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**Modelling Masonry as a Continuum:
Finite Element Analysis using ABAQUS**

ICAR/08 STRUCTURAL MECHANICS

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INTRODUCTION

MOTIVATION

Most of human’s history, heritage, culture and identity are associated with masonry structures that have stood against environmental and anthropogenic challenges. These structures whether they remain in good shape or are now in ruins, a state reached after nothing less than hundreds or thousands of years, have proven to survive the test of time. Engineers aim to ensure that masonry structures remain safe, operational for public or personal use, and meet performance standards throughout their lifespan. The analyses carried out by engineers play a crucial role in the design, assessment, and rehabilitation of masonry buildings and infrastructure.

The structural engineer must always bear in mind his primary duty: establishing a strong and satisfactory state of equilibrium as the cornerstone for his design. Occasionally, a gentle reminder is necessary, as the engineer's thinking may become overly influenced by conventional elastic theory, leading to potential oversight of calculation limitations. While elastic theory can provide a plausible equilibrium state for a structure, it represents merely one among countless possibilities for any statically indeterminate structure (*Heyman, 1982*). The decisions taken based on the results of the engineer’s analyses are of great importance as they specify whether or not remedial actions are to be taken, what type of intervention and to what extent (*Zerlenga et al., 2021*). When approaching such structures it is important to safeguard their architecture and their structural integrity taking into consideration the historical, geographical, spatial and artistic identity. Hence, making decisions that disregard alternative states and exclusively concentrate on a single assumed "true state" of equilibrium may lead to interventions that have a predisposition to transform masonry into something it inherently is not, a monolithic structure capable of withstanding both tensions and compressions (*Huerta, 2008*). The complexity of the process requires different expertise to safeguard the structure specifically since challenges arise for they differ one case from the other (*Cusano et al., 2021*). As engineers strive to ensure the safety and longevity of masonry structures, the choice of analysis methods becomes paramount.

Jaques Heyman's insights on elastic theory echo with the broader challenges faced by engineers. Elastic theory, often based on assumptions of unknown validity, can lead to potential oversights, especially when applied to structures with historical, architectural, and material complexities. In his book *“The Masonry Arch”* (Heyman, 1982), Heyman states the following:

“...Hidden in elastic theory, and in computer packages or finite element analyses based on elastic theory, are ideas of compatibility of deformation. The engineer is forced to supply information, for example, about the conditions at the abutments of an arch, before an elastic analysis can be made to work at all. Now, as is noted in Chapter 2, the engineer does not actually know the conditions at the abutments, nor does he know whether the structure was forced together during construction, so that it is stressed even in the absence of load. He will state, for the purpose of working his computer program, that the structure is initially stress-free, and that the abutments are rigid (or that they have a certain elastic response); but these are assumptions of unknown validity, which can nevertheless have a marked effect on the results produced by the computer.

The analysis of arches by plastic theory shows that their safety does not in fact depend on the precise conditions at the abutments, nor on whether the arches were built originally to fit perfectly within those abutments. The designer does not need information about these geometrical imperfections; he has an alternative method of analysis, which is always, by the theorems of plasticity, safe. And that method can be applied as well to the iron as to the masonry arch....”

The means for assessing tensile cracks have evolved significantly, tracing back to Galileo Galilei's (Galilei, 1638) reflections in the XVII century and A. A. Griffith's original fracture mechanics theory in 1921 (Griffith, 1921). Looking back, in the late 1950s and early 1960s, Barenblatt (Barenblatt, 1959) and Dugdale (Dugdale, 1960) brought in significant ideas by introducing cohesive forces in the crack tip region. Barenblatt stuck to elasticity theory, while Dugdale assumed an elastic-perfectly plastic material behaviour. These contributions aimed to bridge the gap between Solid Mechanics theories: fracture mechanics (FM) and continuum mechanics (CM). With the arrival of digital computers and the Finite Element Model (FEM) in the 1960s, engineers started using these tools for solutions in Structural and Solid Mechanics.

Over time, these programs, originally developed to analyse continuous media such as steel and concrete, are now being extended to encompass quasi-brittle materials, among which masonry is classified.

Certainly, the robust nature of Finite Element (FE) programs is undeniable, and there is a growing trend among engineers to expand their application to masonry structures, particularly for dynamic analyses. In this endeavour, engineers are supposedly expanding the FE models based on Limit Analysis theory, which will be extensively explored in the State-of-the-Art section.

However, there's lingering suspicion within the academic world due to the fact that the accurate representation of cracks poses challenges in computational programs. For instance, FE models such as the Damage Plasticity Model exhibits mesh sensitivity, numerical instabilities, and limitations in capturing complex crack patterns and mixed-mode loading. On the other hand, the Smeared Crack Model faces volume sensitivity, mesh-bias dependence, and struggles with handling strain softening and sharp displacement gradients. This suspicion has fuelled efforts over the last 40 years to investigate and address the drawbacks of these approaches in concrete, casting doubts about their broader applicability.

If there exists concerns about these approaches being suitable for concrete (Lee & Fenves, 1998), those concerns intensify when extended to masonry. Masonry brings in a whole set of complications, like the size of the representative volume elements, the setting of many unknown material parameters and complex boundary conditions, which raises significant challenges. Therefore, trying to use these models for masonry materials faces even more uncertainty, thus amplifying the reservations within the academic community about the feasibility and accuracy of such extensions.

In this dissertation, a new model based on a perfectly plastic material models which allows the adoption of Limit Analysis is introduced and validated while also highlighting its limitations in its current state. Additionally, this dissertation discusses how one can use sensibly existing FEM to analyse masonry structures depending on the desired outcome of the analysis, while ensuring a sound understanding of masonry's behaviour.

THESIS
STRUCTURE

Considering the uncertain nature of the aforementioned approaches, particularly in the context of masonry, the present dissertation aims to contribute to the ongoing debate by exploring alternative methods.

Specifically, it focuses on leveraging ABAQUS (*Abaqus*, 2023), currently recognized as one of the most advanced FE programs for structural analysis under large displacements, to delve into the analyses of masonry structures in FE environment. An associate plastic Mohr-Coulomb (MC) type material is used to model masonry as a continuum. Extreme values of stiffness, cohesion and friction are adopted to mirror Normal Elastic No-tension (NENT) and Normal Rigid No-Tension (NRNT) models. The MC model is first validated and calibrated against established benchmark results by conducting numerical simulations with the finite element program ABAQUS. The Mohr-Coulomb Failure model implemented in ABAQUS is examined in the subsequent chapters, and compared with a widely used damage plasticity model (CDP) also available in ABAQUS.

This dissertation not only presents the application of these methodologies but also critically evaluates their validity and the accuracy of the results obtained. Starting from the comparative analysis of some simple benchmark problems, more complex equilibrium cases are considered. In the final part the dynamic behaviour analysis, using both methods to compare and evaluate their results and assess their limitations is specifically investigated. The objective of these analyses is to strike a balance between computational efficiency and the present uncertainties when it comes to analysing masonry structures through FEM, aiming to contribute to the development of reliable methods for the assessment and design of these structures, as well as contributing to more accurate and responsible assessment and analyses of results.

The “State of the Art” section will begin with an extensive review of masonry mechanics, the existing approaches, its tools for analyses and the modelling strategies. This will be followed by a detailed presentation of the mathematical framework of the benchmark reference approaches used in this thesis.

Subsequently, “Part A” contains an introduction of the implementation of the

MC model in ABAQUS and its use to mirror the NRNT and NENT masonry like material. Validation benchmarks will then be conducted to confirm the method's efficacy to analyse masonry as a continuum. This section will also demonstrate its applicability in comparison to previously introduced reference approaches through two case studies, the Torre del Piano and the Aqueduct de los Cinco Ojos each undergoing different type of analyses.

“Part B” presents a full-on analysis of the vulnerability of a selected monument, within my country Lebanon, namely the Hammam of Beit El-Din. This will be studied under a dynamic time history non-linear analysis to examine the response of its complex structure covered with domes and vaults. Building on the insights gained from earlier sections, the choice of finite element model will be carefully selected based on the analysis's objectives and computational efficiency considerations.

Finally, a “Conclusive Remarks” section summarizes the result of the work and sheds light on further aspects to explore.

STATE OF THE ART

1. STRUCTURAL THEORY OF MASONRY

The process of assembling diverse elements to form a structure is known as building, with masonry standing out as the most effective and vital component in any construction. Masonry's ecological qualities, such as being a regenerative material with minimal environmental impact having the ability to regulate the environment, renews interest in it as a suitable material for sustainable construction amid contemporary concerns for eco-friendly architecture (Almssad et al., 2022). Recently, structural engineers and architectural firms have been increasingly exploring the use of masonry in structural design. due to masonry's ability to be reused and its inherent thermal mass properties. This renewed interest is driven by masonry's potential for sustainability, allowing for the incorporation of recycled materials, demolition waste, and materials with low embodied carbon. Worldwide in construction, masonry structures are known for their exceptional durability. This fact is evident since the majority of buildings and important architectural monuments have surpassed the test of time. Consequently, there is an increasing need for assessments to guarantee the safety and integrity of these structures, along with incorporating their principles into future designs for sustainable construction. Having established the importance of masonry in sustainable construction, it is now evident the attention given nowadays to its structural behaviour, safety and stability. This opens way to discuss the challenges faced in the structural analysis of these structures.

Generally, in structural analysis, the objective is to understand how a structure responds to an external system of loads and its interaction with the environment. This involves understanding the internal forces that must be in equilibrium with the external forces. These internal force resultants induce deformations, which can be calculated based on material properties. This is the focus of strength of materials, which aims at assessing the global behaviour by starting from the local material response. Local deformations are related to structural displacements, and, in equilibrium, we can satisfy three different types of conditions: static balance, material properties (constitutive equations), and deformation-displacement compatibility. However, when considering masonry structures, while the loads can be precisely defined quite precisely, material properties are difficult to be established in detail, the boundary conditions of the structure are often unknown, material properties as well are difficult to gather and the actual path followed by the loads due to the unknown construction history and the associated sequence of forcing among parts remains uncertain. This leads to a high degree of uncertainty in the solution of these usually abundantly hyperstatic problems which evidently present a challenge that requires a different approach. How can such problems be analysed?

For masonry structures, stability is primarily ensured by their form and only secondarily by the strength of the material they are made of. Masonry is a well-ordered piling of bricks or stones, according to different methods called rules of art and for which stereotomy (that is the way the bricks are cut and assembled) plays a fundamental role, where the joints can either be filled with mortar or of dry contact. Stereotomy significantly influences the structural behavior of masonry and plays a crucial role in distributing forces three-dimensionally and preventing sliding. The correct inclination of the cuts between voussoirs, along with staggered patterns, contributes to the overall stability and performance of masonry structures. This material is anisotropic, composite and discontinuous as opposite to today's materials, whose properties are so uniform to make it possible to model them as homogeneous and an isotropic continuum. One may wonder how, from a structural point of view, we can characterize masonry materials in such a way that their properties become common to all masonry. A not so easy problem as generally a masonry structure is a mixture of several types of masonries such as an ashlar façade with a regular masonry facing and a rubble core (Block et al., 2006; Huerta, 2004). After thorough and numerous tests through the 19th century up to the 20th century, attempts to quantify the strength were generally unsuccessful but, as strength failures are rare, and masonry has the ability to resist compressive stresses which are much higher than those usually experienced by walls, it is very weak in tension and mostly sliding is prevented due to the high frictions between the blocks and to masonry's stereotomy, a very simple theory for masonry for which there are not any material parameters needed, emerged. These are in effect the three necessary statements that allows the incorporation of arch theory into the framework of plasticity theory, a work done by Heyman.

In this respect, Couplet had a pivotal contribution to this theory in his second memoir of 1730. Within its introductory section, Couplet highlights two pivotal assumptions regarding material behaviour. Specifically, he observes that, in practical application, friction effectively secures the voussoirs preventing their displacement while simultaneously permitting the separation between them without resistance. It should be noted that, in this theory, there is no reference to the fundamental strength characteristics of the stone of which the voussoirs are made. Consequently, he presumes that ambient stresses remain at minimal levels, thus downgrade the significance of crushing strength to an inferior role (Heyman, 1982).

Building upon the understanding of masonry's structural characteristics, we now turn our attention to the diverse arrangements and elements that

constitute these structures. There exists a huge array of potential arrangements of elements to form a masonry structure. This combination involves an assortment of units like bricks, blocks, ashlar, adobes, irregular stones, and more, all interconnected with mortar or dry assembled. This pliable paste, including materials like clay, bitumen, chalk, lime/cement-based mortar, or others, plays a crucial role in bringing these diverse elements together. However, in historical masonry structures, the influence of mortar is often less significant due to the deterioration of its mechanical properties over time. Thus, we can easily imagine that the accurate knowledge and mechanical characterization of a particular masonry is then connected to the in-situ material proprieties, the quality of the mortar, the workmanship and the phases of construction and assembly.

Given the context provided earlier, it is important to acknowledge that there might not be a one-size-fits-all solution for masonry structures. The possibility to group this large array of combination of masonry constructions into a single material class for a unified and general approach poses great challenge. Another option involves adopting a case-specific model for each individual structure. Hence, the chosen approach for analysing a masonry structure may need to be justified based on the unique characteristics of each case and the desired aim of the analysis.

This dissertation delves into two primary philosophies for modelling masonry. The first philosophy is based on the unilateral model approach, while the second considers complex constitutive models based on sophisticated stress-strain laws.

1.1. MECHANICS OF MASONRY

In the past, master masons were able to construct buildings without relying on tests for construction materials. However, in the contemporary context, compared to ancient times when metaphysics dominated physics (Angelillo et al., 2014), the situation has evolved, and no building project can proceed without an experimental assessment of material performances. Hence, to proceed with structural analyses, both the unilateral model and the complex constitutive model rely on codes and guidelines for the mechanical properties of masonry. The key distinction lies in the number of unknown parameters required for each approach, a subject that will be further investigated in the subsequent sections of this dissertation.

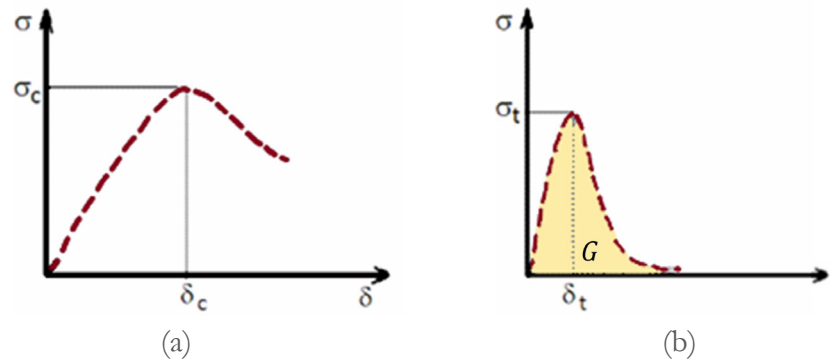
As previously defined, masonry is a heterogeneous assemblage of units and joints, hence it allows for detailed tests on component materials, masonry assemblies, and small masonry structures. Nevertheless, the accurate knowledge and mechanical characterization of a particular masonry are strictly connected to the in-situ material properties, which in some cases is not possible to know..

In the Italian code NTC 2018 (NTC, 2018) the properties of masonry are made dependent on the level of knowledge, specified as LC, linked to a Confidence Factor(FC). For what concerns the Physical and mechanical properties of masonry, Table C8.5.I of the explanatory notes (CNT, 2019) outlines the main parameters of different types of coarse masonry with poor mortar, as detailed in Table 1.

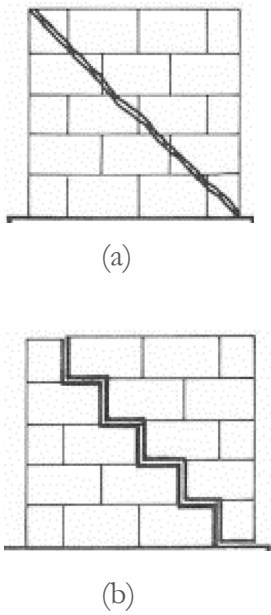
TIPOLOGIA DI MURATURA	F (N/mm ²)	To (N/mm ²)	fvo (N/mm ²)	E (N/mm ²)	g (N/mm ²)	W (kN/m ³)
	min-max	min-max		min-max	min-max	
muratura in pietrame disordinata (ciottoli, pietre erratiche e irregolari)	1,0-2,0	0,018-0,032	- -	690-1050	230-350	19
muratura a conci sbazzati, con paramenti di spessore disomogeneo (*)	2,0	0,035-0,051	- -	1020-1440	340-480	20
muratura in pietre a spacco con buona tessitura	2,6-3,8	0,056-0,074	- -	1500-1980	500-660	21
muratura irregolare di pietra tenera (tufo, calcarenite, ecc.,)	1,4-2,2	0,028-0,042	- -	900-1260	300-420	13 16(**)
muratura a conci regolari di pietra tenera (tufo, calcarenite, ecc.,) (**)	2,0-3,2	0,04-0,08	0,10-0,19	1200-1620	400-500	
muratura a blocchi lapidei squadri	5,8-8,2	0,09-0,12	0,18-0,28	2400-3300	800-1100	22
muratura in mattoni pieni e malta di calce (***)	2,6-4,3	0,05-0,13	0,13-0,27	1200-1800	400-600	18
muratura in mattoni semipieni con malta cementizia (es.: doppio uni foratura <40%)	5,0-8,0	0,08-0,17	0,20-0,36	3500-5600	875-1400	15

[Table 1]
Mechanical properties of various types of masonry according to Table C8.5.I of the explanatory notes of code (NTC, 2018). F is the average compressive strength, To is the average shear strength in the absence of normal stresses, fv0 is the average shear strength in the absence of normal stresses, E is the average value of the normal modulus of elasticity, g is the average value of the tangential modulus of elasticity and W is the average specific weight.

Figure 1 illustrates the typical stress-displacement plots corresponding to simple compression and tension load tests on masonry walls. The symbol G denotes the tensile fracture energy, also known as toughness. It is the surface energy per unit area required to open a crack in elongation.



[Figure 1]
(a) Qualitative stress-displacement diagram for a masonry wall in compression where σ_c denotes the compressive strength of the wall, whilst G represents the tensile fracture energy (toughness) (b) Qualitative stress-displacement diagram for a masonry wall in tension, where σ_t denotes the tensile strength.



[Figure 2]
Fracture through detachment: (a) involving bricks (b) along the joint.

1.1.1. MASONRY FAILURE MODES

In this section, an examination of failure modes of masonry structures is presented. To ensure structural integrity, safety, and longevity of masonry structures, an understanding of their local and global failure modes is important. Local failure modes focus on the behaviour of individual components (units and mortar), while global failure modes address the overall stability of the entire structure.

a. Local Failure Modes

The three main local failure modes of masonry structures are:

- 1- Detachment between parts,
- 2- In plane shear mechanisms, where both detachment and sliding are involved,
- 3- Crushing under compression.

Generally, the first failure mode does not cause concern since it is usually the result of masonry accommodating to some changes in boundary conditions. In fact, low tensile strength of masonry causes a visible detachment between two adjacent and intact parts, see Figure 2. These detachments occur passing through the bricks or along the joints depending on their relative strengths as can be seen in Figure 3. On the other hand, where in-plane shear mechanisms failure mode is involved, it is generally due to high compressive loads and shears. In fact, it is when the load is close to become a collapse load Figure 4. As for the crushing failure it occurs when the compressive stress is close to the compressive strength of the material. Nevertheless, though this failure mode is the least frequent, it is the most dangerous Figure 5 (Angelillo et al., 2014).



[Figure 3]
Detachment failure examples.



[Figure 4]
Example of detachment and sliding failure.



[Figure 5]
Example of crushing of the material, photo to the right from (Angelillo et al., 2014).

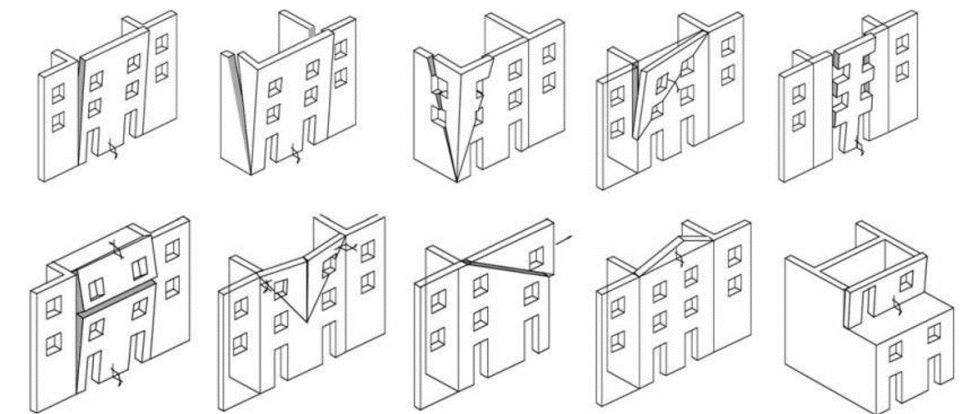
b. Global Failure Modes

Global failure modes can be classified as:

- 1- Out of plane failure
- 2- In plane failure

Generally, these failure modes can lead to the collapse of a masonry structure or a portion thereof. The more frequent one, particularly under seismic loads, is out-of-plane rocking, depicted in Figure 6.

This mechanism occurs because of the self-load or due to the pushing of the roof or the impact of a heavy floor or ceiling. Both crushing failures, previously discussed in Figure 5, and out-of-plane rocking typically result from inadequate design or imprudent modifications to the original construction. To prevent out-of-plane rocking, many regulations specify the maximum distance between two consecutive transverse walls. The demolition of such transverse walls serves as a common example of risky modifications. The third failure mechanism, in-plane shear, is characteristic of well-designed buildings, where structures support horizontal actions through the coordinated resistance of shear-resistant elements (Figure 7). This involves local failure modes of masonry units in their own planes, similar to those shown previously (Angelillo et al., 2014).



[Figure 6]
Example of out-of-plane failure mechanisms (D'Ayala & Speranza, 2002).



[Figure 7]
Example of in-plane failure mechanisms
(Dhakal, 2011).

1.1.2. TWO PHILOSOPHIES ABOUT MASONRY

Two analysis approaches/philosophies for the collapse or near collapse response of masonry structures are commonly applied.

Philosophy a: Limit Analysis Based Solutions

Jacques Heyman understood that masonry material theory can be framed within the modern plasticity theory, and developed the Limit Analysis theory of masonry structures from 1966 to 1999 (Heyman, 1966, 1982, 1992, 1995). It is important to remember that a theory should be capable of explaining what happened, what is happening, and to predict the consequences that are yet to occur; it must fulfil all these aspects adequately.

Limit Analysis in the context of masonry refers to the evaluation of a masonry structure's capacity to withstand various loads and forces without experiencing failure or exceeding design limits. These analyses are crucial in the field of structural engineering to ensure their safety and stability. The material characterized as a unilateral material of infinite compressive strength, no tensile strength, and having no possibility of sliding allows for the use of Limit Analysis.

Based on the Theorems of Limit Analysis, two distinct approaches to addressing structural problems are applicable. The first approach involves the examination of equilibrium principles, specifically statics, wherein thrust lines are meticulously analysed. The second approach, on the other hand, employs deformation principles, particularly mechanisms, to construct patterns of hinges (Bagi & Angelillo, 2023).

The equilibrium solution or safe theorem is based on the fundamental principles of statics. It ensures the stability of the structure if an internal force system, admissible in the sense of the limits posed on the stress exists in equilibrium with the external loads exerted on it. The possible ways to study the equilibrium equations are either analytical or graphical.

For block by block masonry, which holds within it many possible equilibrium solutions, the graphical representation of the transmission of forces within the structure called the “Thrust Line” method has proven to be enormously fruitful (Huerta, 2004). Computerized applications of the previous method are present and attract particular interest which will be briefly explained in the next section “Tools for analyses – Graphic Statics”.

On the other hand, the kinematic theorem, also known as a mechanism-based approach, considers the deformations and movements of the masonry structure, which it can exhibit due to external actions. Starting from Heyman's model, the combination of equilibrium equations and simple material statements are sufficient to assess the stability of even complex mechanical problems while implementing only geometrical boundaries specifically constraints to the intrados and extrados. From a Limit Analysis based energy perspective, complex mechanical problems can thus be solved by modelling the structure as a system of interconnected rigid blocks that can move relative to each other (Angelillo et al., 2010, 2014; Bagi & Angelillo, 2023). In the rigid block model or the Livesley model, the surface of contact between the blocks is ruled by the friction coefficient or in some rare cases by an elastic compressive behaviour (Livesley, 1978). This approach provides insights into the failure mechanisms of masonry structures by understanding the energy dissipation and redistribution of forces in the structure as it deforms plastically.

Limit analysis aims to analyse the overall stability of the constructed work by considering the path of loads within the masonry and their relationship with the geometry of the structure. As mentioned earlier, this theory aligns with what ancient builders sought, which is "correct proportion."

When comparing two masonry structures, such as two-barrel vaults, the first constructed with stronger stone and mortar with higher "toughness" compared to the second, if they share the same geometry, then their structural behaviour is identical, as it is mainly due to form. The only relevant variable is their overall shape, meaning their geometry, and a change in this can produce a stable or an unstable configuration. Two geometrically identical vaults, regardless of their origin, time period, or purpose of the building, will exhibit the same structural behaviour (Huerta, 2004).

To understand the reason of this, it is sufficient to assume the three material assumptions made by Heyman:

- 1) The material does not resist tension.
- 2) Its compressive strength is infinite.
- 3) Failure due to sliding is not possible.

The first hypothesis is maintained on the side of safety because a stone block does have some tensile strength, although its value is very low and practically negligible at the joints, where the mortar has even less tensile strength than the stone.

The second hypothesis could go against safety because evidence demonstrates a great variability in the compressive strength of different mortars and stones. However, this hypothesis helps understand that the safety of the masonry does not depend on its mechanical compression behaviour, as the stresses generated within the masonry are extremely low. The third hypothesis also goes against safety. Nevertheless, frictional forces are high in masonry, and sliding phenomena are rare. This aspect is related to the arrangement of joints between the stones, as the example of the arch tells: the inclination of thrusts inside the masonry must be monitored to avoid that it exceeds the angle of friction.

Regarding the postulate of no sliding, it will be further elaborated within the section "Overview of the Masonry Arch". Mainly, it derives from the assumption that the frictional forces between voussoirs are of a sufficiently elevated magnitude, or that the stones are securely interlocked through alternative means, thereby preventing any relative movement or sliding between them. It should be noted that this assumption, although entirely reasonable, may occasionally manifest instances of slippage within a masonry structure in practical scenarios (Heyman, 1982)

Heyman's 1966 model, as simple as it may seem, captures the principles of traditional masonry design, aligning with the concerns of master builders who prioritized geometry and stone arrangement to insure structural stability against external forces (Huerta, 2008).

Heyman's material restrictions, "No Tensile Strength, Infinite Compressive Strength and no sliding", are crucial for the validity of Limit Analysis theorems. Nevertheless, in modern rules of practice, despite being grounded on Limit States, are often so confused in defining those states, that a clear prioritization of these criteria for each material, hardly emerges. The central importance of evaluating Limit States, where structures lose equilibrium by developing mechanisms, is the aim of any structural analyses. However, the latter is overshadowed in modern structural analysis. Heyman's findings that Limit Analysis remains applicable in this context marks a focal moment in twentieth-century Building Construction Mechanics (Bagi & Angelillo, 2023).

Building upon Heyman's work, a variety of analytical tools have emerged for masonry analyses. Some tools exclusively rely on the safe theorem to assess equilibrium, while others incorporate both the safe and kinematic theorems. These tools are often associated with specific graphical and limit analysis methods.

SIMPLIFIED MODELS

The fundamental response of masonry structures is unilateral, unlike standard structures with bilateral constraints, which undergo elastic deformations and significant internal force modifications in response to minor settlements and eigenstrains. Unilateral structures, even when heavily over-constrained, can exhibit zero energy modes. This unique behaviour allows them to offset relatively large settlements without an increase in internal forces, utilizing mechanisms that involve minimal energy dissipation (Heyman, 1995). The fundamental idea behind this approach is built on the notion that the material behaves unilaterally, where the tensile strength is dismissed, and only compressive stresses are considered. This idea commonly is referred to as the No-Tension assumption. This concept allows us to take into account the main aspect of masonry behaviour: fractures and cracks.

For idealizing the uniaxial masonry-like behaviour three simplified models are introduced and classified based on Angelillo's classification (Angelillo et al., 2014). The three models are: model zero, one, and two in Figure 8.

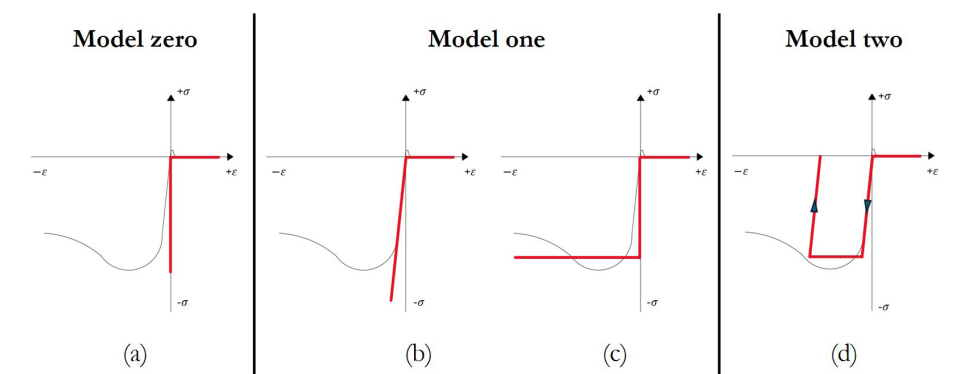
Model zero is the Rigid No-Tension (RNT), now known as the Normal Rigid No-Tension (NRNT), first introduced by (Heyman, 1995) and expanded in Italy by Salvatore Di Pasquale in 1984 (Angelillo et al., 2014). It characterizes the material as indefinitely strong and stiff in compression, incapable of sustaining tensile stresses, and is crucial in interpreting fractures as positive deformations. Its behaviour can be seen in the first row of Figure 8a.

To refine the initial model zero, a subsequent model, **Model one**, also known as the Elastic No-Tension (ENT) is introduced. This step involves adding a mechanical parameter to model zero, allowing for the definition of an ideal uniaxial behaviour using only one parameter. Two potential options are considered, both focusing on the behaviour in compression:

- 1- Finite Stiffness, Infinite Strength: This option results in an elastic response (see Figure 8b)
- 2- Limited Strength, Infinite Stiffness: Another choice involves limiting the strength while assuming infinite stiffness, as depicted in Figure 8c. This option requires a history-dependent behaviour.

Given that model one is generally oriented towards studying masonry structures under working conditions, the first option (Figure 8b) is commonly preferred and more widely adopted. This model is known as the Normal Elastic No-Tension (NENT).

Moving forward, **Model two**, termed the Masonry-Like (ML) material, adds the assumption of limited strength σ_c in compression. This incorporation enables the modelling of failure modes due to crushing. The ML material behaves as perfectly plastic in compression, exhibiting incremental and path-dependent constitutive responses. This peculiar material requires two material parameters, the elastic modulus (E) and the strength in compression σ_c , with tension-related properties being entirely neglected (see Figure 8d).



[Figure 8]
Simplified masonry models.

Revisiting model zero, the NRNT material is Heyman's rigid no-tension model. It has been widely used for analysing the stability of masonry systems (Angelillo et al., 2014). First, these assumptions enabled straightforward solutions for stability assessments of masonry vaults (Huerta, 2001) and the kinematic analysis of masonry buildings (Angelillo et al., 2012; Gesualdo & Monaco, 2015; Iannuzzo et al., 2020; Monaco et al., 2014). Second, Heyman's hypotheses formed a foundation for modern computational methods based on limit analysis. These various methods will be discussed later. These methods rely either on the safe theorem (Block & Ochsendorf, 2007; Fantin & Ciblac, 2016; Maia Avelino et al., 2021a) or the kinematic theorem (Angelillo et al., 2021; Iannuzzo et al., 2020; Montanino et al., 2022). The problem can be addressed through optimization approaches, using genetic algorithms, nonlinear differential equations, or linear/sequential linear programming.

Philosophy b: Detailed stress-strain law based models

In parallel, continuum mechanics is applied for masonry analyses even though masonry is not a continuum. Within this methodology, the masonry is modelled as a continuous medium, and precise equations governing equilibrium, compatibility, and material behaviour are used to describe the material's response to external loads and deformations. It is noteworthy that advocates of this approach assert adherence to Heyman's material restrictions, a critical aspect that will be explored later in this dissertation.

These analyses are generally conducted inside the FE environment. The evolution of equilibrium conditions in a structure is investigated in a step-by-step procedure. This involves dividing the loading and structural response into intervals or increments, referred to as "steps," with iterations performed to achieve equilibrium within each step.

Although the latter is common as well for some limit-based solutions, the difference is that these procedures require a thorough knowledge of the material properties and many parameters, typically unknown and unknowable, have to be assumed. consider mechanical nonlinearity, which is a requirement for a reliable assessment of the collapse and near-collapse behaviour of masonry structures. While linear elastic models have been developed for preliminary assessments of historic masonry structures (Cecchi & Sab, 2002; Cerone et al., 2002), they proved to be ineffective in studying failure modes and structural safety.

These methods are generally applied where large displacements may occur, making geometric nonlinearity a non-negligible factor that should be included in computations.

NONLINEAR STATIC ANALYSIS

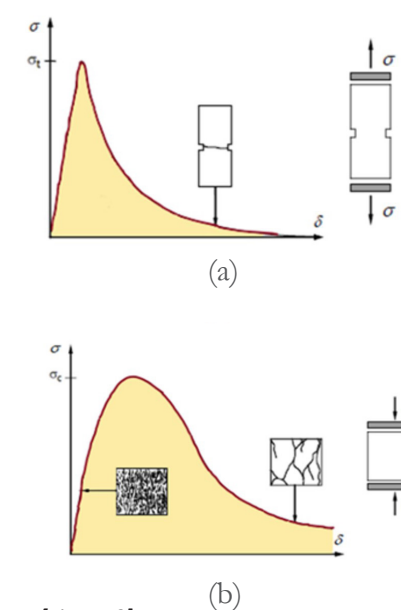
In nonlinear static analyses, the structure undergoes actions step by step until reaching peak-load and beyond, into the post-peak regime. The pseudo-time representing the structural response evolution lacks physical characteristics. Simulations can be performed in load control, displacement control, or event-by-event damage control (DeJong et al., 2009). Given the assumed mechanical nonlinearity of the material, nonlinear differential equations must be solved, transformed into nonlinear algebraic equations, and solved within a numerical framework. Typically, these equations are linearized in a stepwise manner and resolved through iterative procedures.

Notable iterative implicit methods include the Picard iteration, Newton–Raphson iteration methods, and Riks methods (Reddy, 2004). These analyses are commonly used to simulate quasi-static experimental tests on masonry structures and to conduct pushover analyses.

NONLINEAR DYNAMIC (TIME HISTORY) ANALYSIS

In time history nonlinear analysis (THNLA), the structure is step-by-step subjected to time-dependent actions, and the structural response evolves in real time, accounting for inertial and damping effects. Time integration methods are employed to satisfy the equations of motion during each time step. These methods can be explicit or implicit. Explicit methods depend only on quantities from the previous step, while implicit methods include values from the current step, requiring trial values of unknowns refined through successive iterations. Notable time integration methods include the Euler–Gauss procedure, Newmark Beta methods, second central difference formulation, and linear acceleration procedures (Clough & Penzien, 1995).

Nonlinear time history analyses can simulate dynamic actions' effects (e.g., earthquakes, impacts, explosions) on masonry structures. The ability to account for time-dependent loads allows for simulating responses to real accelerograms, shaking table experimental tests, and occasionally, dynamic analyses can be used to simulate quasi-static tests and processes, such as applying loads very slowly (Clough & Penzien, 1995).



[Figure 9]
Non-linear constitutive stress-displacement laws for masonry under (a) tension and (b) compression uniaxial loading (Roca et al., 1998).

COMPLEX CONSTITUTIVE MODEL

As has been mentioned, complex constitutive models are normally used when masonry structures are subjected to seismic loading, environmental effects, or long-term deformations. They consider factors such as nonlinearity, anisotropy, viscoelasticity, damage, and fracture. These considerations require a greater number of material parameters to capture a wider range of behaviours. This allows a simulation of complex phenomena such as stress-softening, strain-rate dependence, or progressive damage. Thus, generally implementing complex constitutive models can be computationally intensive.

A basic modern notion in the mechanical behaviour of masonry is softening, it involves a gradual decrease in mechanical resistance during continuous deformation. It is observed in materials like brick, mortar, stone, or concrete, where failure results from progressive internal crack growth; see Figure 9. Softening is well-identified in tensile and shear failure, with experimental data suggesting the influence of both local and continuum fracturing processes in compressive failure.

Parameters' calibration in these complex constitutive models rely on extensive testing. The properties of masonry, as indicated by compressive strength tests, are complex due to the large array of possible combinations, making the relationship between tensile and compressive strength difficult to determine. Additionally, the unit-mortar interface plays an important role in governing non-linear joint responses, contributing to distinct failure modes.

Various testing methods are essential for numerical simulations. However, in direct tension testing difficulties persist in establishing the bond tensile strength which is mainly influenced by the properties of the unit and mortar forming the masonry. Shear behaviour tests encounter difficulties in maintaining uniform stress, revealing exponential shear softening. Ductility indices for mode II fracture energy depend on the specific combination in addition to the nature of the masonry joint behaviour.

Constitutive models face challenges due to the inadequacy of continuum models for solid masonry strength, necessitating refined approaches, a fact highlighted in (Lourenço & Pina-Henriques, 2006). Despite the availability of formulas, the variability in parameters to be assumed introduces uncertainties in predicting masonry behaviour. The anisotropy of masonry adds to the complexity. Hence the need for comprehensive models that consider anisotropic properties and various influencing factors (Angelillo et al., 2014).

Regarding the two approaches/philosophies, engineers who are expert on masonry behaviour often lean towards the first approach. This inclination derives from the challenge that the parameters required by the second approach, such as effective geometry, specific material behaviour, and various properties and details significantly influencing model predictions, are generally unknown. Furthermore, these parameters are subject to heightened randomness and uncertainty, hence the preference for the first approach.

Therefore, when adopting a unilateral numerical model in masonry, the approach to resolving equilibrium issues for a diverse array of structures requires only minimal material information. While this allows for predictions on actual constructions, it also introduces important approximations.

1.2. TOOLS FOR ANALYSES

1.2.1. GRAPHIC STATIC BASED APPROACHES

The funicular polygon and the thrust line are geometric shapes commonly used for the two-dimensional study of masonry structures. Here we give the definition of each of these geometric objects and highlight the confusions existing between these terms in 19th century literature. Historical information concerning the funicular polygon and the pressure line of this chapter are taken from History of strength of materials (Timoshenko, 1935).

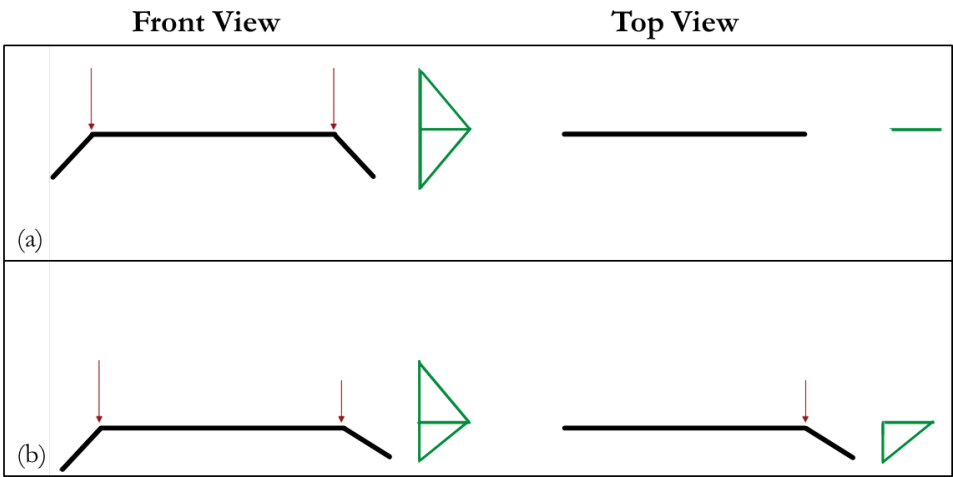
For three forces to be in equilibrium, in both 2D and 3D, it is necessary for their lines of action to be coplanar and pass through a common point, which can be at infinity (Culmann, 1880). However, for more than three forces, these conditions are no longer necessary: four forces can be in equilibrium without their lines of action being coplanar or passing through a common point.

The equilibrium of a voussoir as part of an arch involves three forces: the load and the two forces acting on the joints. Based on the previous property, there exists an intersection point of the forces acting on each voussoir.

Definition 1: A funicular polygon of an arch voussoir is a polygon with vertices at the points of intersection of the lines of action of the forces in equilibrium acting on each voussoir.

When a set of coplanar forces is applied, as it occurs for arches loaded by gravitational forces, then the funicular polygon and the force polygon are planar, see Figure 10a. However, if the forces are not coplanar, then the funicular polygon and the force polygon are not planar, see Figure 10b.

[Figure 10]
Example of funicular polygon
and force polygon for (a) planar
forces and (b) non-planar force.



Historically, La Hire was the first to use the funicular polygon in 1695. The funicular polygon was also employed by Varignon (Varignon, 1725), followed by others. However, it wasn't until Culmann that a systematic presentation of graphical methods for structural analysis was introduced. Culmann published "Die graphische Statik" in 1864 in Zürich (Culmann, 1866).

However, his work was not translated from German to French until 1880, which limited its immediate impact on the French engineering community. During this period, the Méry method, which was the most well-known method for studying masonry arches in France in the 19th century, did not make use of the funicular polygon.

1.2.1.1. THRUST LINE ANALYSIS

An important geometric concept for the study of the equilibrium of arches is, for each joint, the point of application of the resultant contact forces. This point can also be referred to as the point of application of the resultant forces or the centre of pressure. The points of application of the resultant contact forces on the joints form the line of thrust. In the context of graphical statics, these points are constructed based on the funicular polygon.

Definition 2: *A line of thrust of an arch voussoir is the sequence of points of application of the resultant contact forces (or centres of pressure) on its joints.*

As the centres of pressure are determined by the intersections of the lines of action of the resultant contact forces with the joints, the line of thrust is a sequence of intersections of segments or extensions of segments of the funicular polygon with the joints. The concept of the line of thrust was first published by Gerstner in 1831 (Timoshenko, 1935). It was then studied by Moseley (Moseley, 1843), who was the first to distinguish between the line of thrust and the funicular polygon. In France, Méry was the first to use the term "courbe de pression" (curve of pressure) in 1840. According to Dupuit, the 1840 article corresponds to the work written in 1827 (Dupuit, 1870).

There is significant confusion among the terms used to refer to the "funicular polygon" and the "line of thrust", due to false cognates in English and German. Thus, in Culmann's 1880 translation of Culmann's graphical statics treatise into French, the names "funicular polygon" and "line of thrust" are respectively applied to a tension funicular polygon and a compression funicular polygon. In the part of his work where Culmann uses these terms, he only considers the relationships between the funicular polygons.

For example, the term 'lignes de pression' is a false friend, as it is a literal translation of the German term 'Drucklinie,' which can lead to confusion due to its various meanings. Interestingly, Breymann uses the same term 'Drucklinie' to translate the concept of 'line of thrust' (Breymann, 1877).

In English, the "lignes de pression" is translated as "line of thrust", and the "polygone funiculaire" as "line of pressure", which naturally leads to some confusion for translators.

The evolution and historical correspondence of terminologies are presented in Table 2.

Scientist and Era	Line of Pressure	Funicular Polygone
Méry (1840)	“Courbe de Pression”	-
Moseley (1834a)	“Line of Resistance”	“Line of Pressure”
Dupuit (1870)	“Ligne/Courbe de Pression	-
Durand-Claye (1880)	“Courbe de Pression”	-
Culmann (1880)	-	“Polygone Funiculaire, Ligne des Pressions »
Benvenuto (2006)	“Linea di resistenza”	“Linea di Pressione”
Today	Line of Thrust (Eng.) Ligne de Pression (Fr.)	“Funicular Polygon” (Eng.) “Polygone Funiculaire” (Fr.)

[Table 2]
Comparison of the Line of Pressure
and the Funicular Polygon.

The line of thrust and the funicular polygon have distinct definitions. However, in the context of graphical statics, the latter is a necessary step to construct the points of the former, and these two geometric objects are closely related: in most cases, the points on the line of thrust will be located on the segments of the funicular polygon.

