Sharp and unsharp quantum incompatibilities. A comparison and some foundational questions

Roberto Beneduci[1,2], Maria Luisa Dalla Chiara[3], Roberto Giuntini[4], Giuseppe Sergioli[4]

[1]Department of Physics, University of Calabria, Via P. Bucci, cubo 31/C, Rende (CS), 87036, Italy.
[2]National Institute of Nuclear Physics, gruppo coll. Cosenza, via P. Bucci, cubo 31/C, Rende (CS), 87036, Italy.
[3]Dipartimento di Lettere e Filosofia, Università di Firenze, Via della Pergola 60, Firenze, I-50121, Italy.
[4]Dipartimento di Pedagogia, Psicologia, Filosofia, Università di Cagliari, Via Is Mirrionis 1, Cagliari, I-09123, Italy.

Contributing authors: roberto.beneduci@unical.it; marisa.dallachiara@gmail.com; giuntini@unica.it; giuseppe.sergioli@gmail.com;

Abstract

Since its birth quantum mechanics has inspired new logical ideas. The most important "logical revolution" of the theory concerns a divergence between the concepts of *maximal information* and *logically complete information*. This is a natural consequence of Heisenberg's uncertainty principle: any quantum pure state represents a maximal information that cannot be consistently extended to a richer knowledge about the physical system under consideration. At the same time, such information is logically incomplete, since it cannot *decide* all relevant properties of the system in question. Birkhoff and von Neumann's *quantum logic* represents a natural logical abstraction from the algebraic structure of all *events* (that may occur to a quantum system $\mathbf{S}$), mathematically represented as projection operators of the Hilbert space $\mathcal{H}^{\mathbf{S}}$ associated to $\mathbf{S}$. Projections of $\mathcal{H}^{\mathbf{S}}$ represent *sharp* quantum events that satisfy the logical non-contradiction principle. Following Ludwig, the sharp concept of *quantum event* can be generalized to the *unsharp* concept of *quantum effect*, that gives rise to possible violations of the non-contradiction principle. The concepts of *observable* and of *compatibility* between observables have different behaviors in the framework of sharp and of unsharp quantum theory. Canonical examples of observables (say, *position* and *momentum*), that are incompatible in sharp quantum theory, turn out to be *jointly measurable* in the case of unsharp quantum theory. Interestingly enough, a famous thought experiment (proposed by Heisenberg) about the possibility of an approximate simultaneous measurement of position and momentum can be described and justified in the formal language of unsharp quantum theory.

Keywords: keyword1, Keyword2, Keyword3, Keyword4

1 Introduction

Since its birth quantum mechanics has inspired new logical ideas. Strangely enough, logicians did not immediately recognize the most "revolutionary" logical result of quantum theory: the possible divergence between the concepts of *maximal information* and *logically complete information*. As is well known, the *pure states* of a classical physical system **S** (say, a gas-molecule) represent pieces of information that are at the same time *maximal and logically complete*. The information provided by a pure state of **S** cannot be consistently extended to a richer knowledge; at the same time such information *decides* all possible *events* that may occur to **S**. The classical notion of *pure state* seems to be very close to the idea of a *complete concept* investigated by Leibniz: although many properties of an individual object (say, the Moon) may be unknown to finite minds, God knows the complete concept of any object (living either in the actual or in some possible world), and this concept represents a maximal and complete piece of information about the object in question.

An important consequence of Heisenberg's uncertainty principle is that complete concepts (in Leibniz' sense) cannot exist for quantum objects. Consider a quantum system **S** (say, an electron) in a pure state that assigns an exact value to its *momentum* in a given direction. In such a case the *position* of **S** (in the same direction) will be completely *indeterminate*: the object **S** turns out to be non-localized. Quantum objects are, in a sense, "poor", and their "poverty" concerns the number of physical properties that can be satisfied at the same time.

Furthermore, quantum properties behave in a *contextual* way: properties that are completely indeterminate in a given context may become *actual and determinate* in a different context (for instance, after the performance of an appropriate measurement). Hence, the system of properties that are determinate for a given quantum object turns out to be *context-dependent*. It is worth noticing that both "poverty" and "contextuality" are at the basis of the applications of the quantum formalism to cognitive science [1].

Quantum pure states are pieces of information that are at the same time *maximal* (since they cannot be consistently extended to a richer knowledge) and *logically incomplete* (since they cannot decide all relevant properties of the objects under investigation). Apparently, *quantum undecidabilities* turn out to be much stronger than the *syntactical undecidabilities* discovered by Gödel's incompleteness theorems. In the quantum world, undecidability is not only due to the limited proof-theoretic capacities of finite minds: against "Leibniz' dream" even an infinite omniscient mind should be bound to quantum uncertainties.

2 Classical particle mechanics and classical logic

There is a deep coherence between classical physics and classical logic. In both cases the concepts of *maximal information* and of *logically complete information* are equivalent. Consider a set T of sentences, asserting a given piece of information in the framework of a formal language $\mathcal{L}$, and suppose that T is *non-contradictory* according to classical logic (there is no sentence α of $\mathcal{L}$ such that both α and its negation $\neg\alpha$ are *logical consequences* of T). The set T is called

- *maximal* iff for any sentence α (of $\mathcal{L}$) that is not a logical consequence of T, the set $T \cup \{\alpha\}$ is contradictory. In other words, T cannot be consistently extended to a richer information in the language $\mathcal{L}$.
- T is *logically complete* iff for any sentence α of $\mathcal{L}$ either α or $\neg\alpha$ is a logical consequence of T. In other words, T *logically decides* all problems expressed in the language $\mathcal{L}$.

One can prove that, for any non-contradictory set T of sentences,

T is maximal iff T is logically complete.

Of course, contradictory sets of sentences, which logically imply any sentence of the language (because *ex absurdo sequitur quodlibet!*), turn out to be, trivially, maximal and logically complete.

According to an important theorem of classical logic (first proved by the Polish logician Adolf Lindenbaum),

any non-contradictory set T of sentences of a language $\mathcal{L}$ can be extended to a non-contradictory set T' of sentences (of $\mathcal{L}$) that is maximal (and, consequently, logically complete).

Notice that Lindenbaum's theorem does not conflict with the syntactical *undecidabilities* discovered by Gödel. In fact, logically complete extensions of non-contradictory sets of sentences are, generally, *undecidable*. Thus, in agreement with Leibniz, we are entitled to claim that *in mente Dei* all problems are *decided*, in spite of any ignorance of human minds.

As is well known, in classical particle mechanics the "mathematical scenario" where the main concepts of the theory are supposed to live is based on the notion of *phase space*: a convenient power $\mathbb{R}^n$ of the set $\mathbb{R}$ of real numbers (where n depends on the number of particles that are considered). If $Ph_{\mathbf{S}}$ is the phase space of a classical particle system $\mathbf{S}$, any point $\mathbf{s}$ of $\mathbf{S}$ is a possible *pure state* of $\mathbf{S}$, representing a *maximal* piece of information about $\mathbf{S}$. At the same time, the physical properties of $\mathbf{S}$ (say, "the velocity-value in the x-direction belongs to the Borel set Δ"), corresponding to possible *physical events* that may occur to $\mathbf{S}$, can be mathematically represented as subsets X of the phase space $Ph_{\mathbf{s}}$. On this basis, in perfect harmony with classical semantics, the power set of $Ph_{\mathbf{s}}$ can be identified with the set of all possible *physical events* that may hold for pure states $\mathbf{s}$ of $\mathbf{S}$. It is natural to assume that:

the system $\mathbf{S}$ in the pure state $\mathbf{s}$ *verifies* the event X iff $\mathbf{s} \in X$.

What about the algebraic structure of physical events? As is well known, the power set of any set gives rise to a *Boolean algebra*. And also the set $\mathcal{M}(Ph_{\mathbf{S}})$ of all *measurable* subsets of $Ph_{\mathbf{s}}$ (which is more tractable from the measure-theoretic point of view) turns out to have a Boolean structure. Accordingly, one can assume that the "natural" algebraic structure of the physical events that may occur to a classical system $\mathbf{S}$ is the following Boolean algebra:

$$\mathfrak{M}_{\mathbf{S}} = (\mathcal{M}(Ph_{\mathbf{S}}), \cap, \cup, ', \emptyset, Ph_{\mathbf{S}})$$

(where $\cap$, $\cup$, $'$ are the set-theoretic intersection, union and complement).

As a consequence, one immediately obtains that any pure state $\mathbf{s}$ of a system $\mathbf{S}$ *logically decides* any physical event X that belongs to $\mathcal{M}(Ph_{\mathbf{S}})$. We have:

$$\text{either } \mathbf{s} \in X \text{ or } \mathbf{s} \in X'.$$

In this sense, classical particle mechanics is strongly *deterministic*, while pure states of physical systems represent pieces of information that are, at the same time, maximal and logically complete.

3 From the classical to the quantum world

The discovery of Heisenberg's uncertainty relations has brought about a deep change in the mathematical formalism of quantum theory, where phase spaces have been replaced by the more complex Hilbert spaces. As is well known, according to the standard axiomatization of the theory, any quantum system $\mathbf{S}$ (say, an electron, a photon beam, an atom) is associated to a Hilbert space $\mathcal{H}^{\mathbf{S}}$ that represents the "mathematical stage" for $\mathbf{S}$. The possible *pure states* of the system $\mathbf{S}$ are mathematically represented as unit vectors of $\mathcal{H}^{\mathbf{S}}$, usually indicated by the symbols $|\psi\rangle$, $|\varphi\rangle$, $|\chi\rangle$, ... (according to a successful notation introduced by Paul Dirac). Accordingly, any pure state $|\psi\rangle$ can be represented, in infinitely many ways, as a *superposition* of other pure states $|\psi_i\rangle$:

$$|\psi\rangle = \sum_i c_i |\psi_i\rangle,$$

where each c_i is a complex number (called *amplitude*). Since $|\psi\rangle$ is a unit vector, we have: $\sum_i |c_i|^2 = 1$.[1]

How can the concept of *quantum event* be adequately represented in this mathematical scenario? A classical "way of thinking" would suggest identifying the set of

[1]Notice that for any $\theta \in [0, 2\pi]$ the two vectors $|\psi\rangle$ and $e^{i\theta}|\psi\rangle$ are physically indistinguishable. Hence one can say that they represent *the same physical state*.

the events that may occur to a quantum system **S** with the set of all possible sets X of pure states of **S**. In this way, as happens in classical semantics and in the mathematical formalism of classical particle mechanics (based on the notion of *phase space*), a pure state $|\psi\rangle$ should *verify* an event X iff $|\psi\rangle$ belongs to X. At the same time, $|\psi\rangle$ should *falsify* the event X iff $|\psi\rangle$ belongs to the set-theoretic complement X' of X. As a consequence, the excluded-middle principle would hold for quantum events:

any quantum event should be either verified or falsified
by any quantum pure state.

