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, and $Q = q_1^{\beta_1} \cdot p_2^{\beta_2} \cdots q_m^{\beta_m}$, we can determine the degree of pleasure d of $\dfrac{P}{Q}$ with the following formula

$$d = \sum_{i=1}^{n} (\alpha_i p_i - \alpha_i) + \sum_{j=1}^{m} (\beta_j q_j - \beta_j) + 1 \tag{3}$$

Using the formula 3 we have that the degree of pleasure of 15 : 16 is XI, and that one of 32 : 45 is XIV.

Interval	Ratio	Degree
Unison	1 : 1	I
Octave	1 : 2	II
Perfect fifth	2 : 3	IV
Perfect fourth	3 : 4	V
Major sixth	3 : 5	VII
Major third	4 : 5	VII
Minor third	5 : 6	VIII
Minor sixth	5 : 8	VIII
Major second	8 : 9	VIII
Minor seventh	9 : 16	IX
Major seventh	8 : 15	X
Minor second	15 : 16	XI
Tritone	32 : 45	XIV

Table 1: Relations between intervals in just intonation and Euler's degree of consonance

Despite Euler's theory include the interval ratios of just intonation, we note some limits in his classification summarized in table 1. In fact, there are ratios in the same degree that do not correspond to the same level of consonance, neither in his time nor in our day. For instance, in the VIII degree we found the ratios 5 : 6 (minor third), 5 : 8 (minor sixths) and 8 : 9 (major second), but the first two are imperfect consonance, while the last one is a dissonance.

Conclusions and future works

The concept of consonance is not universal in space and time, therefore every intention of analysis of the first four chapters of Euler's *Tentamen* is limited and incomplete. In this work,our aim has been to analyze the first 4 chapters of the *Tentamen*, limiting the concept of consonance as the pleasure of listening to a single chord. We have found that Euler's classification include the ratios of just intonation, but in the same degree of pleasure there are ratios representing intervals that do not have the same level of consonance. Since consonance also depends on the succession of chords, it will be interesting analyze the chapters of *Tentamen* involved on it.

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