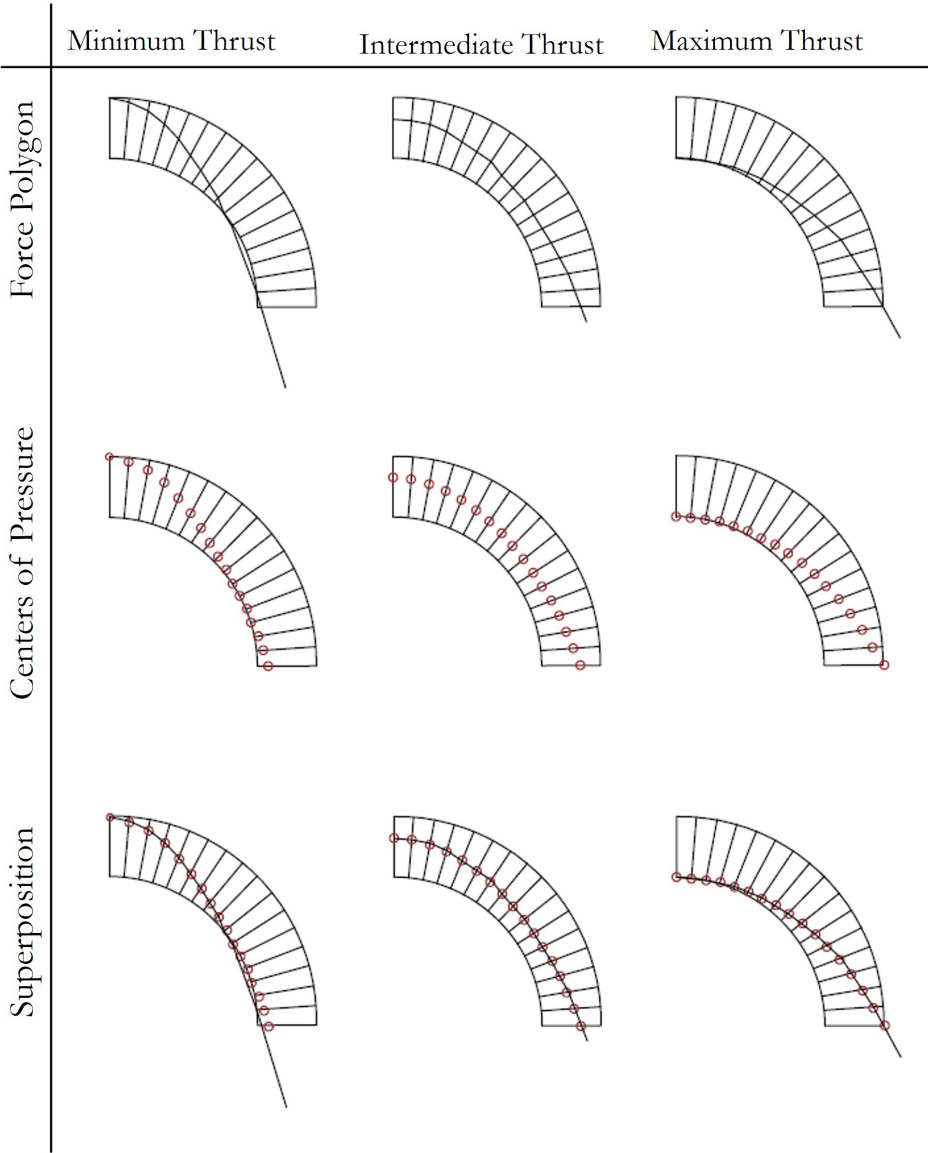
This observation can be illustrated with several examples.

- Figure 11 depicts three equilibrium cases for a semicircular arch. In the first and second columns, two cases are shown where the line of pressure (meaning line of thrust) and the funicular polygon are close. In the third column, a case is shown instead where the line of pressure and the funicular polygon do not overlap near the start of the arch.
- Moseley illustrated this difference in 1843 for a portion of an arch subjected to both horizontal and vertical forces. In Figure 12, the original figure suitably redrawn by the author is provided.
- In the specific case of horizontal joints studied by Dupuit (Figure 13 on the left) it is interesting to note that the funicular polygon reduces to one point, which is the common intersection point of all resultants. Furthermore, Dupuit did not use the concept of the funicular polygon to plot these figures.

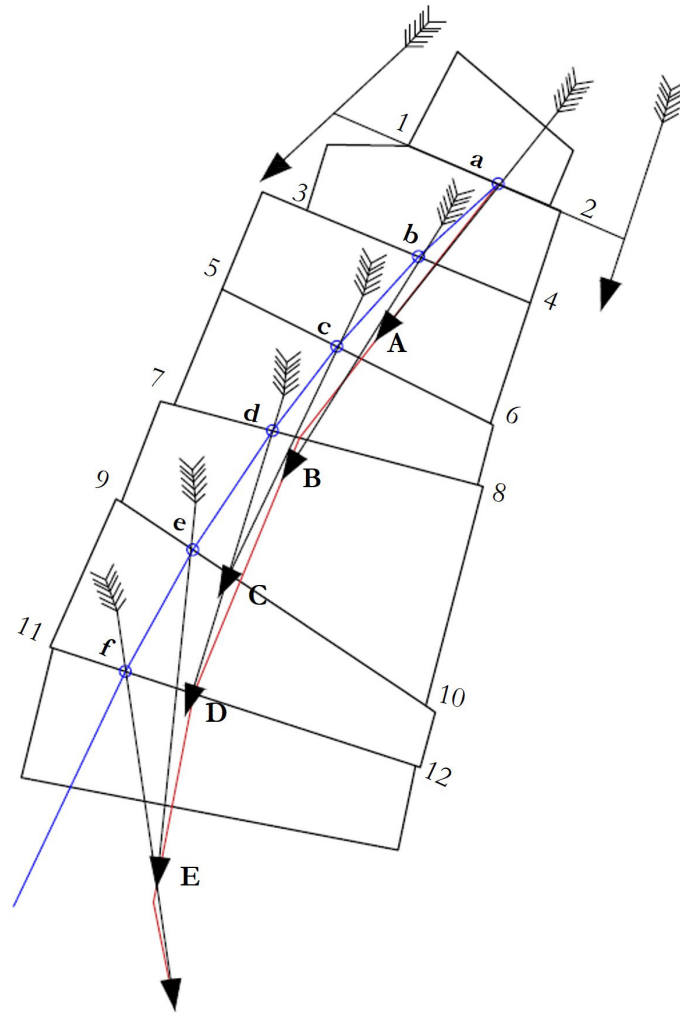
The clear difference between the line of thrust and the funicular polygon in some of these examples can be explained by the following observation: the centres of pressure are not necessarily located on the funicular polygon. By definition, each segment of the funicular polygon corresponds to a specific joint. The intersection of lines extending from a segment of the funicular polygon with the joint provides the centre of pressure. The centre of pressure can be located on the segment of the force polygon or outside of that segment, along the line extending it. For example, in Moseley's figure, the centre of pressure "e" is placed on the line (DE) but not on the segment [DE] (Figure 12).

A corollary of this observation is that the funicular polygon can extend beyond the masonry, while all corresponding centres of pressure are located inside the masonry, or more precisely, within the joints. This observation is essential for finding equilibrium solutions for vaults with specific stereotomy.

Heyman emphasized the differences between the line of thrust and the funicular polygon and the potential for confusion. He also highlighted that these two concepts depend on the stereotomy of the considered arches (Heyman, 2009).

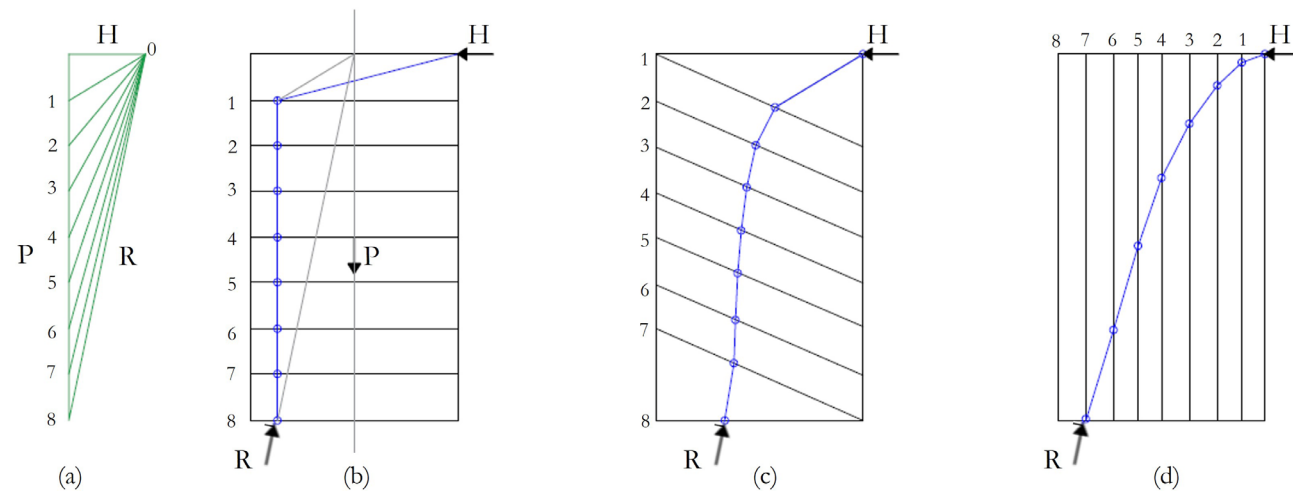


[Figure 11]
Three equilibrium cases for a semicircular arch and the overlapping between the force polygon and line of pressure.



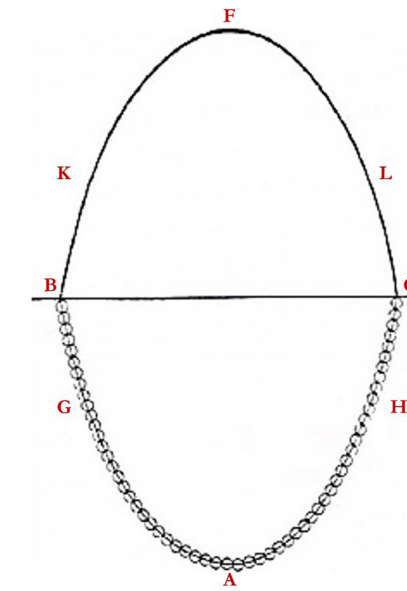
[Figure 12]

Portion of an arch subjected to both horizontal and vertical forces figure redrawn by author based on Moseley (1843). The line of thrust is shown in blue, and the line of pressure, formed by the arrows representing the segments of the funicular polygon, is shown in red.



[Figure 13]

Study of thrust lines in a rectangular abutment, for different families of cutting planes (redrawn according to Dupuit 1870). In all three cases, all blocks have the same weight, and the force polygon drawn is common. Note the significant variations in the shape of the thrust line.



[Figure 14]

Robert Hooke's catenary and arch presentation.

The described procedure can be understood with the analogy of a rope held at two points; it takes the shape imposed by vertical loads, hanging, and forming a catenary (supporting its own weight) in a state of perfect tension equilibrium. By inverting the same shape, it now represents pure compression equilibrium, i.e., a catenary arch (see Figure 14). Robert Hooke was the first to note the relationship between an arch (BKFLC) and a hanging chain (BGAHC), stating in 1675: "As the flexible thread hangs, so, inverted, will the rigid arch stand" (Heyman, 1995).

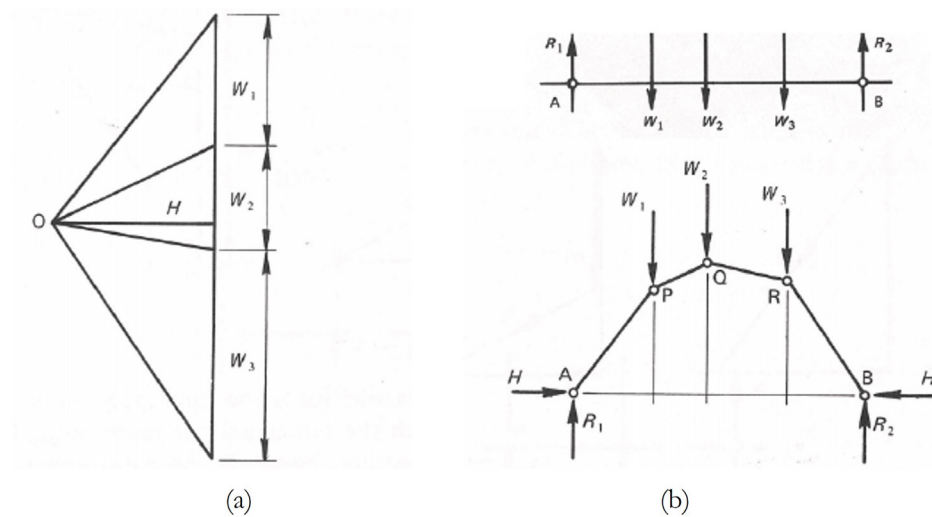
According to (Huerta, 2004), this theory was understood by Professor Emeritus Fernando López Carmona, who, after getting to know Heyman, applied a similar approach. With a scale appropriate to the drawing, these thrust lines are obtained through the division into blocks that correspond to sections of the analysed structure. Using the force polygon, a pole is located, and it is connected to each of the forces, with the resulting angles forming a funicular system of forces.

This procedure is illustrated in Figure 15. It is possible to manipulate this system by moving the pole horizontally and vertically until equilibrium polygons are achieved with the loads, meeting specific conditions. This helps find an equilibrium solution that can explain the transmission of loads within the masonry.

Carmona referred to the predominant role of geometry in structural stability as "strength through form" and referred to it as the "natural form." This natural form is obtained from a system of loads and materials defined by techniques such as the thrust line polygon. The equilibrium approach allows for the application of limit analysis in arches using a thrust line. This thrust line can be found for any other masonry structure. The implementation of graphic statics allows for the determination of this thrust line.

Simply, the self-weight of the structure can be represented as vectors with a scale corresponding to the drawing and obtained by dividing into blocks that correspond to sections of the analysed structure. Using the force polygon, a pole is located, connecting to each force, and the angles formed result in a funicular of the force system, this procedure is illustrated in Figure 15. This makes it possible to manipulate the system by moving the pole horizontally and vertically until equilibrium polygons are obtained with loads that meet specific conditions, finding an equilibrium solution that can explain the transmission of loads within the masonry.

[Figure 15]
Procedure for equilibrium analysis
method demonstrating relation
between force funicular polygon and
load transmission within masonry.

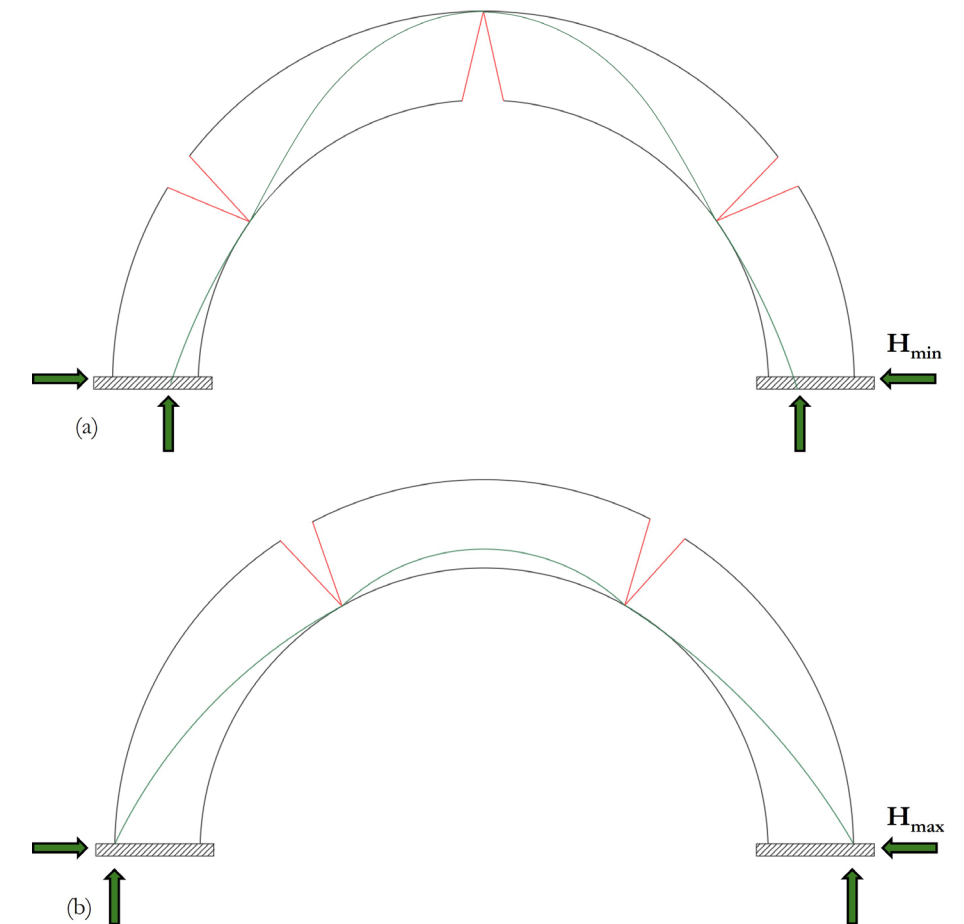


There can be an infinite number of thrust lines passing through the material, but they are bounded by the constructed shape, as they must be contained within the structure. The funicular polygon can determine the maximum and minimum thrust. The importance of analysis through the force polygon lies in its applicability to deformed geometries, geometrically describing how loads are transmitted.

Thanks to Heyman's theory, we can understand that the collapse of a structure occurs not due to the material's lack of strength but because a sufficient number of hinges form, turning the structure into a mechanism. In this way, a hinge (or pivot) corresponds to the moment when the line of thrust approaches the section's edge, causing tensions that the joints cannot absorb, leading to the formation of cracks in the structure. This theory provides a clear and logical interpretation of masonry arches, where the opening of cracks is a normal response to any small movement of their supports. This cracking alters the position of the line of thrust and, therefore, the value of the horizontal thrust: if the movement is an opening of the span, it will adopt the minimum value (see Figure 16a); if it is a closure, it will be the maximum, as illustrated in Figure 16b. The actual line of thrust lies between these two extreme values. This is a way to state the theorem of safety, which asserts that: "as long as a line of thrust remains within the masonry, it will be safe."

This final statement helps us understand the entire historical construction of masonry primarily as a geometric problem. This limit analysis confirms the proportion rules of the ancient builders as they arrived at the same geometric considerations. They clearly grasped this situation without having

[Figure 16]
The two cases limit thrusts (in green)
in semi-circular arches (a): minimum
thrust (b): maximum thrust.

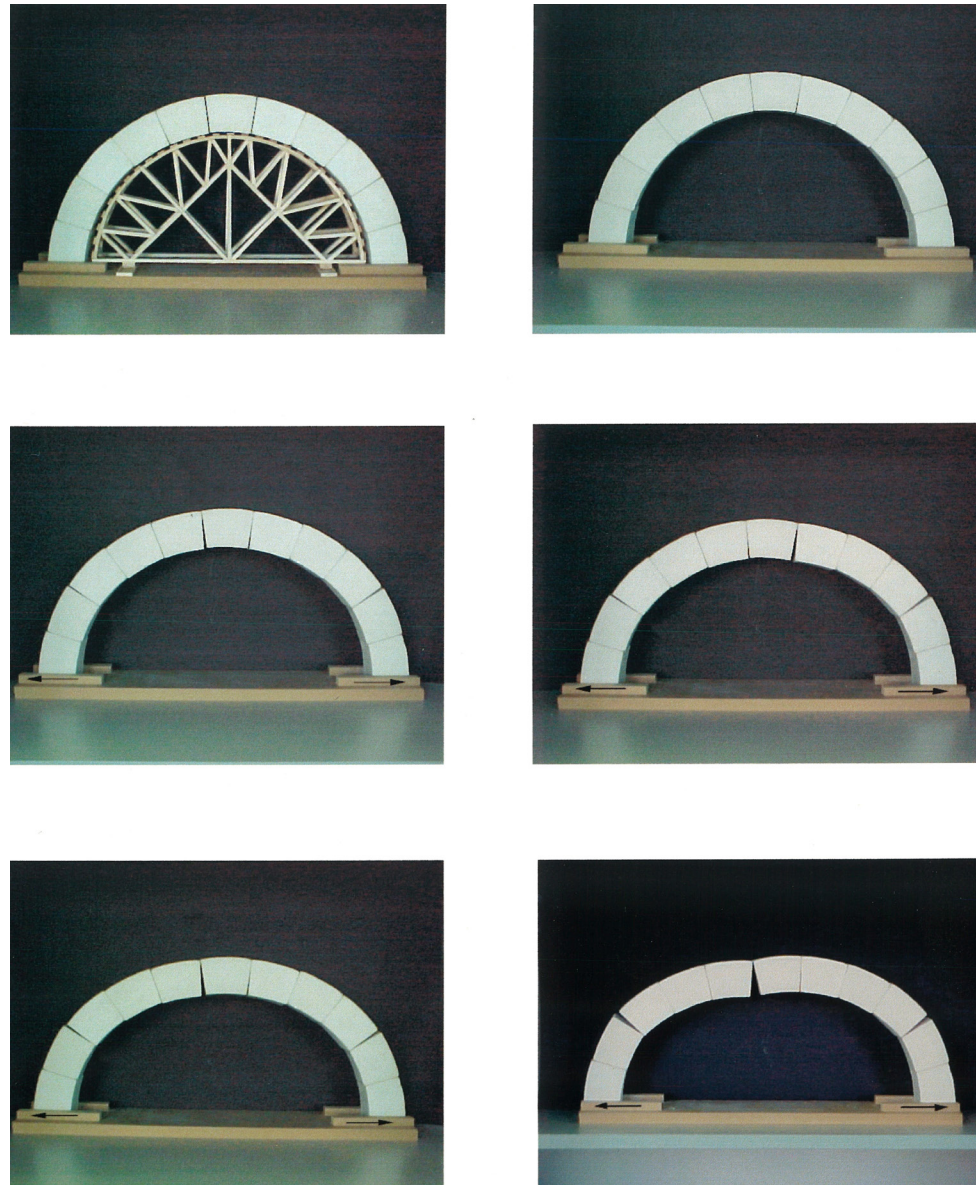


any knowledge of mechanics or structural analysis as we understand it today. Through treaties and manuscripts, we can see that construction was based on the geometry and proportion of its elements: "Its safety depends on stability, and stability depends on proportion" (Huerta, 2004).

The typical hinges explained by Heyman are, in a strict sense, harmless since they constitute the common denominator of these structures. If there are three or fewer hinges, the structure is stable and does not pose a danger. These hinges are the way in which the arch accommodates small movements and settlements.

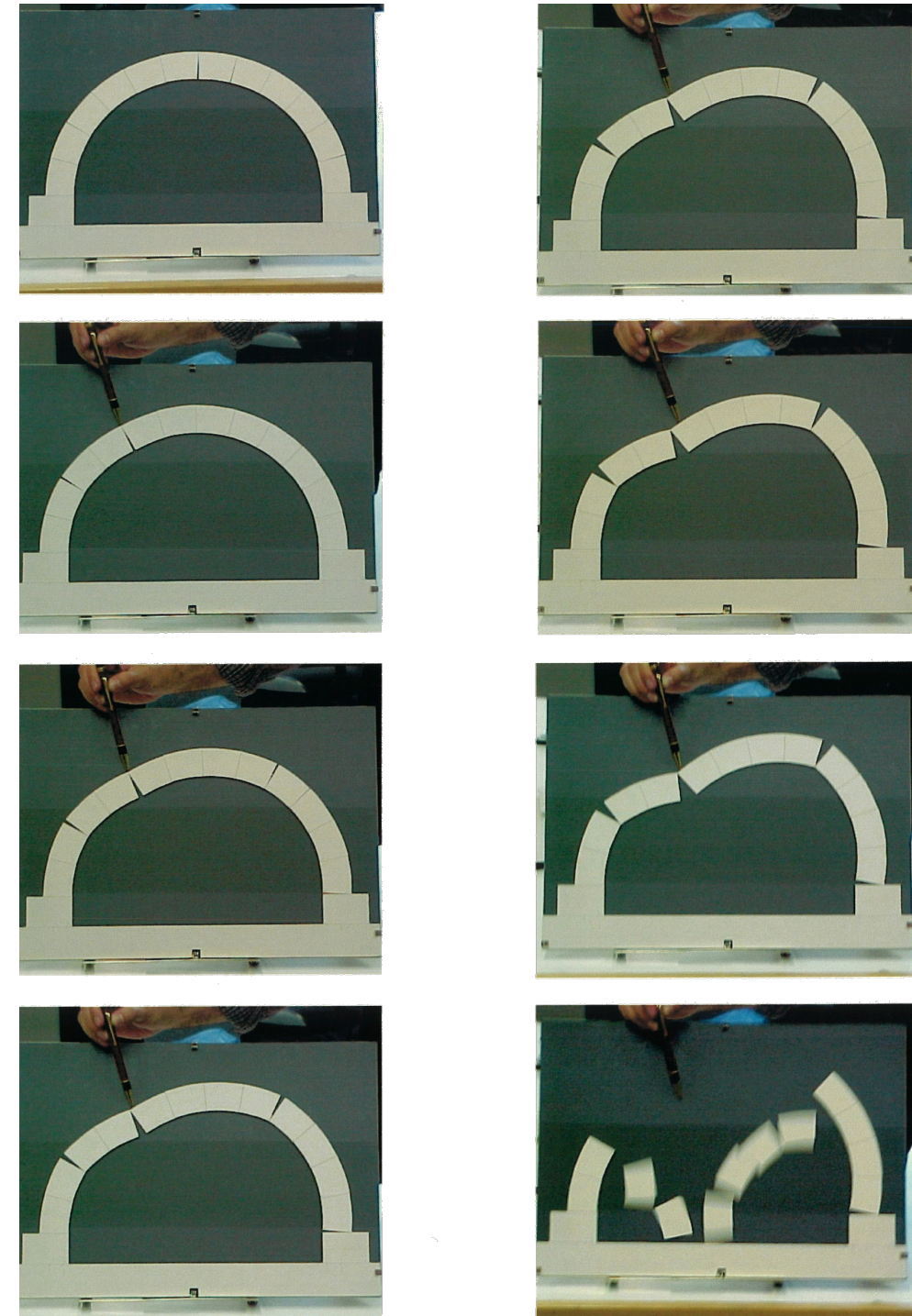
A simple experiment that can be easily reproduced with a two-dimensional or three-dimensional model of a vaulted arch can demonstrate this concept. As shown in Figure 17, in an arch constructed by Prof. Santiago Huerta from gypsum blocks, it can be observed that when there is an outward movement at the supports, the three cracks appear in a nearly consistent manner.

However, when four hinges form in alternation this is when the failure mechanism initiates. This can be observed in Figure 18 through a cupboard model of an arch, as well executed by Prof. Santiago Huerta.



[Figure 17]

A three-dimensional model of a vaulted arch made from gypsum, property of Prof. Santiago Huerta, subject to outward movements at the support demonstrating the three-hinge formation.



[Figure 18]

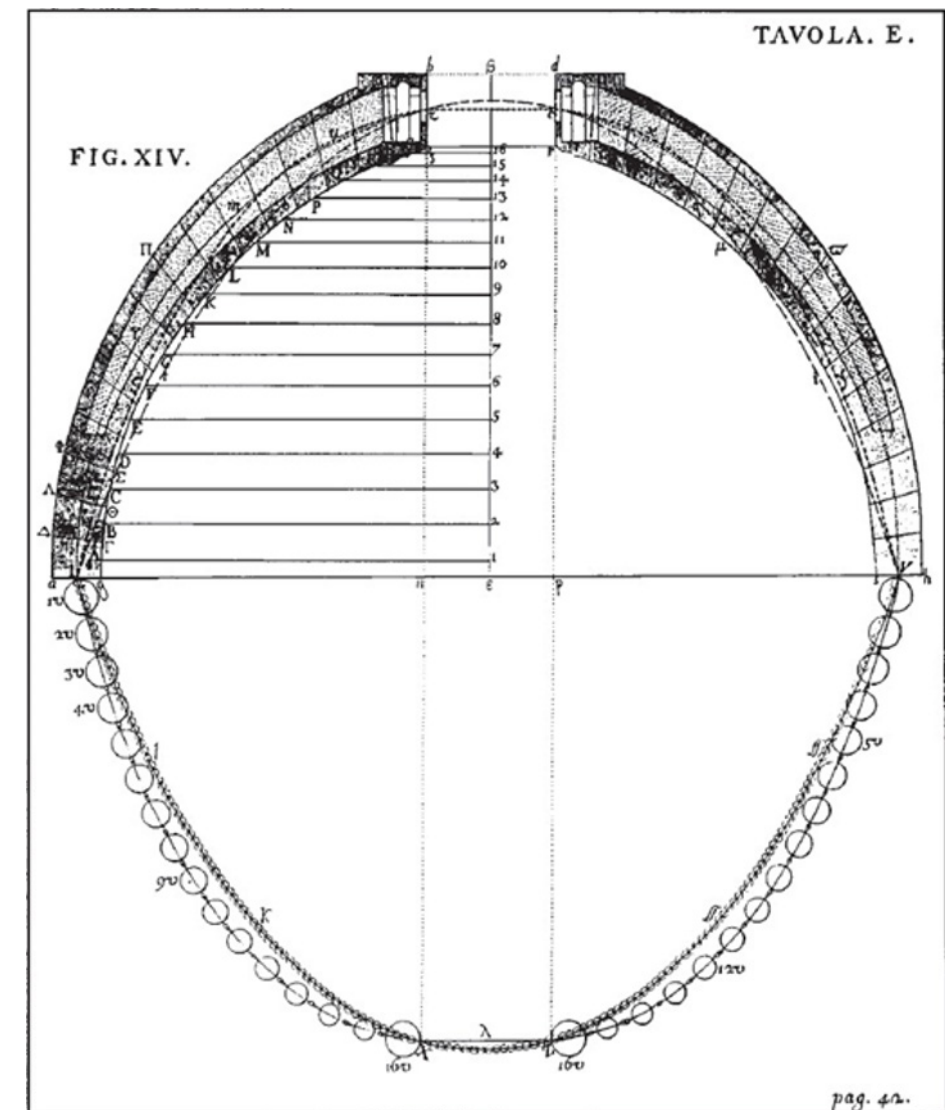
A three-dimensional model of an arch made from cupboard, property of Prof. Santiago Huerta, subject to an asymmetric point load demonstrating the four-hinge formation.

The same analytical approach can be applied to other masonry structures, such as domes. In domes, there are hoop forces, typically in compression at the upper part, which the masonry can absorb, and the thrust line remains along the median as if it were in a state of membrane. However, at the lower part (starting at approximately 52 degrees for semi-spherical domes subjected to their own weight, and higher if the keystone is loaded by a lantern), tensile forces develop that the masonry cannot bear and cracking occurs. Heyman has established that small settlements at the supports can raise the cracks to a colatitude of approximately 25 degrees in domes with a lantern.

Figure 20 shows a spherical dome under gravity, showing meridian and hoop forces. Meridians undergo compression, strengthening masonry and resisting crack propagation. Hoops experience varying compression and tension. As hoop stresses surpass masonry tensile strength, cracks form along meridians, altering the dome's behaviour from rigid to cracked. Dynamic actions like earthquakes and settlement can widen these cracks.

The equilibrium approach for domes was first carried out by Poleni in 1748 to analyse the dome of St. Peter's Basilica in Rome (Huerta, 2001). In his memoir Poleni conducted an analysis of a "lune" employing the principle of a hanging chain by Hook. He conceptualized the weight distribution of a lune by employing 32 spheres of variable masses also considering the loading of the superimposed lantern (see Figure 19). He concluded that transverse cracks do not significantly compromise the stability of the dome. This can be affirmed because the entire surface is cracked, and in the cracked zone, no hoop forces are transmitted since the thrusts follow their path in the same way as in arches.

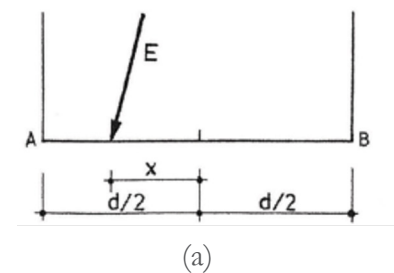
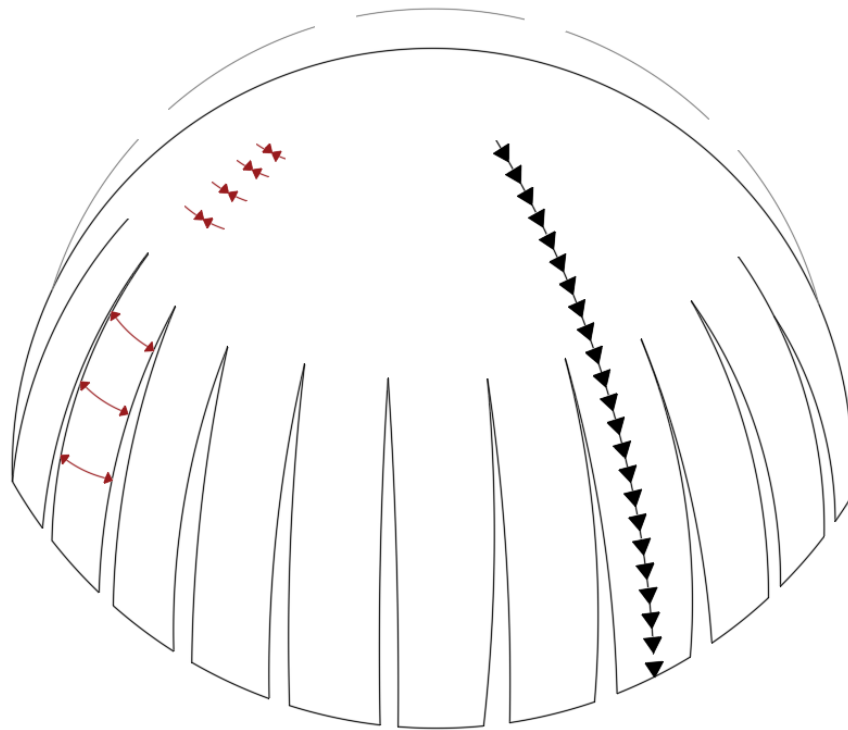
Regarding polygonal domes, Heyman explains that they can be subjected to exceptional stresses at the joints, where ribs are typically located. This results in their "segmented nature," meaning that the dome naturally divides into parts, either due to settlements at the supports or due to some slight movement.



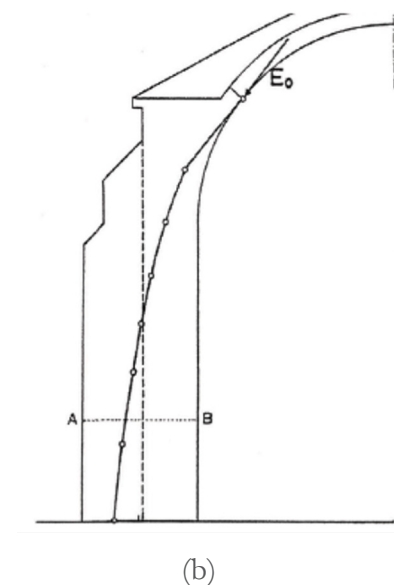
[Figure 19]
Drawing from Poleni's Memoir 1748.

[Figure 20]

The transverse cracks in a dome. The meridional forces in black and the hoop forces in red: compression forces in the dome's top part and tension forces in the lower part.



As for Buttresses they are an essential part of the structural system, and their equilibrium must be analysed globally. The reasoning is the same as that for arches and vaults. If it is possible to find a line of thrust within the section of the masonry for a given state of loads, the structure is stable. However, what is the limit of this safety?



[Figure 21]

Geometric safety coefficient proposed by (Huerta, 2004).

The equilibrium of buttresses is extremely delicate, unlike arches and vaults. If a vault fails, the rest of the structure can still stand; if a buttress fails, the collapse is total, so its stability must be analysed carefully. If the line of thrust approaches the end of the section, the danger of a hinge forming exists. Safety is a geometric factor that quantifies the distance from this limit situation.

The geometric safety coefficient has been sought since the 19th century. According to (Huerta, 2004), Rankine in 1858 introduced the expression of the coefficient of safety, denoted as (q) , that restricts the deviation of the line of thrust from the section's axis and expresses it as a fraction of the section's width. Huerta has proposed another formulation for this coefficient, which he calls " c ," making it consistent with the one used by Heyman for arches and vaults (a value of 2). It is the ratio between half of the section's width ($d/2$) and the deviation of the line of thrust from its axis (x), as shown in Figure 21, so a coefficient greater than 2.5 is considered safe.

1.2.1.2. THRUST NETWORK ANALYSIS

In the domain of two-dimensional structures, compressive force paths result in thrust lines. A concept widely applied to arches, sliced domes, and vaults, often in conjunction with graphic statics methodologies as has been seen in previous sections. Presently recognized as Thrust Line Analysis (TLA), this technique has expanded to address three-dimensional challenges such as domes, vaults, and spiral staircases. Despite TLA's successes, extending lower-bound methods to fully three-dimensional structures, particularly masonry vaults, presents challenges due to increased indeterminacy and complex geometry (Huerta, 2008). Consequently, various lower-bound equilibrium formulations, including discrete approaches, have been developed to assess three-dimensional masonry structures.

Discrete approaches rooted in Limit Analysis, as proposed in (O'Dwyer, 1999) and later refined in (Block & Ochsendorf, 2007), involve modelling compressive forces along a network. This approach considers a singular internal stress field discretized by a network, where compressive forces are modelled as axial forces along the network's edges. Equilibrium equations are linearized, allowing for the investigation of geometric irregularities, concentrated loads, and openings.

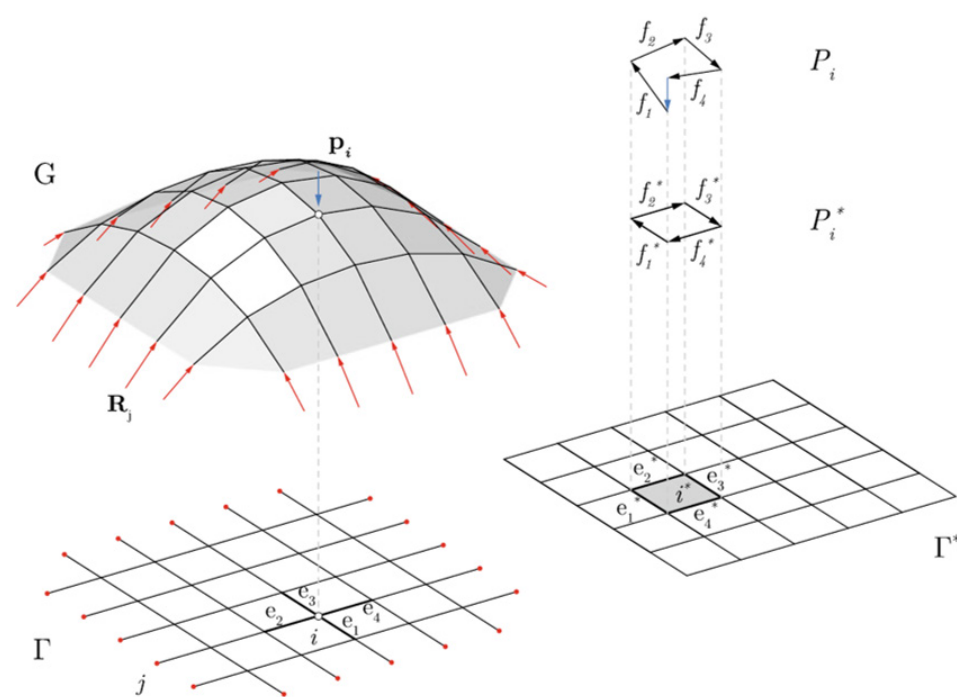
TNA, introduced by Block and Ochsendorf in 2007, integrates discrete approaches with graphic statics, exploring the spatial equilibrium of networks through reciprocal relations of form and force diagrams. Hence, applying the Safe Theorem to masonry within the context of TNA characterizes admissible stress states as compressive force paths within the structural geometry, achieving equilibrium with externally applied loads, providing a comprehensive framework for computing collapse loads or mechanisms in structures. Specifically, TNA relies on geometry-based information rather than conventional mechanical properties such as compressive strength and stiffness, which are frequently challenging to ascertain or remain unknowable. Furthermore, The Force Density Method (Schek, 1974) has further enriched the formulation of TNA, enabling its coupling with optimization procedures for masonry assessment problems (Block & Lachauer, 2014; Maia Avelino et al., 2021a, 2021b).

As described in their book (Bagi & Angelillo, 2023) section 4, subsection 2, the core elements TNA based on Block's work in 2009 are illustrated in Figure 22, where thrust network (G) is conceptualized as a directed and connected

spatial graph, where the edges represent spatial compressive resultants or thrusts within the structure.

The form diagram (Γ) is a planar graph derived from the projection of G , storing the network's topology, connectivity, and planar coordinates. Each vertex in the form diagram corresponds to a vertex in the thrust network, with mapped elevations, representing the horizontal equilibrium of that specific vertex. Similarly, each edge in the form diagram maps to an edge in the thrust network, carrying axial forces. Boundary conditions and loads are applied to the nodes of the network, with support points highlighted in red in the form diagram. Equilibrium is verified by constructing closed force vector polygons. The force diagram (Γ^*) represents the horizontal equilibrium of G , with a reciprocal relationship maintained between the form and force diagrams. It is proposed that the exploration of thrust network geometry involves a reciprocal modification between the form and force diagrams by utilizing parallelization algorithms.

Nowadays, Computational tools like RhinoVault have facilitated this interactive exploration. However, integrating the thrust network within a typically tight structural envelope for masonry assessment requires robust optimization strategies and suitable problem parametrization. Notably, the TNA framework, rooted in graphic statics, is constrained to vertical loading cases, lacking the capability to address horizontal forces (Bagi & Angelillo, 2023).



[Figure 22]

The Thrust Network (G) and its horizontal projection, known as the form diagram (Γ), are represented symbolically. Equilibrium (P_i) of node 'i' under the vertical applied load P_i is emphasized. The horizontal projection of P_i is depicted as the closed polygon, P_i^* . By summing these polygons, the force diagram of the network (Γ^*) is derived. Figure from (Bagi & Angelillo, 2023).

1.2.2. NUMERICAL APPROACHES

The fact that masonry structures can be built with a large variety of materials, as has been discussed in the State-of-the-Art section, poses the challenge of applying a unified general approach, as previously stated: “there is no one-size-fits-all solution for masonry structures”. For this reason, a complementarity between analyses approaches can take place, to accurately analyse a structure of this type.

The numerical modelling and analysis of masonry structures is the most affected by the previously stated challenge. Masonry exhibits linear elastic behaviour at low stress levels but becomes increasingly nonlinear as the applied load intensifies. Hence, such structures require a different approach from that adopted for materials like steel and concrete. This challenge is accentuated by the limited availability of input data and uncertainties that further complicate the modelling process. This complexity is due to several factors, including the presence of joints as a primary source of structural weakness, including brick dimensions, anisotropy, joint width, material properties of bricks and mortar, workmanship quality discontinuity, and nonlinearity.

A comprehensive numerical model must account for two distinct aspects of structural behaviour: (1) the behaviour of masonry units, typically bricks, and (2) the behaviour of the joint material, often mortar.

1.2.2.1 Modelling Scales

Then, in the current scientific context, with regards to modelling masonry it is noticed that there are essentially two scales of approaching the modelling of masonry: macro-scale and micro-scale. In macro-scale, a masonry element is represented as a continuous homogenized entity and implicitly considering the effects of mortar joints.

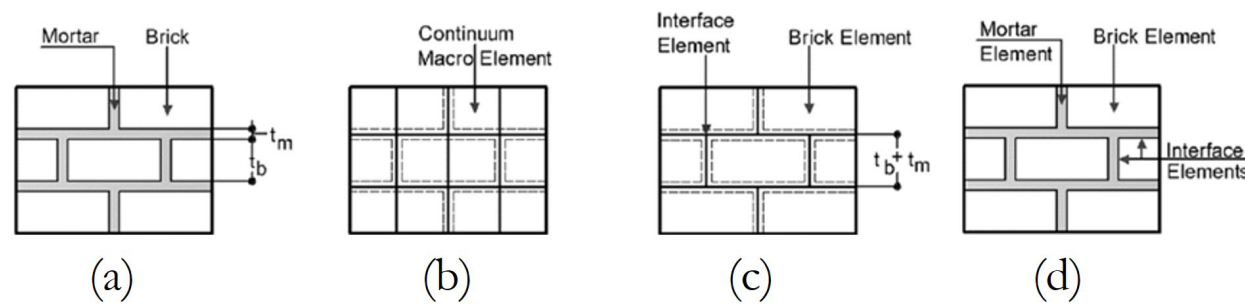
On the other hand, micro-modelling explicitly addresses masonry elements with joint material, capturing the interactions, using interface elements, between masonry bricks and joints.

According to (Asteris & Plevris, 2015; Lourenco et al., 1995, 1995) the aforementioned two scales for masonry models can be categorized into three levels of refinement:

1. Modelling masonry as a one-phase material (Macro-modelling): This approach treats masonry as a homogeneous continuum, ignoring distinctions between individual units and mortar. While suitable for analysing large-scale masonry structures, it may not capture the influence of mortar joints adequately, see Figure 23b.

2. Modelling masonry as a two-phase material (Simplified micro-modelling): In this approach, bricks are represented as expanded elements, and the mortar joint is modelled as an interface with zero thickness. The interface's stiffness is derived from the stiffness of real joints, simplifying computational efforts, see Figure 23c.

3. Modelling masonry as a three-phase material (Detailed micro-modelling): This approach represents units and mortar with continuum elements, while the unit-mortar interface is represented using discontinuous elements. Although this provides accurate results, it is computationally intensive and suitable for small-scale laboratory specimens and structural details, see Figure 23d. A procedure for this problem can be found in (P.G. Asteris & A.D. Tzamtzis, 2003; Sutcliffe et al., 2001).



[Figure 23]

Modelling scales of a masonry unit:
(a) is the masonry unit (b) presents a continuum macro-element, (c) micro-modelling of the unit and (d) detailed micro-modelling.

1.2.2.2 Modelling Strategies

Numerical modelling for masonry structures can be categorized into four main strategies, which can be classified within the main three categories of refinement (D'Altri et al., 2020).

- Macro Modelling, encompasses strategies such as Geometry-based models, Energy-based models, Continuum models and Macro-element models,
- Micro Modelling, includes some approaches of the Block-based models and Energy-based models,
- Detailed Micro Modelling involves other approaches of the Block-based models.

a. Geometry-based models

In geometry-based models, the structure is treated as a rigid body, relying on its geometry as the primary input for modelling, surpassing loading condition specifications. These models predominantly explore structural equilibrium and/or collapse through evolving limit analysis-based solutions, drawing from static or kinematic theorems, with roots often in Heyman's rigid no-tension model (Heyman, 1995). The static theorem-based approaches, demonstrated by Heyman, find applications in masonry arches, vaults, and domes, offering computational developments like O'Dwyer's equilibrium analysis (O'Dwyer, 1999) and the funicular model (Andreu et al., 2007). As well as, the previously discussed TNA (Block & Ochsendorf, 2007), explores the equilibrium analysis of vaulted masonry structures, revealing funicular solutions and accommodating extensions considering joints (Fantin & Ciblac, 2016) or applying structural matrix analysis (Block & Lachauer, 2014).