Of course, such a semantic situation turns out to be strongly in contrast with the quantum uncertainties determined by Heisenberg's principle. Thus, in the quantum framework the very concept of *logical negation* shall be conveniently transformed and weakened (see also [4]). To this aim, a good candidate is the *orthocomplement-operation* that is defined in terms of an important relation of Hilbert-space theory: the *orthogonality relation*.

Definition 1 Orthogonality
Two vectors $|\psi\rangle$ and $|\varphi\rangle$ of a Hilbert space $\mathcal{H}$ are called *orthogonal* ($|\psi\rangle \perp |\varphi\rangle$) iff their inner product $< \psi \mid \varphi >$ is the number 0.

Definition 2 Orthocomplement
For any set X of vectors of a Hilbert space $\mathcal{H}$, the *orthocomplement* $X^\perp$ of $X$is the set of all vectors of $\mathcal{H}$ that are orthogonal to all elements of X.

If X is a set of vectors of $\mathcal{H}$, the orthocomplement $X^\perp$ of X is generally a proper subset of the set-theoretic complement X'. Consequently, the excluded-middle principle turns out to be violated. For some vectors $|\psi\rangle$ and for some sets X of vectors we may have:

$$|\psi\rangle \notin X \ \text{ and } \ |\psi\rangle \notin X^\perp.$$

One can show that for any set X of vectors of $\mathcal{H}$,

$$X = X^{\perp\perp} \ \text{ iff } \ X \ \text{ is a closed subspace of } \ \mathcal{H}$$

(i.e. a subset of $\mathcal{H}$ closed under linear combinations and limits of Cauchy-sequences of vectors).

The nice properties of the sets $X^{\perp\perp}$ have suggested that the closed subspaces of a Hilbert space (which are richer than simple sets) can represent good mathematical representatives for the intuitive notion of *quantum event*.

We recall that, according to the *projection-theorem*, for any choice of a closed subspace X, any vector $|\psi\rangle$ can be uniquely represented as a superposition

$$|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle,$$

where $|\psi_1\rangle \in X$ and $|\psi_2\rangle \in X^\perp$. The two vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ can be regarded as the two *components* of $|\psi\rangle$ that belong to the closed subspaces X and $X^\perp$, respectively.

The set $\mathcal{C}(\mathcal{H})$ of all closed subspaces of a Hilbert space $\mathcal{H}$ gives rise to an algebraic structure that (unlike the case of classical physical events) is not a Boolean algebra [2, 4]. Consider the following structure:

$$\mathfrak{C}_{\mathcal{H}} = (\mathcal{C}(\mathcal{H}),\ \sqcap,\ \sqcup,\ ^\perp,\ \{\mathbf{0}\},\ \mathbf{V}_{\mathcal{H}}),$$

where:

1. $\sqcap$ (the *infimum*) coincides with the set-theoretic intersection $\cap$;
2. $\sqcup$ (the *supremum*) is defined (in terms of $^\perp$ and $\sqcap$) via de Morgan-law:

$$X \sqcup Y := (X^\perp \sqcap Y^\perp)^\perp;$$ [2]

3. $^\perp$ is the orthocomplement;
4. $\{\mathbf{0}\}$ is the singleton of the null vector $\mathbf{0}$ and $\mathbf{V}_{\mathcal{H}}$ is the total closed subspace.

The subspaces $\{\mathbf{0}\}$ and $\mathbf{V}_{\mathcal{H}}$ are, respectively, the *minimum* and the *maximum* with respect to the partial order-relation that is defined as follows:

$$X \sqsubseteq Y \text{ iff } X \sqcap Y = X.$$

One can prove that $\mathfrak{C}_{\mathcal{H}}$ is an *orthomodular lattice*, which is also *algebraically complete* (i.e. closed under infinitary *infimum-operations* and infinitary *supremum-operations*).

Orthomodular lattices $\mathfrak{C}_{\mathcal{H}}$, based on a Hilbert space $\mathcal{H}$ are also called *Hilbert lattices*. An important Boolean property that is generally violated by Hilbert lattices is *distributivity*. We may have:

$$X \sqcap (Y \sqcup Z) \not\sqsubseteq (X \sqcap Y) \sqcup X \sqcap Z).$$

Orthomodularity (the characteristic property of orthomodular lattices) represents a special weakening of distributivity that can be formulated as follows:

$$X \sqsubseteq Y \ \Rightarrow\ Y = X \sqcup (Y \sqcap X^\perp).$$

We know that the orthomodular lattice based on the set $\mathcal{C}(\mathcal{H})$ of all closed subspaces of a Hilbert-space $\mathcal{H}$ is isomorphic to a different structure whose support is the set $\mathcal{P}(\mathcal{H})$ of all *projection operators* (briefly *projections*) P of $\mathcal{H}$. Given a closed subspace X, the projection corresponding to X is the operator P_X that assigns to any vector $|\psi\rangle$ its component $|\psi_1\rangle$ in the space X (according to the projection-theorem). At the same time, given a projection P, the closed subspace corresponding to P is the smallest closed subspace X_P that includes the *range* (the set of all possible values) of P. In the projection-lattice the minimum-element is the null projection $\mathbb{O}$ (which transforms any vector into the null vector $\mathbf{0}$), while the maximum-element is the identity projection $\mathbb{I}$ (which transforms any vector into itself).

[2]Notice that generally $X \sqcup Y$ properly includes $X \cup Y$. We have: $X \sqcup Y = (X \cup Y)^{\perp\perp}$.

As expected, the excluded-middle principle fails for quantum events. Given a pure state $|\psi\rangle$ and a quantum event represented by the closed subspace X (or by the corresponding projection P_X), three cases are possible:

1. $|\psi\rangle \in X$ and $P_X|\psi\rangle = |\psi\rangle$. In such a case one can say that the state $|\psi\rangle$ *certainly verifies* the event represented by X and by P_X.
2. $|\psi\rangle \in X^\perp$ and $P_{X^\perp}|\psi\rangle = |\psi\rangle$. In such a case one can say that the state $|\psi\rangle$ *certainly falsifies* the event represented by X and by P_X.
3. $|\psi\rangle \notin X$, $|\psi\rangle \notin X^\perp$ and $P_X|\psi\rangle \neq |\psi\rangle$, $P_{X^\perp}|\psi\rangle \neq |\psi\rangle$. In such a case one can say that the event represented by X and by P_X is *indeterminate* for the state $|\psi\rangle$.

Thus, quantum pure states are pieces of information that are at the same time *maximal* and *logically incomplete*.

As is well known, the relationship between quantum pure states and quantum events is highly informative, since it essentially involves *quantum probabilities*. Given a quantum event represented by a closed subspace X and by the corresponding projection P_X, any pure state $|\psi\rangle$ assigns to X and to P_X a probability-value (indicated by $\mathtt{p}_{|\psi\rangle}(X)$ and by $\mathtt{p}_{|\psi\rangle}(P_X)$) which is determined by the *Born-rule*:

$$\mathtt{p}_{|\psi\rangle}(X) = \mathtt{p}_{|\psi\rangle}(P_X) := \parallel P_X|\psi\rangle \parallel^2 .$$

Thus, the probability that a quantum system in state $|\psi\rangle$ verifies the event represented by X and by P_X is the number $\parallel P_X|\psi\rangle \parallel^2$ (the squared length of the X-component of $|\psi\rangle$).

We have:

- $\mathtt{p}_{|\psi\rangle}(X) = \mathtt{p}_{|\psi\rangle}(P_X) \in [0,1]$.
- $\mathtt{p}_{|\psi\rangle}(X) = \mathtt{p}_{|\psi\rangle}(P_X) = \mathrm{tr}(P_{|\psi\rangle}P_X)$, where tr is the *trace-functional* and $P_{|\psi\rangle}$ is the projection determined by the one-dimensional closed space that contains the vector $|\psi\rangle$.
- For any closed subspaces of X and Y of $\mathcal{H}$:
  $X \sqsubseteq Y$ iff for any pure state $|\psi\rangle$, $\mathtt{p}_{|\psi\rangle}(X) \leq \mathtt{p}_{|\psi\rangle}(Y)$.
  Hence, the event-partial order turns out to have an important physical meaning.

As a particular case consider a system **S** in a pure state

$$|\psi\rangle \;=\; c_1|\psi_1\rangle + \ldots + c_n|\psi_n\rangle,$$

where all amplitudes c_i are different from 0. According to the Born-rule this system *might* satisfy with probability $|c_i|^2$ the properties that are *certain* for any system whose state is $|\psi_i\rangle$.

Quantum pure states seem to describe a kind of "cloud of potential properties" that are, in a sense, all co-existent. Interestingly enough, such a co-existence (which may appear *prima facie* strange and mysterious) can be used as a powerful resource for different aims. For instance, in quantum computation the parallel computational paths of quantum computers are essentially based on superpositions, where alternative states of a quantum objects *act* at the same time.[3]

So far we have been concerned with pure states, which represent *maximal* pieces of information about the physical systems under investigation. Like classical physics, quantum theory as well cannot avoid the use of *mixed states* (or *mixtures*) that correspond to a *non-maximal knowledge* of the observer.

Quantum mixtures can be expressed in a form that is similar to classical mixtures. Let $\mathcal{H}^{\mathbf{S}}$ be the Hilbert space associated to a quantum system $\mathbf{S}$. Consider an operator of $\mathcal{H}^{\mathbf{S}}$ having the following form:

$$\rho = w_1 P_{|\psi_1\rangle} + \ldots + w_n P_{|\psi_n\rangle},$$

where $|\psi_1\rangle, \ldots, |\psi_n\rangle$ are possible pure states of $\mathbf{S}$, while $w_1, \ldots, w_n$ are positive real numbers such that $w_1 + \ldots + w_n = 1$. Such an operator, which is expressed as a *convex combination* of pure states, represents a possible *mixed state* of $\mathbf{S}$: a *mixture* of the pure states $|\psi_1\rangle, \ldots, |\psi_n\rangle$ with *weights* $w_1, \ldots, w_n$. When an observer has associated to $\mathbf{S}$ the mixed state ρ, the physical interpretation is the following: $\mathbf{S}$, whose state is ρ, *might* be in the pure state $|\psi_i\rangle$ with probability w_i. Of course, pure states $|\psi\rangle$ correspond to special cases of mixed states, having the form $P_{|\psi\rangle}$.