Meanwhile, kinematic theorem-based approaches provide fast and effective assessments, with computational advancements, such as Milani's upper bound approach (Milani, 2015). A combination of an upper bound approach, based on Non-Uniform Rational B-spline (NURBS), and a genetic algorithm was proposed by (Chiozzi et al., 2016, 2017). Hence, IsoGeometric analysis can be extended to model complex 2-D surfaces and shells such as domes and vaults (Cazzani et al., 2016a, 2016b, 2016c; Grillanda et al., 2021), the genetic algorithm (Chiozzi et al., 2017), refining the evaluation of collapse mechanisms and multipliers for masonry structures. Recently, an application of NURBS in depicting the geometric characteristics of an edge-corrugated shallow dome was presented in (Lai et al., 2022).

These models, rooted in geometry and employing static or kinematic theorems, contribute to comprehensive insights into equilibrium states, collapse mechanisms, and vulnerabilities at different scales in the assessment of masonry structures.

b. Energy-based Limit Analysis models

This approach is based on Heyman's theory, it handles masonry as a solid fracturing continuum, this approach is fuelled by the fact that commercial structural software neglect friction, fracture and contact. A set of basic material restrictions, founded upon Limit Analysis, are established at the macroscale to showcase the behaviour of masonry. This leads to a variational formulation, a mathematical approach that formulates the problem in terms of an extremum principle (minima/maxima), of the Boundary Value Problem (BVP). Meaning that numerical methods can be developed to find a function that satisfies the given differential equation of the domain, in this case masonry, while adhering to the conditions on the boundaries, in this case Heyman's restrictions (Bagi & Angelillo, 2023).

In this section, a concise overview of the Normal Rigid No-Tension (NRNT) masonry like material will be presented, while a more comprehensive mathematical representation will be provided in the "Benchmark Theories" Section.

The mathematical treatment NRNT materials is different from conventional equality-based formulations, it aims for inequalities that represent the unilateral characteristics of these materials. Groundbreaking work in the 1960s and 1970s by Fichera in 1973, Lions and Stampacchia in 1967, and Moreau in 1968 lead to the exploration of variational inequalities, linking them to convex analysis and sub-differentiability (Bagi & Angelillo, 2023). Early research focused on refining convex energy functions, addressing the monotonicity in the stress-strain law of NRNT materials. This gave rise to potential and complementary energy minimization problems. As for functional spaces, they were first introduced as Functions of Bounded Variation (BV) for Perfect Plasticity and later as functions of Bounded Deformation (BD) (Temam & Strang, 1980). BV and BD are integral for establishing the uniqueness and existence of the minimum potential and complementary energy.

In the Masonry domain, NRNT material is defined by three relations concerning stress and strain. In this case the solution of the BVP is a triplet of stress, strain and displacement that adhere to the special bounded variation functions (SBV) and special bounded variation measures (SBM) which are the restrictions on displacement and stress fields.

Hence, the search for solutions of BVP for NRNT material can be through the normality rule which presents equivalence between two energies: (1) total potential energy and (2) complementary energy. In the first we look for a minimum of the potential energy in a set of kinematically admissible displacement. In the second we search for the minimum of the complementary energy among the admissible stress states following the equilibrium approach (Gagliardo et al., 2021). This leads to the following numerical methods used to discretize the BVP formulations: displacement- and force-based methods, employing two strategies to approximate singular strains and stresses.

Two approaches exist for the displacement method: one seeking solutions in the space of smooth (at least continuous) functions and another seeking solutions in terms of singular strain distributions. The second approach aligns with 'rigid block' methodologies present in the literature, while the first one addresses convergence issues related to mesh dependency. Both variational formulations are inserted into mathematical programming problems with continuous variables. The methods in this category fall all under discretized numerical approaches either displacement-based approaches or stress-based approaches.

Displacement-based approaches to the minimum of the Total Potential Energy (TPE)

- **Continuum-Discontinuum Fracture (CDF)** method seeks solutions in the space of continuous functions derived from a suitable FE discretization. It assumes strain to be purely regular, allowing cracks to appear smeared inside elements, becoming mesh independent. It perfectly captures a clear subdivision of the structural domain into macro-regions behaving as rigid or quasi-rigid bodies. CDF envisions cracks as continuous deformations within the element (Iannuzzo, Block, et al., 2021).

- **Piecewise-Rigid Displacement (PRD)** is a two dual-energy based Linear Programming (LP) program where LP is a discretization of the dual problem. It optimizes external loads' potential energy with cracks defining rigid partitions. PRD doesn't need any initial assumptions about mechanisms, as fractures and forces are outcomes of optimization problems and rely solely on the chosen discretization. This approach accommodates small-displacement fields as well as large displacement scenarios where it considers both crack evolution and internal forces during prescribed displacements (Iannuzzo et al., 2020; Iannuzzo, Dell'Endice, et al., 2021; Olivieri, Iannuzzo, et al., 2022).

Stress-based approaches for the Total Complementary Energy (TCE)

- **Continuous Airy-based Stress for Singularities (CASS)** method intense stress singularities are avoided. Instead, narrow stress bands that form within the discrete elements represent potential singularities. These bands are similar to how phase fields simulate cracks. Importantly, they don't depend on the mesh size, ensuring the method's reliability (Bagi & Angelillo, 2023; Montanino et al., 2022).

- **Linear Arch Static Analysis (LASA)**, a method that models the masonry material as NRNT material in the sense of the safe theorem of the Limit Analysis. It is built on 1D no-tension filaments forming space linear arches, this efficient design represents compressive elements spiralling downstairs. Horizontal uniaxial compressive forces from exterior wall support maintain filament equilibrium, contained within the masonry. It has been applied to circular and elliptical stairs (Angelillo et al., 2021; Olivieri, Cennamo, et al., 2022; Olivieri, Iannuzzo, et al., 2022).

- **Membrane Equilibrium Analysis (MEA)** method looks for one membrane solution that can be found in equilibrium under the given loads, through a purely compressive stress field, within the volume of the studied masonry dome, vault or stair. The proposed methods looks for the equilibrium of membranes by minimizing an error function with respect to shape parameters and BV, in equilibrium with admissible stress fields (Angelillo, 2015; Montanino et al., 2021; Olivieri, Angelillo, et al., 2021; Olivieri, Fortunato, et al., 2021).

c. Continuum models

In continuum models, the calibration of masonry models often bypasses the need for Representative Volume Element (RVE) based homogenization procedures. Experimental tests, including those derived from analytical strength domains, play a key role in this calibration process (Maier & Nappi, 1990). Despite the theoretically rigorous deduction of homogeneous model properties through homogenization theories, various simplified approaches have proven effective in real applications.

One such family of direct approaches involves a radical idealization of masonry, conceptualizing it as a material devoid of tension. This perfectly no-tension material assumption, indicating an isotropic medium with no capacity for tensile stresses but linear-elastic otherwise, has been employed for preliminary structural analyses (Angelillo, 1994). While this hypothesis has found utility in the stability analysis of structures like masonry vaults and domes within geometry-based models, it has limitations, particularly in simulating post-peak behaviour and representing realistic tensile strengths.

Angelillo contributed to this area by proposing a FE solution based on a complementary energy theorem for elastic no-tension bodies (Angelillo, 1994). A piecewise-linear description of the no-tension material behaviour was developed, simplifying the discretized boundary value problem in finite terms. However, these no-tension approaches, although elegant, have primarily been limited to 2D problems, with recent attempts at extending to 3D structures (Briccola et al., 2018).

Continuum models for masonry structures also explore nonlinear constitutive laws based on fracture mechanics (Hillerborg et al., 1976; Rots & De Borst, 1987), damage mechanics (Dragon & Mróz, 1979; Løland, 1980), or plasticity theory (Lee & Fenves, 1998; Lubliner et al., 1989). Smeared crack, plastic, damage, and plastic-damage models, initially developed for concrete structures, have been adapted for masonry, an aspect that will be further investigated in the next section. Despite their efficiency, widespread use in historic structures, and applicability to 3D geometries, these models have limitations due to the multi-level anisotropy and heterogeneity introduced by mortar joints in masonry (Lotfi & Shing, 1991).

An emerging trend involves the development of orthotropic constitutive laws to capture the anisotropic nature of masonry (Lourenco et al., 1995; Lourenço et al., 1998). These models address the shortcomings of isotropic approaches, providing more accurate representations. Various orthotropic models have been proposed, each aiming to simulate the complex behaviour of masonry structures subjected to different loading conditions (Berto et al., 2002; Lopez et al., 1999; Pelà et al., 2011, 2013). In addition to coupling it recently with localized crack reclosure processing (Parent et al., 2016; Sellier et al., 2013). However, these direct continuum anisotropic approaches, while scientifically sound, present challenges in terms of computational effort and the number of material properties to be characterized (Parent et al., 2016; Pelà et al., 2014).

In addition to direct continuum models, homogeneous FE models employing FE limit analysis or systems of springs (Milani et al., 2012; Reyes et al., 2009). Despite their borderline status due to interfaces or springs, these models are practically represented as continua, justifying their classification in this category (Pantò et al., 2016, 2018).

Conversely, homogenization procedures and multi-scale approaches aim to derive constitutive laws from RVEs. A priori homogenization involves a two-step process, deducing mechanical properties from RVE-based homogenization and then incorporating these properties into the structural-scale model (Anthoine, 1995; Bati et al., 1999). Methodologies for this deduction range from closed-form and quasi-analytical methods to numerical

techniques (Kawa et al., 2008; Wei & Hao, 2009), covering elastic moduli, failure surfaces, and integration of homogenized properties in structural-scale models for real-case studies (Milani et al., 2007; Silva et al., 2017). These approaches demand higher computational efforts and a significant number of material properties compared to isotropic models, limiting their current application in real case studies.

d. Macro-element models

In macro-element models, masonry structures are represented as panel-scale components exhibiting a phenomenological or mechanically based nonlinear response. The primary structural elements identified in these models are piers and spandrels, with a focus on analysing the global seismic response of masonry buildings. Macro-element models assume the prevention of local failure modes, particularly associated with the out-of-plane response of masonry walls (Quagliarini et al., 2017). The global seismic response is intricately linked to the in-plane capacity of walls or the load transfer due to diaphragms.

Global analyses, often incremental-iterative static and/or dynamic, are conducted on 3D models to capture load transfer between bearing walls during horizontal actions. Identifying structural components such as piers and spandrels relies on a priori observations of damage in real buildings. Piers, responsible for carrying vertical or horizontal loads, and spandrels, horizontal sections between vertically aligned openings, play crucial roles in the seismic response. While identification is straightforward for regularly distributed openings, irregular layouts, as seen in historic masonry structures, pose challenges (Sorrentino et al., 2023).

In the category of macro-element models, there are equivalent beam-based and spring-based approaches. Equivalent beam-based models idealize masonry panels as nonlinear beams, with various formulations proposed (Addessi et al., 2015; Raka et al., 2015). Spring-based approaches, an alternative, involve implementing nonlinear springs within a fictitious frame (Vanin et al., 2020; Xu et al., 2018). Each approach has its advantages and is suited to different aspects of masonry structure analysis.

Macro-element models, widely used for seismic assessment due to their computational efficiency and ease of definition, have limitations. They assume the prevention of local out-of-plane failure modes, hindering accurate seismic capacity estimates. Furthermore, they may overlook structural details like toothing between orthogonal walls and face challenges with irregular opening layouts. Despite these drawbacks, their simplicity and wide dissemination make them popular among practitioners.

e. Block-based models

First introduced by Page (A. W. Page, 1978), Block-based models are an assembly of blocks interconnected by mortar joints, thereby influencing mechanical and failure responses. They accurately represent heterogeneity of masonry, its bond, failure modes, and account for anisotropy. They offer 3D solid and 2D shell models to address both in-plane and out-of-plane responses. Block-based models are classified based on interaction formulations between blocks.

Interface Element-Based Approaches

An initial nonlinear interface model for masonry collapse (Lotfi & Shing, 1994) used zero-thickness interface elements for mortar joints and smeared crack elements for expanded masonry units. It was then improved (Lourenço & Rots, 1997) focusing nonlinearities in interfaces, applied in addition to extensions in these studies (Calderón et al., 2017; Oliveira & Lourenço, 2004; Sandoval & Arnau, 2017; Senthivel & Lourenço, 2009). Later a cyclic mortar joint interface model was developed, showcasing brittleness under tension, with frictional dissipation and stiffness degradation during compression (Gambarotta & Lagomarsino, 1997). Further cohesive interfaces for masonry shear wall simulation were discussed in (Alfano & Sacco, 2006; Gambarotta & Lagomarsino, 1997; Parrinello et al., 2009). Rigid block strategies, in the applied element method, employed nonlinear springs for in-plane cyclic behaviour analysis (Malomo et al., 2018).

Lourenço and Rots (Lourenço & Rots, 1997) model was expanded for 3D analysis, incorporating geometric nonlinearity (Macorini & Izzuddin, 2011). This co-rotational approach, applied in (Chisari et al., 2015, 2018), was further adapted for simulating the cyclic response of masonry structures (Minga et al., 2018). Another interface constitutive model (Aref & Dolatshahi, 2013), coupled with elasto-plastic block elements, facilitates explicit cyclic analysis of 3D masonry walls, extensively used for investigating various aspects of masonry mechanics.

Contact-Based Approaches

Block-based modelling based on contact mechanics, is extensively used for accurate masonry structure representation, involving rigid blocks interacting through frictional or cohesive-frictional contacts.

Firstly, the Discrete Element Method (DEM), initially proposed by Cundall and Stack, has been widely applied for analysing granular assemblies, implemented in the UDEC code (Lemos, 2019; Sarhosis et al., 2019). The 3D version of

UDEC, known as 3DEC, has further enhanced these analyses. DEM utilizes contact penalty formulations and explicit integration (Bagi & Angelillo, 2023), leading to applications on real masonry structures (Diaferio et al., 2022; Foti et al., 2018; Galvez et al., 2022; Lemos, 2019; Paris et al., 2021) using rigid or linear elastic blocks. Typically, this method adheres to Heyman's assumptions, employing rigid blocks with infinite compressive strength and zero cohesion, while modeling the stereotomy of the blocks in 3D with the real friction angle to prevent sliding. Furthermore, DEM allows for the application of different constitutive laws to deformable blocks, enabling more versatile modeling of masonry behavior. DEM has been widely used in the assessment of masonry, and numerous scholars have significantly contributed to this field, including Lemos, De Jong, Malomo, Mehrotra, the Block Research Group, and Bagi (Bagi & Angelillo, 2023) and the references therein.

Secondly, the Discontinuous Deformation Analysis (DDA) is an implicit approach considering block deformability, ensuring constraints of no tension and no penetration between blocks. It adheres to Coulomb's law at all contact positions for static and dynamic computations (Bagi & Angelillo, 2023; Sarhosis et al., 2019).

Thirdly, the Non-Smooth Contact Dynamics (NSCD) method (Jean, 1999), employs a direct contact formulation in its non-smooth form, uses implicit integration, and energy dissipation due to blocks' impacts. Although it proved to be successful in some studies (Beatini et al., 2017; Lancioni et al., 2016), NSCD is limited to dry stone masonry structures as it is unable to represent the cohesive responses in mortar joints.

These approaches may not fully address masonry crushing, a crucial aspect in certain mechanical responses. To take this into account, Sarhosis and Lemos (Sarhosis & Lemos, 2018) integrated crushing in the Finite-Discrete Element Method (FDEM), representing units and mortar joints as densely packed particles with zero-thickness interfaces (Smoljanović et al., 2013). In 3D structures this method was expanded to simulate nonlinear block responses (Smoljanović et al., 2013, 2018). A recent 3D block-based model explicitly incorporates mortar layers using implicit integration, a contact penalty method, compressive and tensile damage for the blocks, and rigid-cohesive-frictional contact behaviour (D'Altri et al., 2018, 2019). This FEM-DEM approach has as well been applied in several studies on masonry structures (Pepe et al., 2020; Reccia et al., 2012, 2015).

Textured Continuum-Based Approaches

These models are in FEM framework with nonlinear elements, modelling

blocks separately without any interface with the joints. This allows to represent nonlinear deformations and failure in blocks, mortar, or mortar joints. Ali and Page (Ali & Page, 1988) introduced mesh discretization, distinguishing finite elements (FEs) with block and mortar joint properties. Their approach uses a strength criterion for crack initiation and propagation, employing the smeared crack modelling technique. Advancements employ continuum elements to discretize both units and mortar joints, incorporating a tension/compression damage model (Petracca et al., 2017). Another solution uses a kinematic damage model (Addessi & Sacco, 2016; Serpieri et al., 2017).

Block-Based Limit Analysis Approaches

This approach is precise and powerful for predicting the collapse load and failure mechanisms of masonry structures. 2D and 3D developed approaches, rely on static or kinematic theorems of limit analysis. An initial approach solved the nonlinear programming problem for friction at interfaces through linearized limit analysis (Baggio & Trovalusci, 1998). Then the computation of collapse loads for rigid block systems as a constrained optimization problem was formulated (Ferris & Tin-Loi, 2001). Using linear programming and finite elements the lowest possible limit loads for unreinforced masonry shear walls were calculated, ensuring a strict lower bound on the actual collapse load (Sutcliffe et al., 2001). In (Orduña & Lourenço, 2005a, 2005b) a method for the non-associated limit analysis of rigid block masonry groups was introduced. Subsequently, Milani addressed masonry crushing and mortar joint cohesion in a 3D finite element upper bound limit analyses considering Mohr-Coulomb failure criteria (Milani, 2008). Nevertheless, their computational demand, especially for large-scale real masonry structures, remains high.

Extended Finite Element Approaches

In extended finite element methods (XFEM), few block-based models have been proposed. A 3D surface-based cohesive behaviour model which capture elastic and plastic masonry joint behaviour is introduced in (Abdulla et al., 2017). It incorporates a Drucker-Prager plasticity model to simulate crushing under compression. XFEM is also used to model the cracking behaviour and compressive failure of masonry in infill panels (Zhai et al., 2017).

However, for full-scale masonry structures, the computational demand of block-based models is high and an approximation of the actual bond in existing structures is needed. While advancements in computational capabilities may help overcome the high computational time the complexity of modelling limits its use to academic and high-level consultancy groups.

2. REFERENCE APPROACHES

This section will provide a comprehensive overview of the approaches that will be used as reference points in this thesis, along with an explanation of their mathematical framework. These approaches fall under two categories: discrete computation mechanics and continuum mechanics. The chosen benchmarks for comparison are Piecewise-Rigid Displacement method (PRD) from the discrete computation mechanics category and the finite element model, Concrete Damage Plasticity (CDP), from the continuum mechanics category.

2.1. DISCRETE COMPUTATIONAL MECHANICS

2.1.1. PIECEWISE-RIGID DISPLACEMENT (PRD)

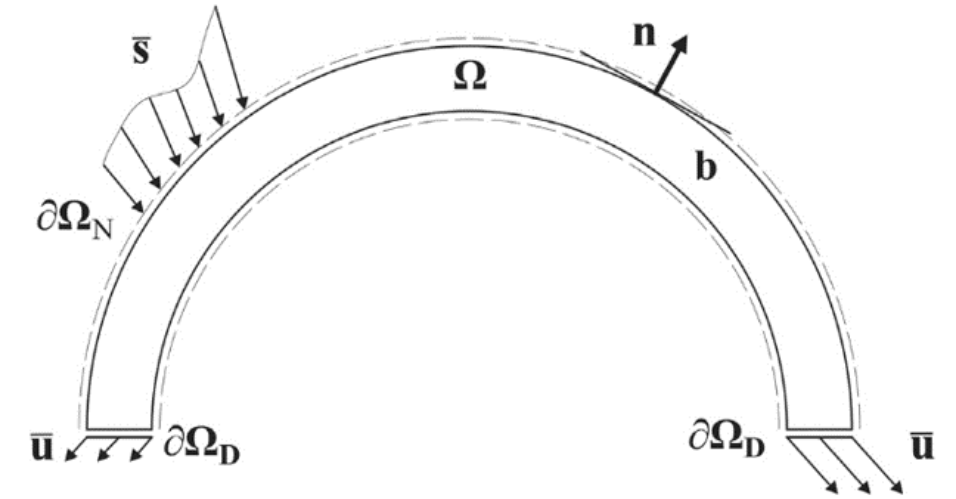
Recalling what has been presented, PRD is a displacement-based energy approach of the NRNT masonry-like material. It is based on the minima of both potential and complementary energy. A representation of the mathematical framework will start with the introduction of the NRNT material, the boundary value problem, the displacement and stress jump and eventually the methodology of this approach.

2.1.1.1 NRNT Material

A masonry structure in two dimensions is represented as a continuum within the Euclidean space $\mathbb{E}^2$, specifically occupying the region Ω . The stress tensor acting on Ω is denoted as $\mathbf{T}$, and $\mathbf{b}$ signifies the body load field. The displacement field is expressed as $\mathbf{u}_{(x)}$, corresponding to the infinitesimal strain tensor $\mathbf{E}$. The vector $\mathbf{n}$ represents the outward normal to the boundary $\partial\Omega$, partitioned into its constrained part $\partial\Omega_D$, where the displacement field $\mathbf{u}_{(x)}$ takes the value $\mathbf{u}$, and its complementary and loaded part $\partial\Omega_N$, subjected to the applied load $\bar{\mathbf{s}}$ Figure 24.

[Figure 24]

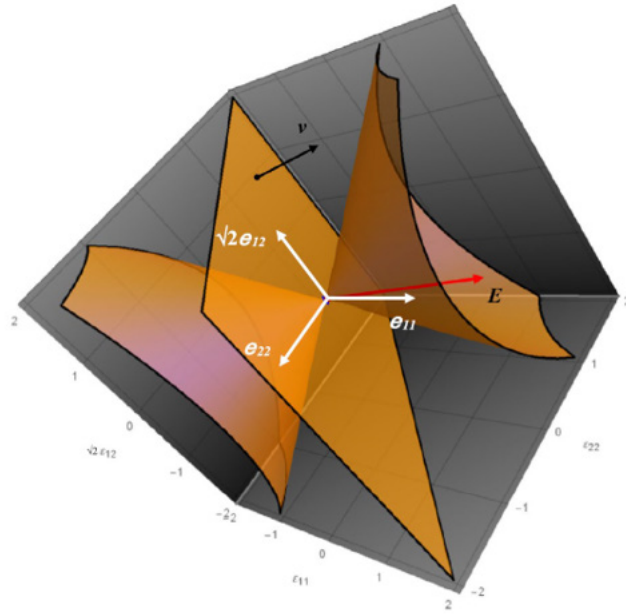
A two-dimensional continuum occupying a region Ω of the Euclidean space. Load tractions on $\partial\Omega_N$ are represented by $\bar{\mathbf{s}}$ whilst prescribed boundary displacement on $\partial\Omega_D$ by $\bar{\mathbf{u}}$. Figure from (Iannuzzo, Dell'Endice, et al., 2021).



The extension of NRNT to masonry based on Heyman's restrictions is based on the following constitutive relations:

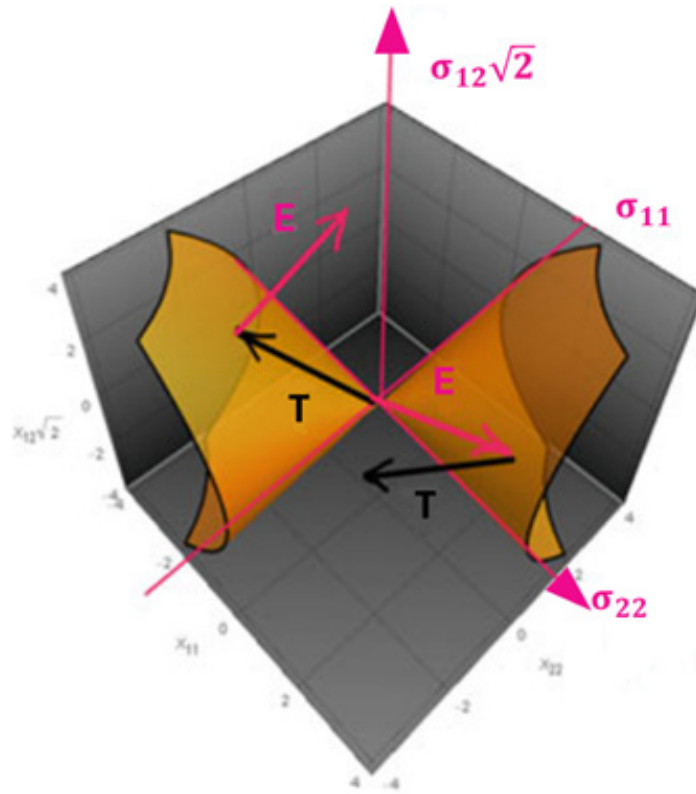
$$\mathbf{T} \in \text{Sym}^-, \mathbf{E} \in \text{Sym}^+, \mathbf{T} \cdot \mathbf{E} = \mathbf{0} \quad (1)$$

such that Sym^- and Sym^+ represent the polar cones of semidefinite negative and positive symmetric tensors (see Figure 25). These three relations are equivalent to the well known normality conditions (see Figure 26) (Bagi & Angelillo, 2023).



[Figure 25]

A symbolic 3D representation of the polar cones of semidefinite negative and positive symmetric tensors (Bagi & Angelillo, 2023).



[Figure 26]

The normality and dual normality conditions. Figure from Prof. Maurizio Angelillo's notes, modified by the author.

2.1.1.2. Boundary Value Problem (BVP)

The following is a summary of the relations of BVP for a continuum of a NRNT material subjected to $\bar{\mathbf{s}}$ and respective displacements $\bar{\mathbf{u}}$

$$\mathbf{E} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T), \mathbf{E} \in \text{Sym}^+, \mathbf{u} = \bar{\mathbf{u}} \text{ on } \delta\Omega_D, \quad (2)$$

$$\text{div} \mathbf{T} + \mathbf{b} = \mathbf{0}, \mathbf{T} \in \text{Sym}^-, \mathbf{T} \mathbf{n} = \bar{\mathbf{s}} \text{ on } \delta\Omega_N, \quad (3)$$

$$\mathbf{T} \cdot \mathbf{E} = 0. \quad (4)$$

Hence, finding the triplet of stress, strain and displacement that adhere to the relations is the solution of the BVP.

The strain tensor $\mathbf{E}$ and stress tensor $\mathbf{T}$ are uncoupled except for their product yielding zero. This allows the introduction of the two sets $\mathcal{K}$ and $\mathcal{H}$, which are the kinematically admissible displacements and the statically admissible stresses respectively:

$$\mathcal{K} = \left\{ \mathbf{u} \in S / \mathbf{E} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \in \text{Sym}^+ \text{ \& } \mathbf{u} = \bar{\mathbf{u}} \text{ on } \partial\Omega_D \right\}, \quad (5)$$

$$\mathcal{H} = \{ \mathbf{T} \in S' / \text{div} \mathbf{T} + \mathbf{b} = \mathbf{0}, \mathbf{T} \in \text{Sym}^-, \mathbf{T} \mathbf{n} = \bar{\mathbf{s}} \text{ on } \partial\Omega_N \}. \quad (6)$$

Where S and S' are two suitable functionals (takes function as input and produces a scalar as output). The search for $\mathbf{u} \in \mathcal{K}$ and $\mathbf{u} \in \mathcal{K}$ are the kinematical problem (KP) and equilibrium problem (EP) respectively.

1. Displacements and stress jumps

The fact that in NRNT the strain and stress fields are bounded measures they can be decomposed into an addition of a regular and singular part each:

$$\mathbf{E} = \mathbf{E}^r + \mathbf{E}^s, \mathbf{T} = \mathbf{T}^r + \mathbf{T}^s \quad (7)$$

The regular part of the strain is disregarded, aligning with Heyman's approach, who never addressed it in his works. The singular parts are depicted by 1D stress fields T^s (e.g., thrust lines) and 'displacement jumps' along specific lines E^s (e.g., hinges).

Usually, Dirac delta functions are employed to represent concentrated masses, point charges or instantaneous events, here it helps represent the singular stress field. Based on the aforementioned, a Dirac delta function $\delta(\Gamma)$ can be identified such that $\mathbf{n}$ and $\mathbf{t}$ are its unit and tangential vectors respectively. Hence the stress field Γ is:

$$\mathbf{T}^s = P\delta(\Gamma)\mathbf{t}\otimes\mathbf{t}, \quad (8)$$

Such that P is the magnitude of the concentrated stress which must be negative according to equation (1). Conversely, due to the displacement jumps $[\mathbf{u}]$ across Γ , a singular strain field arises E^s taking this form:

$$\mathbf{E} = v\delta(\Gamma)\mathbf{n}\otimes\mathbf{n} + \frac{1}{2}w\delta(\Gamma)(\mathbf{t}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{t}), \quad (9)$$

Such that

$$v = [\mathbf{u}] \cdot \mathbf{n}, w = [\mathbf{u}] \cdot \mathbf{t}, \quad (10)$$

knowing that $\mathbf{E} \in \text{Sym}^+$ implies that: $v \geq 0$ and $w = 0$.

2.1.1.3. PRD Methodology

The essential elements at the base of PRD are as follows. A 3D continuum masonry structure $\Omega \subset \mathbb{R}^3$ is partitioned into M elements as:

$$(\Omega_i)_{i \in \{1, \dots, M\}} \quad (11)$$

Following a Mohr-Coulomb friction-associated flow rule the contact among the elements is unilateral. The constrained and loaded part of the boundary of Ω are represented respectively by the two open and disjoint parts $\partial\Omega_U$, $\partial\Omega_F$ while $\partial\Omega_I$ is the union of $\partial\Omega_{Ii}$ for $i \in \{1, \dots, M\}$, i.e. the union of the interfaces shared by two adjacent elements of (11). Prescribed displacements are applied on $\partial\Omega_U$:

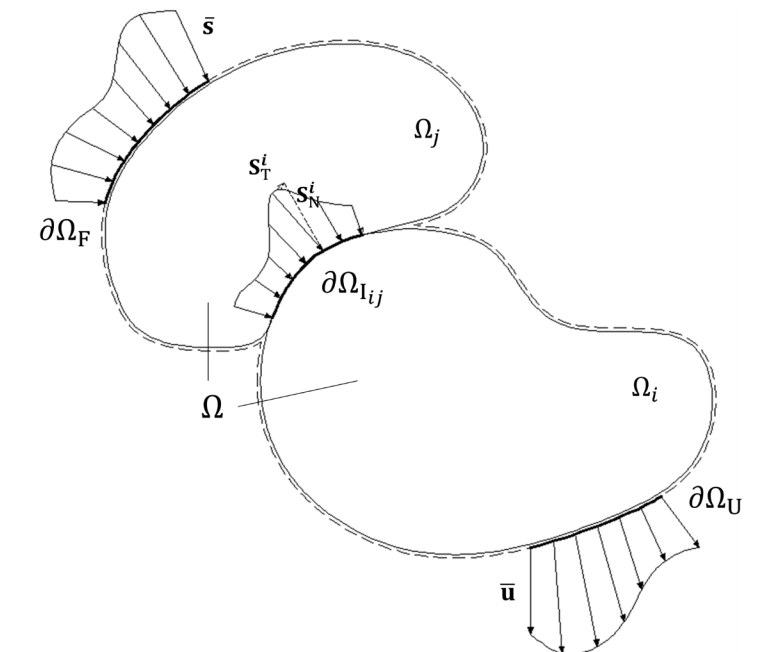
$$\mathbf{u} = \bar{\mathbf{u}}, \quad (12)$$

while on $\partial\Omega_F$:

$$\mathbf{s}(\mathbf{T}) = \bar{\mathbf{s}}, \quad (13)$$

such that $\mathbf{s}(\mathbf{T})$ is the trace of $\mathbf{T}$ on the boundary.

Figure 27 refers to a continuum decomposed into two elements and gives an overview of the problem and its mathematical representation.



[Figure 27]

An elastic body Ω subjected to body forces $\mathbf{b}$, prescribed boundary displacements $\bar{\mathbf{u}}$ on $\partial\Omega_U$, friction boundary condition on $\partial\Omega_I$ with given normal S_N and tangential S_T pressures, and tow surface forces $\bar{\mathbf{s}}$ on the $\partial\Omega_F$ (Olivieri, Iannuzzo, et al., 2022).

Equation (13) applies to internal interface, where the stress and displacement space must accommodate the singularities represented by Dirac delta distributions on $\partial\Omega_I$. The displacement field must exhibit jumps Δu^i on $\partial\Omega_I$, decomposed into Δu_N^i and tangential Δu_T^i components. The friction condition, linked to the behaviour, involves L2 norm of the displacement jump in the tangential direction t , denoted as $|\Delta u_T^i|$:

$$\text{if } |\mathbf{S}_T^i| \leq \mu |\mathbf{S}_N^i| \text{ then } |\Delta \mathbf{u}_T^i| = 0, \quad (14)$$

$$|\mathbf{S}_T^i| = \mu |\mathbf{S}_N^i|, \text{ there exists } \lambda \geq 0 \text{ s.t. } \Delta \mathbf{u}_T^i = -\lambda \mathbf{S}_T^i \text{ and } \Delta \mathbf{u}_N^i = -\lambda \mu \mathbf{S}_T^i, \quad (15)$$

where μ is the friction coefficient, and S_N^i and S_T^i are the unknown, normal and tangential force distribution on Γ_I . The stress tensor is symmetric and abides to this requirement:

$$\mathbf{T} \in \mathbf{C}, \quad (16)$$

where $\mathbf{C}$ is the convex Mohr-Coulomb cone. The set of admissible stress fields is:

$$\mathcal{H} = \left\{ \mathbf{T} \in \text{SBM} \mid \text{div} \mathbf{T} + \mathbf{b} = \mathbf{0}, \mathbf{T} \in \mathbf{C}, \mathbf{s}(\mathbf{T}) = \bar{\mathbf{s}} \text{ on } \partial\Omega_F \right\}. \quad (17)$$

The solution for the BVP is a stress $\mathbf{T} \in \mathcal{H}$ and a displacement $\mathbf{u} \in \mathcal{K}$. Using the Gauss-Green theorem the following variational criterion can be derived:

$$A(\mathbf{T}^*, \mathbf{T} - \mathbf{T}^*) \geq \int_{\partial\Omega_U} \bar{\mathbf{u}} \cdot (\mathbf{T}\mathbf{n} - \mathbf{T}^*\mathbf{n}) d\mathbf{a}, \quad \forall \mathbf{T} \in \mathcal{H}, \quad (18)$$

where $\mathbf{A}$ doubles the complementary elastic energy, and $\mathbf{T}^*$ corresponds to the boundary value problem solution. Relation (20) is equivalent to the following minimisation problem:

$$\mathbf{T}^* = \arg \min \{ \Pi_c(\mathbf{T})_{\mathbf{T} \in \mathcal{H}} \}, \quad (19)$$

with

$$\Pi_c(\mathbf{T}) = \frac{1}{2} \mathbf{A}(\mathbf{T}, \mathbf{T}) - \int_{\partial\Omega_U} \bar{\mathbf{u}} \cdot \mathbf{T}\mathbf{n} d\mathbf{a}, \quad (20)$$

where $\Pi_c(\mathbf{T})$ is the TCE. For an associated flow rule, the dual problem is represented by the total potential energy. Notably, when $\mathbf{C}$ aligns with Sym^+ , the previous formulation results in a classical Heyman problem.

2. Numerical Discretization of Minimum Problem

TCE minimum problem is numerically discretised into the following Second-Order Cone Program adopting a typical concave formulation:

$$\begin{aligned} &\text{Minimise} && -\mathbf{B}_s \mathbf{F}^T \bar{\mathbf{U}} + \mathbf{F}^T \mathbf{D} \mathbf{F} \\ &\text{s.t.} && \mathbf{A}_{\text{eq}} \mathbf{F} + \mathbf{b} = \mathbf{0} \end{aligned} \quad (21)$$

$$\|\mathbf{f}_t^i\|_2 \leq \mu \mathbf{f}_n^i \text{ for } i = 1, \dots, n$$

The meaning of symbols appearing in formulas (21) is the following:

- n is the total number of interfaces' nodes
- $\mathbf{F}$ is the vector collecting the internal nodal normal $\mathbf{f}_n^i$ and tangential $\mathbf{f}_t^i$ are the forces on the node i ;
- $\bar{\mathbf{U}}$ is the vector collecting the boundary displacement of the supports.
- $\mathbf{F}_s = \mathbf{B}_s \mathbf{F}$ is the emerging stress vector obtained using the extract operator $\mathbf{B}_s$
- $\mathbf{D}$ is the compliance (deformability) matrix that takes the elastic block's compressive response into account.
- $\mathbf{A}_{\text{eq}}$ is the equilibrium matrix
- $\mathbf{b}$ is the vector modelling the volume forces as lumped to the blocks' centroid.

Equation (21)² ensures equilibrium via matrix $\mathbf{A}_{\text{eq}}$, while equation (21)³ represents the Mohr-Coulomb relationship between tangential and normal forces, scaled down by the friction coefficient μ . There are n second-order cone constraints, corresponding to the total nodes n , represented by equation (21)³. The objective function (21)¹ represents the TCE, with the linear part representing boundary work and the quadratic function enforcing the

elastic behaviour of the interface using the compliance matrix $\mathbf{D}$, which considers different mechanical properties of the interface and their geometric distribution in the digital model.

Neglecting Equation (21) yields a standard Heyman problem, equivalent to assuming infinite friction capacity. Omitting the quadratic part means rigid contacts, providing the dual formulation of the PRD problem for NRNT material.

2.2. CONTINUUM COMPUTATIONAL MECHANICS

It is essential to discuss the macro modelling continuum approaches in detail before introducing the CDP model. The complexity of modelling historic masonry structures incorporating it with the use of sophisticated methods and computational models, often requires material parameters that are challenging to obtain through experimental testing. In such cases, engineering judgment and the modeler's experience play a crucial role.

Macro-modelling treats masonry as a homogeneous continuum medium, with no differentiation between individual units and mortar. Continuum plasticity (Lourenco et al. 1998) and continuum damage models (Papa et al. 2000) have been successfully developed and implemented for in-plane analysis of masonry shear walls. These models assume average stresses and strains to describe the behaviour of the composite medium, with different elastic and inelastic properties along the material axes. While suitable for large-scale masonry structures, these models may not be valid for small-scale masonry walls or structures with large masonry units exhibiting blocky behaviour.

Damage-plasticity model approach

The plasticity model integrates stiffness degradation within a plasticity framework, offering a comprehensive understanding of material behaviour under various loading conditions. Initially proposed by (Simo & Ju, 1987) and subsequently extended by researchers like (Lubliner et al., 1989) and (Yazdani & Schreyer, 1990), the plastic-damage approach embeds stiffness degradation in a plasticity model. The coupled elastoplastic-damage model incorporates

the effective-stress concept from continuum damage mechanics to represent stiffness degradation. This model allows for the decoupling of damage from plastic deformation, although the calibration of parameters remains challenging due to the reliance on experimental stress data.

The Barcelona model, proposed by (Lubliner et al., 1989), introduces a fracture-energy-based scalar damage variable along with elastic and plastic degradation variables. While this model provides a means to simulate elastic stiffness degradation, the coupled relations lead to a complex and unstable numerical algorithm, causing fake plastic unloading during iteration (Lee & Fenves, 1998). Recognizing the inadequacy of a single damage variable for quasi-brittle materials undergoing cyclic loading, the model proposes a new plastic-damage model. Utilizing fracture-energy-based damage, this model introduces two damage variables for tensile and compressive damage states, allowing for a more accurate representation of material response. The modified yield function facilitates convenient calibration with experimental results, and a thermodynamically consistent stiffness recovery scheme is introduced. This plastic-damage model enables the separate evolution of tensile and compressive strengths, providing a valuable tool for concrete modelling under cyclic loading (Lee & Fenves, 1998).

2.2.1. CONCRETE DAMAGE PLASTICITY (CDP)

Advanced nonlinear Finite Element Modelling (FEM) methods have been developed offering more realistic results by incorporating additional parameters to model tensile strength, softening, and other crucial factors, as demonstrated in various studies. They combine aspects of both continuum plasticity and continuum damage models.

Concrete Damage Plasticity (CDP) model in ABAQUS (Abaqus, 2023) is an example of these FEM techniques. In CDP, the material is considered as a continuum, and it models both the plastic deformation (inelastic behaviour) and the accumulation of damage (cracking and degradation) in the material. This model represents cracks as smeared bands, with the band's width varying as the material undergoes loading, thus incorporating the smeared-crack approach as well. Another technique is the quasi-brittle damage material model in CAST3M and Code_Aster (Parent et al., 2024), which characterizes the fracture mechanism using a single crack line. This line exhibits the ability to cyclically transition between the plastic and elastic regions of the stress-strain diagram, with its behaviour influenced by temperature variations.

While these sophisticated approaches enable modelling the impact of foundation displacement, they come at the cost of increased modelling complexity and data requirements, making the analyses computationally intensive.

Our focus in this thesis is on CDP which is commonly used to analyse the response of quasi-brittle materials like concrete, which can undergo significant deformations and develop microcracks or damage under loading (Lee & Fenves, 1998; Lubliner et al., 1989).

1. Framework of CDP

Constitutive relations, derived from thermodynamic potential functions, maintain consistency within the thermodynamic framework. However, constructing a constitutive system for quasi-brittle materials like concrete based on physical observations is challenging due to the need for well-defined potential functions and the preferred associativity between state variables and conjugate forces. In this study, we directly derive constitutive equations for a plastic-damage model, ensuring thermodynamic consistency while addressing the specific challenges associated with modelling concrete.

In the incremental theory of plasticity, the strain tensor denoted $\boldsymbol{\varepsilon}$ is decomposed into the elastic tensor $\boldsymbol{\varepsilon}^{\text{el}}$ and the plastic tensor $\boldsymbol{\varepsilon}^{\text{pl}}$, presented as

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\text{el}} + \boldsymbol{\varepsilon}^{\text{pl}} \text{ such that } \boldsymbol{\varepsilon}^{\text{el}} = \boldsymbol{\sigma} : \mathbf{E} \quad (22)$$

The elastic stiffness $\mathbf{E}$ is a rank four tensor and $\boldsymbol{\sigma}$ is the stress tensor. The plastic strain represents all irreversible deformations including those caused by microcracks.

From (22)

$$\boldsymbol{\sigma} = \mathbf{E} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\text{pl}}) \quad (23)$$

The stress is mapped into an effective stress $\bar{\boldsymbol{\sigma}}$ by a rank four tensor $\mathbf{D}$, based on the concept of continuum damage theory:

$$\bar{\boldsymbol{\sigma}} = \mathbf{D} : \boldsymbol{\sigma} \quad (24)$$

The effective stress is defined by elastic stiffness yields:

$$\bar{\boldsymbol{\sigma}} = \mathbf{E}_0 : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\text{pl}}) \quad (25)$$

With $\mathbf{E}_0$ the initial elastic-stiffness tensor.

In many cases, for practical reasons, the degradation damage is assumed isotropic. If a scalar degradation damage variable $0 \leq D < 1$ is used to represent the isotropic damage, then:

$$\mathbf{D} = \left(\frac{1}{1-D}\right)\mathbf{I}, \quad (26)$$

Where $\mathbf{I}$ is the rank four identity tensor, and consequently:

$$\begin{aligned} \boldsymbol{\sigma} &= (\mathbf{1} - \mathbf{D})\bar{\boldsymbol{\sigma}}; \\ \bar{\boldsymbol{\sigma}} &= (1 - D)\mathbf{E}_0 : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\text{pl}}) \end{aligned} \quad (27\text{a,b})$$

Comparing (25) and (27a,b) it yields $\mathbf{E} = (1 - D) \mathbf{E}_0$,

As for the plastic strain $\boldsymbol{\varepsilon}^{\text{p}}$ it is evaluated by a flow rule, assumed to be generated from a scalar plastic potential function ϕ . The plastic stain is given by:

$$\dot{\boldsymbol{\varepsilon}}^{\text{pl}} = \dot{\lambda} \nabla \phi(\bar{\boldsymbol{\sigma}}) \quad (28)$$

where λ is a nonnegative function referred to as plastic consistency parameter.

Another damage internal variable set is needed to represent the damage states, in this case $\mathbf{K}$ given as:

$$\dot{\mathbf{K}} = \dot{\lambda} \mathbf{H}(\bar{\boldsymbol{\sigma}}, \mathbf{K}) \quad (29)$$

where the function $\mathbf{H}$ is derived considering plastic dissipation, as will be shown later.

To accommodate for concrete different damage responses in tension and compression, a multi-hardening or multi-softening yield function is employed. This yield function uses two state variables, f_t and f_c , representing the uniaxial tensile strength and compressive strength of the material, respectively. The admissible stress states are constrained by the following condition:

$$\widetilde{F}(\boldsymbol{\sigma}, f_t, f_c) \leq 0 \quad (30)$$

Such that $\widetilde{F}$ is a first-degree homogenous function with respect to all three variables as well as being an isotropic function in the stress space.

In the CDP model f_t and f_c are expressed in function of two damage parameters K_t and K_c . The two state variables can be factored into the degradation damage and effective-stress responses in the same form as equation (5 in lee and fen), giving:

$$\begin{aligned} f_t &= [1 - D_t(K_t)]\bar{f}_t(K_t), \\ f_c &= [1 - D_c(K_c)]\bar{f}_c(K_c) \end{aligned} \quad (31a,b)$$

where D_t and D_c are respectively the tensile and compressive degradation damage responses laying in the ranges $0 \leq D_t < 1$ and $0 \leq D_c < 1$, while $\bar{f}_t$ and $\bar{f}_c$ are the effective-stress responses in the uniaxial tensile and compressive stress functions, respectively. A single degradation damage variable D can be introduced:

$$D = D_{(\mathbf{K})} = 1 - (1 - D_t)(1 - D_c) \quad (32)$$

Hence $\bar{f}_t$ and $\bar{f}_c$ in (31a,b) can be written as:

$$\begin{aligned} f_t &= (1 - D)\bar{f}_t, \\ f_c &= (1 - D)\bar{f}_c \end{aligned} \quad (33a,b)$$

With both definitions of $\widetilde{F}$ and D , substituting the relation (27a,b) and (33a,b) into (30) it yields:

$$\widetilde{F}(\bar{\boldsymbol{\sigma}}, \bar{f}_t, \bar{f}_c) \leq 0 \quad (34)$$

After introduction of a vector function $\mathbf{c} = \mathbf{c}(\mathbf{K})$, the latter function is rewritten as a function of stress and damage variable,

$$F(\bar{\boldsymbol{\sigma}}, \mathbf{K}) \leq 0, \quad (35)$$

where F also is an isotropic, first-degree homogeneous scalar function with multiple-hardening evolution.