We know that any mixture expressed as a convex combination $\rho = w_1 P_{|\psi_1\rangle} + \ldots + w_n P_{|\psi_n\rangle}$ (in a Hilbert space $\mathcal{H}$) belongs to a special class of operators called *density operators* of $\mathcal{H}$. While any convex combination of pure states uniquely determines a density operator, the inverse relation does not hold: generally, a density operator ρ can be represented in many different ways as a convex combination of pure states.

As is well known, the Born-rule can be naturally generalized to all possible states of a quantum system (which may be either pure or mixed). For any event represented by the closed subspace X (and by the projection P_X) and for any state ρ living in the space $\mathcal{H}^{\mathbf{S}}$ (associated to a system $\mathbf{S}$), the probability that the state ρ verifies the event X (and P_X) is determined by the following definition:

$$\mathtt{p}_\rho(X) = \mathtt{p}_\rho(P_X) := \mathrm{tr}(\rho P_X).$$

4 Compatible and incompatible quantum observables

In the framework of an abstract logical approach the notion of *observable* (or *physical quantity*) that can be measured on a quantum system $\mathbf{S}$ can be defined as a *projection-valued measure*, which refers to the set of all possible events that may occur to $\mathbf{S}$.

Consider the set $\mathcal{P}(\mathcal{H}^{\mathbf{S}})$ of all projections of the space $\mathcal{H}^{\mathbf{S}}$ and let $\mathcal{B}(\mathbb{R})$ be the set of all Borel sets of real numbers.[4]

Definition 3 *Observable*
An *observable* for the quantum system $\mathbf{S}$ is a map

$$O : \mathcal{B}(\mathbb{R}) \;\to\; \mathcal{P}(\mathcal{H}^{\mathbf{S}}),$$

[3] See, for instance, [16].

[4] We recall that the algebraic structure of $\mathcal{B}(\mathbb{R})$ is a σ-complete Boolean algebra (closed under infinitary intersections and unions over countable sets of elements).

that satisfies the following conditions:

1. $O(\mathbb{R})$ is the identity operator $\mathbb{I}$.
2. For any countable set $\{\Delta_i\}_{i\in I}$ of pairwise disjoint Borel sets,

$$O(\bigcup\{\Delta_i\}_{i\in I}) = \bigsqcup\{O(\Delta_i)\}_{i\in I}$$

(where $\bigsqcup$ is the infinitary supremum operation).

The intuitive meaning of the concept of *observable* is quite clear. Any observable O associates to any Borel set Δ a particular event that will be naturally interpreted as the following state of affairs: "the value of the observable O lies in the Borel set Δ". As expected, it is required that the map O preserves some structural properties of the σ-Boolean algebra of all Borel sets.

The definition of *observable* can be generalized by assuming that the domain of O is any $\mathcal{B}(\mathbb{R}^n)$ (with $n \geq 1$).

On this basis one can define the notions of *expected value* and of *variance* (or *dispersion*) of a given observable O.

Definition 4 Expected value
The *expected value* of O with respect to a state ρ, is defined as follows:

$$Exp_\rho(O) := \int_{-\infty}^{+\infty} x\mathrm{tr}(\rho O(dx)),$$

provided the integral exists.

Definition 5 Variance
The *variance* of O with respect to a state ρ, is defined as follows:

$$Var_\rho(O) := \int_{-\infty}^{+\infty} [x - Exp_\rho(O)]^2 \mathrm{tr}(\rho O(dx)),$$

provided the integral exists.

A crucial quantum question (which does not arise in classical physics) concerns the notion of *compatibility* between different observables.

Definition 6 Compatible observables
Let O_1 and O_2 be two observables for a quantum system $\mathbf{S}$ (with associated Hilbert space $\mathcal{H}^{\mathbf{S}}$). O_1 and O_2 are called *compatible* iff for any Borel sets Δ and Γ,

$$O_1(\Delta)O_2(\Gamma) = O_2(\Gamma)O_1(\Delta)$$

(in other words, the two projections $O_1(\Delta)$ and $O_2(\Gamma)$ commute in the space $\mathcal{H}^{\mathbf{S}}$).

One can prove the following Lemma.

Lemma 4.1. *Two observables O_1 and O_2 are compatible iff for any Borel sets Δ and Γ, $O_1(\Delta)O_2(\Gamma)$ is a projection.*

Compatible observables turn out to be *jointly measurable.*

Definition 7 *Joint measurability*
Two observables $O_1 : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathcal{H}^S)$ and $O_2 : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathcal{H}^S)$ are called *jointly measurable* iff there exists an observable $O : \mathcal{B}(\mathbb{R} \times \mathbb{R}) \to \mathcal{P}(\mathcal{H}^S)$, such that for any $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R})$:

$$O_1(\Delta_1) = O(\Delta_1 \times \mathbb{R}) \text{ and } O_2(\Delta_2) = O(\mathbb{R} \times \Delta_2).$$

The observable O is called the *joint observable* of O_1 and O_2, while O_1 and O_2 are called the *marginals* of O.

One can prove that the existence of a joint observable characterizes the compatibility-relation.

Theorem 1 *Two observables O_1 and O_2 are compatible iff O_1 and O_2 are jointly measurable.*

Interestingly enough, the notion of compatibility between observables for a quantum system **S** can be also characterized in terms of an algebraic property of the Hilbert lattice of all possible events that may occur to **S**.

Theorem 2 *Let O_1 and O_2 be two observables for a quantum system* **S** *and consider the Hilbert lattice $\mathfrak{P}(\mathcal{H}^{\mathbf{S}})$ based on the set of all projections of the space $\mathcal{H}^{\mathbf{S}}$.*
O_1 and O_2 are compatible iff for any Borel sets Δ and Γ, the following equality holds in the lattice $\mathfrak{P}(\mathcal{H}^{\mathbf{S}})$:

$$O_1(\Delta) = [O_1(\Delta) \sqcap O_2(\Gamma)] \sqcup [O_1(\Delta) \sqcap O_2(\Gamma)^{\perp}].$$

Thus, the compatibility-relation between two observables can be characterized by means of a special case of distributivity that concerns particular triplets of quantum events.

As is well known, the existence of pairs of incompatible observables can be mathematically proved. Canonical examples are represented by the pairs:

- *position* in a given direction and *momentum* in the same direction.
- *spin* in a given direction and *spin* in a different direction.

Some important pairs of incompatible observables (like *position* and *momentum*) satisfy a general form of an *uncertainty principle*, that can be expressed as follows:

$$\exists \epsilon > 0 \text{ such that for any pure state } |\psi\rangle, Var_{P_{|\psi\rangle}}(O_1) Var_{P_{|\psi\rangle}}(O_2) > \epsilon.$$

As a consequence, the dispersion of either observable shall increase with the decreasing of the dispersion of the other observable.

It may happen that, although O_1 and O_2 do not commute, there exists a pure state $P_{|\psi\rangle}$ such that

$$Var_{P_{|\psi\rangle}}(O_1)Var_{P_{|\psi\rangle}}(O_2) = 0.$$

As an example, consider the two observables O^{σ_z} and O^{σ_x} corresponding to the two self-adjoint operators σ_z and σ_x that represent the spin in the z-direction and the spin in the x-direction (respectively) for a given quantum system. Let $\{|+\rangle, |-\rangle\}$ be an orthonormal basis of eigenvectors of σ_z such that: $\sigma_z|+\rangle = \frac{1}{2}|+\rangle$ and $\sigma_z|-\rangle = -\frac{1}{2}|-\rangle$. We have:

$$\sigma_z = \frac{1}{2}|+\rangle\langle+| - \frac{1}{2}|-\rangle\langle-| \text{ and } \sigma_x = \frac{1}{2}|+\rangle\langle-| + \frac{1}{2}|-\rangle\langle+|.$$

Consider now the pure state $P_{|+\rangle}$. We obtain:

$$Var_{P_{|+\rangle}}(O^{\sigma_z}) = 0 \text{ and } Var_{P_{|+\rangle}}(O^{\sigma_x}) = \frac{1}{2}.$$

Consequently: $Var_{P_{|+\rangle}}(O^{\sigma_z})Var_{P_{|+\rangle}}(O^{\sigma_x}) = 0$.

At first sight, this seems to contradict Heisenberg's uncertainty principle. However, this is not the case, because an *uncertainty* equal to $\frac{1}{2}$ represents a *maximal uncertainty* for the spin-observable in the x-direction. In fact, the expected value of O^{σ_x} for the state $P_{|+\rangle}$ is 0, while σ_x can only assume the values $\frac{1}{2}$ and $-\frac{1}{2}$. In other words, we know that the spin of the system in the z-direction is $\frac{1}{2}$; at the same time, we do not have any information about the spin in the x-direction. The two observables O^{σ_z} and O^{σ_x} are also called *complementary*.[5]

5 Birkhoff and von Neumann's quantum logic

The mathematical structures that arise in the quantum theoretic formalism have inspired the development of different forms of non-classical logic, termed *quantum logics*. The prototypical example of quantum logic is Birkhoff and von Neumann's quantum logic (first proposed in their celebrated article "The logic of quantum mechanics" [11]). This logic (that will be indicated by $\mathbf{QL^{BN}}$) represents a natural logical abstraction from the class of all Hilbert lattices:

$$\mathfrak{C}_{\mathcal{H}} = (\mathcal{C}(\mathcal{H}),\ \sqcap,\ \sqcup,\ ^{\perp},\ \{\mathbf{0}\}\,,\ \mathbf{V}_{\mathcal{H}})$$

(where $\mathcal{H}$ may be the Hilbert space $\mathcal{H}^{\mathbf{S}}$ associated to a quantum system $\mathbf{S}$).

Consider a sentential language $\mathcal{L}$ with atomic sentences (which may assert that "the value of given observable lies in a given Borel set") and with the following logical connectives: the negation $\neg$, the conjunction $\wedge$ and the disjunction $\vee$.

In the Birkhoff and von Neumann's semantics a *model* of $\mathcal{L}$ can be defined as a pair

$$\mathcal{M} = (\mathfrak{C}_{\mathcal{H}}, v),$$

[5] For a deep analysis of the relationships between *complementarity* and *uncertainty* see [12, 13].

where $\mathfrak{C}_{\mathcal{H}}$ is a Hilbert lattice and v is a valuation that assigns to every sentence of $\mathcal{L}$ a *meaning* represented by a quantum event X in $\mathfrak{C}_{\mathcal{H}}$. The map v shall preserve the logical form of all sentences, satisfying the following conditions (for any sentences α, β of $\mathcal{L}$):

1. $v(\neg\alpha) = v(\alpha)^{\perp}$;
2. $v(\alpha \wedge \beta) = v(\alpha) \sqcap v(\beta)$;
3. $v(\alpha \vee \beta) = v(\alpha) \sqcup v(\beta)$.

On this basis one can define the concepts of *truth*, *logical truth* and *logical consequence*:

Definition 8 Truth
Let $\mathcal{M} = (\mathfrak{C}_{\mathcal{H}}, v)$ be a model of $\mathcal{L}$.

1. A sentence α is called *true* for a state ρ of $\mathcal{H}$ (abbreviated as $\models_\rho \alpha$) iff $\mathtt{p}_\rho(v(\alpha)) = 1$.
2. A sentence α is called *true* in the model $\mathcal{M}$ ($\models_{\mathcal{M}} \alpha$) iff $v(\alpha)$ is the whole space $\mathbf{V}_{\mathcal{H}}$ (hence, for all states ρ of $\mathcal{H}$: $\mathtt{p}_\rho(v(\alpha)) = 1$).

Definition 9 Logical truth and logical consequence in the logic $\mathbf{QL}^{\mathbf{BN}}$

1. A sentence α is called a *logical truth* ($\models_{\mathbf{QL}^{\mathbf{BN}}} \alpha$) iff α is true in any model $\mathcal{M}$.
2. β is called a *logical consequence* of α ($\alpha \models_{\mathbf{QL}^{\mathbf{BN}}} \beta$) iff for any model $\mathcal{M} = (\mathfrak{C}_{\mathcal{H}}, v)$, $v(\alpha) \sqsubseteq v(\beta)$ (hence, for all states ρ of $\mathcal{H}$: $\mathtt{p}_\rho(v(\alpha)) \leq \mathtt{p}_\rho(v(\beta))$).
3. α is called a *logical consequence* of a set T of sentences ($T \models_{\mathbf{QL}^{\mathbf{BN}}} \alpha$) iff for some sentences $\beta_1, \ldots, \beta_n \in T$, $\beta_1 \wedge \ldots \wedge \beta_n \models_{\mathbf{QL}^{\mathbf{BN}}} \alpha$.

It is interesting to recall some important logical truths and logical arguments that hold in the logic $\mathbf{QL}^{\mathbf{BN}}$:

1. $\alpha \models_{\mathbf{QL}^{\mathbf{BN}}} \neg\neg\alpha$, $\neg\neg\alpha \models_{\mathbf{QL}^{\mathbf{BN}}} \alpha$ (the double-negation principle).
2. $\models_{\mathbf{QL}^{\mathbf{BN}}} \neg(\alpha \wedge \neg\alpha)$ (the logical non-contradiction principle).
3. $\models_{\mathbf{QL}^{\mathbf{BN}}} \alpha \vee \neg\alpha$ (the logical excluded-middle principle).
4. $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \models_{\mathbf{QL}^{\mathbf{BN}}} \alpha \wedge (\beta \vee \gamma)$ (weak distributivity).

As expected, a characteristic classical argument that is generally violated in Birkhoff ad von Neumann's quantum logic is the strong distributivity principle. We have:

$$\alpha \wedge (\beta \vee \gamma) \quad \not\models_{\mathbf{QL}^{\mathbf{BN}}} \quad (\alpha \wedge \beta) \vee (\alpha \wedge \gamma).$$

Interestingly enough, the validity of the logical excluded-middle principle does not imply the semantic *tertium non datur* (for any sentence α, either α or its negation $\neg\alpha$ is true in any model). While any sentence whose form is $\alpha \vee \neg\alpha$ is true in any model, there are sentences γ that are indeterminate in some models, where (unlike classsical models) we may have for some pure states ρ of $\mathcal{M}$:

$$\not\models_\rho \gamma \text{ and } \not\models_\rho \neg\gamma.$$

The divergence between the logical excluded-middle principle ($\models_\rho \alpha \vee \neg\alpha$, for any state ρ) and the *semantic excluded-middle principle* (either $\models_\rho \alpha$ or $\models_\rho \neg\alpha$, for any state ρ) is a peculiar feature of quantum logic.

Another important characteristic property of quantum logic concerns a strong failure of Lindenbaum's theorem. Unlike classical logic, generally in quantum logic a non-contradictory set of sentences cannot be consistently extended to a non-contradictory set of sentences that is logically complete. Only a weaker version of Lindenbaum's theorem can be proved.[6]

Theorem 3 *Weak Lindenbaum's theorem*
Any non-contradictory set T of sentences can be consistently extended to a maximal *non-contradictory set T' that satisfies the following condition for any sentence α:*
$T' \not\models_{\mathbf{QL}^{\mathrm{BN}}} \alpha \;\Rightarrow\; T' \cup \{\alpha\}$ *is contradictory (i.e. for some sentence β, both β and $\neg\beta$ are logical consequences of $T' \cup \{\alpha\}$).*

As happens in the case of quantum pure states, the concepts of *maximality* and of *logical completeness* diverge for pieces of information that are represented by quantum-logical sets of sentences. Interestingly enough, the failure of Lindenbaum's theorem has played an important role in the discussions about the logical possibility of *non-contextual hidden variable theories* in quantum theory.[7]

6 From sharp to unsharp quantum theory

Closed subspaces X and projections P of a Hilbert space $\mathcal{H}^{\mathbf{S}}$ (associated to a quantum system $\mathbf{S}$) represent *sharp events* that satisfy the non-contradiction principle:

$$(X \sqcap X^{\perp})^{\perp} = \mathcal{C}(\mathcal{H}^{\mathbf{S}}) \text{ and } (P \sqcap P^{\perp})^{\perp} = \mathcal{P}(\mathcal{H}^{\mathbf{S}})$$

(where $\mathcal{C}(\mathcal{H}^{\mathbf{S}})$ and $\mathcal{P}(\mathcal{H}^{\mathbf{S}})$ represent the *certain event* for $\mathbf{S}$).
Thus, contradictions are *impossible* and the negation of any contradiction is *certain*. Accordingly (as we have seen in the previous Section), the non-contradiction principle ($\neg(\alpha \wedge \neg\alpha)$) is a logical truth of Birkhoff and von Neumann's quantum logic.

Furthermore, any event that is not impossible is *certainly verified* by at least one pure state. We have:

- $X \neq \emptyset \implies$ for some pure state $|\psi\rangle, |\psi\rangle \in X$ and $\mathtt{p}_{|\psi\rangle}(X) = 1$.
- $P \neq \mathbb{O} \implies$ for some pure state $|\psi\rangle, P|\psi\rangle = |\psi\rangle$ and $\mathtt{p}_{|\psi\rangle}(P) = 1$.

A natural question arises: to what extent do $\mathcal{C}(\mathcal{H}^{\mathbf{S}})$ and $\mathcal{P}(\mathcal{H}^{\mathbf{S}})$ represent an *optimal* choice for the mathematical representatives of quantum events? Does $\mathcal{P}(\mathcal{H}^{\mathbf{S}})$ represent the largest set of Hilbert-space operators to which a probability-value can be assigned (according to the Born-rule)? The answer to this question is negative. One can easily recognize the existence of bounded linear operators E that are not projections and that satisfy the following condition:

[6] See, for instance, [15].
[7] See, for instance, [19].

$$\text{for any state } \rho, \operatorname{tr}(\rho E) \in [0,1].$$

From an intuitive point of view, this means that such operators E "behave as possible events", because any state assigns to them a probability-value.

An interesting example of this kind is represented by the operator $\frac{1}{2}\mathbb{I}$. One immediately realizes that $\frac{1}{2}\mathbb{I}$ is a linear bounded operator that is not a projection, because

$$\frac{1}{2}\mathbb{I}\frac{1}{2}\mathbb{I} = \frac{1}{4}\mathbb{I} \neq \frac{1}{2}\mathbb{I}.$$

(hence $\frac{1}{2}\mathbb{I}$ fails to be idempotent.). At the same time, for any state ρ we have:

$$\operatorname{tr}(\rho\frac{1}{2}\mathbb{I}) = \frac{1}{2}.$$

Thus, $\frac{1}{2}\mathbb{I}$ seems to be a totally *indeterminate* event, to which each state assigns probability $\frac{1}{2}$. Apparently, $\frac{1}{2}\mathbb{I}$ plays the role that, in fuzzy set theory, is played by the *semitransparent* fuzzy set $\frac{1}{2}\mathbf{1}$ such that for any object x of the universe:

$$\frac{1}{2}\mathbf{1}(x) = \frac{1}{2}.$$

This situation suggests that we liberalize the notion of *quantum event* and extend the set $\mathcal{P}(\mathcal{H}^{\mathbf{S}})$. Following Ludwig[8], the elements of this new set have been called *effects*.

Definition 10 Effect
An *effect* of $\mathcal{H}$ is a bounded linear operator E that satisfies the following condition, for any state ρ:

$$\operatorname{tr}(\rho E) \in [0,1].$$

We denote by $\mathcal{E}(\mathcal{H})$ the set of all effects of $\mathcal{H}$. Clearly, $\mathcal{E}(\mathcal{H})$ properly includes $\mathcal{P}(\mathcal{H})$. Because:

- any projection satisfies the definition of effect;
- there are examples of effects that are not projections (for instance, the effect $\frac{1}{2}\mathbb{I}$, that is usually called *the semitransparent effect).*

By definition effects represent a kind of *maximal* representative for the notion of quantum event in agreement with the basic statistical rule of quantum theory (the Born-rule).

Unlike projections, effects represent quite general mathematical objects that describe at the same time events and states. Let E be any effect in $\mathcal{E}(\mathcal{H})$. The following conditions hold:

- E represents a sharp event (in $\mathcal{P}(\mathcal{H})$) iff E is idempotent ($EE = E$).
- E is a density operator representing a state iff $\operatorname{tr}(E) = 1$

[8] See [25].

- E represents a pure state iff E is at the same time a projection and a density operator.