With the previously described framework, the elastoplastic responses in CDP are described in terms of $\bar{\boldsymbol{\sigma}}$ and $\mathbf{K}$

$$\begin{aligned} \bar{\boldsymbol{\sigma}} &= \mathbf{E}_0 \oslash (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{pl}) \in \{\bar{\boldsymbol{\sigma}}/F(\bar{\boldsymbol{\sigma}}, \mathbf{K}) \leq 0\} \\ \dot{\boldsymbol{\varepsilon}}^{pl} &= \dot{\lambda} \nabla \phi(\bar{\boldsymbol{\sigma}}) \\ \dot{\mathbf{K}} &= \dot{\lambda} \mathbf{H}(\bar{\boldsymbol{\sigma}}, \mathbf{K}) \end{aligned} \quad (36a,b,c)$$

Additionally, from Kuhn-Tucker complementary conditions and the plastic consistency condition are used as loading/unloading condition,

$$\dot{\lambda} \geq 0; \dot{\lambda} F = 0; \dot{\lambda} \dot{F} = 0 \quad (37a,b,c)$$

From (32) the total stress is determined by evaluating the degradation damage

$$\boldsymbol{\sigma} = [1 - D(\mathbf{K})\bar{\boldsymbol{\sigma}}] \quad (38)$$

The relations from (36a,b,c) to (38), have an effective numerical implementation, since the solution for the elastoplastic response is separate from the degradation damage process.

2. Evolution of Damage

In quasi-brittle materials damage is defined through the evaluation of the fracture energy required to generate microcracks. In the Barcelona model (Lubliner et al., 1989), one scalar damage variable represents all damage states (compressive and tensile). This scalar variable simulates the monotonic loading problem. Concrete experiences different damage process under tension and compression, therefor one scalar variable is not enough to account for its cyclic behaviour. Hence CDP introduces two damage variables, one for tensile damage and the other for compressive damage, each variable is factorized into the effective-stress response and stiffness degradation response parts.

Considering a uniaxial tensile and compressive stress state, a state variable $\mathbf{x} \in \{t, c\}$ is introduced.

In the Barcelona model the relation between the uniaxial stress $\sigma_{\mathbf{x}}$ and scalar plastic strain ϵ^p is:

$$\bar{\sigma}_{\mathbf{x}} = f_{\mathbf{x}0}[(1 + a_{\mathbf{x}})\exp(-b_{\mathbf{x}}\epsilon^p) - a_{\mathbf{x}}\exp(-2b_{\mathbf{x}}\epsilon^p)] \quad (39)$$

Where $f_{\mathbf{x}0}$ is the initial yield stress (maximum yield stress without damage); and $a_{\mathbf{x}}$ and $b_{\mathbf{x}}$ as constants.

A factorization following equation (31a,b) while assuming that the degradation takes an exponential form yield:

$$1 - D_{\mathbf{x}} = \exp(-d_{\mathbf{x}}\epsilon^{pl}) \quad (40)$$

such that $d_{\mathbf{x}}$ is a constant.

Hence the effective stress is denoted as

$$\bar{\sigma}_{\mathbf{x}} = f_{\mathbf{x}0} \left[(1 + a_{\mathbf{x}})(\exp(-b_{\mathbf{x}}\epsilon^{pl}))^{1-(\frac{d_{\mathbf{x}}}{b_{\mathbf{x}}})} - a_{\mathbf{x}}(\exp(-b_{\mathbf{x}}\epsilon^{pl}))^{2-(\frac{d_{\mathbf{x}}}{b_{\mathbf{x}}})} \right] \quad (41)$$

For the uniaxial version of the damage variable denoted as $0 \leq \mathbf{K}_{\mathbf{x}} \leq 1$ it is expressed as

$$\mathbf{K}_{\mathbf{x}} = \frac{1}{g_{\mathbf{x}}} \int_0^{\epsilon^{pl}} \sigma_{\mathbf{x}}(\epsilon^{pl}) d\epsilon^{pl} \quad (42)$$

such that

$$g_{\mathbf{x}} = \int_0^{\infty} \sigma_{\mathbf{x}}(\epsilon^{pl}) d\epsilon^{pl} \quad (43)$$

$g_{\mathbf{x}}$ represents the dissipated energy density during the entire process of microcracking; since it cannot be given as a material property it has to be derived from other known material properties such as the fracture energy.

The fracture energy in uniaxial tensile state $G_{\mathbf{x}}$ and its counterpart in uniaxial compressive state are denoted as material properties $g_{\mathbf{x}}$ is the normalized specific fracture energy divided by the localization zone size $\iota_{\mathbf{x}}$, defined as $g_{\mathbf{x}} = G_{\mathbf{x}}/\iota_{\mathbf{x}}$ (Lubliner et al., 1989). $\iota_{\mathbf{x}}$ should be assumed as a material property to maintain objective results at the structural level, but the counterpart of the fracture energy remains an open issue (W.-F. Chen & Saleeb, 1994).

Doing the needed substitutions of (39) into (43) and combining (39) with (42); the following expressions defining the relationship between the constants, the

uniaxial stress and the effective stress and degradation variables are obtained:

$$g_N = \frac{f_{N0}}{b_N} \left(1 + \frac{a_N}{2}\right) \quad (44)$$

$$\begin{aligned} \sigma_N &= f_N(K_N) \\ \sigma_N &= \frac{f_{N0}}{a_N} [(1 + a_N)\sqrt{\phi_N(K_N)} - \phi_N(K_N)] \end{aligned} \quad (45a,b)$$

such that, $\phi_N = 1 + a_N(2 + a_N)K_N$

Similarly, effective stress and degradation variable are written in terms of K_N

$$\begin{aligned} \bar{\sigma}_N &= \bar{f}_N(K_N) \\ \bar{\sigma}_N &= f_{N0} \left[\left(\frac{1}{a_N}\right) (1 + a_N - \sqrt{\phi_N(K_N)}) \right]^{1 - \left(\frac{d_N}{b_N}\right)} \sqrt{\phi_N(K_N)} \end{aligned} \quad (46a,b)$$

$$D_N = 1 - \left[\left(\frac{1}{a_N}\right) (1 + a_N - \sqrt{\phi_N(K_N)}) \right]^{\left(\frac{d_N}{b_N}\right)} \quad (47)$$

To model the damage states under compressive and tensile loading

$$\mathbf{K} = [K_t K_c] \quad (48)$$

Hence the damage evolution, from (42) and (45a,b), of a uniaxial state is given by

$$\dot{K}_N = \frac{1}{g_N} f_N(K_N) \dot{\epsilon}^{pl} \quad (49)$$

the scalar plastic strain rate $\dot{\epsilon}^p$ in (49) is assumed to be evaluated in the general three-dimensional case to extend the uniaxial formulation to a multidimensional case:

$$\dot{\epsilon}^p = \delta_{tN} r(\hat{\boldsymbol{\sigma}}) \dot{\epsilon}_{max}^{pl} + \delta_{cN} (1 - r(\hat{\boldsymbol{\sigma}})) \dot{\epsilon}_{min}^{pl} \quad (50)$$

where δ is Kronecker delta; and $\dot{\epsilon}_{max}^{\hat{p}}$ and $\dot{\epsilon}_{min}^{\hat{p}}$ are the algebraically maximum and minimum eigenvalues, respectively, of the plastic strain tensor $\dot{\boldsymbol{\epsilon}}^p$. The scalar quantity $r(\hat{\boldsymbol{\sigma}})$ is a weight factor, such that $0 \leq r \leq 1$.

Substituting the previous equation in the equation of $\dot{K}_N$ gives an evolution equation:

$$\dot{\mathbf{K}} = \mathbf{h}(\hat{\boldsymbol{\sigma}}, \mathbf{k}): \widehat{\dot{\boldsymbol{\epsilon}}^{pl}} \quad (51)$$

where the matrix

$$\mathbf{h}(\hat{\boldsymbol{\sigma}}, \mathbf{k}) = \begin{bmatrix} r(\hat{\boldsymbol{\sigma}}) f_t(K_t)/g_t & 0 & 0 \\ 0 & 0 & (1 - r(\hat{\boldsymbol{\sigma}})) f_c(K_c)/g_c \end{bmatrix} \quad (52)$$

As for the eigenvalue matrices $\hat{\boldsymbol{\sigma}}$ and $\widehat{\dot{\boldsymbol{\epsilon}}^{pl}}$, it is assumed that the diagonal elements are the eigenvalues in algebraic order.

Hence the flow rule in equation (28) can be modified to:

$$\widehat{\dot{\boldsymbol{\epsilon}}^{pl}} = \lambda \nabla_{\hat{\boldsymbol{\sigma}}} \phi(\hat{\boldsymbol{\sigma}}) \quad (53)$$

By substituting the new flow rule (53) in the damage evolution equation (51), we get

$$\dot{\mathbf{K}} = \lambda \hat{\mathbf{H}}(\hat{\boldsymbol{\sigma}}, \mathbf{K}) \quad (54)$$

where

$$\hat{\mathbf{H}} = \mathbf{h} \nabla_{\hat{\boldsymbol{\sigma}}} \phi(\hat{\boldsymbol{\sigma}}). \quad (55)$$

3. Rate-Independent Plasticity

The yield function for quasi-brittle material can be rewritten as:

$$\bar{F}(\sigma) - c = 0 \quad (56)$$

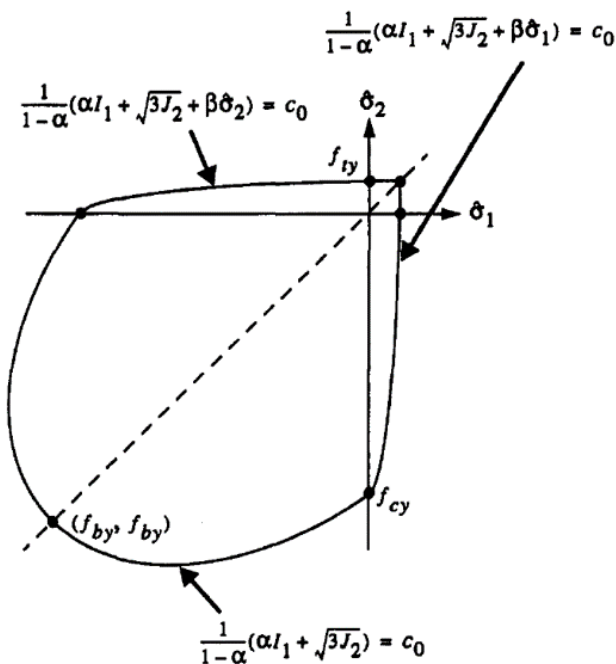
where c is the cohesion and $\bar{F}$ is the scalar function of invariants of the stress tensor.

To make yield surface more realistic for concrete material the algebraically largest principal stress $\hat{\sigma}_1$ is included in the Barcelona model. This provides a yield function that better represents the behavior of both tensile and compressive regions. The yield function provided in the Barcelona model is:

$$\bar{F} = \frac{1}{1-\alpha}(\alpha I_1 + \sqrt{3J_2} + \beta(\hat{\sigma}_{max})) \quad (57)$$

where α and β are dimensionless constants and $\hat{\sigma}_{max}$ denoting the algebraically maximum principal stress.

Figure 28 shows a plane stress cross section of the yield surface in the principal stress space.



[Figure 28]
Biaxial concrete yield surface in plane stress space (Lubliner et al., 1989).

The parameters in equation (57) are evaluated so that they can accurately represent the experimental data (Lee & Fenves, 1998). In the original model, three independent material strength values for the plane stress condition are used to determine the following parameters.

- Initial cohesion c_0 ,

$$c_0 = -f_{c0}, \quad (58)$$

by assuming the uniaxial compressive loading state, such that f_{c0} is the uniaxial initial yield compressive stress.

- Coefficient α

$$\alpha = \frac{f_{b0} - f_{c0}}{2f_{b0} - f_{c0}}, \text{ such that } 0 \leq \alpha \leq 0.5 \quad (59)$$

by considering the biaxial initial yield compressive stress f_{b0} with .

- Coefficient β

$$\beta = \frac{f_{c0}}{f_{t0}}(\alpha - 1) - (1 + \alpha) \quad (60)$$

by considering the uniaxial initial yield tensile stress f_{t0} , usually assumed equal to the maximum tensile strength, used with f_{c0} .

The biaxial tensile strength is dependent on α and β , usually equivalent to less than 1.3% of the uniaxial strength. In the Barcelona model, the cohesion c represents only material strength and the cohesion is only affected by the damage variable consequently (56) can be expressed as

$$\bar{F}(\sigma) = c(K) \quad (61)$$

The latter equation models isotropic hardening behaviour, which yields accurate results for monotonic loading. On the other hand, it is not adequate for cyclic loading of concrete due to the fact that the evolution of one strength does not greatly affect the evolution of the other one. Therefore, it is necessary to use two cohesion variables. A yield function featuring two cohesion variables is derived by defining β , a constant in the Barcelona model,

and defining the cohesion accordingly, such that:

$$\beta = \beta(\mathbf{K}) \quad (62)$$

$$\beta = \frac{c_c(\mathbf{K})}{c_t(\mathbf{K})}(1 - \alpha) - (1 + \alpha) \quad (63)$$

$$c = \frac{c_c(\mathbf{K})}{c_t(\mathbf{K})} \quad (64)$$

Where c_c and c_t are the compressive and tensile cohesion, respectively.

Assuming $c_t = \bar{f}_t(K_t)$ and $c_c = \bar{f}_c(K_c)$, the yield function becomes:

$$F(\boldsymbol{\sigma}, \mathbf{K}) = \frac{1}{(1 - \alpha)} [\alpha I_1 + \sqrt{3J_2} + \beta(\mathbf{K})(\hat{\sigma}_{max})] - c_c(\mathbf{K}) \quad (65)$$

where the invariants are functions of the effective stress.

The flow rule relates plastic flow direction and strain rate. In contrast to metals, in modelling frictional materials, non-associative flow rules are crucial for handling dilatancy (W. F. Chen & Han, 2012). Given that the yield surface in equation (65) combines shapes from two Drucker-Prager functions, the plastic potential function in CDP is a Drucker-Prager type.

$$\phi = \sqrt{2J_2} + \alpha_p I_1; \phi = \|s\| + \alpha_p I_1 \quad (66a,b)$$

such that

$$\|s\| = \sqrt{\frac{s}{s}}, \quad (67)$$

denotes the norm of the deviatoric stresses, in addition α_p is chose to give proper dilatancy.

Hence from (28) and (66a,b) the plastic strain yields:

$$\dot{\epsilon}^{pl} = \dot{\lambda} \left(\frac{s}{\|s\|} + \alpha_p \mathbf{I} \right) \quad (68)$$

2.2.1.1. SIMPLIFIED CDP APPLIED TO MASONRY

As has been seen the CDP model was introduced to study concrete structures with reduced reinforcement (quasi-brittle material) under cyclic loading (Hafezolghorani et al., 2017; Lee & Fenves, 1998; Lubliner et al., 1989). Nevertheless, as masonry is considered a quasi-brittle material, CDP model has been widely adopted to describe the nonlinear behaviour of unreinforced historical masonry acted upon by seismic loads (Acito et al., 2016; Bertolesi et al., 2017; D'Altri et al., 2017; Fiore et al., 2013; Milani & Clementi, 2021; Porcu et al., 2021).

In the context of masonry, CDP can be applied as a material model to simulate the behaviour of the mortar, concrete blocks, or stones that make up the masonry. It tracks how the masonry deforms under load, predicts where cracks and damage are likely to initiate, and captures the evolving material properties as damage accumulates. As the analysis progresses, CDP helps predict Crack Initiation and Propagation. It provides insights into when and where cracks may emerge within the masonry structure. Furthermore, it models how these cracks propagate and influence the overall structural response. Here's a coherent explanation of how this model is applied to masonry:

The process begins with Material Characterization, material testing is conducted to determine specific parameters for these components, such as the elastic modulus, Poisson's ratio, yield strength, ultimate strength, strain at peak stress, and damage parameters. This data forms the foundation for the subsequent analysis. Once the materials are characterized, a Mesh Generation process is employed. A finite element model is created to represent the masonry structure. This model subdivides the structure into discrete elements, considering the individual bricks or stone blocks, as well as the mortar joints. In addition, Boundary Conditions and Loads are introduced to the model.

In practice, the analysis is often an Iterative Process, a fine-tuning of the material parameters is required. The calibration is critical to align the model's predictions with real-world observations, ensuring that it accurately represents the behaviour of the masonry under various loading conditions.

But the question that rises here is: what to do in the case where testing is not a possibility?

Figure 29 recalls the uniaxial compressive and tensile mechanical behaviour of the material assumed by the CDP model. In this model, the nominal tensile stress σ_t and the nominal compression stress σ_c in the inelastic range are given by:

$$\sigma_t = (1 - d_t)E_0(\varepsilon_t - \varepsilon_t^{pl}) \quad (69)$$

$$\sigma_c = (1 - d_c)E_0(\varepsilon_c - \varepsilon_c^{pl}) \quad (70)$$

Here E_0 denotes the initial value of Young's modulus, assumed to have the same value both in compression and tension, while d_t and d_c are two scalar damage variables ranging from 0 (undamaged) to 1 (fully damaged). The latter variables allow the stiffness degradation to be considered through the modified elastic moduli.

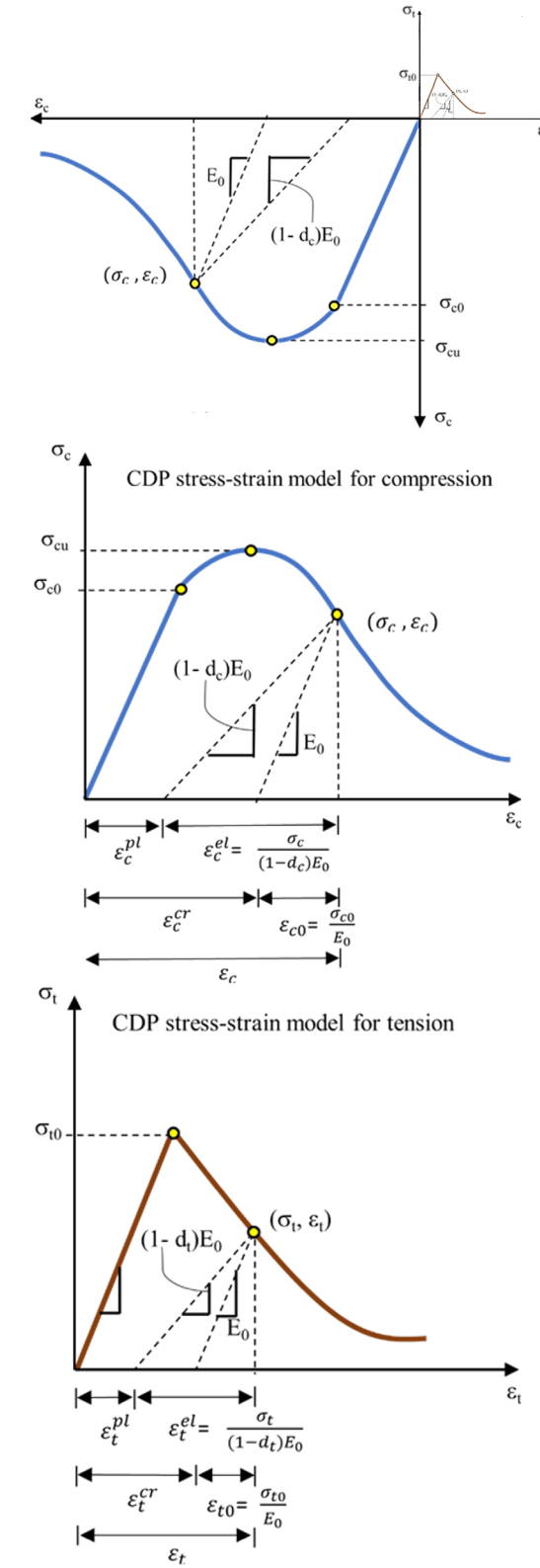
$$E_{dt} = (1 - d_t)E_0 \text{ and } E_{dc} = (1 - d_c)E_0, \quad (71)$$

which respectively apply to the tensile and compressive plastic range. Accordingly, the total strain ε_t and ε_c are given by the sum of an

$$\varepsilon_t^{cr} = (\varepsilon_t - \varepsilon_{0t}^{el}) = (\varepsilon_t - \sigma_t/E_0), \quad (72)$$

$$\text{or } \varepsilon_c^{cr} = (\varepsilon_c - \varepsilon_{0c}^{el}) = (\varepsilon_c - \sigma_c/E_0). \quad (73)$$

Since masonry material is greatly stronger in compression than in tension, the strain in tension is usually higher in magnitude than that in compression. Furthermore, the energy dissipated in compression is greatly larger than that dissipated in tension since generally the strain in tension is usually higher in magnitude than the strain in compression. As a result, both tension and compression are depicted in Figure 29, with adjusted scales to ensure clarity in formula depiction.



[Figure 29]
Compression and tension uniaxial
stress-strain in the CDP theoretical model.

2.2.1.2. PARAMETERIZATION OF THE CDP MODEL FOR MASONRY

To adequately choose the values of the CDP parameter for masonry some literatures were exploited. A parametric study was conducted using the ABAQUS code to investigate the damage plasticity constitutive model, examining several key parameters: dilation angle ψ , eccentricity ϵ , ratio f_{b0} / f_{c0} of biaxial to uniaxial compressive yield strength, coefficient K_c , viscosity parameter χ (Fiore et al., 2013). In addition another parametric study was conducted by Rainone et al. (Rainone et al., 2023), building upon the experimental findings in (Vermeltfoort et al., 1993), which as well aimed to assess the influence of key parameters on masonry behaviour within the CDP model.

Fiore et al. (2013) found that parameters such as ϵ , f_{b0} / f_{c0} , and K_c were consistent with values typically adopted for concrete and also used for masonry (Jankowiak & Lodygowski, 2005). The latter K_c parameter, which is based on the stress invariant in a three-dimensional stress state, modifies the Drucker Prager DP surface; it distorts the DP surface in the deviatoric plane from a circle to a surface more like a Mohr–Coulomb one. K_c physically represents the ratio between distance from the hydrostatic axis of the maximum compression and tension respectively. Hence K_c is better kept equal to 2/3, a value suggested by the user's Guide to well approximate a Mohr–Coulomb failure criterion as used for concrete see Figure 30 (Acito et al., 2016).

As for the eccentricity value ϵ since it has been found by Fiore et al that it is consistent with the concrete values as well it is recommended to use the default value 0.1 as smaller values might lead to convergence issues, particularly under low confining pressures due to the local curvature of the flow potential where it intersects the hydrostatic axis (Acito et al., 2016). In addition, in ABAQUS, a smaller eccentricity closer to zero is typically preferred for defining the flow potential in stress-strain models. A slower rate of convergence towards the asymptote promotes controlled material response and aligns with the structural requirements of masonry to withstand different load scenarios while maintaining integrity. This behaviour is crucial for accurately simulating masonry behaviour in structural analyses using, as discussed in the ABAQUS Theory Manual Section 4.4.2 (*Abaqus Analysis User's Guide*, 2016) on models for granular or polymer behaviour. Hence, to avoid convergence problems and to be close to zero the default value 0.1 is preferred.

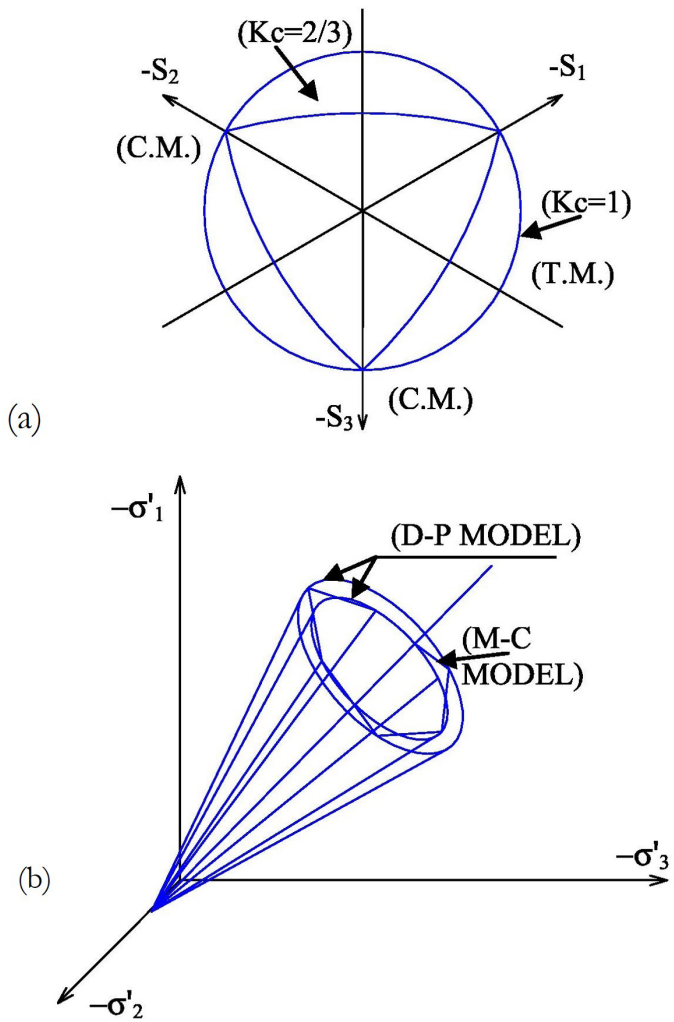
The study conducted in 2023 found that variations in the ratio between compressive yield stresses for uniaxial and biaxial states, exhibited minimal impact on force-displacement curves (Rainone et al., 2023) which aligns with (Fiore et al., 2013). Furthermore, the value of 1.16 usually used for concrete, representing the ratio between biaxial and mono-axial compression strength, is as well referenced in established literature regarding the behaviour of clay-brick masonry in compression (A. Page, 1981).

Now concerning the viscosity parameter the study by Fiore et al (Fiore et al., 2013) found that a viscosity parameter value of 0.004 was necessary to achieve good agreement between experimental and numerical results. As for Rainone et al. (Rainone et al., 2023), building upon previous experimental findings, found that higher values of the viscosity parameter resulted in smoother curves but also led to an overestimation of material strength which is something undesirable. This suggests that the viscosity parameter has a notable impact on the mechanical response of masonry structures. According to (Hafezolghorani et al., 2017) the viscosity should be kept very low to speed convergence during the softening process. Additionally as far as ABAQUS user's guide is concerned [(*Abaqus Analysis User's Guide*, 2016)] the viscosity parameter allows the model to attain convergence and preserve accurate results for post-peak softening behaviour if a sufficiently small value is assumed. Nevertheless, setting the value to zero makes numerical convergence more difficult. Thus, the value adopted for viscosity parameter through this thesis was set as an infinitesimal number 5*E-05 to strike a balance between numerical stability and negligible impact on material behaviour.

Finally, sensitivity analyses of the dilation angle ψ on material behaviour revealed that finite element models exhibited ductile behaviour within a range of 30° to 37°, contrasting with brittle behaviour observed at near-zero values (Fiore et al., 2013). In addition, a low dilation angle facilitated alignment between experimental and numerical data in (Rainone et al., 2023). Since actually the model should assume a non-associated behaviour for the elasto-plastic deformation part, maintaining a dilation angle (ψ) of 10° for the elasto-plastic component of inelastic deformation is the best choice as it aligns with experimental data by van der Pluijm (Pluijm, 1993) under moderate pre-compression levels, as it reflects typical conditions encountered in masonry structures a value generally adopted in literature (Acito et al., 2016; Gerges et al., 2023c; Jain et al., 2020; Milani & Clementi, 2021; Milani & Valente, 2015). Finally, a summary of the values adopted for the CDP model throughout this thesis can be found in Table 3.

[Table 3]
Assumed values for parameters of the
CDP model in ABAQUS.

Dilatation angle ψ	Eccentricity ϵ	f_{b0}/f_{c0}	K_c	Viscosity χ
10°	0.1	1.16	0.667	5.00E-05



[Figure 30]
(a) K_c parameter to approximate a MC
failure criterion with a DP failure criterion,
in a smooth way, (b) 3D strength domain
adopted in the principal stresses space
(DP: Drucker-Prager failure criterion; MC:
Mohr-Coulomb failure criterion).

PART A

THE MOHR-COULOMB MODEL FOR MASONRY MODELLED AS A CONTINUUM

1. OVERVIEW OF THE MASONRY ARCH

The masonry arch, an ancient element, that constitutes a revolutionary idea in construction history. It embodies key characteristics, influencing complex masonry structures both conceptually and geometrically as well as contributing significantly to structural development and human progress.

It is commonly defined as a curved structure spanning a space, potentially bearing weight. Arches date back to the 2nd millennium BC in Mesopotamian brick architecture, with the Ancient Romans pioneering systematic usage across various structures. From the arch, the evolution led to masonry vaults, continuous arches forming roofs. Therefore, analysing the arches is a fundamental cornerstone that serves for an in-depth analysis of the behaviour of ancient masonry structures.

As previously discussed on the State-of-the-Art section, in the 19th century, masonry arch equilibrium analysis relied on 'lines of thrust' derived from La Hire's funicular polygon concept (Heyman, 1998). Robert Hooke's 1675 chain theory confirmed that a flexible line's inverted form represents a rigid arch. Real arches may deviate from this perfect shape, such as having a circular profile, hence the safe theorem suggests sufficient masonry thickness to contain Hooke's inverted chain within the arch ring.

A framework for the application of the theorems of Limit Analysis to masonry, was given by Heyman by introducing these three necessary assumptions (Heyman, 1995):

- (i) masonry has no tensile strength,
- (ii) masonry has infinite compressive strength,
- (iii) sliding does not occur.

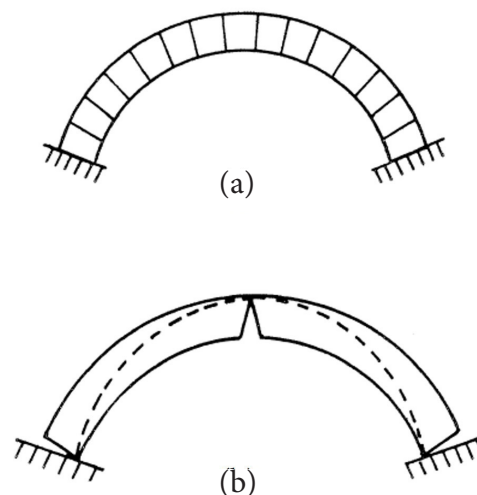
The initial assumption is conservative yet reasonable, recognizing the strength of individual stone blocks in tension. However, the assumption of unlimited

compressive strength, while generally accurate for average stresses, can lead to stress concentration and localized distress, like splitting or surface spalling. This may not typically result in overall building failure, except in cases of apparently solid walls that may have a rubble fill between two skins.

In some instances, slippage of individual stones is observed, but the masonry structure often maintains its shape well. A minimal compressive prestress seems sufficient to prevent slip and preserve cohesion. A detailed discussion on sliding is later provided in this section.

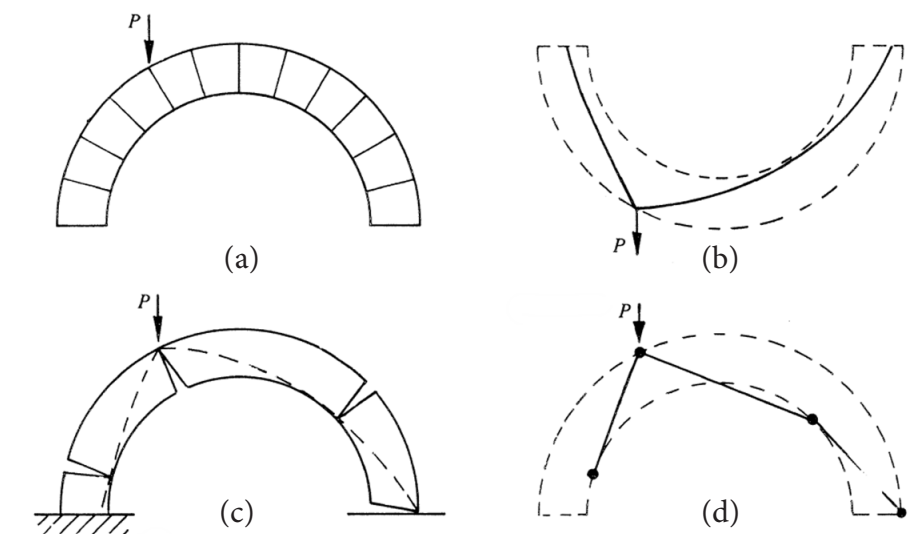
Consider the case of the arch in Figure 31a, constructed with wedge-shaped voussoirs without mortar. Once the centring is removed, the arch exerts horizontal thrust, causing the abutments to slightly yield. Under such conditions, any structural theory must satisfy the equations of equilibrium and ensure that internal forces in the arch balance external loads, including self-weight. Statements about the structural state of the arch must adhere to Heyman's three structural assumptions: infinite strength in compression, inability to withstand tension, and no sliding (Huerta, 2004).

Hence, the voussoirs are considered rigid blocks, incapable of deformation, which eliminates the possibility of obtaining an elastic solution for the arch. Moreover, assumptions about boundary conditions, specifically the movements of the abutments, are generally unknown. If the abutments shift slightly, a geometric mismatch occurs, and to prevent collapse, the arch must adjust, potentially leading to cracking as depicted in Figure 31b. While the voussoirs cannot slip or deform internally, they may turn about points of contact on the extrados or intrados forming thus hinges.



[Figure 31]
Horizontal thrust of an arch, constructed with wedge-shaped voussoirs, upon removal of centring and abutment movement leading to potential cracking (Huerta, 2004).

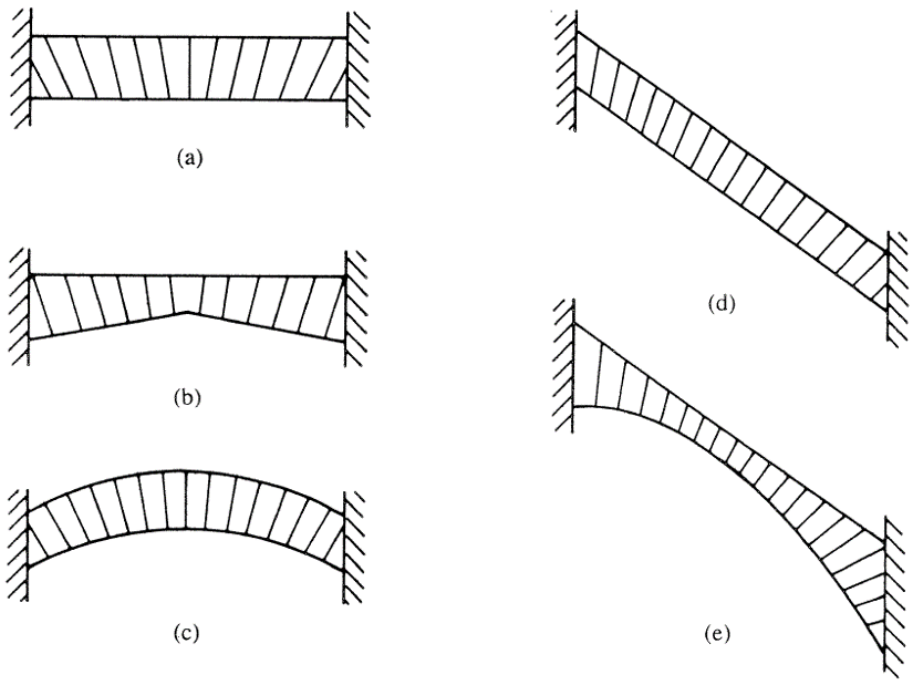
Additionally, consider the semi-circular arch subjected to its self-weight and an additional load P in Figure 32a. In this scenario, as P increases, Hooke's hanging chain deforms from the catenary to the shape depicted in Figure 32b. The inverted chain or thrust line fits less within the arch, and at a certain value, the thrust line can only just be contained, as illustrated in Figure 32c. This limit stage induces the formation of four hinges as shown in Figure 32d, initiating the collapse mechanism a geometrical constraint unrelated to the arch's initial state.



[Figure 32]
Semi-circular arch: (a) subjected to a point load, (b) deformation of the catenary shape, (c) limit thrust reached and (d) formation of four hinges, initiating the collapse mechanism.

It's worth noting that certain arch forms, within the current assumptions, may lack a collapse mechanism such as the flat arches, quasi-flat arches and flying buttresses shown in Figure 33.

These arch forms will only fail if they are loaded to a point where overall crushing occurs or if one of the stones slides out. This brings us to a discussion about sliding failure and Heyman's third assumption.

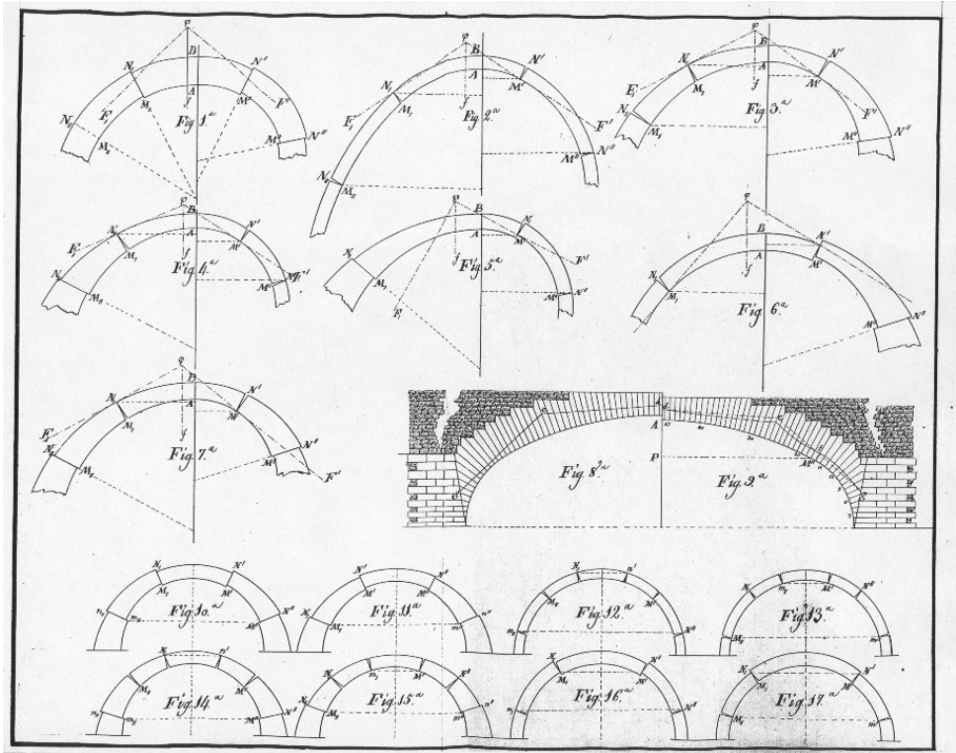


[Figure 33]
Arch forms that lack a collapse
mechanism (Huerta, 2004).

Charles-Augustin de Coulomb, an 18th-century physicist and engineer, made significant contributions to the understanding of friction and the stability of structures, which are highly relevant to the theoretical possibility of a collapse mechanism that includes failure by slip in some joints.

In the context of masonry structures, Coulomb's consideration in his 1773 memoir was crucial yet general. He recognized that friction played a significant role in holding stones or blocks together in arches, walls, and other architectural elements. He understood that if the frictional resistance at the joints was overcome, it could lead to the sliding or slipping of stones, potentially compromising the structural stability, but did not provide examples.

It is worth noting here that Joaquín Monasterio's contribution, around 1800, examined seven basic collapse mechanisms (see Figure 34) that combine joints and sliding. He identified the possible mechanisms using thorough analyses and considering the compatibility of the mechanism as a basic requirement (Huerta, 2004).



[Figure 34]
The seven basic collapse mechanisms
examined by Joaquín Monasterio in 1800.

In his book Huerta provides a comprehensive account of the scholarly contributions made by prominent researchers in the 19th century concerning the investigation of structural stability. Notably, in 1826 Navier analysed the sliding collapse mechanisms of two distinct occurring within joints. Petit in 1835 conducted experiments that affirmed the stability of stirrups at their base. In 1875 Michon further studied mixed mechanisms, his work was later translated by Saavedra in 1860 (Huerta, 2004). However, in the 19th century the exploration of sliding mechanisms remained a theoretical curiosity, until 1982 when Parland formulated a comprehensive theory for the analysis of structural failures in masonry.

Huerta summarized Parland's theoretical framework explaining that it originates from classical mechanics of continuous media and considers the potential for sliding failures. This proposed theory is characterized as a linear analysis of the stability of a system composed of rigid solids in a state of dry and full contact(Huerta, 2004).

These hypotheses form the base of the theoretical framework:

1. The assembly is made of an infinite number of rigid elements ($E = \infty$; $\partial \text{rot} = \infty$). Within the mechanics of continuous media. This assumption implies that two distinct material points cannot occupy the same spatial position.

2. There exists no tensile strength within the joints. Which allows to define specific contact conditions within the joints:

(a) Joints within the rigid solids exhibit a compatible indentation, characterized by a perfectly smooth surface, which enables a simulation of friction between the contacting solids. Any deformation is confined within a convex cone defined by the indentation.

(b) The resultant vectors of stresses on the surfaces must satisfy the following constraints: they must be less than zero (indicative of compression), and they must remain within the bounds of the friction cone.

(c) The cones representing both friction and deformation are polar, meaning that their generators are orthogonal to each other.

Based on the previous in combination with virtual work theorem, Parland establishes the criteria for stable equilibrium concerning an arc. Specifically, Parland proves that an arc is stable in the presence of a particular load configuration, provided that there exists at least one line of thrusts wholly confined within the arc. Additionally, the tangential orientation of these thrusts must consistently fall within the constraints of the friction cone at each joint. This statement remains valid under the assumption of "geometric friction," which is realized through the incorporation of indentations in the structural components.

In cases where friction is considered as a real and energy-dissipating phenomenon, Parland's theory is applicable only under specific circumstances. This theory becomes applicable when the coefficient of friction either approaches zero or infinity, where the latter provides identical results to those formulated by Heyman.

It is worth noting that it is rare to have a thrust line that contravenes the stability condition associated with friction, as per Parland's analysis. Such instances are predominantly confined to exceptionally thick arches, examples of such arches can be seen in Figure 35.

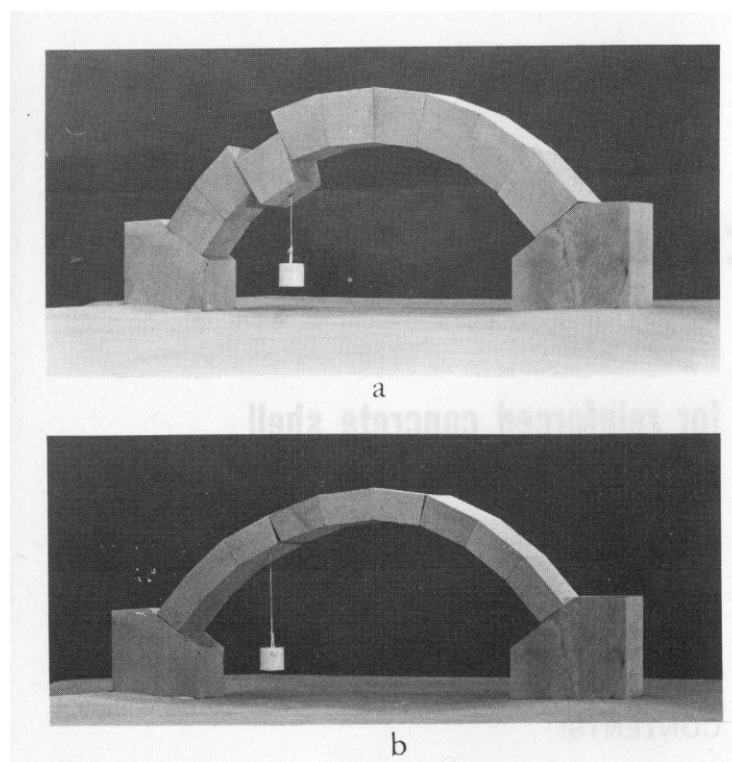


[Figure 35]
Examples of thick arches where sliding can be noticed (pictures taken in Segovia, Spain).

Parland's experiment in Figure 36 consisted of a segmental arch having two variable thicknesses subjected to the same point load. The experiment proved that the event of sliding can be avoided by decreasing the thickness of the arch (see Figure 36b). In the context of the thickest arch, collapse is induced by the phenomenon of pure sliding, see Figure 36a. The latter is due to the substantial thickness of the arch, which prevents the possibility of formation of the four-hinge mechanism.

On the other hand, in the case of the thinnest arch collapse resulting from sliding is avoided. This is attributed to the formation of joints that are first predisposed to engage in the classical collapse mechanism, which prevents the occurrence of sliding-induced failure. Parland's analysis does not give a clear definition regarding the condition under which an arch can be viewed "definitely unstable".

It appears an arch can be categorized as unstable if a kinematically admissible collapse configuration can be identified. In cases where the specified friction condition is not met, the system will not experience a collapse unless an associated kinematically admissible mechanism is identified. In other words, the potential sliding of a voussoir between two inclined joint planes will not happen unless there is a plausible scenario for them to separate or detach from each other.



[Figure 36]
Parland's experiment of two segmental
arches of different thicknesses
subjected to the same point load.

In his book Huerta was able to verify through a series of essays on an educational arc at the Science Museum in the small town of Gamla Linköping, in Sweden. Subsequent repetitions of these tests with a cardboard model that gave identical results. It is important to highlight that the thorough analysis of these mechanisms is characterized by a high level of complexity.

Numerous structural guidelines, especially those related to elements where sliding may occur, are designed with the specific intent of preventing such phenomenon. This is exemplified by the incorporation of columns or arches beneath the heads of flying buttresses and the deliberate adjustments made to the inclination of courses in the abutments of arch bridges play an important role in significantly reducing the possibility of sliding (Huerta, 2004). In other cases, conventional measures, such as the cut of the voussoirs normal to the intrados curve, serve to simplify construction and effectively handle the risk of slippage in most cases. However, in the event of an earthquake, landslides may occur requiring other measure, such as the use of staples and tenons among the ashlar of the Greek temples.

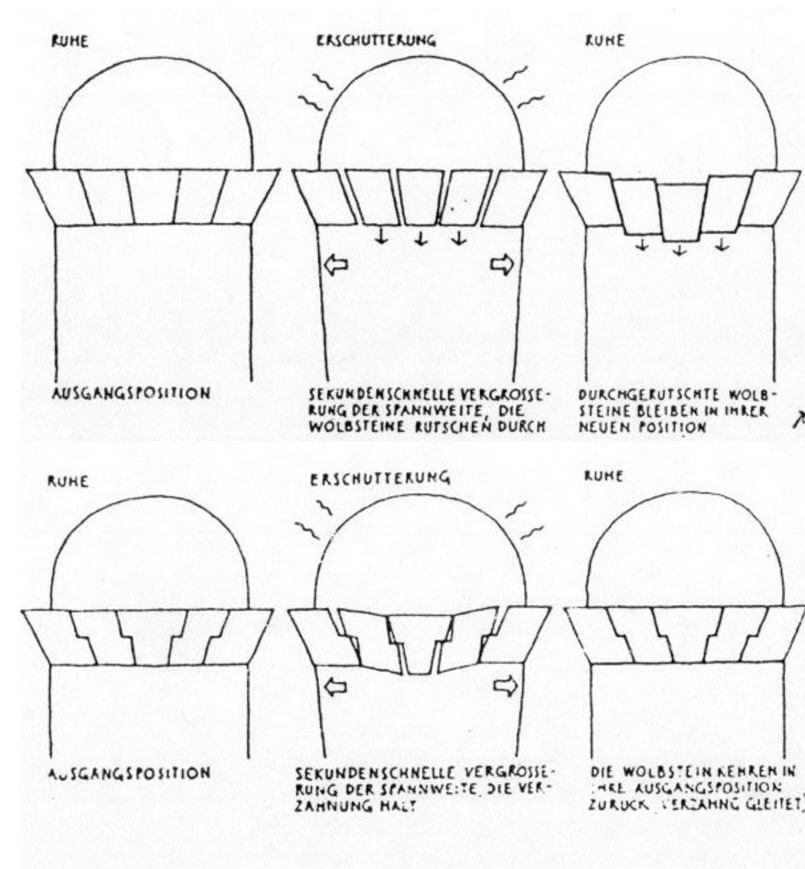
Slip failure in voussoirs was further examined by Cejka in 1978. He examined various aspects which included the voussoir's non-planar joints, ravines, and other forms Figure 37.

While interlocking vaulting stones are not commonly seen in true barrel vaults, they are a distinct feature of Islamic architecture closely linked to vaulting. Surprisingly, the literature lacks comprehensive discussions about their structural significance, despite their acknowledged beauty. Consequently, Cejka undertook a more detailed examination of their role in construction (Cejka, 1978).

The earliest instances of interlocking stones were identified by Creswell in association with the early Islamic structure at Qasr al-Kharana (Creswell & Allan, 1958). However, Creswell's findings primarily establish the long-standing presence of these stones prior to the Islamic conquest, without delving into their function or the reasons for their creation. Creswell does not emphasize the two distinct uses of these interlocking stones: they are used for the interlocked arch stones of an archivolt and in the construction of segmental arches. The earliest datable examples feature interlocking in segmental arches, making them particularly intriguing to Cejka. Nevertheless, the question of their origin and constructive significance remains of high importance.

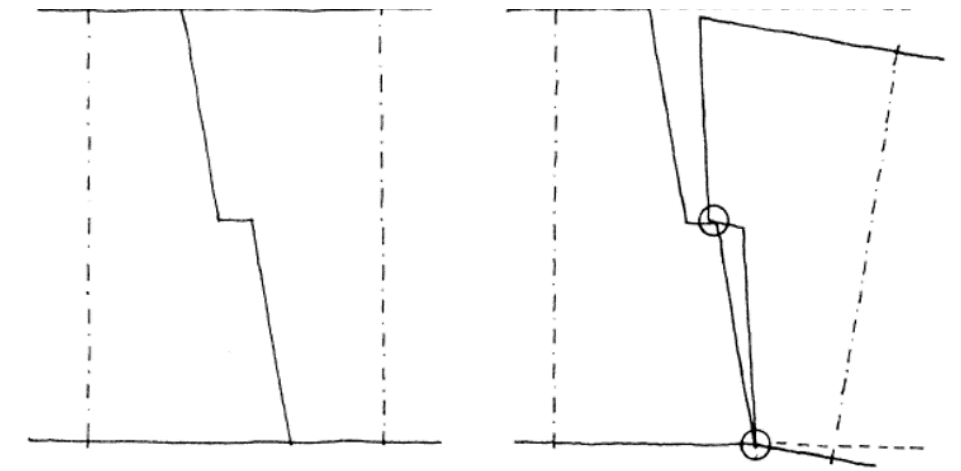
One possible cause of their formation is attributed to thermal or seismic effects on the buildings. Notably, the most significant earthquakes were primarily concentrated in the southern latitudes of the Roman Empire. North of the Alps, interlocked voussoirs are absent. According to Cejka, this geographical distribution holds specific significance. In regions exposed to substantial temperature fluctuations or seismic influences, structures were more vulnerable. These phenomena caused the voussoirs to shift, potentially disrupting the arch's equilibrium (Cejka, 1978).

Segmental arches were particularly at risk (see Figure 37), but even true arches faced danger when they were narrow and comprised a ring of unclamped voussoirs. In a barrel vault with overlapping joints, the risk was reduced due to the alternation of joints and increased friction surfaces of the keystone, ensuring the synergy of the entire vault. In a narrow arch, however, voussoirs could slip down one after another, even with slight vibrations causing the joints to briefly open. Cejka explains that dovetail joints were a way to prevent this sliding process. It is worth noting that large temperature fluctuations can also have similar consequences. Interlocking vaulting stones can be helpful in these scenarios because they are held in place through their interlockings (Cejka, 1978).



[Figure 37]
Slipping of arch stones during
vibration hand drawn by Cejka 1978
(Cejka, 1978).

However, it's essential to note that their resistance also has limits. The voussoirs can only maintain their position as long as the interlockings engage while simultaneously supporting the respective outer wedge stone edges (see Figure 38).



[Figure 38]
Interlocking resistance mechanism
(Cejka, 1978).

A typical example illustrating this kind of interlocking can be found in Figure 39.



[Figure 39]
Interlocking vaulting stones examples
in Beit-El-Din, Lebanon a 19th century
Ottoman masonry structure.

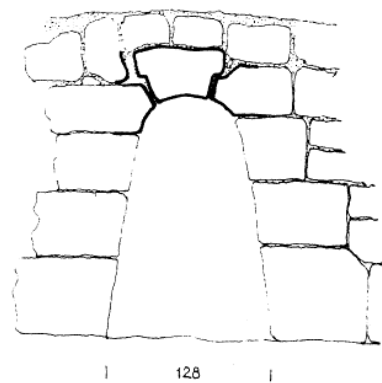
According to Cejka, the need for addressing structural challenges in narrow, single-arch constructions led to the development of interlocked vaulting stones. This concept likely emerged as a solution to a common problem in ancient construction where arch masons typically created semi-circular openings, while carpenters produced rectangular doors and gates.

To bridge this gap, a horizontal beam was introduced to close the door opening beneath the arch. The semi-circular space between the arch and the beam, acting as a lintel, could either remain open or be filled. This combination of a straight lintel with a round relieving arch gained popularity in the Roman Near East, with structures like the 2nd-century AD stage exits in the theatre of Bostra and gate structures in the temple precinct of Hatra featuring such lintels.

Subsequently, the use of single-piece stone lintels with a horizontal orientation appeared in Byzantine constructions, as observed in the main entrance and side entrances of the 6th-century AD Basilica in Qalb Louzeh and above openings in Qasr ibn Wardan from 564 AD.

Similar examples can be seen in early Islamic architecture, with the first Islamic example of a monolithic lintel found in the western portal of the Dome of the Rock in Jerusalem, which still reflects the Byzantine tradition.

The development of interlocked keystone arch stones, particularly in skew arches, was driven by the possibility of deformation of traditional arches during disturbances, including seismic activity. The origin of this innovation is uncertain, but it likely emerged in response to damage caused by seismic events, possibly in the Vesuvius volcanic region (Cejka, 1978).



[Figure 40]
Primitive Etruscan dovetail joint.

The Romans introduced dowels into keystones with alternating "teeth" for interlocking, and they implemented measures to stabilize skew arches, such as metal bars. The concept of interlocking, which initially involved dowels and clamps hidden from external view, eventually became visible on facades, as seen in primitive interlocking in ancient Italian structures Figure 40.

Interlocking was applied to skew arches with increasing sophistication, including dovetail joint arrangements, as evidenced in the Roman Theater of Orange. Diocletian's Palace gates in Split and the Nativity Church in Bethlehem also featured interlocked skew arches with variations. The transition from skew arch interlocking to voussoir arches likely occurred in Samma (Cejka, 1978).

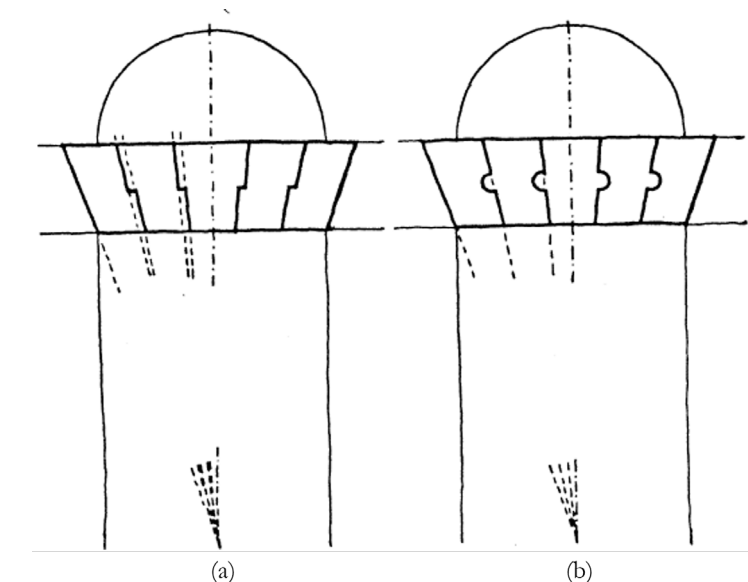
Early Islamic architecture adopted these developments, with structures like Qasr al-Hair al-Sharqi using interlocked voussoirs in larger curtain walls to withstand vibrations and potential attacks. However, knowledge of interlocking

in early Islamic architecture was not uniform. Skew arches, particularly those with dovetail interlocking, played a pivotal role in the evolution of interlocked voussoirs in early architectural practices, see Figure 41a.

Until this point, interlocking methods were primarily reliant on dovetail joints, where the joints were staggered along their course. However, this method had its limitations as the staggered joints caused visual disruption in the facade. Architects attempted to conceal these dovetail joints by incorporating them into the architrave's profile. Aside from the aesthetic concerns, this approach demanded a high level of precision in working with the voussoirs to maintain the desired angles on each of the four inclined surfaces.

In the Near East, a technique emerged that enabled interlocking of voussoirs without the need for joint staggering. Early indications of this method were observed in Rome, with examples such as concealed or mentioned tombs on the Via Appia. However, the method further developed in the Islamic context. This approach involved interlocking with a tongue and groove, where the joint runs in the same line both before and after the interruption.

The "joints" are simply incised into a monolithic beam. In Figure 41b one finds an example of the tongue and groove. This means that by this time, such solutions were not only known but also widespread when there were artisans ready to replicate them in the facades.



[Figure 41]
(a) Interlocking with dovetail joint,
(b) interlocking with tongue and
groove (Cejka, 1978).

Due to their appealing aesthetic effect, the newer solution was already suitable for decorative application from the beginning, while the older one was more for constructive purposes. Since dovetail interlocking couldn't bring significant innovations, the new solution was mainly adopted and widely varied. As the years passed, the developments in stone interlocking techniques became clearer. Interlocking of the older type, with staggered joints or dovetail joints, were still used. However, when aesthetics was a concern, the tongue and groove solutions came to the forefront. This was the case with portals of cultic and sophisticated secular buildings. Countless examples of the decorative application of tongue and groove interlocking can be found both in the Syrian-Anatolian and Egyptian regions. The joints created in this way, with a decorative effect, are usually emphasized in Anatolia through the alternation of light and dark stones (Cejka, 1978).

In the 14th century, tongue and groove joints weren't limited to arches but were also employed in piers, even when not structurally necessary. The Ottomans inherited the use of groove and tongue voussoirs, particularly the older type with staggered radial joints, from their predecessors. While this design may seem unnecessary from a structural perspective, it was often used for its decorative appeal. In some cases, a thin decorative marble layer may have concealed the actual structural elements. However, it's possible that a similar, smaller console arch design with notching could provide a more scaffolding-saving and functional solution.

In grand Ottoman buildings in Istanbul from the 16th and 17th centuries, tongue and groove joints made a resurgence. These joints were initially found in familiar locations, such as the segmental arches of entrances. However, the original structural purpose was not entirely forgotten. Segmental arches, which have a higher risk of sliding voussoirs compared to pointed arches, retained tongue and groove joints that ran the full depth of the arch. This tradition remained until the 19th century, and it is seen in Beit El-Din as well as can be seen in Figure 42.

In Figure 43 an examination of two distinct joint configurations is portrayed, with one aimed at mitigating slippage and the other designed to concurrently function as a practical solution and a decorative element within the adjacent lintels of windows. In summary, the non-planar joints in addition to their practical use to avoid sliding have become ornamental as well.



[Figure 42]
Tongue and groove joints example in Beit El-Din, Lebanon
a 19th century Ottoman masonry structure.



[Figure 43]
Example of dual usage of the interlocking of voussoirs aimed for
practical solution and decorative element in Beit El-Din, Lebanon a
19th century Ottoman masonry structure.

2. MOHR-COULOMB (MC) MODEL

The most simple, though efficient, way to describe failure is to assume that the material does not yield or fail as long as the stress is sufficiently small. Mathematically this condition can be stated by assuming that the material is safe as long as the stress lies inside a -usually convex- set of admissible stresses in the stress space, called the saturation domain.

In most applications the material is assumed to be linear elastic inside the saturation domain; even though, for many practical purposes, considering that the material is rigid inside the saturation domain is a feasible option, especially if the goal is to study the collapse behaviour of the structure.

Excluding hardening and softening, the saturation domain, a convex domain D in the space of symmetric tensors, is fixed in time, and considering the material rigid inside D , the material can deform only if the stress is limit, that is attains a value which is at the boundary ∂D (also called the yield surface) of D .

For standard materials the flow rule is associate and the rate of deformation is orthogonal to the saturation domain at the stress limit. In other words, given any stress at the yield surface the corresponding strain rate has a direction and a verse which is determined by the surface ∂D but whose magnitude is otherwise arbitrary.

For standard materials the theorems of Limit Analysis are valid and the safety of the structure can be assessed either through the equilibrium theorem or through the kinematic theorem. The vast majority of construction materials can be modelled as standard, though non-associate flow rules are often necessary when modelling friction phenomena such as the sliding of granular matter or sand.

Many macroscopic failure criteria, which are easy to use and understand -since they require only the setting of few parameters- have been developed. Some of the most popular failure criteria are von Mises and Tresca for ductile materials, Rankine for brittle materials, Drucker-Prager and Mohr-Coulomb for pressure-dependent and cohesive-frictional soils.

Among these simple failure criteria we can put the most elementary -since it requires the setting of zero parameters- namely the unilateral No-Tension material of Heyman.

Here we focus specifically on Heyman No-Tension and Mohr-Coulomb materials, since Heyman material can be seen as a limit of Mohr Coulomb material when the material parameters are pushed to their limits.

The Mohr-Coulomb (MC) model, is a fundamental constitutive law in soil mechanics and structural engineering. The MC model, which has evolved over time spanning from the 18th to the early 20th century, is one of the first formulated failure theories and it is used up to the present day, especially for brittle materials, although it has also the potential to be extended to ductile materials (Bai & Wierzbicki, 2010; Christensen, 2005).

Richard M. Christensen noted that “*Probably the most misunderstood but definitely the most prominent of all failure concepts is that of the Coulomb–Mohr type.*” (Christensen, 2013). Charles de Coulomb's work in the late 18th century was fundamental as it emphasized that a material fails when “a proper combination of shear and normal stresses is realized”(Andrianopoulos & Manolopoulos, 2012). Coulomb’s well-known failure criterion, says that failure occurs across an internal surface when the shear stress τ and the normal stress σ acting on that surface satisfy a condition dependent on the cohesion c and friction angle φ of the material. The maximal principal stress criterion known as the Rankine criterion emerges as a special case of the MC criterion for $\varphi = 90^\circ$ (Cowin, 1974; Ebnoether & Mohr, 2013; Jiang & Xie, 2011).

Early in the 19th century Karl Culmann introduced for the first time the representation of stress at a point as a circle in the (σ, τ) plane, that is the 2d space of all the possible components of normal and tangential stress across an internal surface (Karaoulanis, 2013). In 1882 Otto Mohr presented the well-known graphical representation known today as Mohr's circle. He was the first who used systematically stress circles for biaxial or triaxial state of stress problems. Subsequently in 1900, by working in the (σ, τ) plane, Mohr refined the understanding of material failure by conjecturing the concept of a stress envelope delimiting the fracture limits. The envelope is tangent to all Mohr's circles of diameter $\sigma_1 - \sigma_3$ representing limit states of stress at which failure of material (yielding or fracture) occurs. This idea is usually called “Mohr's yield condition” (Cowin, 1974; Jiang & Xie, 2011).

The mathematical model of Mohr-Coulomb generally describes the response of materials with compression/tension asymmetry to shear stress as well as normal stress. In structural engineering, Coulomb's friction hypothesis is used to determine the stress combination leading to material fracture. Mohr's

circle aids in identifying the principal stresses causing this combination of shear and normal stress and determines the orientation of the fracture plane. According to the principle of normality, the strain rate introduced at failure is perpendicular to the stress envelope. This allows both to predict the failure load and, that is when a displacement fracture occurs, and determine the angle of fracture.

Just like in Mohr-Coulomb that the normal to the failure plane aligns with the plane containing the maximum and minimum principal stresses, this assumption can also be derived from standard Elasto-Plasticity Theory (EPT). Both MC and EPT predict the same orientation of the failure planes for associative failure theories, this orientation being independent of the elastic parameters in EPT.

Incorporating isotropic cohesion hardening, the MC framework allows the material to soften or harden isotopically. In 2002, Weir and Young present a new (second-order) Rigid-Plastic Theory (RPT) of failure and show that it also agrees with EPT for associative failure theories where the experimental results on the orientation of failure planes in unconsolidated sands are consistent with the predictions of RPT (Weir & Yzoung, 2002).

We have to point out that, for some kinds of materials, such as sand, assuming an associate behaviour is often too far from reality since the normality assumption tends to dramatically overestimate the dilatancy angle ψ of the material, that is the deviation of the strain rate from the pure shear response. With the normality assumption this dilatancy angle ψ coincides with the friction angle φ , but in reality for fine and dry granular assemblies it can be as small as zero.

In conclusion, the standard MC criterion, besides the parameters defining the elastic behaviour (if any) for stress states which are inside the stress envelope, requires the setting of two parameters, the friction angle φ and the cohesion c . If a non-standard behaviour is assumed then a third parameter, the angle of dilatancy ψ is required.

We notice that the Heyman No-Tension material can be obtained as the limit of a standard Mohr-Coulomb material by letting φ tend to 90° and c tend to zero. In this case the saturation domain becomes a cone in the 6-dimensional space of symmetric second order tensors and the dilatancy ψ becomes 90° , meaning that there is no possibility of sliding along a failure plane.

2.1. MC ANALYTICAL FRAMEWORK

2.1.1. YIELD BEHAVIOUR

The MC criterion assumes that failure on a given internal surface is controlled by the maximum shear stress τ , which is a value dependent on the normal stress σ :

$$\tau = c + \sigma \sin \varphi \quad (74)$$

where σ (as it is usual in Solid Mechanics) is negative in compression and positive in tension.

This can be visualized by plotting the Mohr's circle for states of stress at failure in terms of the maximum and minimum principal stresses. By plotting the shear strength of a material against the applied normal stress, the MC failure criterion is represented as the linear envelope summarizing these relationships (Budhu, 2011; Das, 2013). The Mohr-Coulomb failure line is the optimal straight line that is tangent to these Mohr's circles as can be seen in Figure 44.

In terms of the radius s of the circle and of the position σ_m of its center along the σ axis, we can write

$$\tau = s \cos \varphi, \quad (75)$$

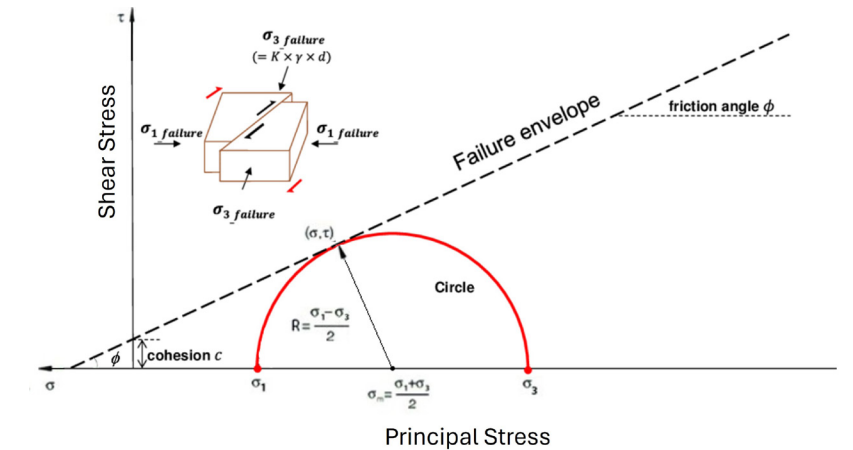
$$\sigma = \sigma_m + s \sin \varphi. \quad (76)$$

Following substitutions in (74) and multiplying both sides by $\cos \varphi$ we get the following equation:

$$s + \sigma_m \sin \varphi - c \cos \varphi = 0 \quad (77)$$

in which s (the radius of the Mohr circle) is the maximum shear stress where $s = \frac{1}{2}(\sigma_1 - \sigma_3)$ and σ_m is the average of the maximum and minimum principal stresses $\sigma_m = \frac{\sigma_1 + \sigma_3}{2}$.

[Figure 44]
Mohr-Coulomb failure model.



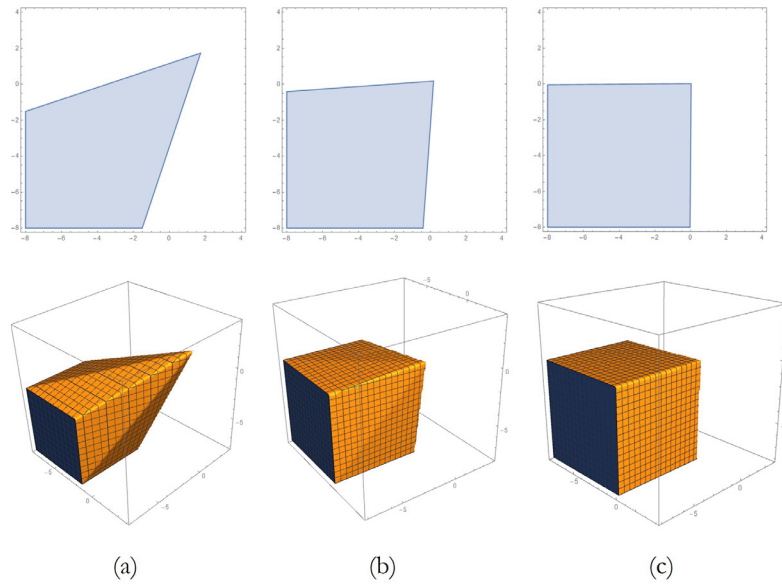
If we wish to express the MC criterion in the space of the principal stress components, we can introduce the coordinates

$$\begin{aligned} \xi_1 &= \frac{\sigma_2 + \sigma_3}{2}, \eta_1 = \frac{\sigma_2 - \sigma_3}{2}, \\ \xi_2 &= \frac{\sigma_3 + \sigma_1}{2}, \eta_2 = \frac{\sigma_3 - \sigma_1}{2}, \\ \xi_3 &= \frac{\sigma_1 + \sigma_2}{2}, \eta_3 = \frac{\sigma_1 - \sigma_2}{2}. \end{aligned} \quad (78a,b,c)$$

and write the saturation domain $\mathcal{D}$ as the convex domain defined by the following inequalities:

$$\begin{aligned} \eta_3 &> \xi_3 \sin[\varphi] - c \cos[\varphi], & \eta_3 &< -\xi_3 \sin[\varphi] + c \cos[\varphi], \\ \eta_2 &> \xi_2 \sin[\varphi] - c \cos[\varphi], & \eta_2 &< -\xi_2 \sin[\varphi] + c \cos[\varphi], \\ \eta_1 &> \xi_1 \sin[\varphi] - c \cos[\varphi], & \eta_1 &< -\xi_1 \sin[\varphi] + c \cos[\varphi]. \end{aligned} \quad (79a,b,c)$$

The pictures of the domain $\mathcal{D}$ for different values of the friction angle φ and of the cohesion c depicted in Figure 45, show how the saturation domain changes as the friction angle tends to 90° and the cohesion tends to zero.



[Figure 45]
The domain $\mathcal{D}$ for different values
of the friction angle φ and of the
cohesion c . (a): $\varphi=30^\circ, c=1$; (b):
 $\varphi=60^\circ, c=0.3$; (c): $\varphi=80^\circ, c=0.1$.

When formulated in the space σ, τ the MC criterion appears as extremely simple, but, for practical purposes, we need to express it in terms of the components of the Cauchy stress σ .

The following traditional decomposition of the stress σ into its pressure and deviatoric parts is particularly useful to distinguish volumetric and shear stress

$$\sigma = S + p I \quad (80)$$

p being the equivalent pressure, such that $p = \frac{1}{3} \text{tr}(\sigma)$ and $S = \sigma - p I$, the deviatoric stress.

By noticing that the MC failure criterion is isotropic, we can express it in terms of the stress invariants

$$\begin{aligned} i_1(\sigma) &= \text{tr}(\sigma) = 3p, \\ i_2(\sigma) &= \frac{1}{2}[(\text{tr} \sigma)^2 - \sigma : \sigma], \\ i_3(\sigma) &= \det(\sigma), \end{aligned} \quad (81a,b,c)$$

or, what is the same, in terms of the invariants

$$\begin{aligned} p &= \frac{1}{3} \text{tr}(\sigma) \\ J_2 &= -i_2(S) = \frac{1}{2}(S : S) \\ J_3(\sigma) &= i_3(S) = \det(S) = \frac{1}{3}((S : S) : S) \end{aligned} \quad (82a,b,c)$$

With reference to Figure 46, we can write

$$F = R_{mc} q - p \tan \varphi - c = 0 \quad (83)$$

where q is the Mises equivalent stress, $q = \sqrt{3 J_2}$,

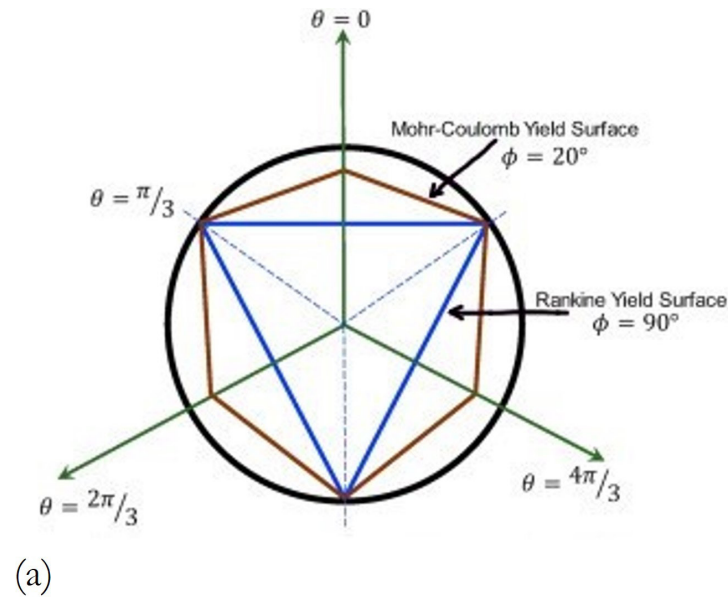
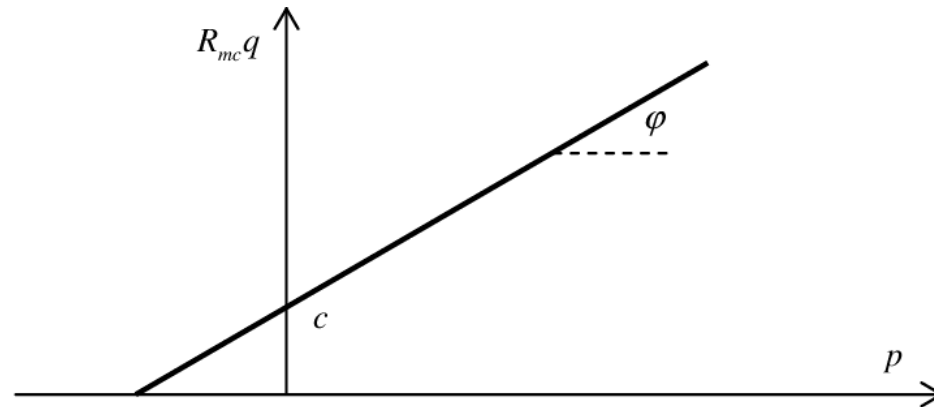
$$\text{and } R_{mc}(\theta, \phi) = \frac{1}{\sqrt{3} \cos \varphi} \sin\left(\theta + \frac{\pi}{3}\right) + \frac{1}{3} \cos\left(\theta + \frac{\pi}{3}\right) \tan \varphi \quad (84)$$

θ being the deviatoric polar angle defined as $\cos(3\theta) = \left(\frac{r}{q}\right)^3$

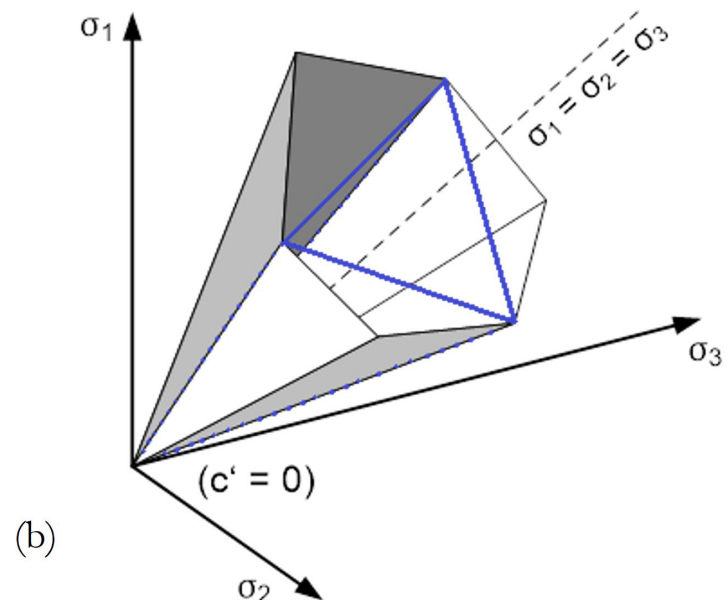
$$\text{such that } r = 3\left(\frac{1}{2} J_3\right)^{\frac{1}{3}}.$$

The shape of the yield surface in the deviatoric plane is function of the angle of friction ϕ Figure 47(a), which values are $0^\circ \leq \phi < 90^\circ$. the Mohr-Coulomb model reduces to the pressure-independent Tresca when $\phi = 0^\circ$ with a perfectly hexagonal deviatoric section. In the case of $\phi = 90^\circ$ the Mohr-Coulomb model reduces to the “tension cut-off” Rankine model with a triangular deviatoric section and $R_{mc} = \infty$. A three-dimensional representation of MC and Rankine ($\phi = 90^\circ$) is represented in Figure 47(b).

[Figure 46]
Mohr-Coulomb meridional plane



(a)



(b)

[Figure 47]
Mohr-Coulomb representation (a)
The deviatoric plane (b) The 3D
representation of Mohr-Coulomb
and Rankine (blue).

2.1.2. FLOW RULE

Initially, within the concept of plastic flow rules in materials, we encounter the Levy-Mises flow rule, which describes plastic deformation as a function of stress increments. This rule provides insights into how materials deform under loading conditions. However, we can generalize this approach by introducing the notion of a plastic multiplier determined by a hardening rule. This broader perspective allows us to define a wide range of material behaviours, encompassing practical scenarios. In this context, we introduce the concept of a plastic potential, a scalar function representing the relationship between stresses and resulting plastic strains (Budhu, 2011; Das, 2009).

Potential flow is assumed, so G is chosen as a hyperbolic function in the meridional stress plane and the smooth elliptic function proposed by Menetrey and Willam (Menetrey & William, 1995) in the deviatoric stress plane. G can be written as:

$$G = \sqrt{(\epsilon c|_0 \tan \psi)^2 + (R_{mw} q)^2} - p, \quad (85)$$

where the meridional eccentricity parameter ϵ dictates the speed at which the hyperbolic function approaches its asymptote, wherein the flow potential tends toward a straight line as the eccentricity approaches zero and R_{mw} is defined as:

$$R_{mw}(\theta, e) = \frac{4(1 - e^2) \cos^2 \theta + (2e - 1)^2}{2(1 - e^2) \cos \theta + (2e - 1)} R_{mc}\left(\frac{\pi}{3}, \varphi\right) \quad (86)$$

such that φ and θ are as previously defined,

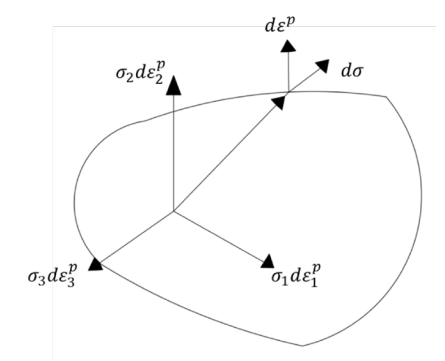
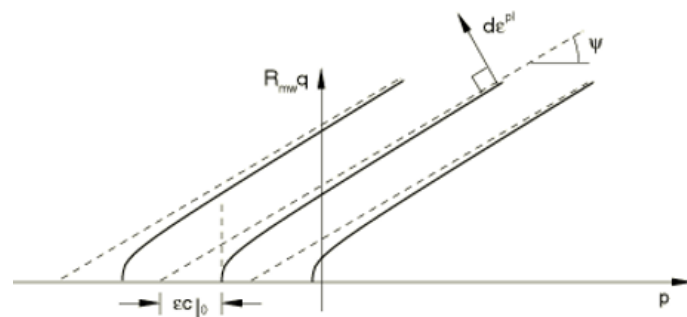
$$R_{mc}\left(\frac{\pi}{3}, \varphi\right) = \frac{3 - \sin \varphi}{6 \cos \varphi} c|_0 \quad (87)$$

is the initial cohesion yield stress defined as $c|_0 = c|_{\epsilon p|_0=0}$. In addition, e refers to the deviatoric eccentricity, which quantifies the deviation from perfect roundness in the deviatoric section in terms of the ratio between the shear stress along the extension meridian ($\theta = 0$) and the shear stress along the compression meridian ($\theta = \frac{\pi}{3}$).

Generally, in ABAQUS (Abaqus, 2023) the meridional eccentricity ϵ is attributed a default value of 0.1 and the deviatoric eccentricity is calculated as $e = \frac{3 - \sin \varphi}{3 + \sin \varphi}$. The previous calculation corresponds to aligning the flow potential, within the deviatoric plane, with the yield function surface both

in the triaxial tension and the compression. However, ABAQUS allows the user to consider e as an independent parameter where the convexity and smoothness of the elliptic function requires that $\frac{1}{2} < e \leq 1$. The upper limit corresponds to $\phi=0^\circ$ and the lower limit corresponds to $\phi=90^\circ$. The latter lower limit is not permitted within the MC model in ABAQUS. Figure 48 displays a family of hyperbolic potentials in the meridional stress plane.

[Figure 48]
Hyperbolic MC flow potential
function (Abaqus Analysis User's
Guide, 2016).



[Figure 49]
Yield surface for an associated flow rule.

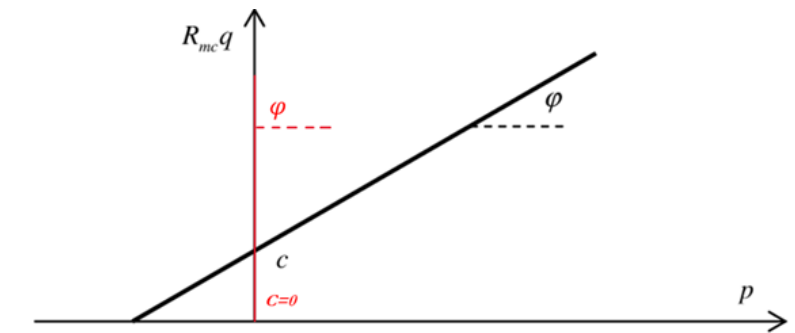
For most materials an associated flow rule is adopted for which the plastic potential coincides with the yield function, providing a specific characterization of plastic flow behaviour aligned with a particular yield criterion, namely the yield surface $f(\sigma_{ij}) = 0$ (see Figure 49).

By considering a plastic potential independent of the yield surface, we can develop a non-associated flow rule for modelling material behaviour. In the case of MC materials this is done by setting a dilatancy angle different from the friction angle and on introducing a plastic potential different from the yield surface and depending parametrically on ψ .

Figure 49 represents the problem into the space of principal stress and principal plastic strain, considered coincident due to the isotropic nature of the material. The direction of the normal to the yield surface aligns with the gradient of the yield function with respect to the stresses, and, consequently, in the case we assume an associated flow rule, the plastic strain increment vector is perpendicular to the yield surface, as depicted in the figure. This fundamental concept is commonly referred to as the normality rule.

2.2. MOHR-COULOMB VALIDATION IN ABAQUS

Considering the three Heyman assumptions: (a) No tensile strength (b) Infinite Compression (c) No sliding, one can calibrate MC to model masonry. When assuming a very low tensile strength, which is equivalent to negligible cohesion, and a friction angle equal to 90° , the MC failure model becomes close to the simplified models specifically model zero: NRNT masonry-like material. For a cohesion equal to zero and an infinite compressive stress the MC model becomes as can be seen in red in Figure 50.



[Figure 50]
Idealized Mohr-Coulomb model for
NRNT masonry-like material.

2.2.1. MOHR-COULOMB vs. RIGID COMPRESSION MODELS IN MASONRY

It is worth noting that it is evident that the stress-strain relationship of the associated Mohr-Coulomb type that is adopted moving forward, with values of friction angle greater than 89° and of cohesion of order less than 10-3 of the mean stress, represents the closest approximation to the unilateral Heyman material that can be digested by a Finite Element displacement based code, such as ABAQUS (Abaqus, 2023).

Therefore, the material has a saturation domain which is essentially the unilateral semi-cone of negative semidefinite symmetric tensors and, due to the associated flow rule, anelastic sliding is effectively forbidden.

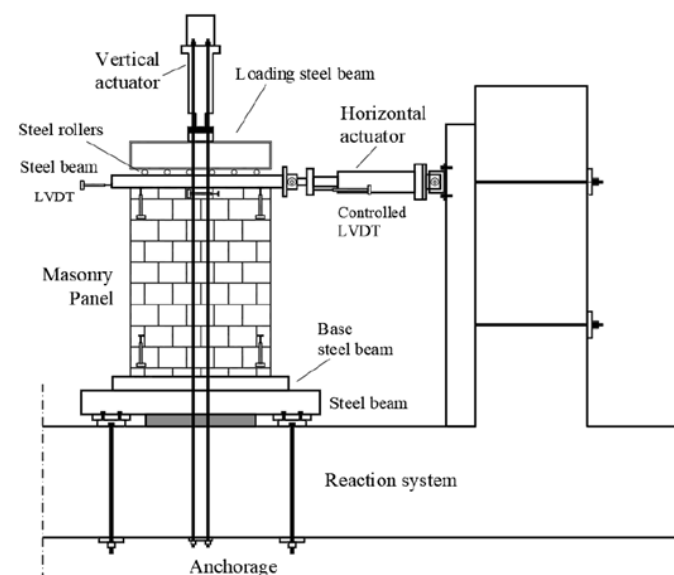
Inside this saturation domain the material is elastic and here a value of 1.4 GPa is mostly adopted. ABAQUS is a powerful and efficient computer code performing finite deformation analyses, meaning that the effect of large deformations, and namely of geometrical nonlinearities, is accounted for.

When comparing the Force-Displacement graphs of a structure made of unilateral but ideally rigid in compression material with those of an identical structure made of unilateral but elastically deformable in compression material, salient similarities and appreciable differences are expected.

These similar and different features can be elucidated by a simple example, namely that of a rectangular masonry wall loaded by a fixed vertical load and subject to a gradually increasing imposed horizontal force/displacement. The main outcome of this “test” is a force T versus displacement U diagram in which the role of cause and effect between T and U can be switched depending on the soft/hard arrangement of the test.

The study of the state of compression and shear that builds up in rectangular panels is crucial since they, being the main structural elements composing a typical wall of a masonry building, provide the way in which the loads are transferred to the foundation under the effect of the weight and of horizontal forces.

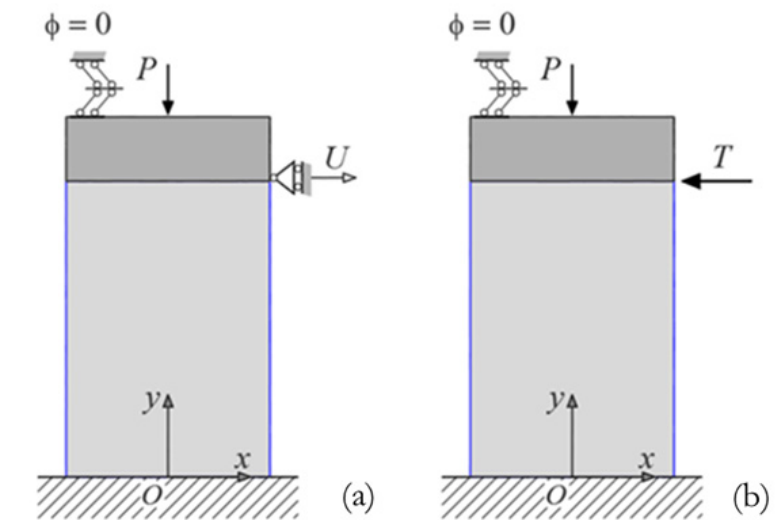
The testing of such masonry elements is the more correct and exhaustive way for assessing masonry structures experimentally. Despite being a difficult task due to the large forces necessary to realize the test under realistic geometrical and loading conditions, it has been the object of many experimental investigations in the last fifty years. We refer to (Gesualdo, 2022) for the detailed description of such a kind of experiments and for an exhaustive list of references. A typical test arrangement is shown in Figure 51 for clarity.



[Figure 51]
Experimental setup of a typical shear test on a masonry panel (thickness of the wall greater than 0.1 the height of the panel).

The analytical problem to be studied concerns a rectangular wall made of isotropic Masonry-Like material loaded by a fixed vertical force P through a rigid beam placed at the top of the panel. The rigid beam is assumed as perfectly restricted against rotation, and a class of corresponding equilibrium BVP's is depicted in Figure 52. Our aim is to obtain the horizontal force-horizontal displacement plot ($T - U$ plot), and this can be done in different ways.

It is clear that the easiest way to obtain the $T-U$ plot experimentally is through a displacement controlled test (hard device: Figure 52a), in which the horizontal displacement is gradually increased and the corresponding horizontal reaction force is measured.



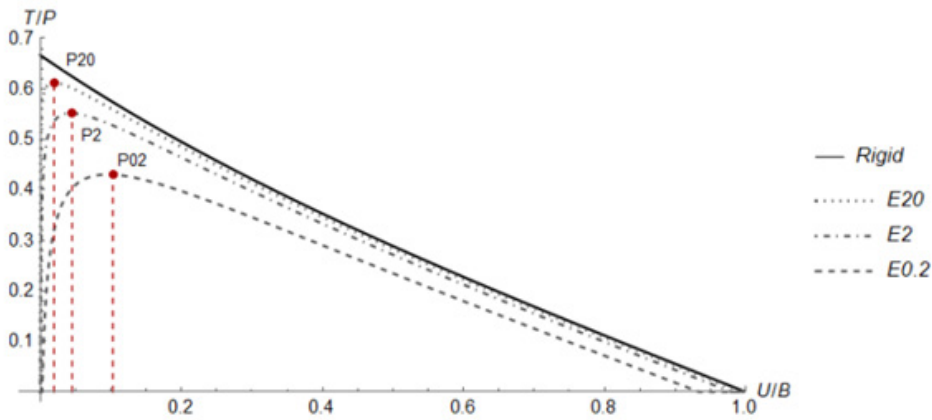
[Figure 52]
Basic BVP's for a ML panel for which the rotation of the upper rigid beam is inhibited. (a): $\phi = 0$, U given, P given. (b): $\phi = 0$, P , T given.