From an intuitive point of view, we could say that moving to an unsharp approach represents an important step towards a kind of "second degree of fuzziness". In the framework of a sharp approach, any physical event P can be regarded as a kind of "clear" property. Whenever a state ρ assigns to P a probability-value different from 1 and 0, one can think that the semantic uncertainty involved in such a situation totally depends on the ambiguity of the state (first degree of fuzziness). And we know that in quantum theory even a pure state does not represent a *logically complete information* that is able to decide any possible physical event. In the unsharp approaches, instead, one can take into account also "genuinely ambiguous properties". We will see that this second degree of fuzziness can be regarded as depending on the accuracy of the measurement which tests the property in question, and also on the accuracy involved in the operational definition for the physical quantities which the property refers to.

One immediately realizes that $\mathcal{E}(\mathcal{H})$ can be naturally structured as an *involutive regular poset*:

$$\mathfrak{E}(\mathcal{H}) = (\mathcal{E}(\mathcal{H})), \sqsubseteq, {}', \mathbb{O}, \mathbb{I}),$$

where

1. $\sqsubseteq$ is the *natural* order determined by the set of all states. In other words: $E \sqsubseteq F$ iff for any state ρ, $\mathrm{tr}(\rho E) \leq \mathrm{tr}(\rho F)$.
   (i.e. any state assigns to E a probability-value that is less or equal than the probability-value assigned to F).
2. $E' = \mathbb{I} - E$ (where $-$ is the standard operator difference).

One can easily check that:

- $\sqsubseteq$ is a partial order;
- $'$ is an involution, satisfying the double negation law ($E'' = E$) and the contraposition law ($E \sqsubseteq F \;\Rightarrow\; F' \sqsubseteq E'$);
- $\mathbb{O}$ and $\mathbb{I}$ are respectively the minimum and the maximum with respect to $\sqsubseteq$;
- the regularity condition holds:

$$E \sqsubseteq E' \text{ and } F \sqsubseteq F' \;\Rightarrow\; E \sqsubseteq F'.$$

One can prove that,unlike the event algebra $\mathfrak{C}(\mathcal{H})$, the effect poset $\mathfrak{E}(\mathcal{H})$ is not a lattice: there are pairs of effects E, F for which the infimum $E \sqcap F$ does not exist.[9]

7 Unsharp observables

In the previous section we have asked whether $\mathcal{P}(\mathcal{H}^{\mathbf{S}})$ represents an optimal choice for the mathematical representation of quantum events. Our answer has been negative: $\mathcal{P}(\mathcal{H}^{\mathbf{S}})$ is not the largest set of Hilbert space operators to which a probability-value can be assigned, because the set $\mathcal{E}(\mathcal{H}^{\mathbf{S}})$ properly includes $\mathcal{P}(\mathcal{H}^{S})$. We will see how further evidence for the thesis according to which $\mathcal{E}(\mathcal{H}^{\mathbf{S}})$ represents an optimal choice comes

[9] See, for instance, [16] and [18].

from the study of *operational definitions* of quantum physical quantities. This will also provide a deeper insight about the question: how does the concept of *compatibility* changes when $\mathcal{P}(\mathcal{H}^{\mathbf{S}})$ has been replaced by $\mathcal{E}(\mathcal{H}^{\mathbf{S}})$?

The natural unsharp generalization of the notion of *observable* is the concept of *effect-valued measure*, also called *positive-operator valued measure*.

Definition 11 *Effect-valued measure*
An *effect-valued measure* for the quantum system **S** is a map

$$\mathfrak{O} : \mathcal{B}(\mathbb{R}) \to \mathcal{E}(\mathcal{H}^{\mathbf{S}}),$$

that satisfies the following conditions:

1. $\mathfrak{O}(\mathbb{R}) = \mathbb{I}$.
2. For any countable set $\{\Delta_i\}_{i\in I}$ of pairwise disjoint Borel sets,

$$\mathfrak{O}(\bigcup\{\Delta_i\}_{i\in I}) = \sum_{i\in I} \mathfrak{O}(\Delta_i).$$

   where the series $\{\mathfrak{O}(\Delta_i)\}_{i\in I})$ converges in the weak operator topology.

In the following effect-valued measures will be also called *unsharp observables*. Like in the sharp case, the definition of *effect-valued measure* can be generalized by assuming that the domain of $\mathfrak{O}$ is any $\mathcal{B}(\mathbb{R}^n)$.

Definition 12 *Self-commuting effect-valued measure*
An effect-valued measure $\mathfrak{O}$ is called *self-commuting* iff for any $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R})$,

$$\mathfrak{O}(\Delta_1)\mathfrak{O}(\Delta_2) = \mathfrak{O}(\Delta_2)\mathfrak{O}(\Delta_1). \tag{1}$$

While sharp observables are always self-commuting, unsharp self-commuting observables turn out to be a proper subset of the set of all observables.

Interestingly enough, effects and effect-valued measures emerge in a natural way in the analysis of *measurement-processes*. Unsharp observables have a deep physical motivation that is encoded in laboratory-procedures. In order to understand this connection, let us briefly examine measurement-processes from an *operational* point of view. A quantum system **S** to be measured is first prepared in a state that is mathematically represented by a density operator ρ of the Hilbert space $\mathcal{H}^{\mathbf{S}}$. Then, a physical interaction between **S** and a measurement-apparatus $\mathcal{A}$ is performed. Just this interaction represents a *measurement* M on the system **S**. Finally the measurement's outcomes are registered. This last step is connected to an idea that is usually called "quantum objectification", according to which every registration-process leads to a *definite result* [12] (as is illustrated in Figure 1, item a)).

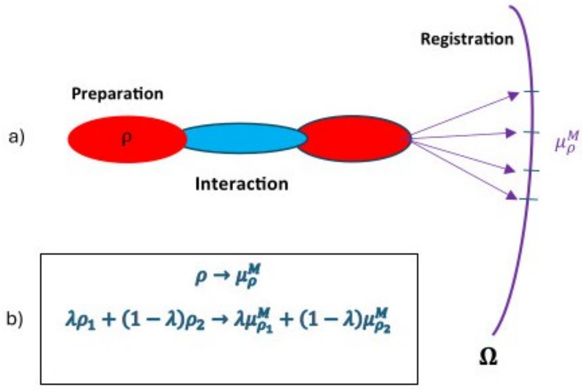

Fig. 1: a) A qualitative illustration of the measurement process, b) the measurement process as an affine map

Generally, repetitions of one and the same measurement M on a system $\mathbf{S}$ in a given state ρ give rise to different unpredictable results. Quantum systems do not have a *deterministic* behavior. In spite of this, the construction of a physical theory is possible, because a *deterministic regularity* can be replaced by a *statistical regularity*, represented by a *probability-measure* defined on the measurable space $\Omega \subseteq \mathcal{B}(\mathbb{R})$ of all possible measurement-outcomes. In order to obtain such statistical regularity, one shall repeat the measurement in question several times. From a mathematical point of view, this procedure can be represented as a map from states to probability-measures:

$$\rho \;\mapsto\; \mu_\rho^M.$$

Such map shall satisfy the *affinity-property*. This means that for any mixed state ρ having the form:

$$\rho = \lambda\rho_1 + (1-\lambda)\rho_2,$$

the associated probability-measure μ_ρ^M shall be a convex combination of probability-measures, having the form:

$$\mu_\rho^M = \lambda\mu_{\rho_1}^M + (1-\lambda)\mu_{\rho_2}^M.$$

(See Figure 1, item b).)

Two measurements M_1 and M_2 are said to be *equivalent* ($M_1 \equiv M_2$) iff $\mu_\rho^{M_1} = \mu_\rho^{M_2}$ for any state ρ. *Operational observables* can be then defined as equivalence- classes $[M]_\equiv$, where $[M]_\equiv = \{M' \mid M' \equiv M\}$.

The following theorem asserts an important strong correlation between affine maps μ_ρ^M and unsharp observables $\mathfrak{O}$.

Theorem 4 ([23]) *There is a one-to-one correspondence between the set of all affine maps $\rho \mapsto \mu_\rho^M$ and the set of all unsharp observables $\mathfrak{O}$. For any $\Delta \in \mathcal{B}(\mathbb{R})$, we have:*

$$\mu_\rho^M(\Delta) = \mathrm{tr}[\rho \mathfrak{O}(\Delta)].$$

Theorem 4 confirms that effect-valued measures represent an appropriate mathematical tool for representing quantum observables, since they are strongly connected with the description of measurement-processes. The number $\mathrm{tr}(\rho\, \mathfrak{O}(\Delta))$ (based on the Born-rule) represents the probability that the outcome of a measurement M gives a result in Δ, when the system $\mathbf{S}$ is prepared in the state ρ.

In this framework, effects can be represented as special cases of unsharp observables, called *dichotomic observables*.

Definition 13 *Dichotomic observable*
Let f be a function from $\{\{0\}, \{1\}\}$ to $\mathcal{E}(\mathcal{H}^{\mathbf{S}})$ such that

$$f(\{0\}) = \mathbb{I} - f(\{1\}).$$

One can prove that f can be uniquely extended to an effect-valued measure $\mathfrak{O}^f : \mathcal{B}(\mathbb{R}) \to \mathcal{E}(\mathcal{H}^{\mathbf{S}})$, such that for any $\Delta \in \mathcal{B}(\mathbb{R})$:

$$\mathfrak{O}^f(\Delta) = \begin{cases} f(\{1\}), \text{ if } 1 \in \Delta \text{ and } 0 \notin \Delta; \\ f(\{0\}), \text{ if } 0 \in \Delta \text{ and } 1 \notin \Delta; \\ \mathbb{I}, \text{ if } 0 \in \Delta \text{ and } 1 \in \Delta; \\ \mathbb{O}, \text{ if } 0 \notin \Delta \text{ and } 1 \notin \Delta. \end{cases}$$

The map $\mathfrak{O}^f$ is called a *dichotomic* unsharp observable.

From an intuitive point of view, a dichotomic observable $\mathfrak{O}^f$ can be regarded as a *question* that admits two possible answers: *YES* and *NO* (corresponding to the numbers 1 and 0, respectively).

Apparently, moving from "sharpness" to "unsharpness" is an important step that allows us to include in the language of quantum theory *operational concepts*, whose statistics could not be described in terms of sharp events and of sharp observables.[10]

We will now analyze the relations of *commutativity* and of *compatibility* in the more general framework of effects and effect-valued measures.