For definiteness, the results of a particular case studied in in (Gesualdo, 2022) are here considered for comparison. The main data of the case study are:

- Base B and height H are set to 2 m and 3 m,
- Thickness S is set to 0.3 m
- Vertical compressive load P is fixed to 300 kN.

The numerical results obtained in (Gesualdo, 2022) through an approximated large displacement analysis are graphically represented in Figure 53 and compared with the unilateral-rigid response of a rocking block.

[Figure 53]
Force-displacement, T-U relation for the panel of Fig.2, for three different values of the elastic modulus E: E20=20 GPa, E2=2 GPa, E02=0.2 GPa. The Horizontal force T is normalized with respect to the vertical force P and the horizontal displacement U is normalized with respect to the width B. The solid line represents the T-U graph for the rigid panel.



We notice that the $T - U$ curves that are obtained in [Gesualdo] approximate the ideal rigid-unilateral behaviour closer and closer as the Young's modulus is increased. A smooth transition from stable (hardening) to unstable (softening) regimes can be seen in Figure 53, in which the limit displacements marking this transition: $U_{\max}(E02)/B = 0.102$ m , $U_{\max}(E2)/B = 0.045$, $U_{\max}(E20)/B = 0.018$, are evidenced with dashed vertical red lines. The corresponding peak stresses are $T_{\max}(E02)/T_{\max}(\text{Rigid})=0.64$, $T_{\max}(E2)/T_{\max}(\text{Rigid})=0.81$, $T_{\max}(E20)/T_{\max}(\text{Rigid})=0.90$.

From these results we expect analogous reductions of the horizontal force capacity when we compare rigid in compression and elastic in compression outcomes of more complex problems.

This discrepancy suggests that the rigid body model predicts failure at substantially higher loads or forces compared to the Mohr-Coulomb model. Therefore, an observed difference in the load at peak for failure for the values of Young's modulus here adopted, can be roughly quantified in percentage between the two models generally between 70 to 80%.

2.2.2. CALIBRATION

In this study, dynamic analysis methods provided by ABAQUS were explored to simulate the collapse behavior of structures using MC material model. The two primary approaches for dynamic simulations are explicit and implicit methods. Explicit dynamics involve solving equations of motion at each time step, making them well-suited for capturing rapid and transient events, such as collapse phenomena. In contrast, implicit dynamics solve equations of motion through iterative techniques to achieve equilibrium progressively at each time step, suitable for quasi-static or slowly varying processes.

During the implementation of the MC model, it was discovered that using extreme parameter values (such as cohesion, friction angle, dilation angle) led to significant challenges with implicit analysis methods. The latter is due to the inherent numerical instability associated with these extreme parameter settings. This highlights a critical limitation of implicit methods when dealing with highly nonlinear material behaviors, where certain parameter combinations can require high computational demand, disturb the progress of the simulation, or prevent the initialization of the analysis.

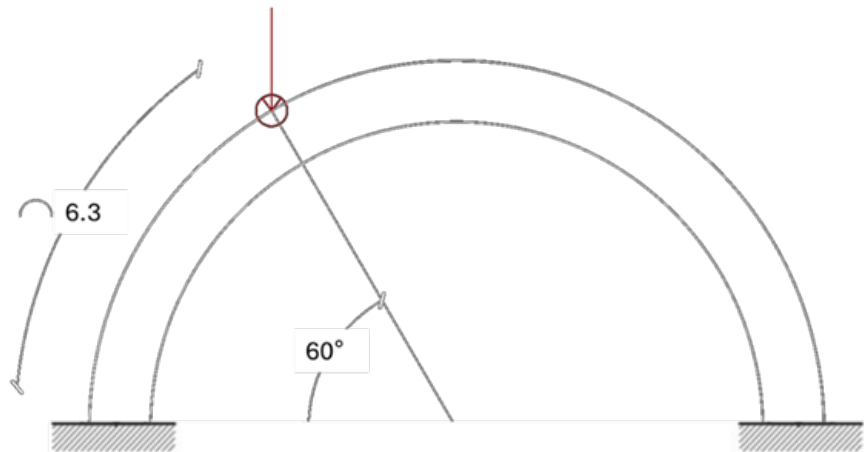
Given these findings, it was determined that the MC model with extreme parameter values in ABAQUS was only capable of running successfully and efficiently using an explicit dynamic step. Consequently, the selected analysis approach must be explicit dynamic for the simulations to execute within ABAQUS. Therefore, the primary fixed parameter in our modeling framework is the choice of analysis type, specifically dynamic explicit.

Then a series of trials for calibration followed by a parametric study were conducted for a masonry arch to adequately determine functioning parameters for the MC model applied to masonry inside the FE environment.

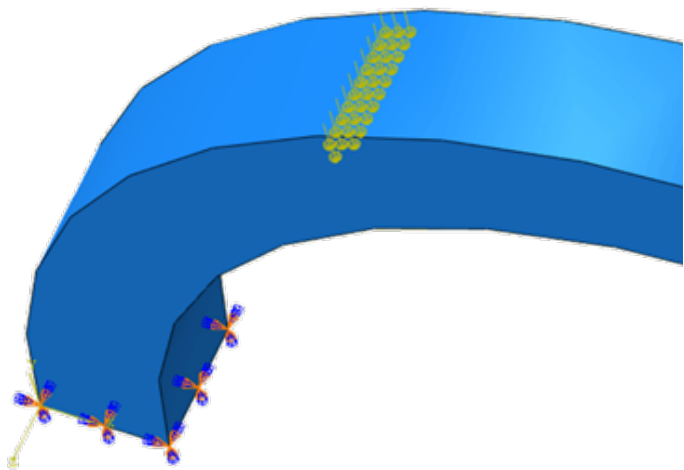
A semi-circular 3D arch having a 10m span and a 1m thickness was loaded with an increasing point load along the 1m arch depth at a 60° angle (see Figure 54 and Figure 55) following the values in Table 4.

[Table 4]
The increase of the applied point load with respect to time.

Increase [kN]	1	2	3	4	5	...	37	38	39	40	41
Time [s]	0	0.025	0.05	0.075	0.1	...	0.9	0.925	0.95	0.975	1



[Figure 54]
Point load applied to the semi-circular
arch model at an angle of 60°.



[Figure 55]
Three-dimensional model in ABAQUS showing
the distribution of the point load along the 1m
thickness of the arch.

[Table 5]
Material Properties adopted for masonry
based on (CNT, 2019; NTC, 2018).

Material Properties		
ρ	E	ν
1800 Kg/m ³	1400*10 ⁶	0.1

The arches are modelled as limestone masonry arches, incorporating elastic material properties specified for 'Regular masonry of soft stone (tuff, limestone, etc.)' according to Table 1. The highest Young's modulus value, along with ρ the mass density, and a Poisson ratio ν set to 0.1 as per (Angelillo, 2014; Angelillo et al., 2010), were adopted for the analysis and are presented in Table 5.

Two approaches were compared for the MC model, one using a tension cut-off and another adopting associative flow.

Since the MC model for $\phi=90^\circ$ simplifies to the “tension cut-off” Rankine model, the initial trial was conducted with a tension cut off. A friction angle of 89° was selected in ABAQUS because attributing a value of 90° is prohibited.

In addition, the cohesion value is set according to the provided range of mean stresses calculated following Table C8.5.I presented in Table 1 of this thesis. Having the interval of compressive stress between 1 MPa and 8 MPa, the tensile stress of masonry is calculated to consider all possible ratios of compressive to tensile stress which are 1/10, 1/30 and 1/60.

Hence the mean stress $\sigma_{\text{mean}}=(\sigma_c+\sigma_t)/2$ will be spanning from the following intervals, respectively to the ratios previously stated, [0.55 MPa, 4.6 MPa], [0.516 MPa, 4.133 MPa] and [0.508 MPa, 4.065 MPa] so we can assume σ_{mean} within the following interval 0.52 MPa to 4.26 MPa for masonry.

Given these ranges we assume a cohesion value of 1 Pa. When it is converted to the same units as the mean stress, where 1 Pa = 1 N/m² and 1 MPa = 1,000,000 (Pa) with the lower bound of mean stress: 0.52 MPa=520,000 Pa and the Upper bound of mean stress: 4.26 MPa = 42,600,000Pa. We can then compare the cohesion value to the range of mean stresses:

-Lower bound: $\frac{1 \text{ N/m}^2}{520000 \text{ N/m}^2} = 1.9 * 10^{-6} = 0.19 \mu\text{strain}$

- Upper bound: $\frac{1 \text{ N/m}^2}{42600000 \text{ N/m}^2} = 2.3 * 10^{-8} = 0.02 \mu\text{strain}$

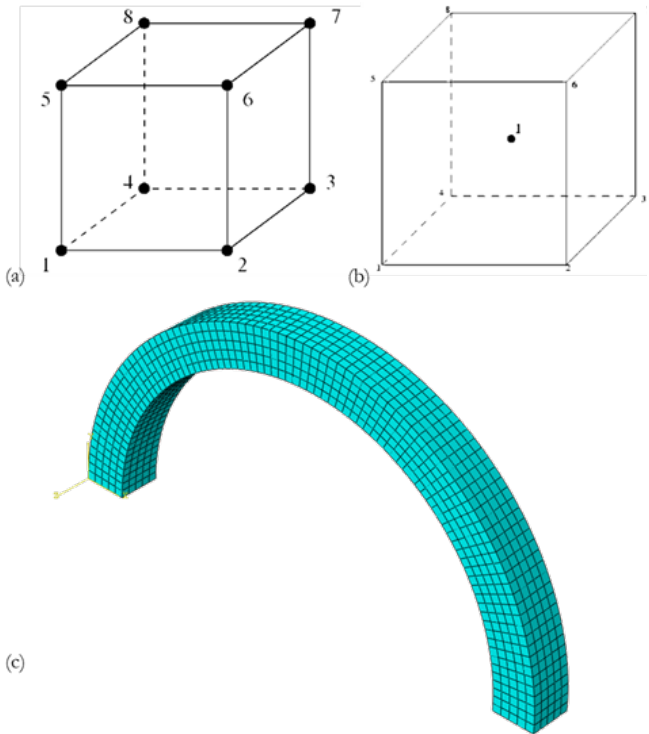
So, with a cohesion value of 1 Pa, the strain falls within the range of mean stresses for masonry, but it's still on the lower end of the stress scale.

Consequently, the preliminary parameters considered for the MC model are presented in Table 6.

[Table 6]
Preliminary parameters adopted for calibration of Mohr-Coulomb (MC) model in ABAQUS.

Mohr-Coulomb Parameters	
φ	c
89°	1 Pa

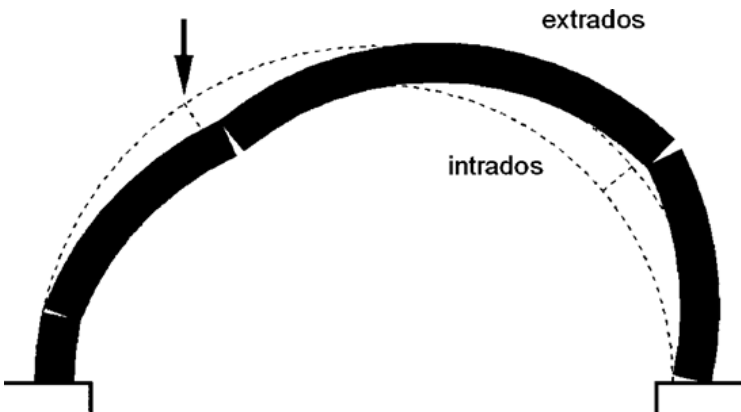
The attributed mesh is a C3D8R characterized by an 8-node linear brick element with reduced integration and hourglass control as represented in Figure 56a and Figure 56b. The global size of mesh attributed was 0.2 with number of elements equal to 2150 as can be seen in Figure 56c.



[Figure 56]
Meshing of the semi-circular arch in ABAQUS: (a) mesh element adopted, (b) integration point of the mesh element and (c) the meshed arch.

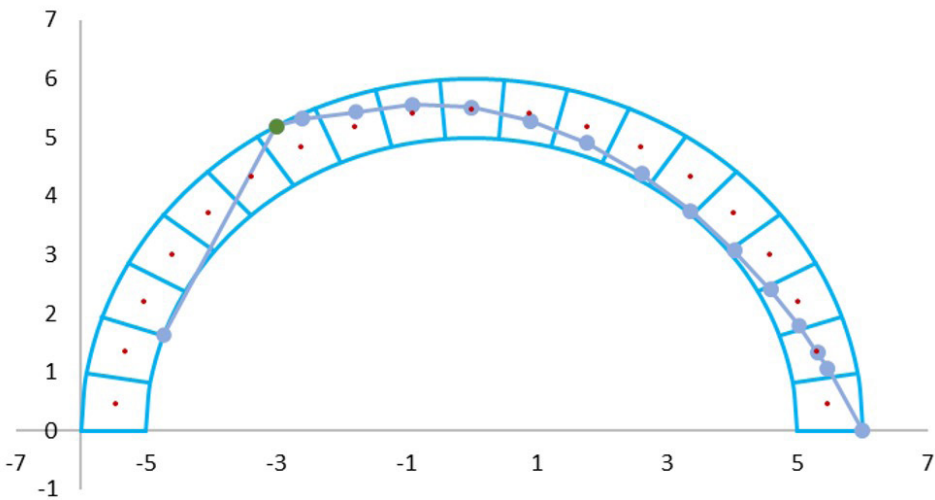
To reiterate, the concept of the thrust line is the theoretical line that represents the path of compressive forces passing through the structure. Hence, the results for the minimum principal stress (compressive stress) are presented in Figure 59. The black band, in the figure, indicates the concentration of highest compressive stress values, representing the “thrust line.”

[Figure 57]
Expected analytical solution determining hinge location at the moment of initiation of the collapse mechanism.



A file in Excel was created to solve generic cases and simulate the thrust line with an increasing point load. With reference to Figure 57 three hinge locations can be determined and fixed in order to create the fourth hinge by increasing the point load.

The solution for the semi-circular arch of the adopted geometry in Figure 58 presents the four hinges at the points where the thrust line (in lavender) touches the intrados and extrados. The limit point load that initiates the collapse mechanism was determined as 29 kN load. This load occurs at 0.6s according to Table 4. Hence the visualization of results in ABAQUS is set at the step time equivalent to this value.

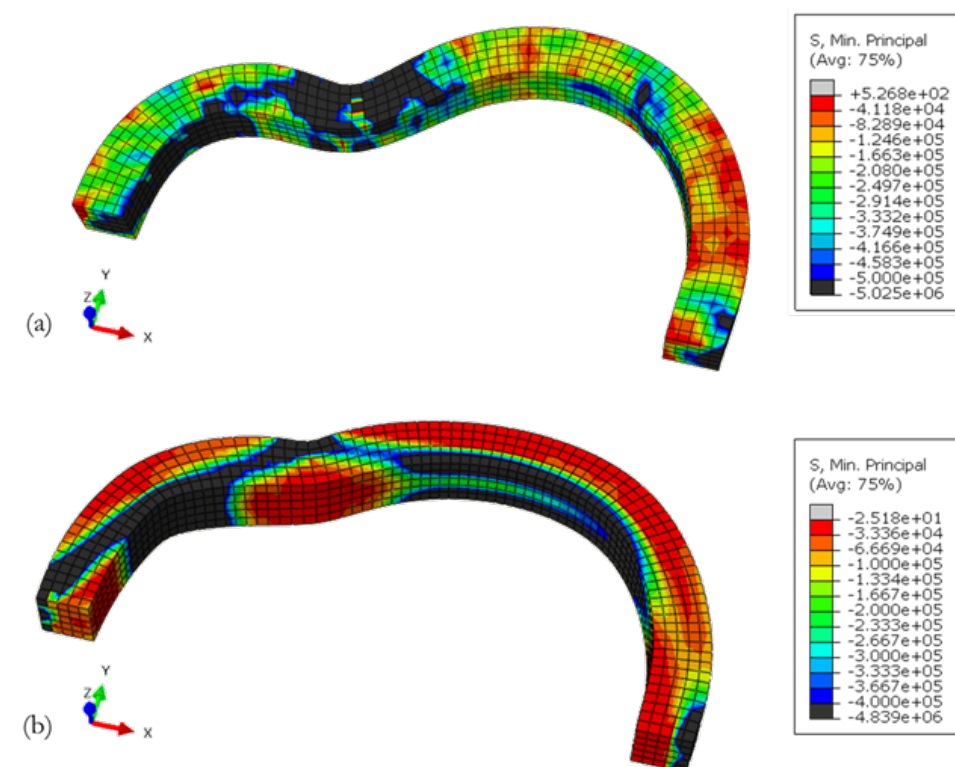


[Figure 58]
Semi-circular arch under incremental point loads at the green dot, with lavender-colored thrust line forming four hinges.

In Figure 59a the results of the analysis carried out by using a MC model with tension cut-off show some similarity with the expected thrust line. Nevertheless, the results are not satisfactory enough as the compressive zones are still rather smeared and unconcentrated in a compressive band.

The second trial consisted of eliminating the tension cut-off and adopting an associated flow rule, where the dilatancy angle is equal to the friction angle $\psi=\varphi=89^\circ$ and the cohesion is equal to 1 Pa. And indeed, as expected applying the associated flow rule, which essentially ensures that plastic deformation is normal to the yield surface, gave more valuable insights into the behaviour of masonry elements under loading. It can be seen in Figure 59b that a better adequate thrust line can be visualized.

Hence MC with associated flow rule following the parameters in Table 6 generates good results in comparison to the analytical results in Figure 57 and Figure 58.

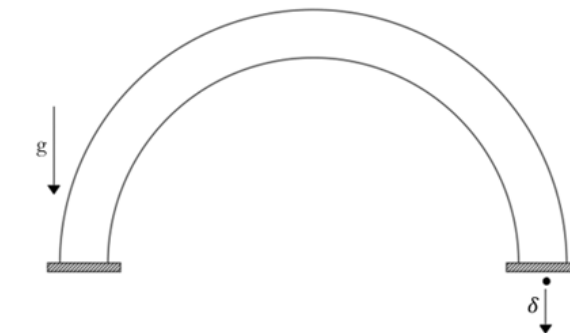


[Figure 59]
Minimum principal stress results for the MC model of the semi-circular masonry arch: (a) using tension cut-off and (b) adopting an associated flow-rule.

2.2.3. PARAMETRIC STUDY

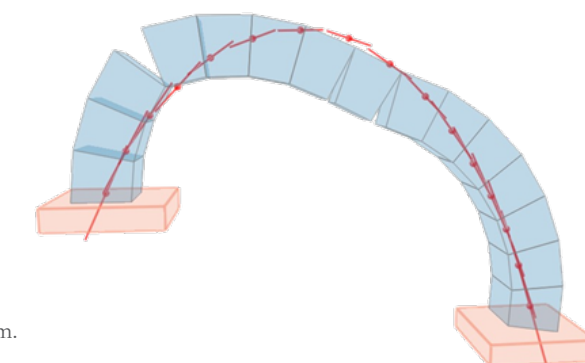
After calibration of the adequate criteria concerning the MC model and choosing an associative flow rule due to masonry's normality rule, a parametric study is necessary to specify which are the values to adopt throughout this thesis for accurate results.

In this section, a vertical settlement analysis is conducted in ABAQUS on a semi-circular arch model with a radius of 1m and a thickness of 0.30m using the MC model (see Figure 60).



[Figure 60]
Loading conditions applied to a semi-circular arch, having a radius $r=1\text{m}$ and a thickness $s=0.3\text{m}$.

When a support of an arch settles (vertically in the example considered) a stable three hinge mechanism is formed and the structure moves from the original equilibrium state to a new, comfortable equilibrium state. The term comfortable refers to the fact that the structure still sits in a minimum state of its potential energy. As the displacement is increased, and the effect of the geometrical nonlinearities is taken into account this minimum state gets less and less protected by an energy barrier, until strong minimality is lost, a four hinge mechanism is formed and the structure becomes unstable. All this can be detected with a sequentially linear PRD analysis. The results of this PRD analysis are presented in Figure 61, where a visible deformation is observed along with the hinge locations identified at a value of the settlement $\delta=0.85\text{m}$, that is abundantly in the geometrically nonlinear phase. The value of the settlement producing the formation of the unstable four hinge mechanism was detected at $\delta=1.00\text{m}$.



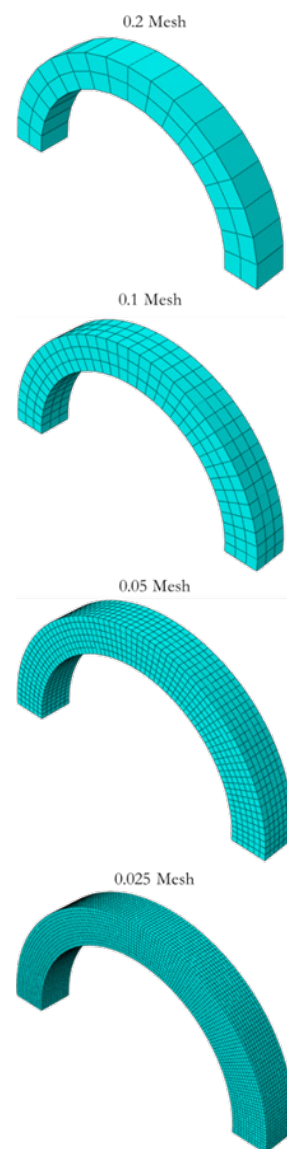
[Figure 61]
Result of the PRD analysis of a vertical settlement applied to a semi-circular arch, having a radius $r=1\text{m}$ and a thickness $s=0.3\text{m}$.

The parametric study explores the effect of the variation of mesh size, cohesion value, and friction angle on the results of the finite element analysis using the MC model. Moreover, the results are discussed and compared against the expected behaviour according to the PRD method.

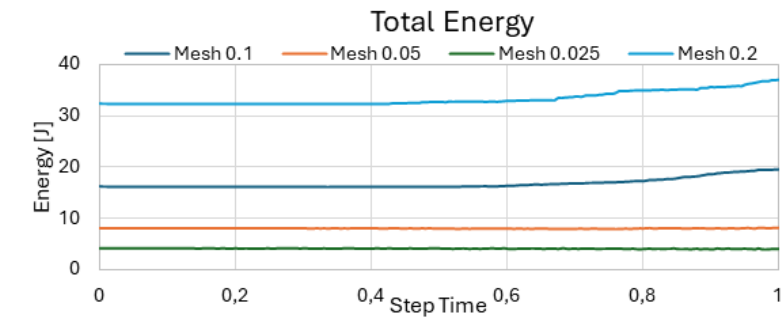
2.2.3.1. Sensitivity to mesh size

A reduction of the global mesh size by 50% is adopted for the previously described arch starting from a 0.2 mesh (36 elements), 0.1 mesh (324 elements), 0.05 mesh (2592 elements) to a 0.025 mesh (20736 elements) as can be seen in Figure 62.

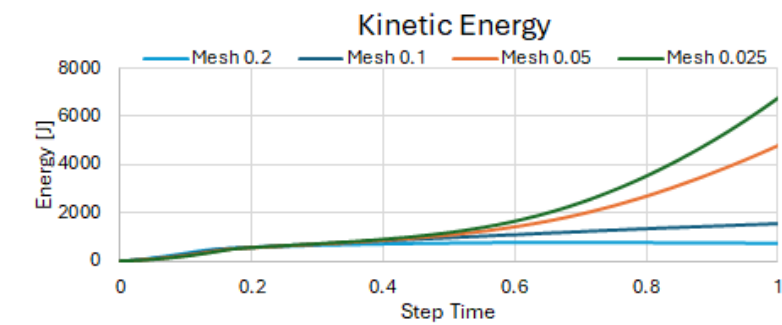
The friction angle is fixed at $\varphi=89^\circ$ and the cohesion value is fixed at 1Pa. A comparison of the energies is provided in the subsequent graphs (see Figure 63 to Figure 65).



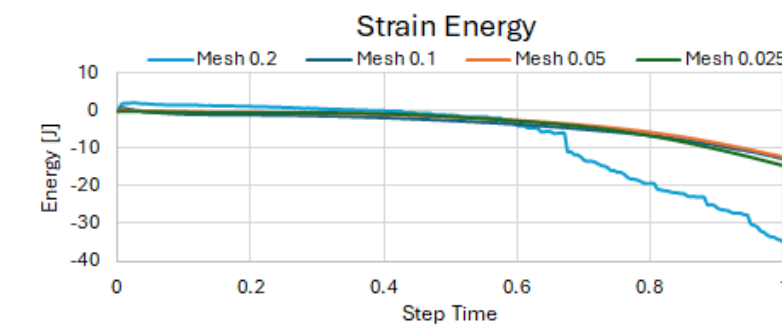
[Figure 62]
Visualization of the meshes attributed to the semi-circular arch in ABAQUS, showing the progressive reduction in mesh size.



[Figure 63]
Graph showing the Total Energy in function of the step time, which is equivalent to the displacement, for all the mesh variations.



[Figure 64]
Graph showing the Kinetic Energy in function of the step time, which is equivalent to the displacement, for all the mesh variations.

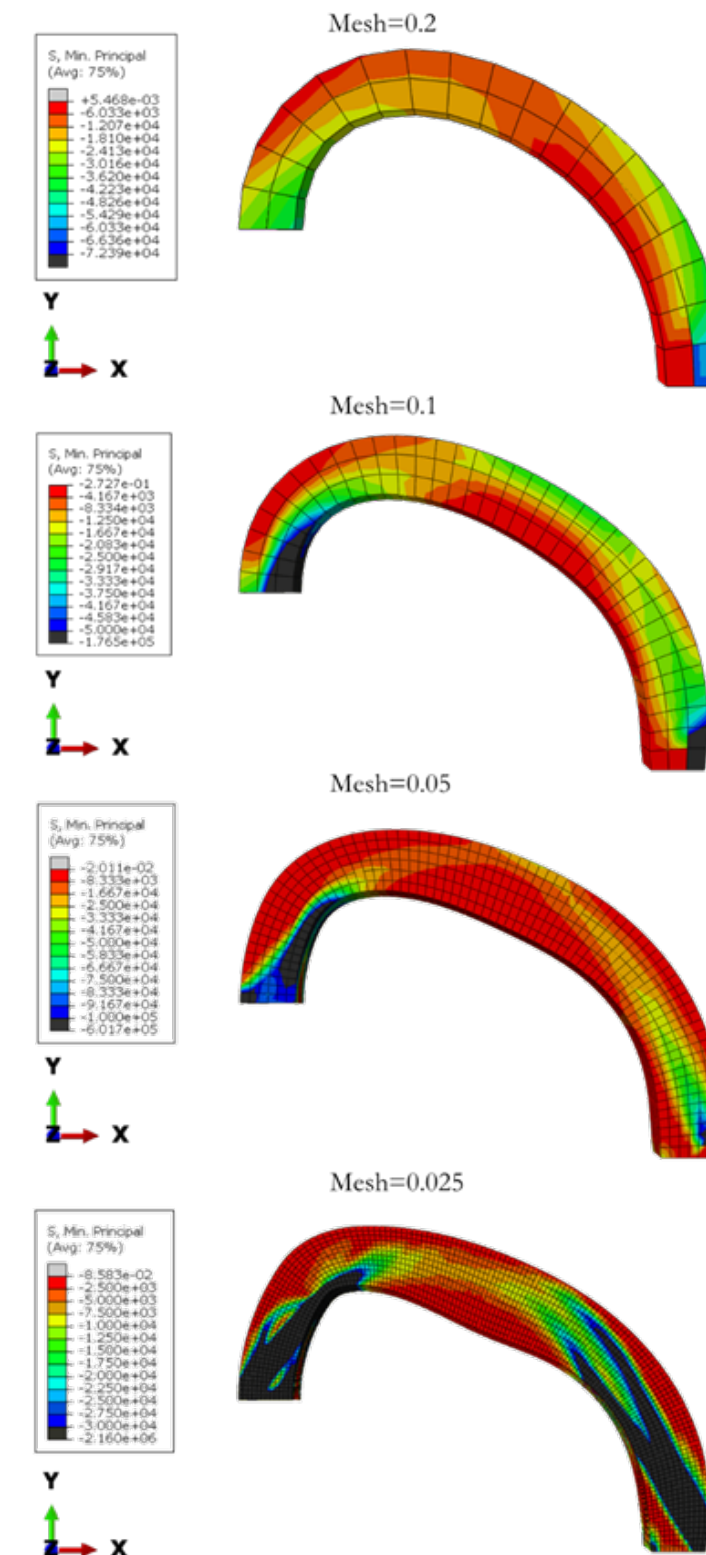


[Figure 65]
Graph showing the Strain Energy in function of the step time, which is equivalent to the displacement, for all the mesh variations.

The previous graphs show identical results for the different mesh sizes, in the Kinetic and Strain energies, up to a particular point, after which they begin to diverge. It is evident that as the mesh size decreases, the disparity in energy values between different meshes decreases, particularly evident when comparing the 0.05 and 0.025 mesh sizes. Furthermore, finer mesh sizes exhibit reduced graph oscillations as can be seen in the strain graph in Figure 64.

The thrust line showing the four hinge mechanism appears in the visualization module at the vertical settlement $\delta=0.85\text{m}$ which corresponds to an 85% reduction compared to the collapse settlement equal to 1.00m found through PRD. The latter is due to the rigid body behaviour and Mohr-Coulomb model which exhibits elasticity in compression. This discrepancy has been addressed in the section “Elastic vs. Rigid Compression Models in Masonry”. Additionally, the thinner the mesh the more evident the thrust line. Accordingly, a thinner mesh is more desirable as can be seen in Figure 66.

For the more refined mesh, having a 0.025 element size, a phenomenon of bulging can be seen at the crown of the arch. This is a sign of loss of equilibrium at a level of displacement lower than that predicted with the coarser mesh of 0.05 element size. The finer mesh size is just more accurate in predicting the exact level of loss of minimality.



[Figure 66]

Minimum principal stress results for the varying mesh sizes of the MC model of the semi-circular arch, having a radius $r=1\text{m}$ and a thickness $s=0.3\text{m}$, at the limit vertical settlement $\delta=0.85\text{m}$.

We move now to the comparison of computational time present in Table 7. Considering the speed, the previously presented graphs and visualization of results, the optimal and most efficient mesh size is 0.05. The latter value will be the one adopted throughout the parametric study.

[Table 7]
Computational time of the mesh sensitivity analyses for a MC model under vertical settlement for $\varphi=89^\circ$ and $c=1\text{Pa}$ (in red the optimal choice of mesh).

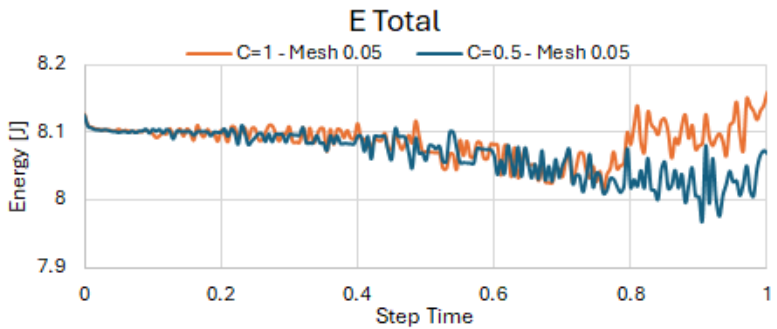
Mesh size	0.2	0.1	0.05	0.025
Computational Time [s]	6	61	1201	16296

2.2.3.2. Sensitivity to cohesion

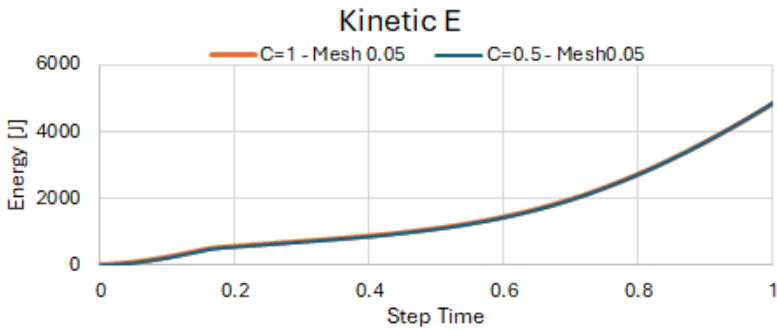
To determine the effect of the reduction of the cohesion parameter, the mesh size selected based on the previous discussion is 0.05 (2592 elements) and the friction angle remains fixed at $\varphi=89^\circ$.

Now given the calculated range of mean stresses for masonry, which spans from 0.52 MPa to 4.26 MPa, and assuming a cohesion value of 1 Pa initially, it becomes evident that the strain resulting from this cohesion value falls within the acceptable range when compared to the mean stresses. However, to provide a more nuanced understanding of the material behaviour, particularly in relation to the mean stress, a cohesion value of 0.5 Pa might be more appropriate. This represents a reduction of 50% from the initially assumed value. With a smaller cohesion value, the resulting strain would remain within the range of mean stresses for masonry.

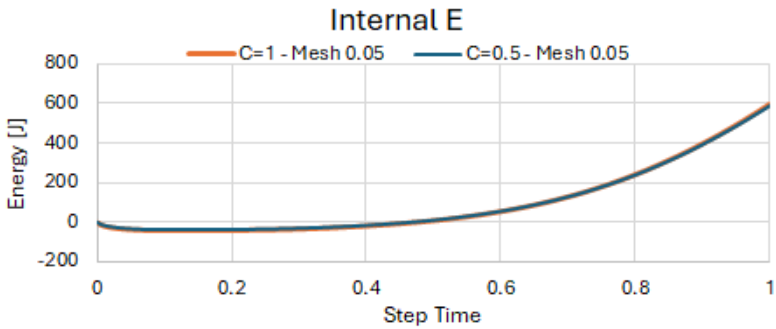
As previously, a comparison of the energies is provided in the subsequent graphs to visualize the effect of this reduction on the model’s behaviour and on the computational time.



[Figure 67]
Graph showing the Total Energy in function of the step time, which is equivalent to the displacement, for cohesion values equal to 1Pa and 0.5Pa.

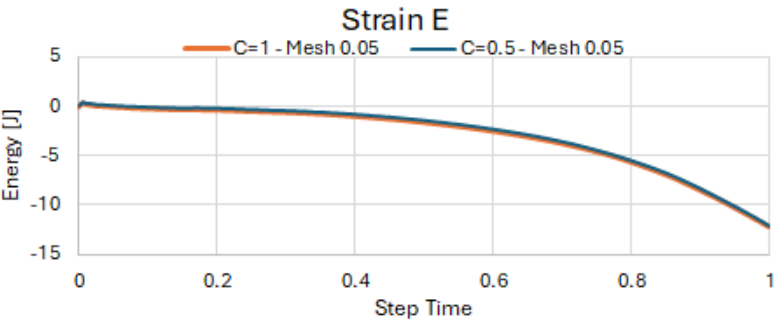


[Figure 68]
Graph showing the Kinetic Energy in function of the step time, which is equivalent to the displacement, for cohesion values equal to 1Pa and 0.5Pa.



[Figure 69]

Graph showing the Internal Energy in function of the step time, which is equivalent to the displacement, for cohesion values equal to 1Pa and 0.5Pa.



[Figure 70]

Graph showing the Strain Energy in function of the step time, which is equivalent to the displacement, for cohesion values equal to 1Pa and 0.5Pa.

[Table 8]

Computational time of the cohesion sensitivity analyses for a MC model under vertical settlement for a mesh size equal to 0.05 and $\varphi=89^\circ$ (in red the optimal choice of cohesion).

c	1Pa	0.5Pa
Computational Time [s]	1201	1008

As can be seen from Figure 67 to Figure 70, it is evident that the results remain identical for both cohesion values. However, a notable divergence occurs in the total energy graph (see Figure 67) around the step time 0.8, which is equivalent to a settlement of 0.8m. The latter value found using the MC model in ABAQUS present an 85% reduction compared to the collapse settlement equal to 1m found through PRD. The latter is due to the rigid body behaviour and Mohr-Coulomb model which exhibits elasticity in compression. This matter has been addressed in the section “Elastic vs. Rigid Compression Models in Masonry”.

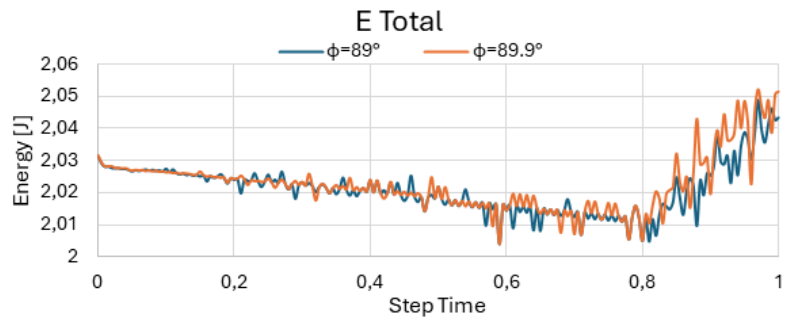
We proceed to compare the computational time; a comparison is presented in Table 8. It is evident that the cohesion value of 0.5 Pa results in faster computation. Therefore, the optimal combination appears to be a mesh size of 0.05 and a cohesion value of 0.5 Pa.

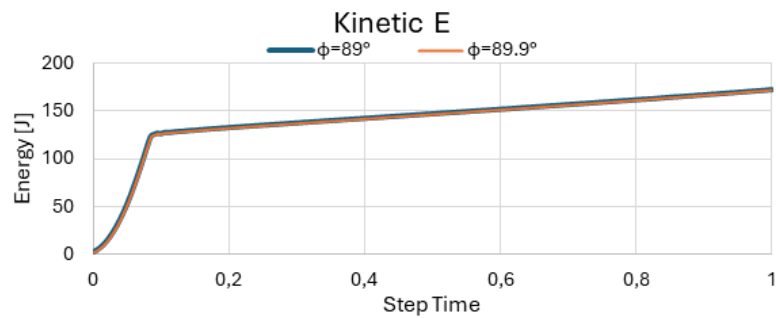
2.2.3.3. Angle of friction increase

After completing our examination of mesh size and cohesion reduction and establishing optimal values (cohesion = 0.5 Pa, mesh size = 0.05 with 2592 elements), we delve into the effect of increasing the angle of friction from $\varphi=89^\circ$ to $\varphi=89.9^\circ$.

[Figure 71]

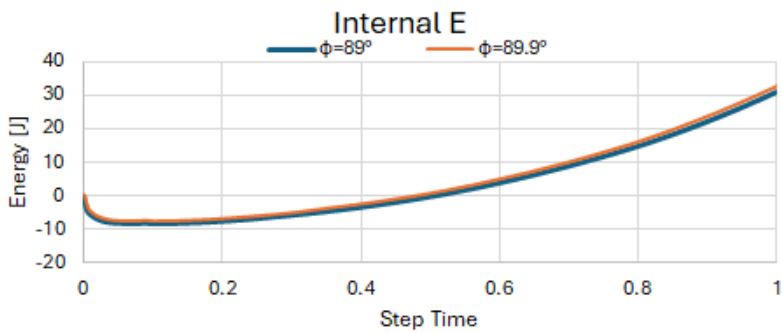
Graph showing the Total Energy in function of the step time, which is equivalent to the displacement, for friction angle values equal to $\varphi=89^\circ$ and $\varphi=89.9^\circ$.





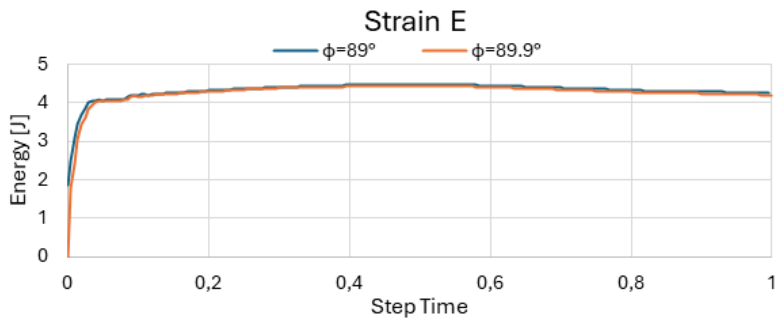
[Figure 72]

Graph showing the Kinetic Energy in function of the step time, which is equivalent to the displacement, for friction angle values equal to $\varphi=89^\circ$ and $\varphi=89.9^\circ$.



[Figure 73]

Graph showing the Internal Energy in function of the step time, which is equivalent to the displacement, for friction angle values equal to $\varphi=89^\circ$ and $\varphi=89.9^\circ$.



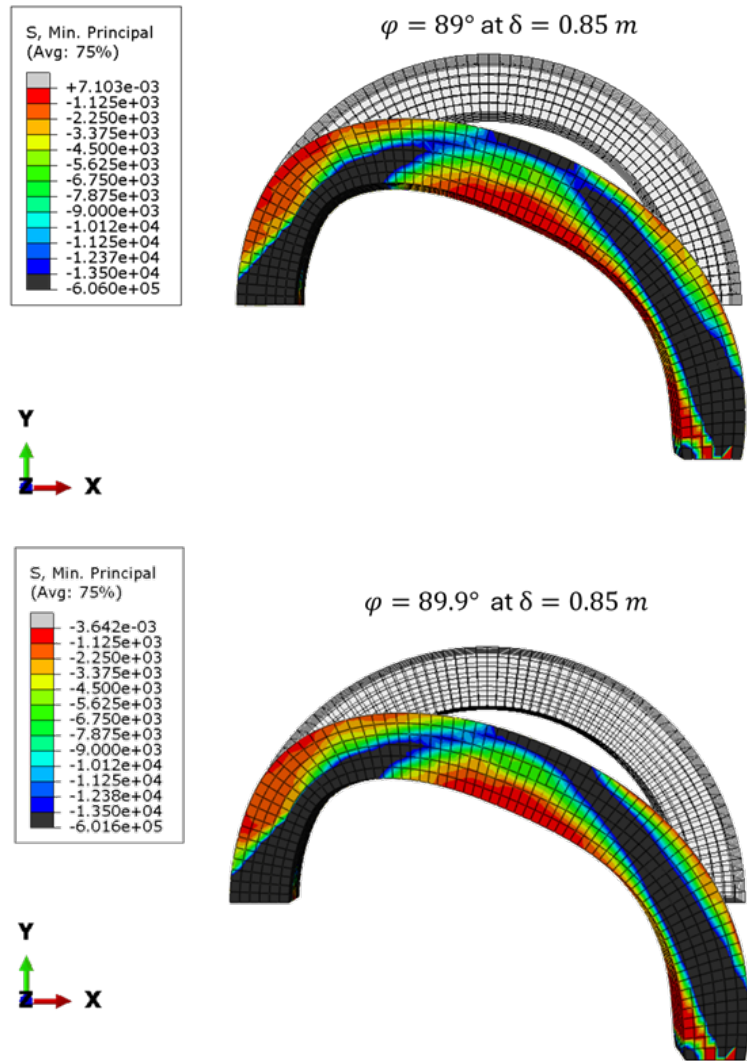
[Figure 74]

Graph showing the Strain Energy in function of the step time, which is equivalent to the displacement, for friction angle values equal to $\varphi=89^\circ$ and $\varphi=89.9^\circ$.

[Table 9]

Computational time of the angle of friction sensitivity analyses for a MC model under vertical settlement for a mesh size equal to 0.05 and a $c=0.5$ (in red the optimal choice of friction angle).

φ	89°	89.9°
Computational Time [s]	1183	1008



[Figure 75]

Minimum principal stress results for the Mc model of the semi-circular arch, having a radius $r=1\text{m}$ and a thickness $s=0.3\text{m}$, at the limit vertical settlement for $\varphi=89^\circ$ and $\varphi=89.9^\circ$.

As can be noticed the variations in the energy graphs are negligible.

Consequently, we proceed to compare the minimum principal stress (compressive stress) at the moment of initiation of the mechanism of collapse, where $\delta=0.85\text{m}$.

The results are visualized in Figure 48 revealing significant similarity, with a clear thrust line and locations of the four hinges identical to those in Figure 61. Thus far, it appears that variations in the angle of friction do not significantly impact the results.

Finally, the computational time is considered and presented in Table 9 to determine the optimal choice. Indeed $\varphi=89.9^\circ$ has been found to be more efficient and closely approximates the ideal model at $\varphi=90^\circ$.

2.3. BENCHMARKS

In this section we validate the MC model in ABAQUS (Abaqus, 2023) by comparing the results with those obtained through the reference approaches described in sub-section 2 of the “State of the Art”, and eventually specifying and clarifying the possible identification of the moment of collapse.

This is done through presenting some applications with reference to the following cases:

1. the assessment of the horizontal static multiplier for:

Case A. round arches of various thicknesses.

Case B. circular arch with a springing angle $\beta = 20^\circ$.

2. a round arch subjected to settlement with a support subjected to a given vertical bottom settlement.

The material properties adopted for the masonry through all the FEA analyses are presented in Table 5. The parameters adopted for the MC model with an associative flow rule are shown in Table 10 and were chosen based on the previously conducted parametric study.

Mohr-Coulomb Parameters	
φ	c
89.9°	0.5 Pa

[Table 10]
Adopted values for the parameters of the MC model.

The values adopted for the parameters of the CDP model are provided in Table 3 and Table 12. The choice of such values agrees with the literature, see section 2.2.1.a. The assumption of taking a tensile strength equal to 1/30 of the compressive strength is justified in Appendix A (sub-section A.1). Interestingly, the comparison not only revealed that the choice of analysis method has minimal impact on the results but also demonstrated that the explicit dynamic approach is significantly faster computationally.

[Table 11]
Material properties adopted for the CDP model of the arches.

Material Properties			
Mass Density ρ	Young modulus E	Poisson ratio ν	Compression resistance f_m
1800Kg/m³	1400Mpa	0.1	2MPa

[Table 12]
Values modelling the anisotropic stress-strain behavior of masonry for the CDP model.

Compression		Tension ($f_t=f_c/30$)	
σ [MPa]	ϵ [%]	σ [MPa]	ϵ [%]
0	0	0	0
2	0.143	0.066	0.004762
3.2	0.4	0.00333	0.15
0.35	0.5	0.00333	0.4
0.35	0.6		

2.3.1. BENCHMARK 1: HORIZONTAL CAPACITY OF CIRCULAR ARCHES

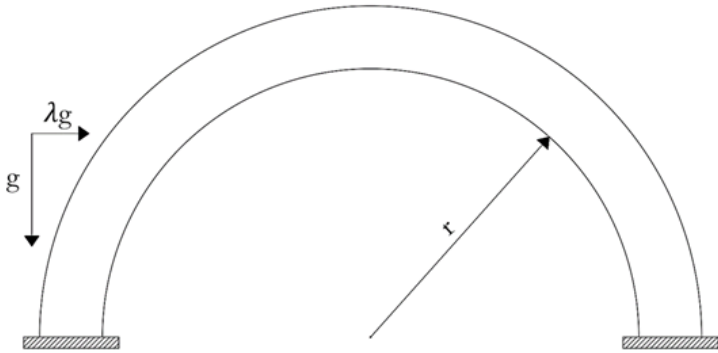
2.3.1.1. Case A: Round arches of various thicknesses

MC compared to PRD

In this section, an arch is analysed under horizontal forces, specifically seismic action, employing the MC model previously described in ABAQUS. We then compare these results with those obtained through the PRD method.

In this context, let's examine an arch exposed to self-weight and horizontal incremental action, where forces are directly proportional to mass and escalate via the loading parameter λ (see Figure 76). Hence subjected to a horizontal load equal to λg , where g is the gravity load. The goal of these analyses is to pinpoint the location of the four hinges denoted by angles θ_i for $i \in \{1, \dots, 4\}$ and determine the multiplier λ_c value.

[Figure 76]
Loading conditions applied to a round
arch (i.e. $\beta = 0^\circ$) of varying thickness S ,
with internal fixed radius $r = 1\text{m}$.



In PRD, each voussoir is conceptualized as being loaded at its centroid by vertical and horizontal forces. When the scale parameter λ , denoted as λ_c , reaches a certain value, four hinges form, and the arch becomes a mechanism.

In ABAQUS, being a finite element program, every element in the mesh experiences both vertical gravity loading and horizontal forces.

With regards to the multiplier, it can be found in MC by analysing the total energy x-y data values. From the latter, the moment when the total energy ceases to remain constant and begins to increase exponentially can be identified as the moment of initiation of the collapse mechanism.

Subsequently, the results found using the MC model in ABAQUS are compared to those found by PRD presented in Table 13. What has been noticed is that the λ_c determined through MC is typically 75% of that found through PRD. Am=n expected discrepancy due to the disparity in collapse loads at peak between rigid body behaviour and Mohr-coulomb model. This matter has been addressed in the section “Elastic vs. Rigid Compression Models in Masonry”. The results of λ_{cMC} compared to λ_{cPRD} , for the different thicknesses are presented in Table 14.

s [m]	λ_{cPRD}	θ_4 [°]	θ_3 [°]	θ_2 [°]	θ_1 [°]
0.20	0.236	162.83	103.26	43.69	0.00
0.22	0.279	165.92	105.59	45.26	0.00
0.24	0.320	168.73	107.74	46.75	0.00
0.26	0.359	171.29	109.73	48.17	0.00
0.28	0.396	173.63	111.58	49.53	0.00
0.30	0.431	175.79	113.31	50.83	0.00
0.32	0.465	177.77	114.93	52.09	0.00
0.34	0.497	179.60	116.45	53.30	0.00
0.36	0.529	180.00	117.88	54.47	0.00
0.38	0.560	180.00	119.21	55.60	0.00
0.40	0.590	180.00	120.46	56.71	0.00
0.42	0.620	180.00	121.63	57.78	0.00
0.44	0.649	180.00	122.74	58.83	0.00
0.46	0.677	180.00	123.78	59.86	0.00
0.48	0.705	180.00	124.78	60.87	0.00
0.50	0.732	180.00	125.72	61.86	0.00

[Table 13]

Results through PRD of the horizontal collapse multiplier λ_c , the hinges position through θ_1 , θ_2 , θ_3 , θ_4 , as function of the thickness s , for a round arch (i.e. $\beta = 0^\circ$) with internal radius $r = 1\text{m}$.

s [m]	λ_{cPRD}	λ_{cMC} graphically	$\lambda_{cPRD} * 0.75$
0.20	0.236	0.16	0.17
0.22	0.279	0.2	0.21
0.24	0.320	0.23	0.24
0.26	0.359	0.245	0.27
0.28	0.396	0.31	0.30
0.30	0.431	0.325	0.32
0.32	0.465	0.335	0.35
0.34	0.497	0.365	0.37
0.36	0.529	0.385	0.39
0.38	0.560	0.415	0.42
0.40	0.590	0.445	0.44
0.42	0.620	0.460	0.46
0.44	0.649	0.470	0.49
0.46	0.677	0.505	0.50
0.48	0.705	0.5302	0.53
0.50	0.732	0.555	0.55

[Table 14]

Comparison of the results through PRD and MC model of the horizontal collapse multiplier λ_c , as function of the thickness s , for a round arch (i.e. $\beta = 0^\circ$) with internal radius $r = 1\text{m}$.

However, in the visualization module the four-hinge formation, i.e. failure mechanism initiation, occurs at a value of the horizontal multiplier very close to that found through the PRD analysis, as opposed to the graphically determined multiplier λ_{cMC} which exhibits a 0.75 discrepancy.

A reduction of the statical collapse load multiplier due to elasticity in compression and geometrical nonlinearity is obviously expected, as found for example in the simulation of a simple shear test of a rectangular panel by Gesualdo in (Gesualdo, 2022), where a similar percentage reduction is detected.

The fact that the load in the simulation, before the hinges’ formation, reaches a value which is 25% bigger than the collapse load is due to the circumstance that the analysis is a dynamical one. Then, at any time t , the load that the structure is carrying is not completely balanced by the reactions, its effect being partly dispersed inside the structure, and then producing a lag -depending on the chosen loading speed which is adopted in the explicit dynamical analysis- between the time of failure and the initiation of the non-equilibrium phase.

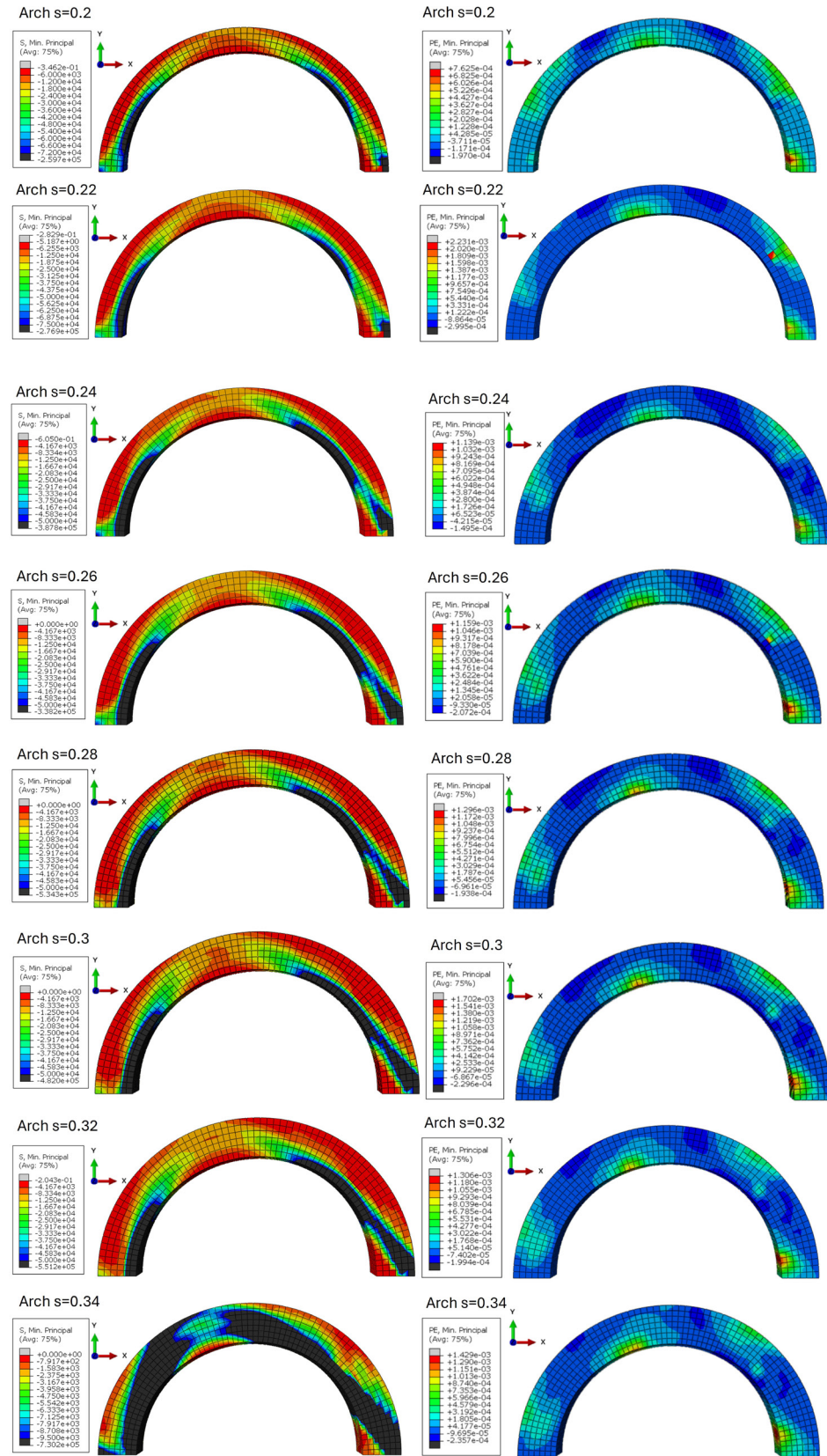
A visualization of the results at λ equal to λ_{cPRD} are presented for each arch thickness in the following figures.

Figure 77 and Figure 78, show on the right the minimum principal stress S_{Min} where the thrust line can be seen through the thickness of the arch compared to the minimum plastic strain PE on the left, that delineate the position of the hinge's opening, where separation occurs opposite the hinge location. Similarly, Figure 79 and Figure 80 reiterate the coherence of the results by showing the vonMises stress results (to the right) in comparison to the minimum PE.

Finally, Figure 81 to Figure 84 illustrate the minimum plastic strain, for the various arch thicknesses. Furthermore, to determine the hinge locations, the calculation of the angles θ_i involves measuring the angle between the hinge positioned at the right base of the arch and the centres of other concentrated plastic strains, as shown in Figure 85. The results for the different thicknesses are present in Table 15.

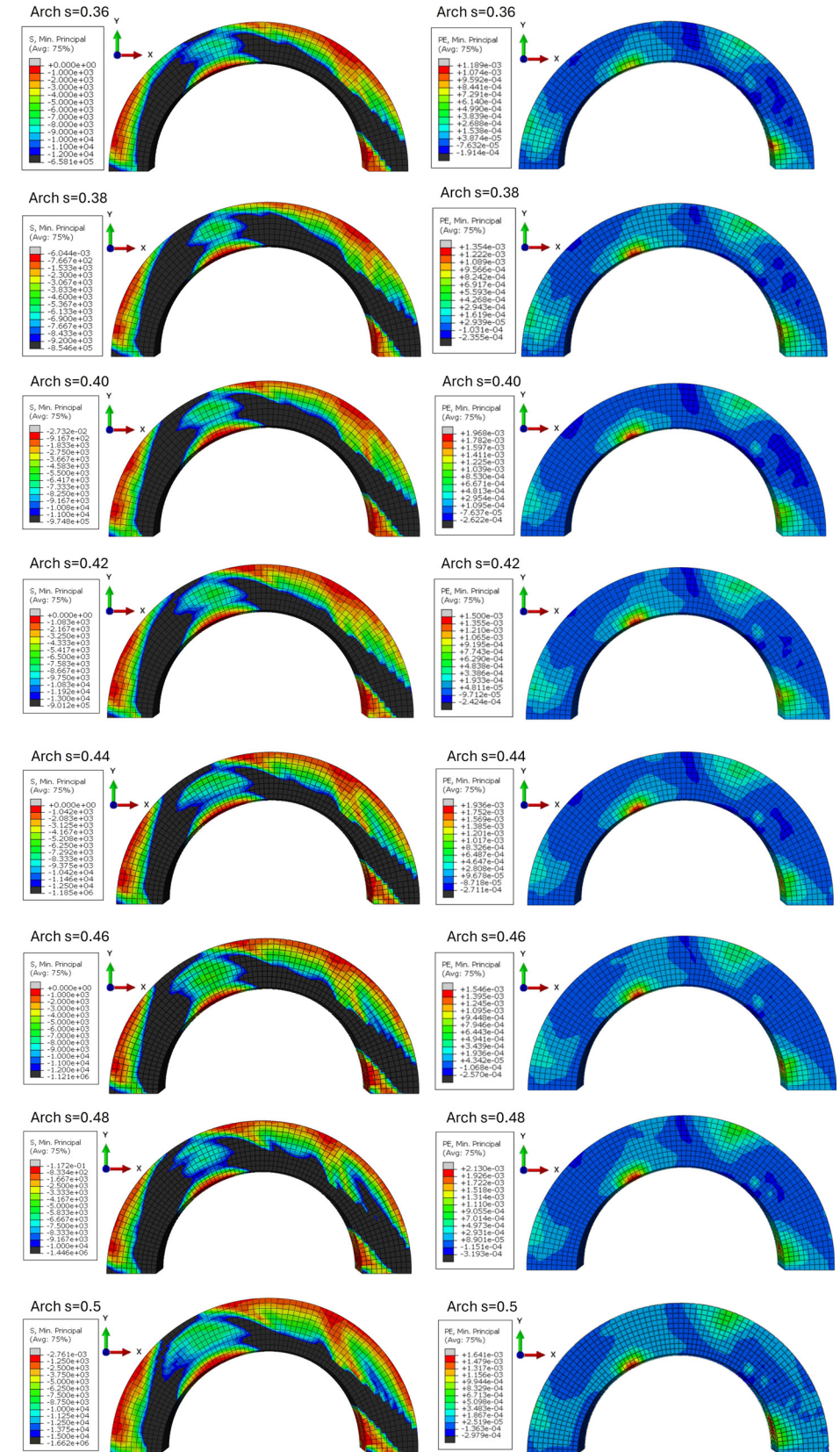
s [m]	λ_{cMC} graphically	θ_4 [°]	θ_3 [°]	θ_2 [°]	θ_1 [°]
0.20	0.16	154	97	36	0
0.22	0.2	161	101	37	0
0.24	0.23	162	100	42	0
0.26	0.245	162	104	46	0
0.28	0.31	165	106	48	0
0.30	0.325	165	107	50	0
0.32	0.335	165	108	52	0
0.34	0.365	166	108	54	0
0.36	0.385	166	109	62	0
0.38	0.415	166	111	62	0
0.40	0.445	166	113	62	0
0.42	0.460	167	113	65	0
0.44	0.470	167	115	66	0
0.46	0.505	167	115	68	0
0.48	0.5302	168	116	68	0
0.50	0.555	168	117	71	0

[Table 15]
Results through MC model of the horizontal collapse multiplier λ_c ., the hinges position through $\theta_1, \theta_2, \theta_3, \theta_4$, as function of the thickness s ., for a round arch (i.e. $\beta = 0^\circ$) with internal radius $r = 1m$.



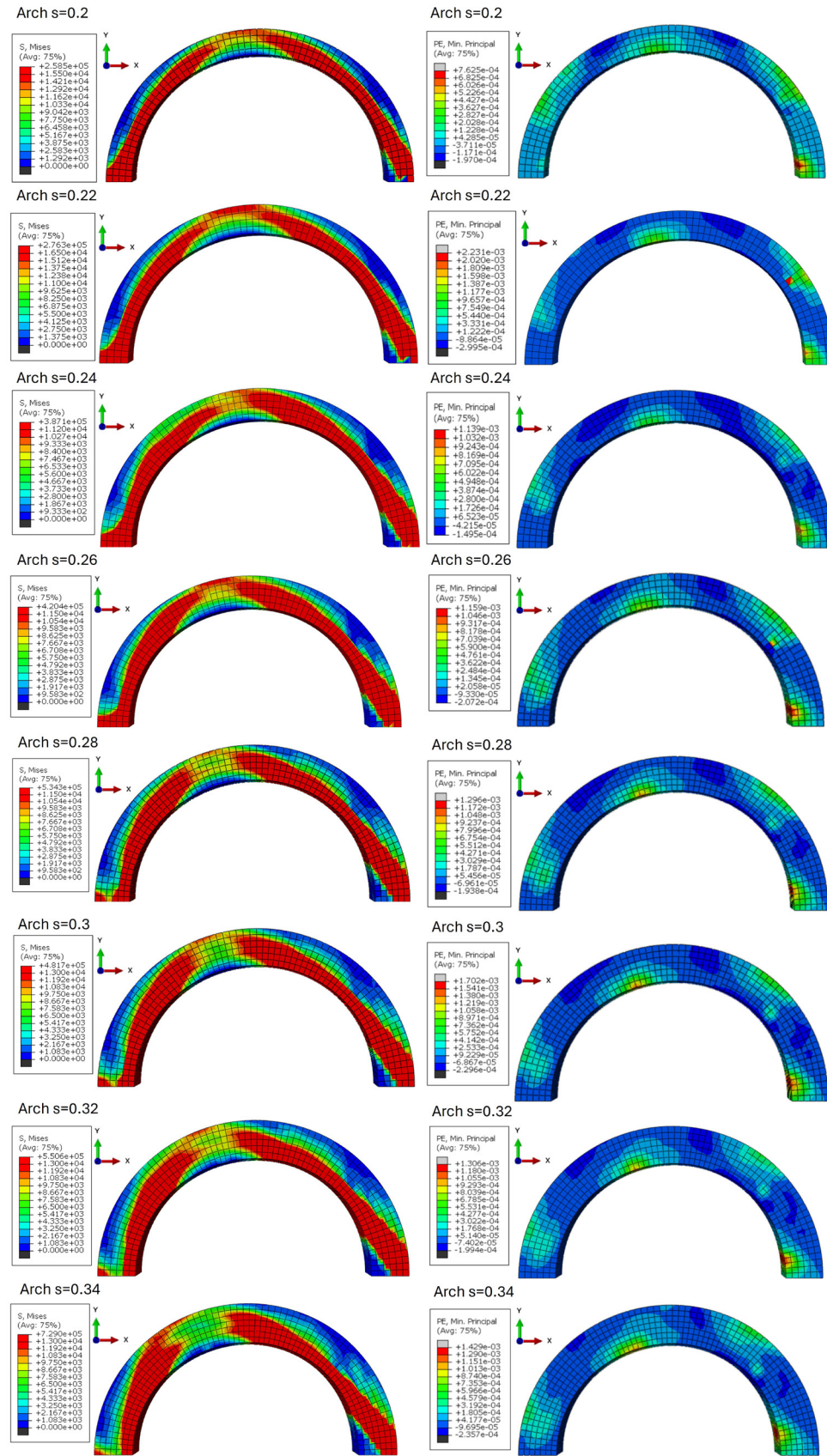
[Figure 77]

A comparison of the results found through the MC model of the minimum principal stress S_{Min} (left) and the minimum plastic strain PE_{Min} (right), at $\lambda_{MC} = \lambda_{cPRD}$.



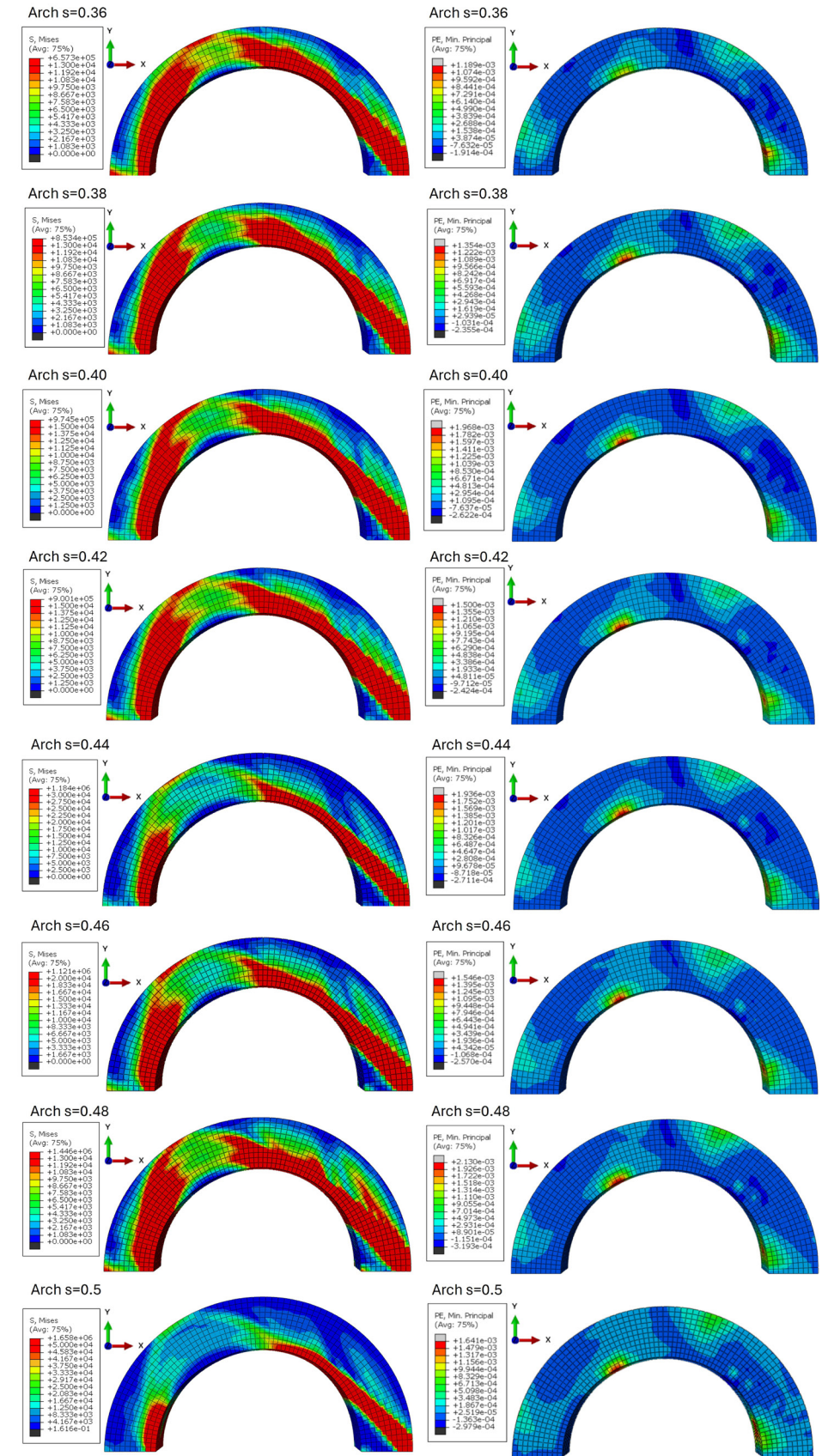
[Figure 78]

A comparison of the results found through the MC model of the minimum principal stress S_{Min} (left) and the minimum plastic strain PE_{Min} (right), at $\lambda_{MC} = \lambda_{cPRD}$.



[Figure 79]

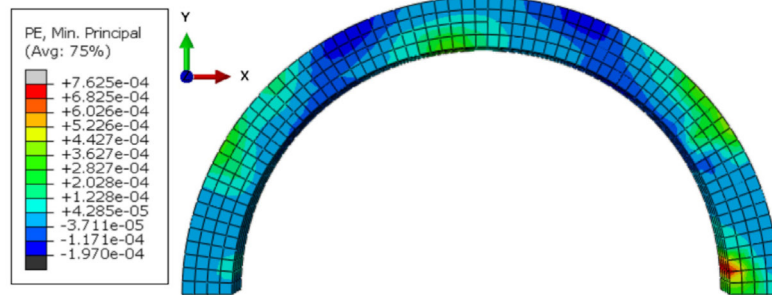
A comparison of the results found through the MC model of the minimum principal stress S_{Min} (left) and the minimum plastic strain PE (right), at $\lambda_{MC}=\lambda_{cPRD}$.



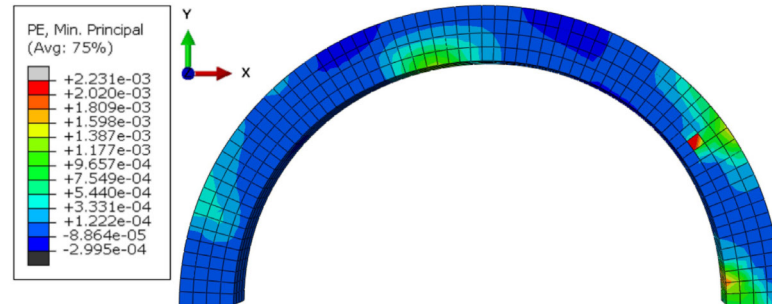
[Figure 80]

A comparison of the results found through the MC model of the vonMises stress S_{Mises} (left) and the minimum plastic strain PE (right), at $\lambda_{MC}=\lambda_{cPRD}$.

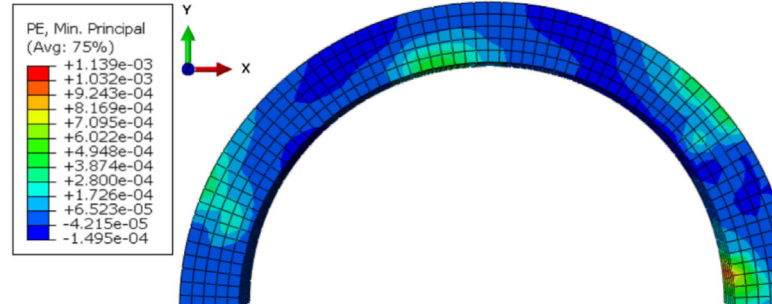
Arch s=0.2



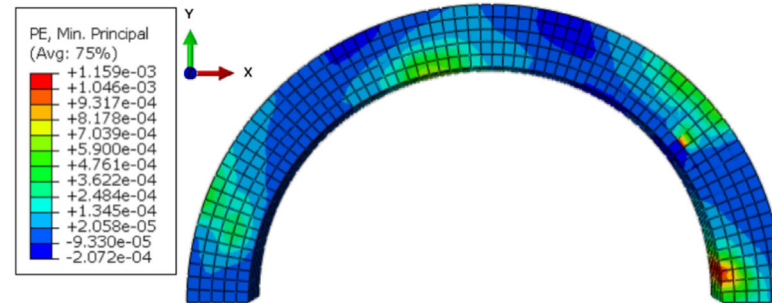
Arch s=0.22



Arch s=0.24



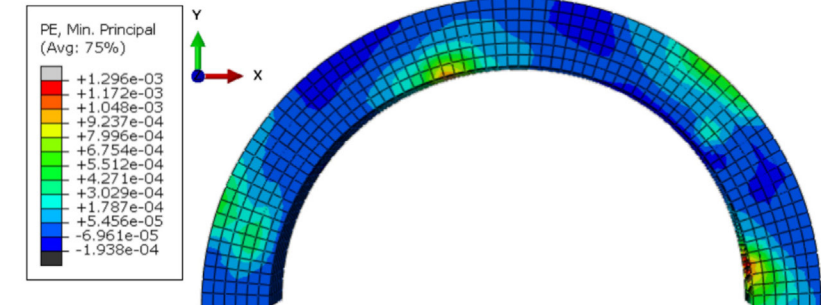
Arch s=0.26



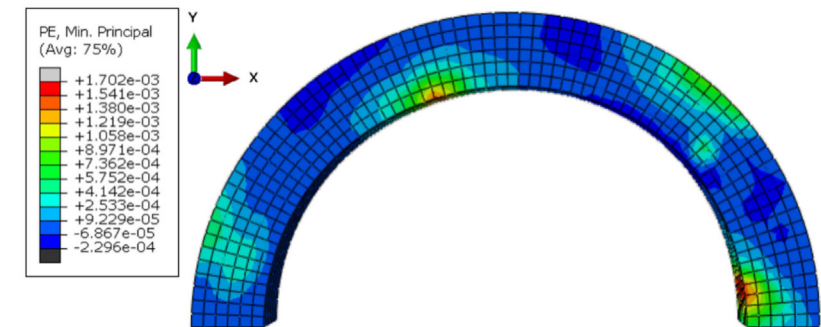
[Figure 81]

Minimum principal plastic strain results found through the MC model, at $\lambda_{MC}=\lambda_{cPRD}$, from arch thickness $s = 0.2m$ through $s = 0.26m$.

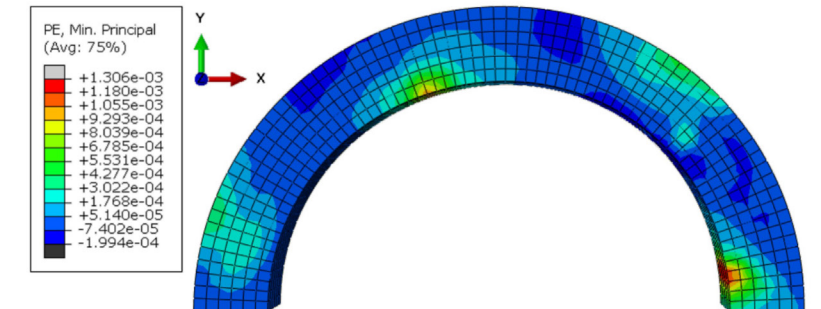
Arch s=0.28



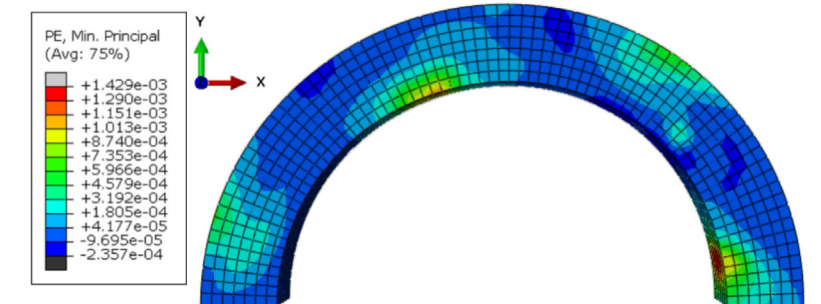
Arch s=0.3



Arch s=0.32

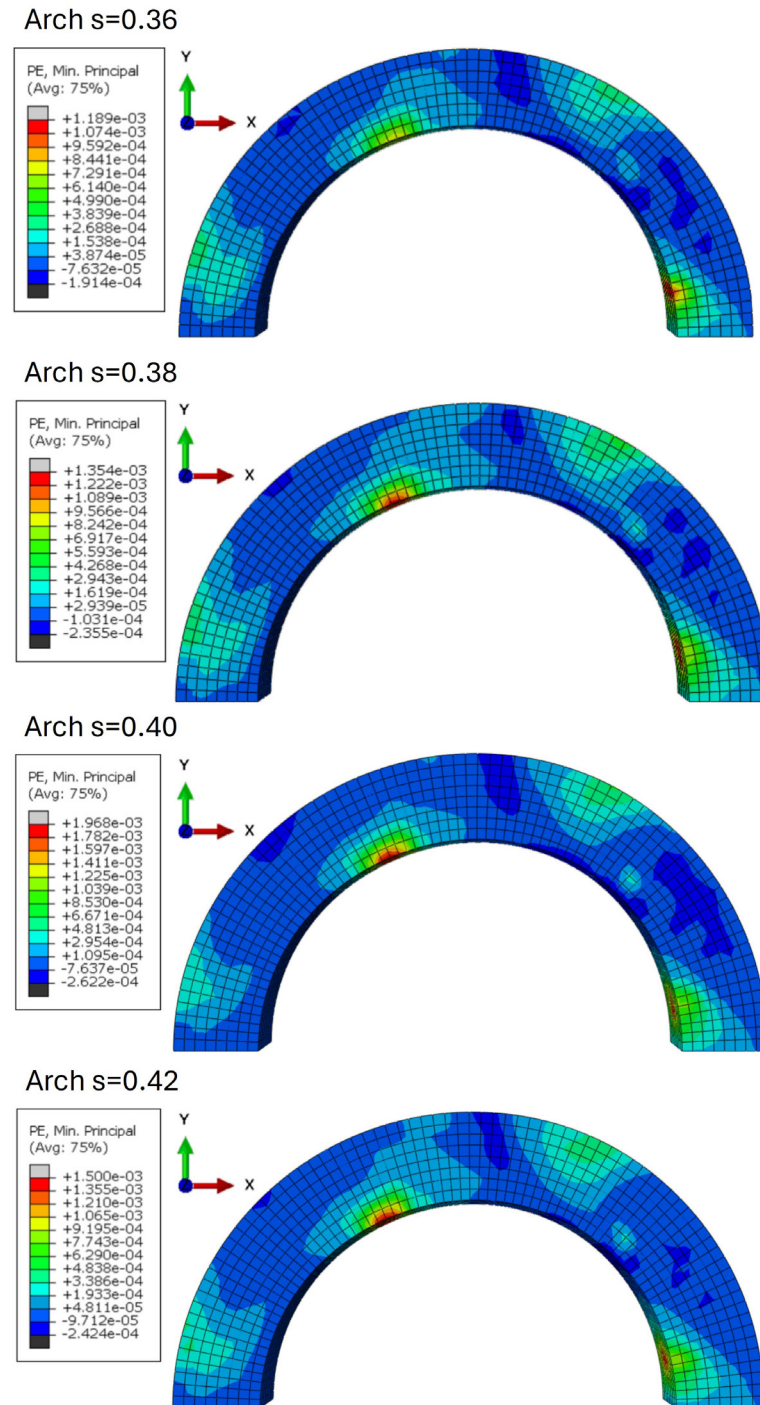


Arch s=0.34



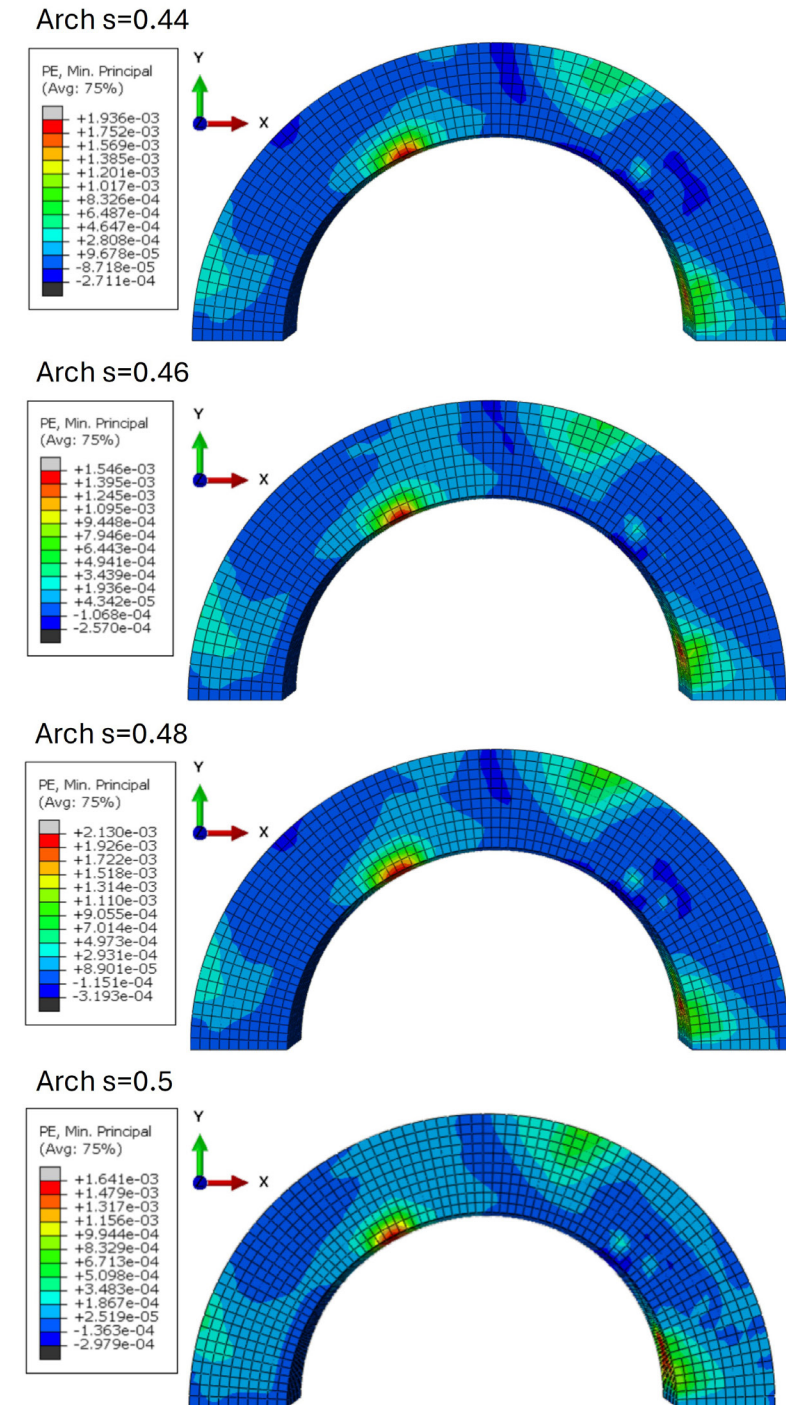
[Figure 82]

Minimum principal plastic strain results found through the MC model, at $\lambda_{MC}=\lambda_{cPRD}$, from arch thickness $s = 0.28m$ through $s = 0.34m$.



[Figure 83]

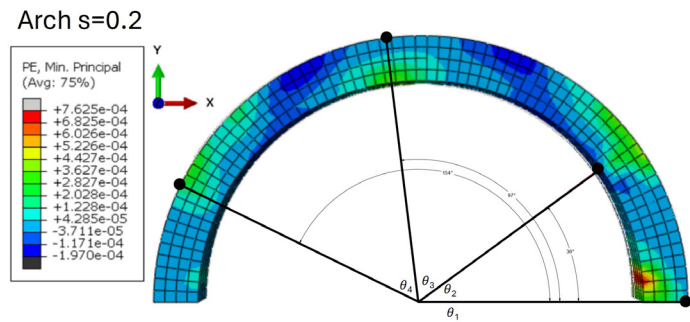
Minimum principal plastic strain results found through the MC model, at $\lambda_{MC}=\lambda_{cPRD}$, from arch thickness $s = 0.36m$ through $s = 0.42m$



[Figure 84]

Minimum principal plastic strain results found through the MC model, at $\lambda_{MC}=\lambda_{cPRD}$, from arch thickness $s = 0.44m$ through $s = 0.5m$.

[Figure 85]
Example of the procedure followed by the author of this thesis to determine the unknowns of the problem which are the positions of the hinges (i.e. the angle θ_i for $i \in \{1, \dots, 4\}$).



MC compared to CDP

As for CDP, when applying a horizontal force equal to λg for the various arch thicknesses the multiplier is undetectable from the energy graphs, even after smoothing of the data to remove the oscillation of the graphs, there is no evident indicator to quantify the right multiplier. In addition, the visualization of the openings of the hinges occurs way later than the accurate λ_c , in Table 16 the values of the multipliers corresponding to each visualization shown from Figure 90 to Figure 93 is presented. It can be pointed out that λ_{cCDP} is quasi double λ_{cPRD} .

s [m]	λ_{cPRD}	λ_{cCDP}
0.20	0.236	0.5
0.22	0.279	0.5
0.24	0.320	0.65
0.26	0.359	0.7
0.28	0.396	0.75
0.30	0.431	0.85
0.32	0.465	0.95
0.34	0.497	1
0.36	0.529	1
0.38	0.560	1
0.40	0.590	1
0.42	0.620	1
0.44	0.649	1
0.46	0.677	1
0.48	0.705	1
0.50	0.732	1

[Table 16]
Comparison of the results through PRD, MC and CDP model of the horizontal collapse multiplier λ_c , as function of the thickness s , for a round arch (i.e. $\beta = 0^\circ$) with internal radius $r = 1m$.

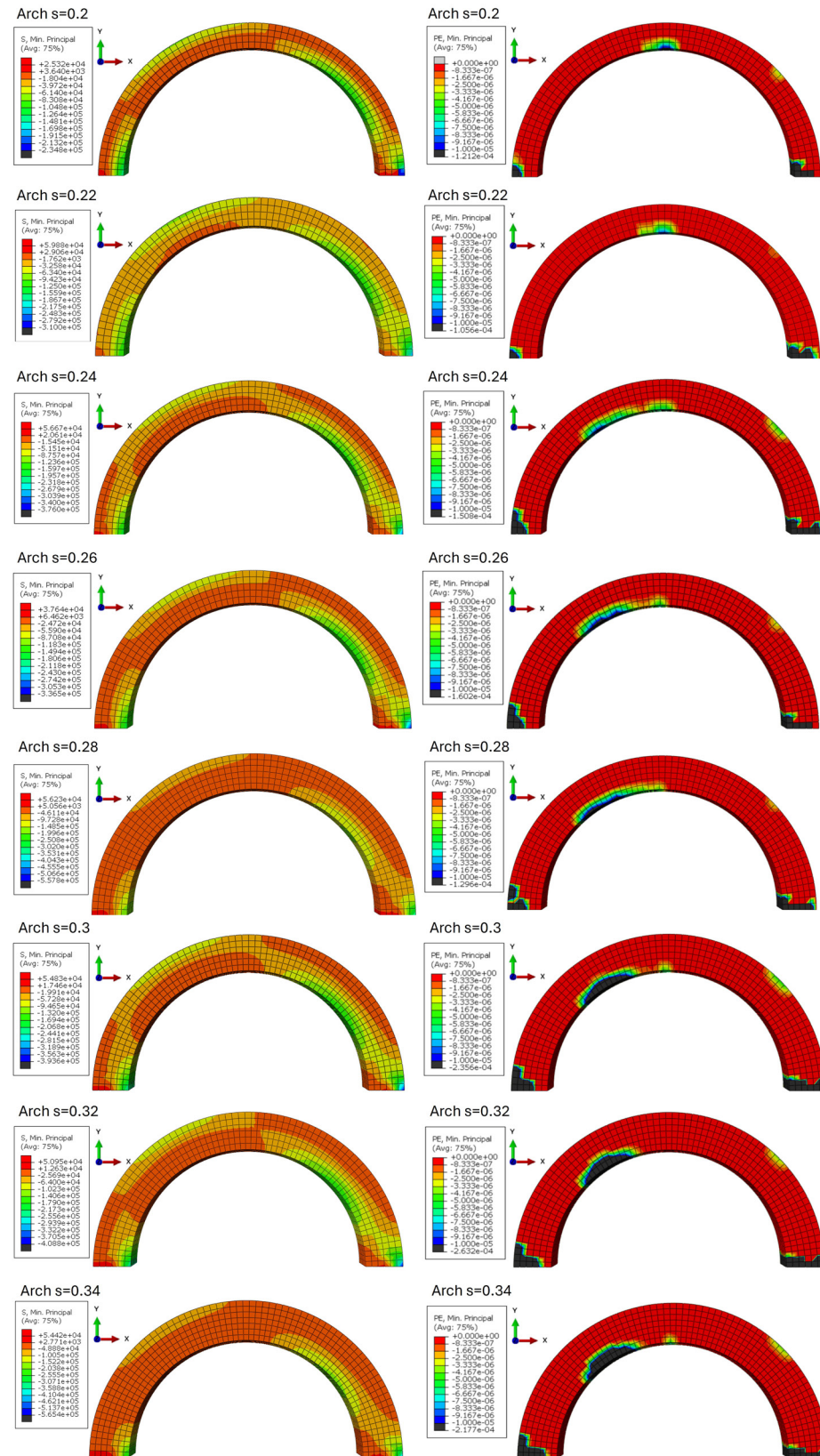
It's important to acknowledge that unlike in MC, where the minimum principal stress indicates a visible thrust line (the path of highest compressive forces), in CDP, as illustrated in Figure 86 and Figure 87, the highest negative compressive stress value are smeared. However, their position aligns with the position where the hinges should form. The minimum principal plastic strain portrays the opening of hinges opposite to the highest negative value of the minimum principal stress. Similarly, the highest values of the vonMises stress accurately depict the location of hinges themselves as can be seen in Figure 88 and Figure 89.

Briefly, the visualization of the stress results, though at a different value of the horizontal multiplier, gives in synch visualization of where the hinge location should be present even though it is a smeared depiction, but it still can give insight. Nevertheless, when looking at the minimum plastic strain a very peculiar instance should be pointed out, in addition to an incorrect value of the horizontal multiplier, it can be noticed that starting from the thickness $s = 0.4m$ and on till $s = 0.5m$ only 3 hinge openings are detectible.

Table 17 presents the hinges position according to the CDP model with respect to the thickness of the arch.