Definition 14 *Commutativity*
Two effect-valued measures $\mathfrak{O}_1 : \mathcal{B}(\mathbb{R}) \to \mathcal{E}(\mathcal{H}^{\mathbf{S}})$ and $\mathfrak{O}_2 : \mathcal{B}(\mathbb{R}) \to \mathcal{E}(\mathcal{H}^S)$ are called *commuting observables* iff for any $\Delta, \Gamma \in \mathcal{B}(\mathbb{R})$:

$$\mathfrak{O}_1(\Delta)\mathfrak{O}_2(\Gamma) = \mathfrak{O}_2(\Gamma)\mathfrak{O}_1(\Delta).$$

Definition 15 *Compatibility*
Two effect-valued measures $\mathfrak{O}_1 : \mathcal{B}(\mathbb{R}) \to \mathcal{E}(\mathcal{H}^S)$ and $\mathfrak{O}_2 : \mathcal{B}(\mathbb{R}) \to \mathcal{E}(\mathcal{H}^S)$ are called *compatible* (or *jointly measurable*) iff there exists an effect-valued measure $\mathfrak{O} : \mathcal{B}(\mathbb{R} \times \mathbb{R}) \to \mathcal{E}(\mathcal{H}^S)$, such that for any $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R})$:

$$\mathfrak{O}_1(\Delta_1) = \mathfrak{O}(\Delta_1 \times \mathbb{R}) \text{ and } \mathfrak{O}_2(\Delta_2) = \mathfrak{O}(\mathbb{R} \times \Delta_2).$$

[10] For more details see [12].

The observable $\mathfrak{O}$ is called the *joint observable* of $\mathfrak{O}_1$ and $\mathfrak{O}_2$, while $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are called the *marginals* of $\mathfrak{O}$.

Interestingly enough, unlike the case of sharp observables, compatible unsharp observables are not necessarily commuting observables. While *commutativity* implies *compatibility*, the inverse implication does not generally hold.

As an example, let us consider two dichotomic observables corresponding to the two following effects:

$$F_1 = \begin{pmatrix} \frac{1}{4} & \frac{1}{4}(1-i) \\ \frac{1}{4}(1+i) & \frac{1}{2} \end{pmatrix}, \quad F_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4}(1-i) \\ \frac{1}{4}(1+i) & \frac{1}{2} \end{pmatrix}.$$

One can easily show that F_1 and F_2 do not commute.
The following Lemma can be proved:

Lemma 7.1. *Consider two dichotomic observables $\mathfrak{O}_1$ and $\mathfrak{O}_2$ corresponding to the effects E_1 and E_2, respectively. $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are compatible, if there is an effect E such that*

$$\begin{aligned} E &\sqsubseteq E_1 \\ E &\sqsubseteq E_2 \\ E_1 + E_2 - E &\sqsubseteq \mathbb{I} \end{aligned} \tag{2}$$

The effects F_1 and F_2 (defined above) turn out to satisfy condition (2) of Lemma 7.1 with $E = \begin{pmatrix} \frac{3}{4} & \frac{1}{4}(1-i) \\ \frac{1}{4}(1+i) & \frac{1}{2} \end{pmatrix}$. Consequently, they represent two compatible non-commuting observables.

A relevant example of non-commuting observables that are compatible is represented by the *unsharp position* and the *unsharp momentum*, which can be obtained by means of a "fuzzification" of the corresponding sharp position and momentum. We will illustrate this procedure in the next Section.

8 From sharp observables to unsharp observables and back

A *fuzzification* of a sharp observable can be realized by using the concept of *Markov-kernel*.

Definition 16 *Markov-kernel*
A *Markov-kernel* is a map $\mu : \mathcal{B}(\mathbb{R}) \times \mathbb{R} \to [0,1]$ that satisfies the following conditions:

1. for any $\Delta \in \mathcal{B}(\mathbb{R})$, the function $\mu_\Delta : \mathbb{R} \to [0,1]$ such that

$$\forall \lambda \in \mathbb{R} : \mu_\Delta(\lambda) := \mu(\Delta, \lambda)$$

is a measurable function.

2. For any $\lambda \in \mathbb{R}$, the function $\mu_\lambda : \mathcal{B}(\mathbb{R}) \to [0,1]$ such that

$$\forall \Delta \in \mathcal{B}(\mathbb{R}) : \mu_\lambda(\Delta) := \mu(\Delta, \lambda)$$

is a probability-measure with domain $\mathcal{B}(\mathbb{R})$.

From an intuitive point of view, we can refer to two possible interpretations of a Markov-kernel μ. When the outcome of a measurement of a given sharp observable is the number λ, then $\mu_\Delta(\lambda)$ can be interpreted as the probability that the actually registered outcome belongs to Δ. Such uncertainty is supposed to depend on the measurement-inaccuracies (which might be unavoidable). Alternatively, the function $\mu_\Delta : \mathbb{R} \to [0,1]$ can be interpreted as a *fuzzy event* (or as a *fuzzy Borel set*).

On this basis, we can now define the concept of *fuzzification* of a given sharp observable.

Definition 17 *Fuzzification*
Let $O : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathcal{H})$ be a sharp observable and let $\mu : \mathcal{B}(\mathbb{R}) \times \mathbb{R} \to [0,1]$ be a Markov-kernel. The observable defined by the integral

$$\mathfrak{O}(\Delta) = \int_{\mathbb{R}} \mu_\Delta(\lambda)\, dO(\lambda) \tag{3}$$

(where $\Delta \in \mathcal{B}(\mathbb{R})$) is called the *fuzzification* of O by means of the Markov-kernel μ.

The following Lemma can be proved:

Lemma 8.1. *The observable $\mathfrak{O}$ defined in Definition 17 is a self-commuting effect-valued measure.*

Fuzzification of sharp observables (via Markov-kernels) provides a technique that allows us to transform any sharp observable into a self-commuting unsharp observable. Interestingly enough, one can prove that every self-commuting unsharp observable is the fuzzification of a sharp one.

Theorem 5 ([5, 24]) *Let $\mathfrak{O}$ be a self-commuting effect-valued measure. Then, there exist a projection-valued measure O and a Markov-kernel μ such that $\mathfrak{O}(\Delta) = \int_{\mathbb{R}} \mu_\Delta(\lambda)\, dO(\lambda)$, for any $\Delta \in \mathcal{B}(\mathbb{R})$.*

The projection-valued measure O corresponding to the effect-valued measure $\mathfrak{O}$ is called the *sharp version* of $\mathfrak{O}$.

Equation (3) suggests two possible interpretations of $\mathfrak{O}$, which depend on the two possible interpretations of a Markov-kernel (mentioned above).

- If μ is interpreted as a description of the inaccuracies of a measurement of O, then $\mathfrak{O}$ can be interpreted as a random version of O.

- If $\mu_\Delta : \mathbb{R} \to [0,1]$ is interpreted as a fuzzy event, then, for every pure state $|\psi\rangle$ of $\mathcal{H}^{\mathbf{S}}$, the real number $\langle\psi|\mathfrak{O}(\Delta)\psi\rangle$ can be interpreted as the probability of the fuzzy event μ_Δ for a system in the pure state $|\psi\rangle$. In other words, the fuzzy event represented by the fuzzy set μ_Δ is the fuzzification of the sharp event represented by the crisp set χ_Δ. The self-commuting unsharp observable $\mathfrak{O}$ gives the probability of the fuzzy events μ_Δ; while its sharp version O gives the probability of the crisp events χ_Δ (for any $\Delta \in \mathcal{B}(\mathbb{R})$).

Another theorem that plays an important role for the connections between sharp and unsharp observables is Naimark's *dilation-theorem.*

Theorem 6 *[3, 7, 26, 31, 32] Let $\mathfrak{O}$ be an effect-valued measure on a Hilbert space $\mathcal{H}$. Then, there exists an extended Hilbert space $\mathcal{H}^+$ (of $\mathcal{H}$) and a projection-valued measure O^+ on $\mathcal{H}^+$ such that for any $\Delta \in \mathcal{B}(\mathbb{R})$ and for any pure state $|\psi\rangle$ of $\mathcal{H}$,*

$$\mathfrak{O}(\Delta)|\psi\rangle = PO^+(\Delta)|\psi\rangle,$$

where P is the projection operator of $\mathcal{H}^+$ whose range is the subspace $\mathcal{H}$.

In other words, Naimark's theorem asserts that unsharp observables can be represented as projections of sharp observables.

The following Corollary, provides an interpretation of families of unsharp events as families of projections of sharp events.

Corollary 8.1. *Let $\{E_n\}_{n\in\mathbb{N}}$ be a family of effects in the Hilbert space $\mathcal{H}$ such that $\sum_n E_n = \mathbb{I}$. There exists an extended Hilbert space $\mathcal{H}^+$ (of $\mathcal{H}$) and a family of projection operators $\{P_n^+\}_{n\in\mathbb{N}}$ in $\mathcal{H}^+$ such that*

1. *$P_i^+ P_j^+ = \mathbb{O}$ (for $i \neq j$);*
2. *$\sum_n P_n^+ = \mathbb{I}$;*
3. *$E_n = PP_n^+P$ (for every E_n).*

Corollary 8.1 asserts that every family of effects that is a resolution of identity (consisting of pairwise orthogonal effects) corresponds to a family of pairwise orthogonal sharp events (projections).

In order to properly understand the meaning of Corollary 8.1 it is expedient to recall that the concept of *orthogonality* behaves differently in the case of sharp events (represented by projections) and in the case of unsharp events (represented by effects)[16]. We know that two projections P_1 and P_2 are *orthogonal* iff $P_1 \sqsubseteq P_2^\perp$, where $^\perp$ is the *orthocomplement-operation* that satisfies the *non-contradiction principle* ($[P \sqcap P^\perp]^\perp = \mathbb{I}$). At the same time, two effects E_1 and E_2 are *orthogonal* iff $E_1 \sqsubseteq E_2' = \mathbb{I} - E_2$, where the operation $'$ is an *involutive complement*, that gives rise to possible violations of the non-contradiction principle. For instance, we have:

$$\left(\frac{1}{2}\mathbb{I} \sqcap (\frac{1}{2}\mathbb{I})'\right)' = \left(\frac{1}{2}\mathbb{I} \sqcap \frac{1}{2}\mathbb{I}\right)' = \left(\frac{1}{2}\mathbb{I}\right)' = \frac{1}{2}\mathbb{I} \neq \mathbb{I}.$$

We have seen that, unlike the case of sharp observables, there are examples of non-commuting unsharp observables that are compatible. A natural question arises: is there any characterization of compatibility for pairs of unsharp observables? A positive answer to this question is given by the following theorem, which is based on Naimark's dilation theorem.

Theorem 7 ([6]) *Two effect-valued measures $\mathfrak{O}_1 : \mathcal{B}(\mathbb{R}) \to \mathcal{E}(\mathcal{H})$ and $\mathfrak{O}_2 : \mathcal{B}(\mathbb{R}) \to \mathcal{E}(\mathcal{H})$ are compatible iff there exist an extended Hilbert space $\mathcal{H}^+$ (of $\mathcal{H}$) and two Naimark dilations $O_1^+ : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathcal{H}^+)$ and $O_2^+ : \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\mathcal{H}^+)$ such that*

$$O_1^+(\Delta)O_2^+(\Delta) = O_2^+(\Delta)O_1^+(\Delta),$$

for any $\Delta \in \mathcal{B}(\mathbb{R})$.

The theorem is illustrated in the following diagram.

$$\begin{tikzcd}
O_1^+ \arrow[r, leftrightarrow, "c"] \arrow[d, leftrightarrow, "P"] & O_2^+ \arrow[d, leftrightarrow, "P"'] \\
\mathfrak{O}_1 \arrow[r, leftrightarrow, "c"] & \mathfrak{O}_2
\end{tikzcd}$$

where the double arrow $\xleftrightarrow{c}$ denotes compatibility, while $\xleftrightarrow{P}$ denotes the relationship between an unsharp observable and its dilation (as expressed by Naimark's theorem). The *compatibility* between the unsharp observables $\mathfrak{O}_1$ and $\mathfrak{O}_2$ corresponds to the *commutativity* of the sharp observables O_1^+ and O_2^+, whose projections coincide with $\mathfrak{O}_1$ and $\mathfrak{O}_2$, respectively.

Example 1 Let $\mathcal{H} = \mathbb{R}^2$ and $\mathcal{H}^+ = \mathbb{R}^3$. Consider the projection operator $P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ that projects the vectors of $\mathbb{R}^3$ into the plane generated by the vectors $(1,1,0)$ and $(0,0,1)$. Consider now the two following operators:

$E_1 = \frac{1}{4}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \frac{1}{8}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$

One can show that E_1 and E_2 are effects in the space $\mathbb{R}^2$. For, they can be represented by the two following matrices in the orthonormal basis
$\mathfrak{B} = \{e_1 = (0,0,1), e_2 = \frac{1}{\sqrt{2}}(1,1,0), e_3 = \frac{1}{\sqrt{2}}(1,-1,0)\}$. We have:

$$E_1^{\mathfrak{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2^{\mathfrak{B}} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4\sqrt{2}} & 0 \\ \frac{1}{2\sqrt{2}} & \frac{1}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

corresponding to the 2×2 matrices $E_1^{\mathfrak{B}} \equiv \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad E_2^{\mathfrak{B}} \equiv \begin{pmatrix} \frac{1}{4} & \frac{1}{4\sqrt{2}} \\ \sqrt{2} & \frac{1}{4} \end{pmatrix}.$

The effects E_1 and E_2 do not commute. At the same time they are compatible. Because:

- they are the projections of the two following sharp events:
$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (respectively).
We have: $E_1 = PP_1P$ and $E_2 = PP_2P$.
- The two sharp events P_1 and P_2 commute.

9 Heisenberg's principle

Consider the sharp observables $O^{\mathbf{Q}}$ and $O^{\mathbf{P}}$ corresponding to the physical quantities *position* and *momentum*, respectively. According to Heisenberg's principle there are events $O^{\mathbf{Q}}(\Delta_q)$ and $O^{\mathbf{P}}(\Delta_p)$ that cannot be simultaneously *verified with certainty* by any quantum system $\mathbf{S}$. In particular, as a consequence of Paley-Wiener theorem, for any pair of Borel sets Δ_q and Δ_p of finite Lebesgue measure, we have ([12]):

$$O^{\mathbf{Q}}(\Delta_q) \sqcap O^{\mathbf{P}}(\Delta_p) = \mathbb{O}.$$

Thus, there is no state ρ of $\mathbf{S}$ (in the space $\mathcal{H}^{\mathbf{S}}$) such that

$$\mathrm{tr}(\rho O^{\mathbf{Q}}(\Delta_q)) = \mathrm{tr}(\rho O^{\mathbf{P}}(\Delta_p)) = 1.$$

Consequently, one immediately obtains that a *joint observable* $O^{\mathbf{QP}}$ of $O^{\mathbf{Q}}$ and $O^{\mathbf{P}}$ cannot exist. In fact, should a joint observable $O^{\mathbf{QP}}$ exist, we would have:

$$O^{\mathbf{QP}} : \mathcal{B}(\mathbb{R} \times \mathbb{R}) \rightarrow \mathcal{P}(\mathcal{H}^{\mathbf{S}}).$$

Hence, $O^{\mathbf{QP}}(\Delta_q \times \Delta_p)$ would be a sharp event. And we know that for any sharp event represented by a projection P (different from the null projection) there exists a pure state ρ such that:

$$\mathrm{tr}(\rho P) = 1.$$

Thus, we would obtain that for some state ρ:

$$\mathrm{tr}(\rho O^{\mathbf{QP}}(\Delta_q \times \Delta_p)) = 1 = \mathrm{tr}(\rho O^{\mathbf{Q}}(\Delta_q)) = \mathrm{tr}(\rho O^{\mathbf{P}}(\Delta_p)),$$

against Heisenberg's principle!

As is well known, the idea that a quantum particle cannot simultaneously have a definite position and a definite momentum is a basic principle of the *Copenhagen-interpretation* of quantum theory[11]. But Heisenberg went beyond the formalism, proposing his celebrated *microscope-thought experiment* [21]. He suggested that approximate versions of *position* and *momentum* could be sequentially measured, giving an approximate information, according to the uncertainty-relation:

$$Var_\rho(\mathbf{Q}) Var_\rho(\mathbf{P}) \geq \frac{1}{4}$$

[11] There are exceptions to this property in the case of unbounded sets. See [13]

(where $\hbar$ is normalized to 1).

Interestingly enough, the unsharp quantum formalism turns out to be particularly efficient for representing such approximate measurements of $\mathbf{Q}$ and $\mathbf{P}$. As we have seen, there is a strong relationship between self-commuting unsharp observables and sharp observables: the former can be represented as *fuzzy versions* of the latter, via Markov-kernels.

A relevant result in the unsharp formulation of quantum mechanics is the existence of *fuzzy versions* of $\mathbf{Q}$ and $\mathbf{P}$ that are compatible [12, 17, 23, 29]. As is well known, the sharp observables *position* ($\mathbf{Q}$) and *momentum* ($\mathbf{P}$) are defined in the Hilbert space $L^2(\mathbb{R})$ of square integrable functions. We have:

$$[\mathbf{Q}\psi](x) = x\psi(x) \text{ and } [\mathbf{P}\psi](x) = -i\frac{\partial \psi}{\partial x}, \quad \textit{for almost all } x.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $\int_{-\infty}^{\infty} g(x)\,dx = 1$ and let $\hat{g}$ denote its Fourier transform. Consider the two following effect-valued measures [17]:

$$\mathfrak{O}^{\mathbf{Q}}(\Delta_q) = \int \mu^{\mathbf{Q}}_{\Delta_q}(x)\,dO^{\mathbf{Q}}(x), \text{ where } \mu^{\mathbf{Q}}_{\Delta_q}(x) = \int_{-\infty}^{\infty} \chi_{\Delta_q}(x-y)\,g(y)\,dy;$$
$$\mathfrak{O}^{\mathbf{P}}(\Delta_p) = \int \mu^{\mathbf{P}}_{\Delta_p}(x)\,dO^{\mathbf{P}}(x), \text{ where } \mu^{\mathbf{P}}_{\Delta_p}(x) = \int_{-\infty}^{\infty} \chi_{\Delta_p}(x-y)\,\hat{g}(y)\,dy$$

The observables $\mathfrak{O}^{\mathbf{Q}}$ and $\mathfrak{O}^{\mathbf{P}}$ turn out to be self-commuting unsharp versions of $\mathbf{Q}$ and $\mathbf{P}$ (respectively), where the *fuzzification* is realized by the Markov-kernels $\mu^{\mathbf{Q}}$ and $\mu^{\mathbf{P}}$. Notice that that $\mathfrak{O}^{\mathbf{Q}}$ and $\mathfrak{O}^{\mathbf{P}}$ are compatible (since they admit a joint observable [5, 12, 17]), although they do not commute. Moreover, for every pure state ρ of $\mathcal{H}^{\mathbf{S}}$, we have:

$$Var_\rho(\mathfrak{O}^{\mathbf{Q}}) = Var_\rho(\mathbf{Q}) + Var(g); \tag{4}$$
$$Var_\rho(\mathfrak{O}^{\mathbf{P}}) = Var_\rho(\mathbf{P}) + Var(\hat{g}). \tag{5}$$

As a consequence, for any state ρ of $\mathcal{H}^{\mathbf{S}}$, we obtain[12]:

$$Var_\rho(\mathfrak{O}^{\mathbf{Q}})Var_\rho(\mathfrak{O}^{\mathbf{P}}) \geq 1. \tag{6}$$

(where $\hbar$ is normalized to 1).

It is worth-while remarking that (6) is stronger than the usual version of the uncertainty relation for sharp position and momentum [12], according to which:

$$Var_\rho(\mathbf{Q})Var_\rho(\mathbf{P}) \geq \frac{1}{4}.$$

Apparently, the uncertainty principle, that currently appears in most physical textbook, refers to *preparation-uncertainties* an not to *measurement-uncertainties* (as happens in the case of Heisenberg's thought experiment).[12] Moreover, equations (4) and (5) show ([12]) that the function g encodes our information about the question: how do the unsharp observables $\mathfrak{O}^{\mathbf{Q}}$ and $\mathfrak{O}^{\mathbf{P}}$ approximate $\mathbf{Q}$ and $\mathbf{P}$ (respectively)?

How to interpret the joint measurability of $\mathfrak{O}^{\mathbf{Q}}$ and $\mathfrak{O}^{\mathbf{P}}$ in the framework of unsharp quantum theory? We know that (unlike the case of sharp events) there are unsharp events E (different from the null event $\mathbb{O}$) such that $\mathrm{tr}(\rho E) < 1$ for every state ρ. This is exactly what happens in the case of some unsharp events $\mathfrak{O}^{\mathbf{QP}}(\Delta_q \times \Delta_p)$ (where $\mathfrak{O}^{\mathbf{QP}}$ is the joint observable of $\mathfrak{O}^{\mathbf{Q}}$ and $\mathfrak{O}^{\mathbf{P}}$), whose probability is less than 1 for every state ρ.[13] One is dealing with special examples of unsharp events that cannot be verified with certainty by any quantum state. In this framework, the number $\mathrm{tr}[\rho\, \mathfrak{O}^{\mathbf{QP}}(\Delta_q \times \Delta_p)]$ represents the probability that a simultaneous measurement of the fuzzy position and of the fuzzy momentum gives a result in $\Delta_q \times \Delta_p$. Notice that such information would be forbidden in the sharp formulation of quantum theory, where Heisenberg's thought experiment only plays a heuristic role, that cannot be "expressed" in the language of the theory. One is dealing with *meta-linguistic* assertions, that become *linguistic* only in the framework of unsharp quantum theory (where Heisenberg's thought experiment can be formulated in a quantitative way, while an optimal approximate joint measurement of position and momentum can be determined [14]).[14]

The study of the relationships between sharp and unsharp quantum concepts have naturally inspired some deep discussions about foundational and philosophical problems of quantum theory. At first sight, Naimark's theorem and its consequences seem to suggest a kind of "supremacy" of the sharp approach. One might conjecture that, from a theoretic point of view, all fundamental uncertainties that characterize the quantum world can be entirely represented in the framework of sharp quantum theory. At the same time, *quantum unsharpness* might be described as a non-fundamental aspect, which is essentially due to the unavoidable limits and approximations of measurement-procedures. However, such a conclusion seems to be an over-simplification. As we have seen, Naimark's theorem guarantees that any unsharp observable of a Hilbert space $\mathcal{H}^{\mathbf{S}}$ (associated to a quantum system $\mathbf{S}$) can be represented as a sharp observable living in a larger space $\mathcal{H}^{\mathbf{S}^+}$. One can naturally ask the following question: given a set of unsharp observables of a Hilbert space $\mathcal{H}^{\mathbf{S}}$, is there any *unique* Hilbert space $\mathcal{H}^{\mathbf{S}^+}$, where the elements of this set can be simultaneously represented as sharp observables ? This question admits different answers, which depend on the choice of the quantum observables we are taking into consideration. Suppose we are only considering the unsharp versions of the observables that play an essential role in the "concrete " quantum mechanics that is currently used by working physicists; one is dealing with the set of all observables that can be defined as functions of the two basic observables *position* and *momentum*. In such a case one can prove that all unsharp observables belonging to this set admit sharp representatives, all living in a unique Hilbert space.[8] However, these sharp observables turn out to be pairwise compatible! Thus, the result is a formulation of classical mechanics

[12] For more details, see [4, 12, 14, 23, 28].
[13] See Theorem 3.4 in Ref. [10].
[14] These questions can be discussed in a more precise way, by introducing the concept of *instrument*. See [12, 14, 27].

in a Hilbert-space language [8, 9]. Of course, such a formulation cannot be regarded as a satisfactory sharp representation of a "fictitious" unsharp quantum world. As we have seen, one of the main features of unsharpness is that, in some cases, unsharp versions of incompatible sharp observables turn out to be compatible, allowing approximate simultaneous measurements of incompatible sharp observables. This is precisely the content of Heisenberg's thought experiment, which can be only formalized in the framework of unsharp quantum theory. We recall that, according to an *operational approach*, every physical theory can be described as consisting of two parts: the *theoretic formalism* and the *correspondence-rules* that connect the formalism with the experiments. In this perspective, the unsharp formalism seems to represent (as far as we know) the best tool that allows us to connect the quantum-theoretic formalism with the experimental world. Other deep motivations in favour of unsharp observables have been suggested by *fuzzy quantum logics*, which still represent an open research-field [15, 30].

Suppose, instead, we are referring to the set of *all* possible unsharp observables of a quantum system $\mathbf{S}$ (with associated Hilbert space $\mathcal{H}^{\mathbf{S}}$). In such a case our question (*is there any* unique *Hilbert space* $\mathcal{H}^{\mathbf{S}^+}$*, where* all *unsharp observables of* $\mathbf{S}$ *can be simultaneously represented as sharp observables ?*) has a negative answer: *joint measurability* in the unsharp world cannot be fully represented in the sharp world. A counterexample is given by the case of three unsharp observables $\mathfrak{O}_1$, $\mathfrak{O}_2$, $\mathfrak{O}_3$ (which are pairwise compatible in a Hilbert space $\mathcal{H}^{\mathbf{S}}$), that do not admit any sharp pairwise compatible Naimark-dilations O_1^+, O_2^+, O_3^+, living in a unique extended Hilbert space $\mathcal{H}^{\mathbf{S}^+}$ [20, 22].

Apparently, foundational and philosophical discussions about the relationships between quantum sharpness and quantum unsharpness and about the theoretic role of unsharp concepts turn out to be strongly connected with these important mathematical results.

Acknowledgments

R. G. and G. S. are partially supported by the project PRIN-PNRR "Quantum Models for Logic, Computation and Natural Processes (Qm4Np)", funded by Italian Ministry of University and Research (code: F53D23011170001). R. G. and G. S. are partially supported by the project PRIN2022 "CORTEX The COst of Reasoning: Theory and EXperiments", funded by Italian Ministry of University and Research (code: F53D23004940006).

R. G. is partially funded by the TÜV SÜD Foundation, the Federal Ministry of Education and Research (BMBF) and the Free State of Bavaria under the Excellence Strategy of the Federal Government and the Länder, as well as by the Technical University of Munich-Institute for Advanced Study.

R.B.: The present work was realized under the auspicies of the Istituto Nazionale di Alta Matematica INDAM

References

[1] Diederik Aerts, *Why the Disjunction in Quantum Logic is Not Classical*, Foundations of Physics, 30 (9) (2000), 1473-1480.

[2] Diederik Aerts, Massimiliano Sassoli de Bianchi, Sandro Sozzo, Tomas Veloz, *Modelling Human Decision Making: An overview of the Brussels Quantum*

Aprroach, Foundations of Science, 26 (2021) 27-54.

[3] Akhiezer N.I., Glazman I.M., *Theory of Linear operators in Hilbert Spaces*, Dover Publication inc., New York, 1993.

[4] Beltrametti E., Cassinelli, G., *The Logic of Quantum Mechanics*, Encyclopedia of Mathematics and it Applications, vol. 15, Addison-Wesley Publishing Company, 1981.

[5] Beneduci R., *Joint measurability through Naimark's dilation theorem*, Rep. Math. Phys. 79 (2017), 197-213.

[6] Beneduci R., *Positive Operator Valued Measures and Feller Markov Kernels*, J. Math. Anal. Appl. 442 (2016), 50-71.

[7] Beneduci R., *Notes on Naimark's dilation theorem*, J. Phys.: Conf. Ser. 1638, 012006 (2020).

[8] Beneduci R., Brooke J., Curran R., Schroeck F.E., *Classical Mechanics in Hilbert Spaces, part I*, Int. J. Theor. Phys. 50 (2011), 3682-3696.

[9] Beneduci R., Brooke J., Curran R., Schroeck F.E., *Classical Mechanics in Hilbert Spaces, part II*, Int. J. Theor. Phys. 50 (2011), 3697-3723.

[10] Beneduci R., Schroeck F.E., *A note on the relationship between localization and the norm-1 property*, J. Phys. A 46 305303 (2013).

[11] Birkhoff, G., von Neumann, J., *The Logic of Quantum Mechanics*, Annals of Mathmatics 37 (1936), 823–843.

[12] Busch P., Grabowski M., Lahti P., *Operational Quantum Physics*, Lecture Notes in Physics 31, Springer, 1995.

[13] Busch P., Lahti P., *To what extent do position and momentum commute?*, J.Phys. A 115 (1986), 259-264.

[14] Busch P., Lahti P., Pellonpaa J. P., Ylinen K, *Quantum measurement*, Springer, 2016.

[15] Dalla Chiara, M.L., Giuntini, R., Greechie, R., *Reasoning in Quantum Theory. Sharp and Unsharp Quantum Logics*, Kluwer, Dordrecht, 2004.

[16] Dalla Chiara, M.L., Giuntini, R., Leporini, R., Sergioli, S., *Quantum Computation and Logic. How Quantum Computers have inspired Logical Investigations*, Springer, 2018.

[17] Davies E.B., *Quantum mechanics of Open Systems*, Academic Press, London, 1976.

[18] Dvurečenskij, A., Pulmannová, S., *New Trends in Quantum Structures*, Kluwer, Dordrecht, 2000.

[19] Giuntini, R., *Quantum Logic and Hidden Variables*, Bibliographisches Institut, Mannheim, 1991.

[20] Heinosaari T., Reitzner D., Stano P., *Notes on Joint Measurability of Quantum Observables*, Found. Phys. 38 (2008) 1133-1147.

[21] Heisenberg, W., Zeitschr. Phys. 43, 172 (1927).

[22] C. Heunen, T. Fritz , M.L. Reyes, *Quantum theory realizes all joint measurability graphs*, Phys. Rev. A 89, 032121 (2014).

[23] Holevo A.S., *Probabilistic and statistical aspects of quantum theory*, North Holland, Amsterdam, 1982.

[24] Jenčová A., Pulmannová S., *Characterizations of Commutative POV Measures*, Found. Phys. 39 (2009), 613-624.

[25] Ludwig G., *Foundations of Quantum Mechanics I*, Springer, New York, 1983.

[26] Naimark M.A., *Spectral functions of symmetric operators,* IZv. Akad. Nauk SSSR, Ser. Mat. 4 (1940), 277-318.

[27] Ozawa M., *Quantum measuring processes of continuous observables*, J. Math. Phys. 25 (1984), 79-87.

[28] Ozawa M. *Uncertainty relations for joint measurements of noncommuting observables*, Phys. Lett. A 320 (2004), 367-374.

[29] Prugovečki E., *Stochastic Quantum Mechanics and Quantum Spacetime*, Reidel Publishing Company, Dordrecht, 1984.

[30] Pykacz J., *Quantum Physics, Fuzzy Sets, and Logic*, Springer, 2015.

[31] Riesz F., Nagy B.S., *Functional Analysis*, Dover, New York, 1990.

[32] Schroeck F.E., *Quantum Mechanics on Phase Space*, Kluwer Academic Publishers, Dordrecht, 1996.