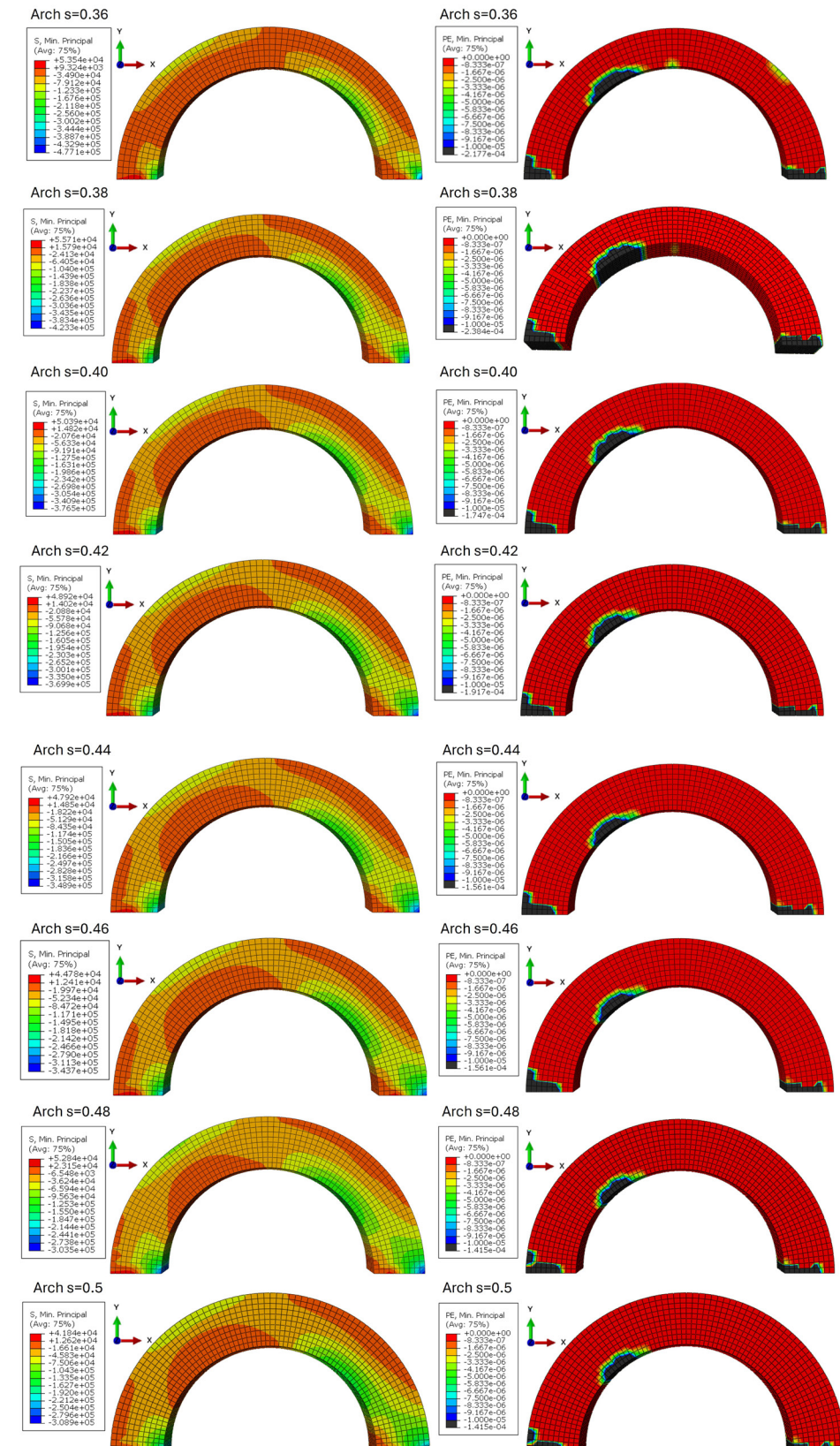
s [m]	λ_{cCDP}	θ_4 [°]	θ_3 [°]	θ_2 [°]	θ_1 [°]
0.20	0.5	180	90	42	0
0.22	0.5	180	92	44	0
0.24	0.65	180	120	44	0
0.26	0.7	180	122	43	0
0.28	0.75	180	122	43	0
0.30	0.85	180	122	44	0
0.32	0.95	180	125	46	0
0.34	1	180	125	47	0
0.36	1	180	125	45	0
0.38	1	180	126	90	0
0.40	1	180	126	-	0
0.42	1	180	128	-	0
0.44	1	180	128	-	0
0.46	1	180	128	-	0
0.48	1	180	128	-	0
0.50	1	180	128	-	0

[Table 17]
Results through CDP model of the horizontal collapse multiplier λ_c , the hinges position through $\theta_1, \theta_2, \theta_3, \theta_4$, as function of the thickness s , for a round arch (i.e. $\beta = 0^\circ$) with internal radius $r = 1m$.



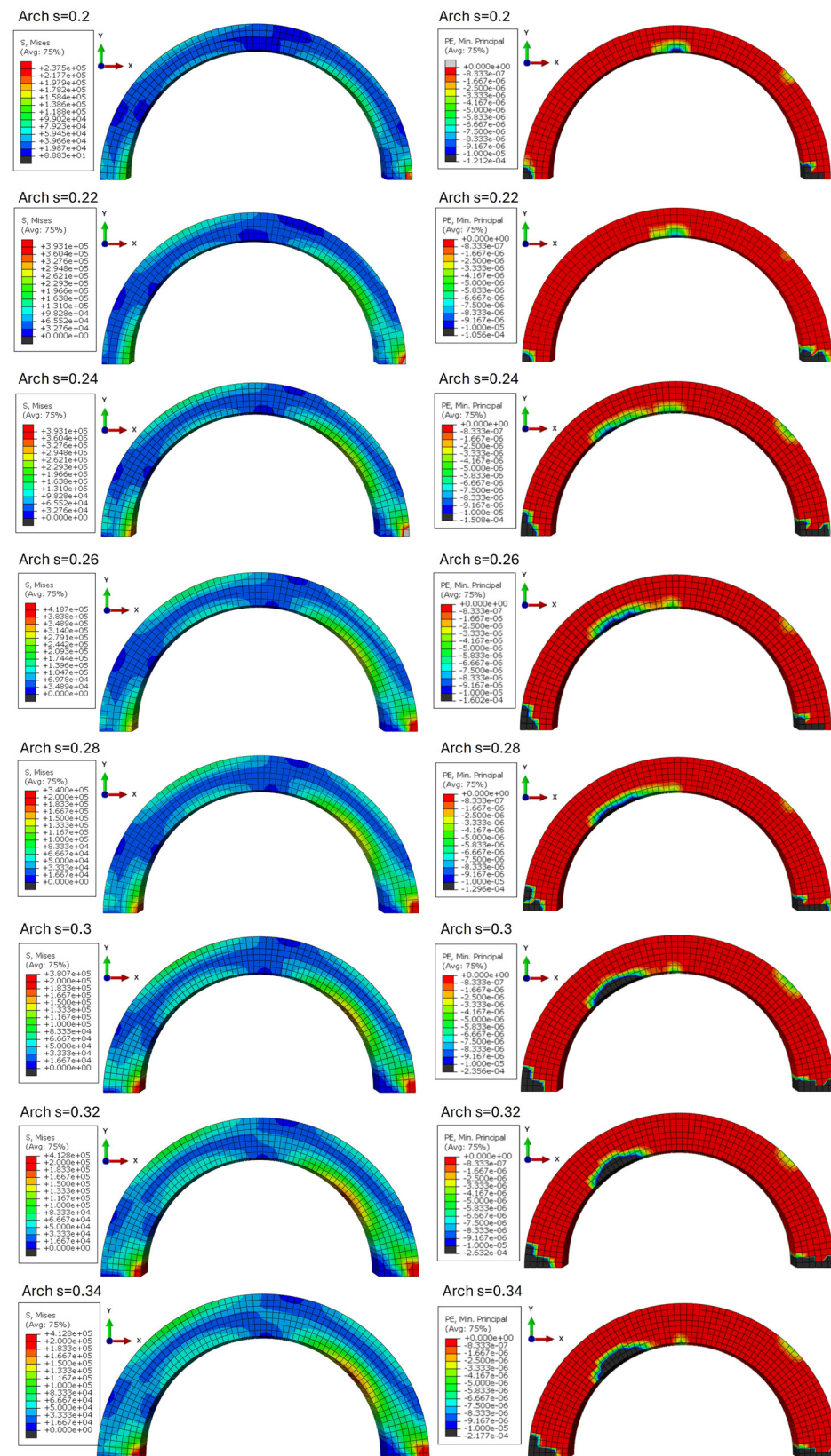
[Figure 86]

A comparison of the results found through the CDP model of the minimum principal stress S Min (left) and the minimum plastic strain PE (right), at λ_{CDP} .



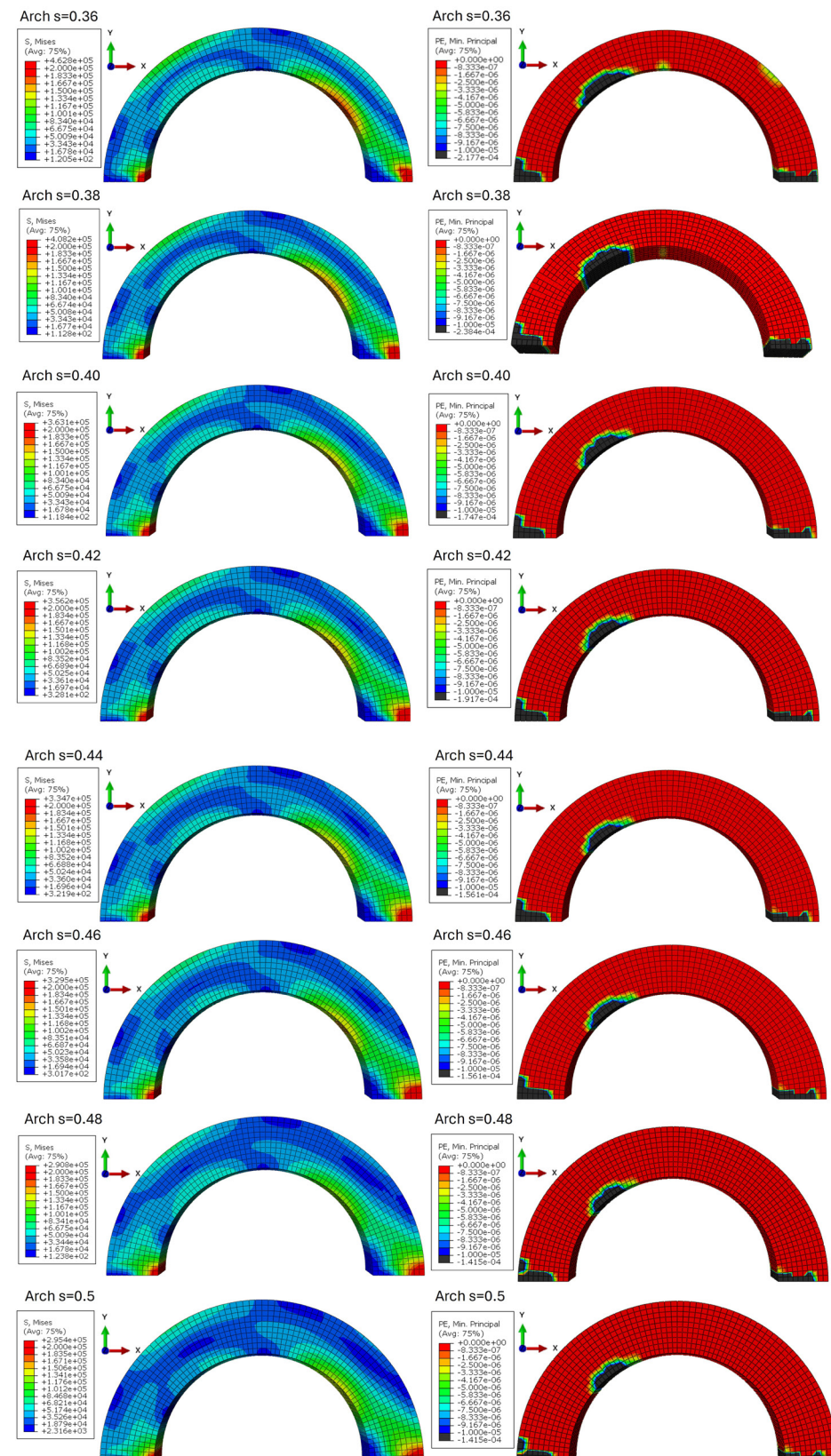
[Figure 87]

A comparison of the results found through the CDP model of the minimum principal stress S Min (left) and the minimum plastic strain PE (right), at λ_{CDP} .



[Figure 88]

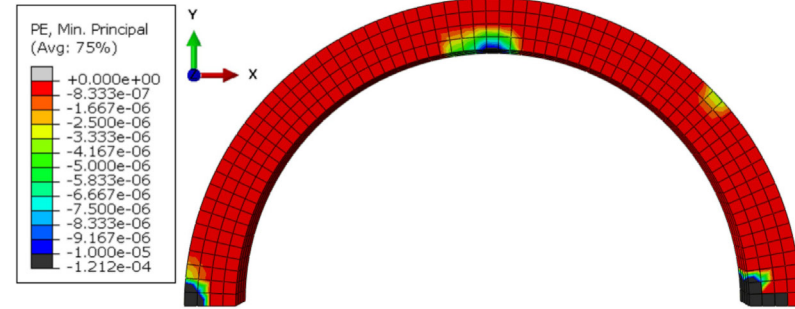
A comparison of the results found through the CDP model of the vonMises stress (left) and the minimum plastic strain PE (right), at λ_{CDP} .



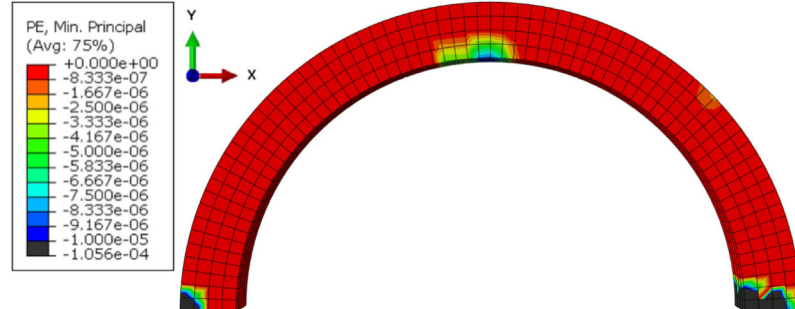
[Figure 89]

A comparison of the results found through the CDP model of the vonMises stress (left) and the minimum plastic strain PE (right), at λ_{CDP} .

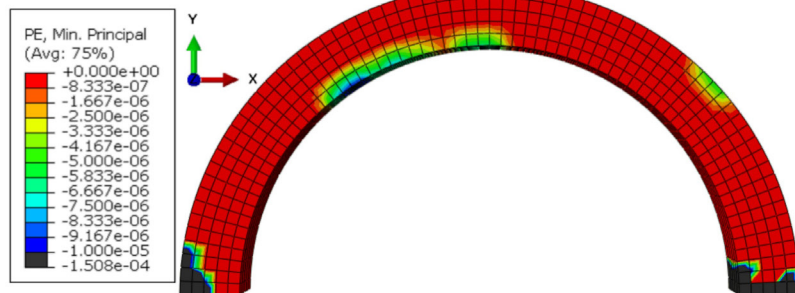
Arch $s=0.2$



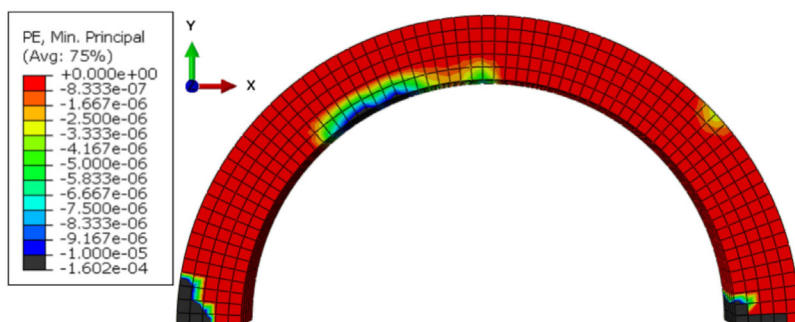
Arch $s=0.22$



Arch $s=0.24$



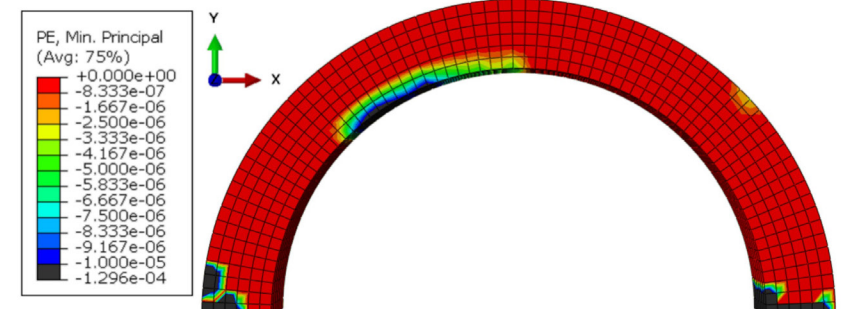
Arch $s=0.26$



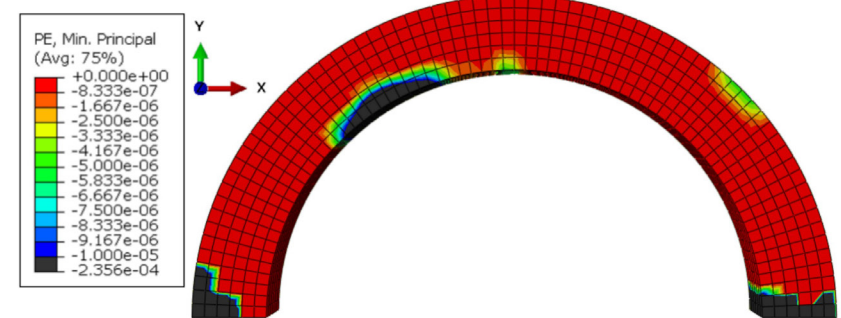
[Figure 90]

Minimum principal plastic strain results found through the MC model, at λ_{cDP} , from arch thickness $s = 0.2m$ through $s = 0.26m$.

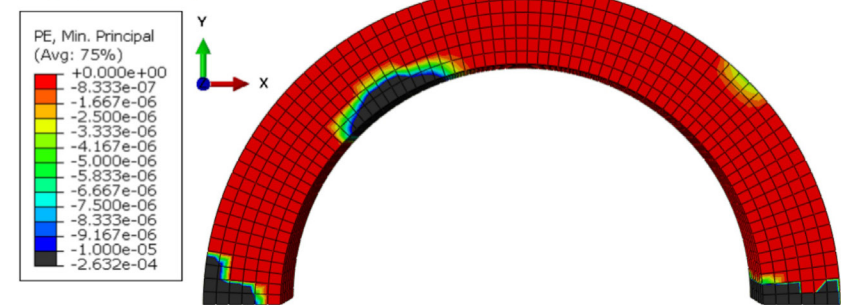
Arch $s=0.28$



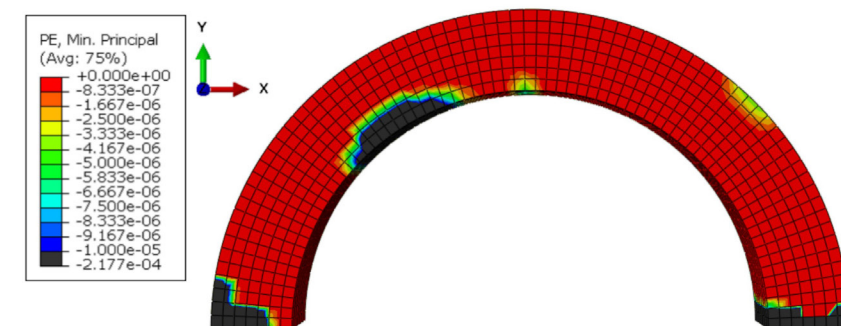
Arch $s=0.3$



Arch $s=0.32$



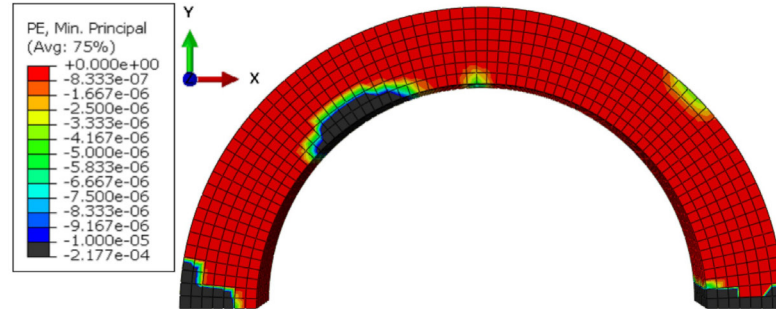
Arch $s=0.34$



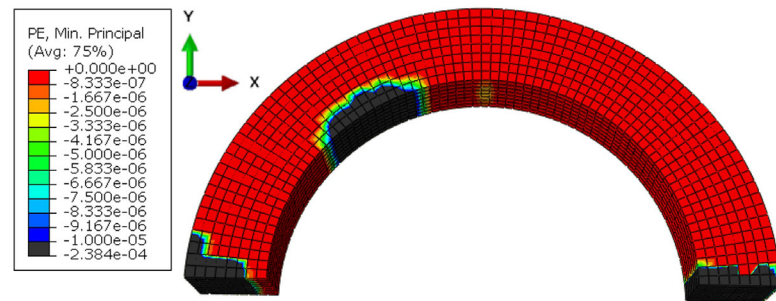
[Figure 91]

Minimum principal plastic strain results found through the MC model, at λ_{cDP} , from arch thickness $s = 0.28m$ through $s = 0.34m$.

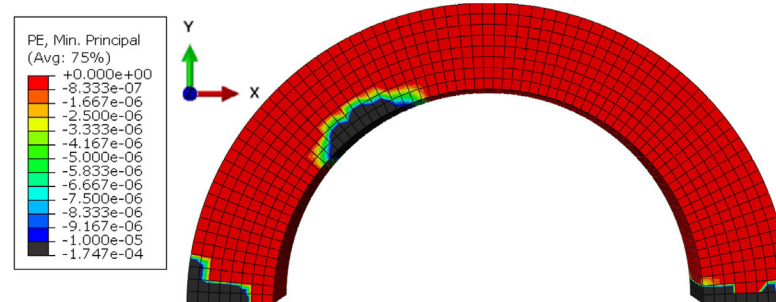
Arch $s=0.36$



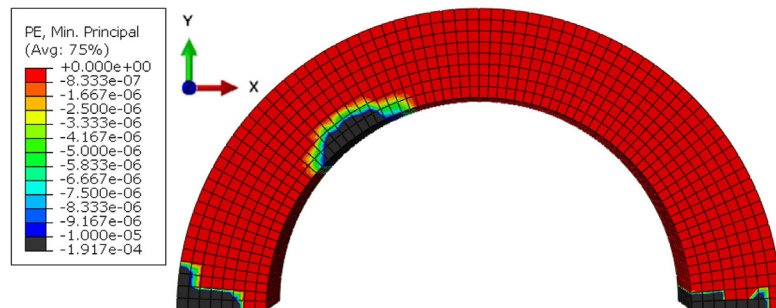
Arch $s=0.38$



Arch $s=0.40$



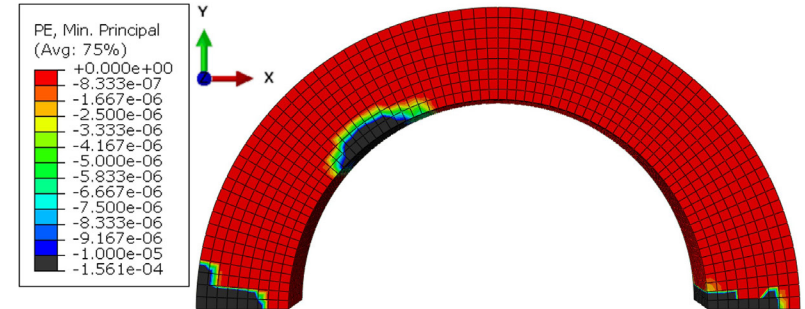
Arch $s=0.42$



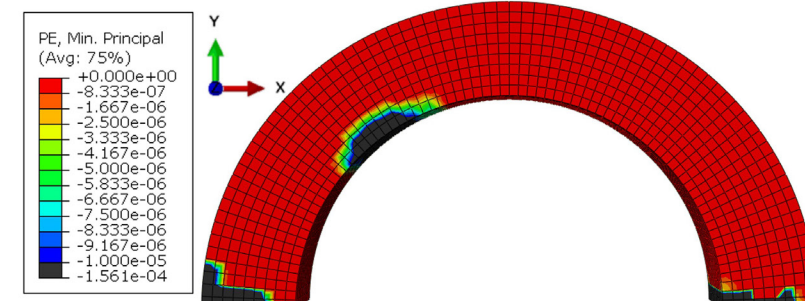
[Figure 92]

Minimum principal plastic strain results found through the MC model, at λ_{cDP} , from arch thickness $s = 0.36m$ through $s = 0.42m$.

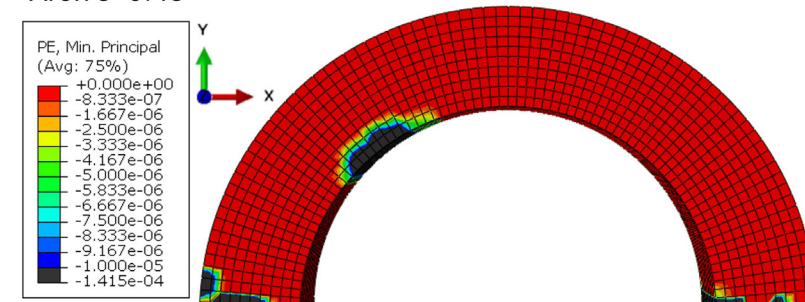
Arch $s=0.44$



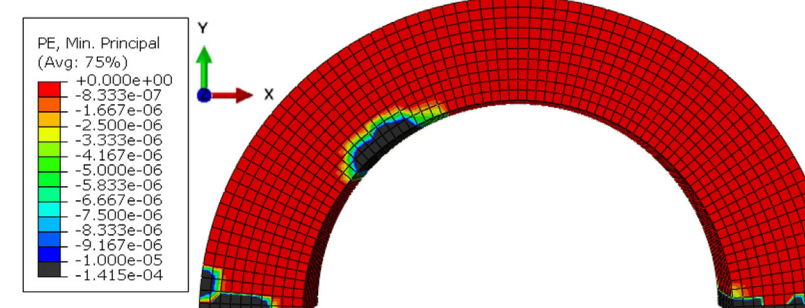
Arch $s=0.46$



Arch $s=0.48$



Arch $s=0.5$



[Figure 93]

Minimum principal plastic strain results found through the MC model, at λ_{cDP} , from arch thickness $s = 0.44m$ through $s = 0.5m$.

To summarize, in comparing MC with PRD and MC with CDP for horizontal capacity analyses, significant differences in the determination of multipliers and hinge locations were observed.

Through the MC model the collapse multiplier is identifiable. From the x-y data of the total energy graphs one is able to identify the value of the multipliers that are 75% of those from PRD due to disparities in collapse loads and model behaviors discussed in previous sections (see Table 18). Furthermore, within the visualization module, it is possible to accurately identify the precise multiplier (matching that determined by PRD) when the thrust line intersects the arch at four specific locations, thereby initiating the collapse mechanism.

Conversely, CDP presents challenges in detecting multipliers from energy graphs. However, stress visualizations in CDP provide valuable insights into accurate hinge locations despite the discrepancies, similarly to the MC model as can be seen in Table 19.

Table 20 presents a comparison of the computational time for the finite element models. The CDP model significantly surpasses the MC model is terms of computational efficiency.

[Table 18]
Comparison of the results through PRD, MC and CDP model of the horizontal collapse multiplier λ_c , as function of the thickness s , for a round arch (i.e. $\beta = 0^\circ$) with internal radius $r = 1\text{m}$.

s [m]	$\lambda_{c\text{PRD}}$	$\lambda_{c\text{MC}}$ graphically	$\lambda_{c\text{PRD}} * 0.75$	$\lambda_{c\text{CDP}}$
0.20	0.236	0.16	0.17	0.5
0.22	0.279	0.2	0.21	0.5
0.24	0.320	0.23	0.24	0.65
0.26	0.359	0.245	0.27	0.7
0.28	0.396	0.31	0.30	0.75
0.30	0.431	0.325	0.32	0.85
0.32	0.465	0.335	0.35	0.95
0.34	0.497	0.365	0.37	1
0.36	0.529	0.385	0.39	1
0.38	0.560	0.415	0.42	1
0.40	0.590	0.445	0.44	1
0.42	0.620	0.460	0.46	1
0.44	0.649	0.470	0.49	1
0.46	0.677	0.505	0.50	1
0.48	0.705	0.5302	0.53	1
0.50	0.732	0.555	0.55	1

s [m]	$\theta 4$ [°]			$\theta 3$ [°]			$\theta 2$ [°]			$\theta 1$ [°]		
	PRD	MC	CDP	PRD	MC	CDP	PRD	MC	CDP	PRD	MC	CDP
0.20	162.83	154	180	103.26	97	90	43.69	36	42	0	0	0
0.22	165.92	161	180	105.59	101	92	45.26	37	44	0	0	0
0.24	168.73	162	180	107.74	100	120	46.75	42	44	0	0	0
0.26	171.29	162	180	109.73	104	122	48.17	46	43	0	0	0
0.28	173.63	165	180	111.58	106	122	49.53	48	43	0	0	0
0.30	175.79	165	180	113.31	107	122	50.83	50	44	0	0	0
0.32	177.77	165	180	114.93	108	125	52.09	52	46	0	0	0
0.34	179.6	166	180	116.45	108	125	53.3	54	47	0	0	0
0.36	180	166	180	117.88	109	125	54.47	62	45	0	0	0
0.38	180	166	180	119.21	111	126	55.6	62	90	0	0	0
0.40	180	166	180	120.46	113	126	56.71	62	-	0	0	0
0.42	180	167	180	121.63	113	128	57.78	65	-	0	0	0
0.44	180	167	180	122.74	115	128	58.83	66	-	0	0	0
0.46	180	167	180	123.78	115	128	59.86	68	-	0	0	0
0.48	180	168	180	124.78	116	128	60.87	68	-	0	0	0
0.50	180	168	180	125.72	117	128	61.86	71	-	0	0	0

[Table 19]
Comparison of the position of hinges for the different approaches.

s [m]	Nb. Elements	Time [s]	Time [s]
		MC-Explicit	CDP-Explicit
0.2	1104	407	9
0.22	1120	394	9
0.24	1750	599	13
0.26	1775	584	14
0.28	2556	808	20
0.3	2592	784	19
0.32	2628	772	20
0.34	3577	1018	27
0.36	3626	983	28
0.38	4800	1277	38
0.4	4800	1226	37
0.42	4864	1188	38
0.44	6156	1445	49
0.46	6273	1404	50
0.48	7800	1800	64

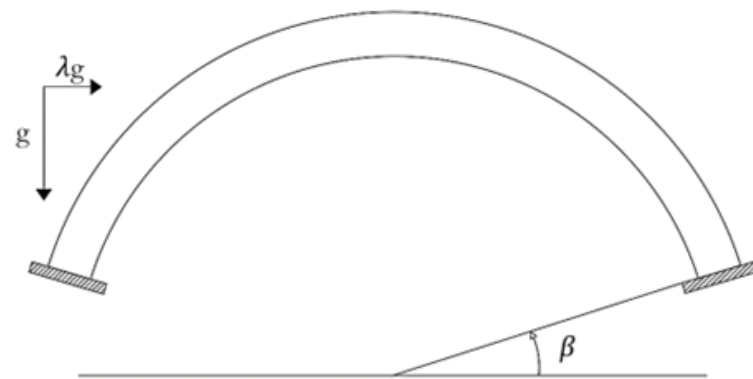
[Table 20]
Comparison of the computational time for the horizontal capacity analyses, for various arch thicknesses, between an MC model and a CDP model.

2.3.1.2. Case B: Circular arch with a springing angle $\beta = 20^\circ$

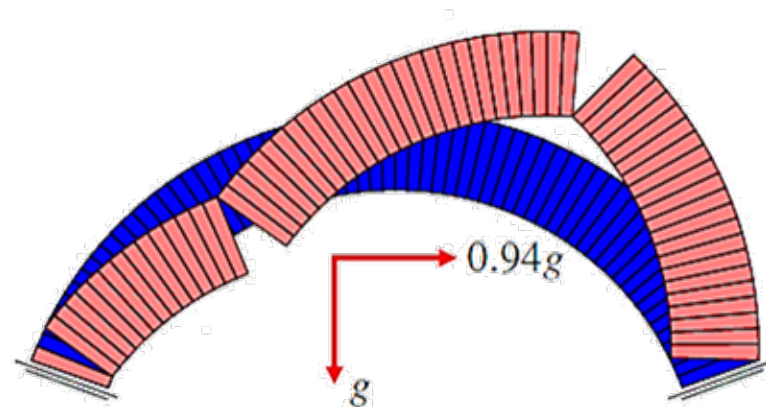
MC compared to PRD

Another horizontal capacity analysis was done on a semi-circular arch with a springing angle of 20° , with a radius of 1.5m and a 0.4m thickness it was studied in a paper by Iannuzzo and Olivieri (Iannuzzo et al., 2019). Similarly to the previous analysis the arch is subjected to its self-weight and horizontal incremental action, where forces are directly proportional to mass and escalate via the loading parameter λ see Figure 94. Figure 95 showcases the results found in (Iannuzzo et al., 2019), indicating that through PRD the initiation of the mechanism starts at $\lambda_c=0.94$.

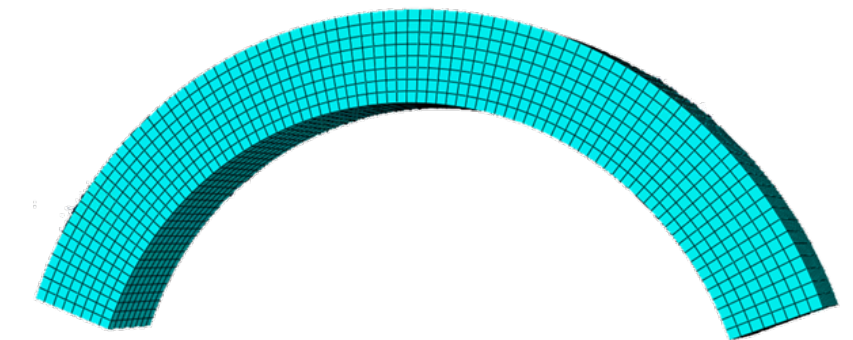
[Figure 94]
Example of the loading conditions applied to a circular arch with a springing angle β .



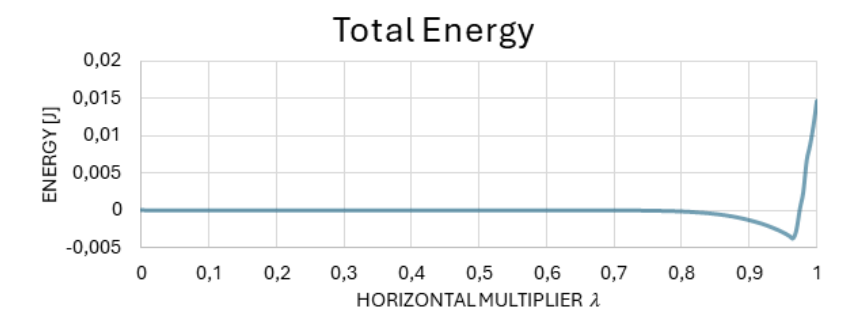
[Figure 95]
Results of the PRD analysis of the circular arch, having a radius $r = 1.5\text{m}$ and springing angle $\beta = 20^\circ$. Figure from (Iannuzzo et al., 2019).



[Figure 96]
Mesh attributes for the circular arch, having a radius $r = 1.5\text{m}$ and springing angle $\beta = 20^\circ$ in ABAQUS.



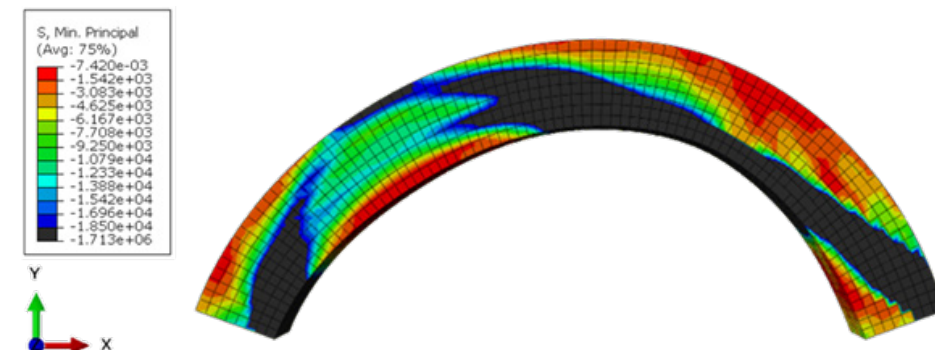
[Figure 97]
Graph showing the Total Energy of the MC model in function of the step time, which is equivalent to the incrementation of the horizontal multiplier λ .



At $\lambda_c=0.94$, Figure 98 clearly depicts the thrust associated with the highest compressive forces. This is clear from the visualization of the minimum principal stress, which intersects both the intrados and extrados at four points corresponding to the locations of hinges. Similarly in Figure 99 the minimum plastic strain PE is illustrated, highlighting the openings of the hinges. A comparison of these results with those of Figure 95 reveal an evident resemblance.

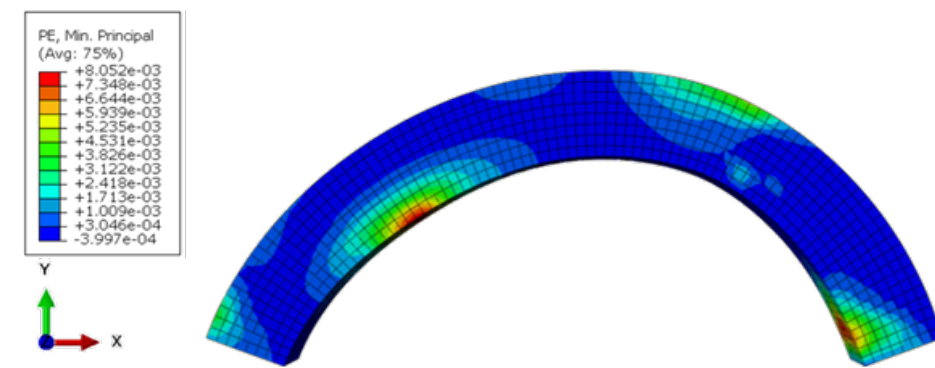
[Figure 98]

Minimum principal stress results of the MC model for the circular arch, having a radius $r = 1.5\text{m}$ and springing angle $\beta = 20^\circ$.



[Figure 99]

Minimum principal plastic strain results of the MC model for the circular arch, having a radius $r = 1.5\text{m}$ and springing angle $\beta = 20^\circ$.



MC compared to CDP

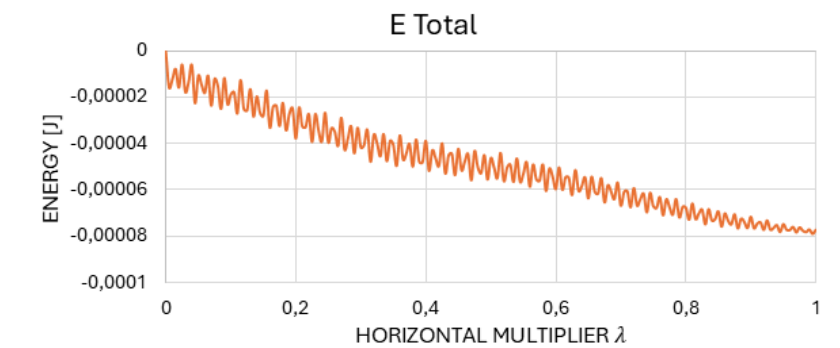
The horizontal capacity analysis was conducted on the arch with springing angle $\beta = 20^\circ$ using the same previously described CDP model.

When analysing the energy graphs within this model, pinpointing the moment of collapse wasn't straightforward. Despite referencing PRD which suggested it should occur at 0.94, there was no distinct indication or observable pattern around this value in the graphs or their retrieved x-y data. Oscillations are evident across all graphs, as illustrated in the total energy graph provided in Figure 100. Usually, the presence of this kind of oscillations is a sign of the bad quality of the numerical solution, generally indicative of too much artificial damping to stabilize the solution. Even after smoothing, by reducing the fluctuations or noise in the data resulting in a more coherent and continuous representation of the pattern of the energy, specifying the collapse multiplier was not possible.

In addition to the oscillation observed in the total energy graph of the MC model in Figure 97 compared to the total energy graph of the CDP model in Figure 100, there is also a significant difference in magnitude. This arises because the CDP model accounts for strain softening, damage evolution, and fracture energy dissipation, which are key behaviours in materials like brick, mortar, stone, and concrete. These behaviours, detailed in sub-section 2.2.1.2, "CDP methodology," result in higher energy requirements and dissipation

[Figure 100]

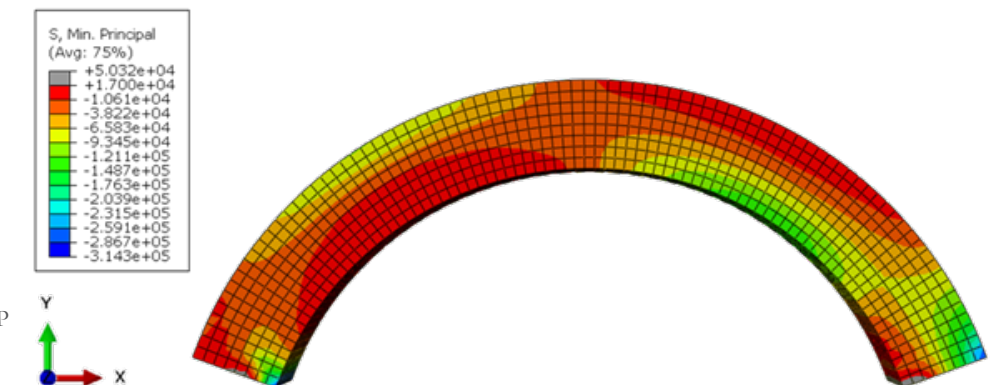
Graph showing the Total Energy of the CDP model in function of the step time, which is equivalent to the incrementation of the horizontal multiplier λ .



Even though the collapse horizontal multiplier cannot be clearly identified through the x-y data graphs. In Figure 101 at $\lambda_c = 0.94$, from the visualization of the minimum principal stress, the thrust associated with the highest compressive forces is not visible as a line but rather as smeared stress on both the intrados and extrados at four points (in green) quasi-corresponding to the locations of hinges. On the other hand, in Figure 102 the min plastic strain PE is shown at $\lambda_c = 0.95$. One might anticipate observing four concentrated plastic strain locations, corresponding to the openings of hinges. However, only three are evident.

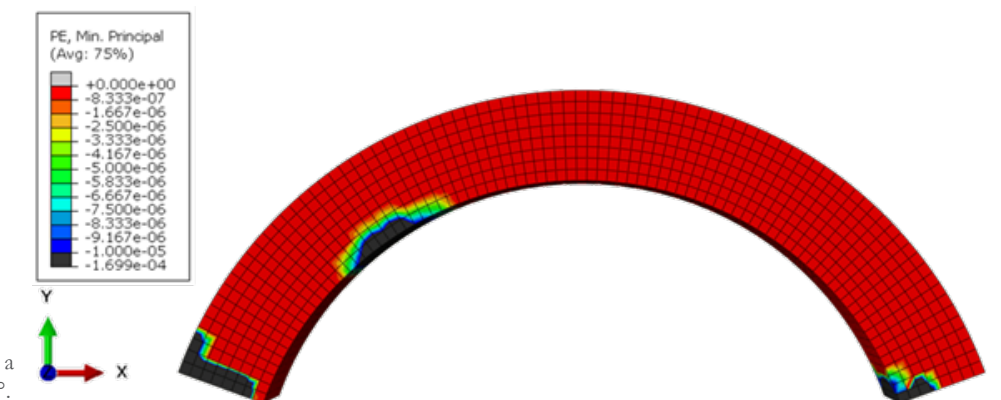
[Figure 101]

Minimum principal stress results of the CDP model for the circular arch, having a radius $r = 1.5\text{m}$ and springing angle $\beta = 20^\circ$.



[Figure 102]

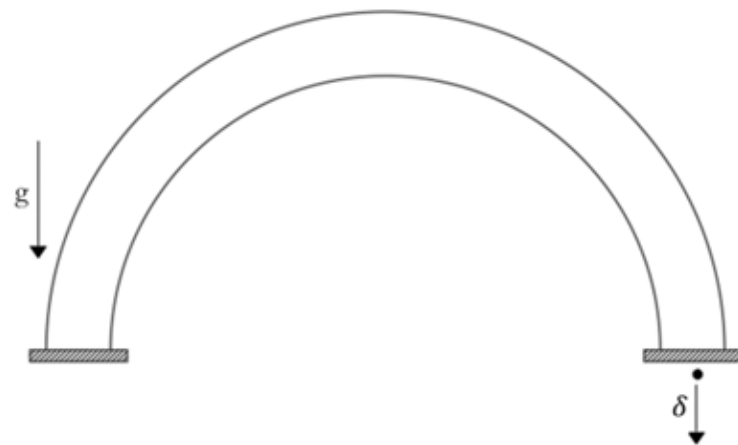
Minimum principal plastic strain results of the CDP model for the circular arch, having a radius $r = 1.5\text{m}$ and springing angle $\beta = 20^\circ$.



2.3.2. BENCHMARK 2: VERTICAL SETTLEMENT OF CIRCULAR ARCH

MC compared to PRD

In the previous section 2.2.3, the arch used for the parametric study has been reintroduced in ABAQUS to investigate a different aspect. It was subjected to its self-weight and an incremental settlement δ as can be seen in Figure 103.



[Figure 103]
Loading conditions applied to a semi-circular arch, having a radius $r=1\text{m}$ and a thickness $s=0.3\text{m}$.

In this section, our focus shifts from observing the impact of parameter variations to investigating how one can determine the collapse load solely from the graphs, without relying on a reference method like for instance PRD. Although PRD indicates a vertical settlement $\delta=1.00\text{m}$ to initiate the collapse mechanism, the graphs resulting from a FEA, with one explicit dynamic step considering an increased settlement, do not distinctly facilitate the identification of a specific collapse indicator.

Therefore, the analysis was split into two dynamic explicit steps. During the initial step of duration equal to 1, the model was only subjected to gravity load which propagates to the second step. Subsequently, in the second step of step time equal 1 as well, the arch experiences vertical settlement, starting from the equilibrium established in the first step. Another factor that was considered is the application of settlement using ramps.

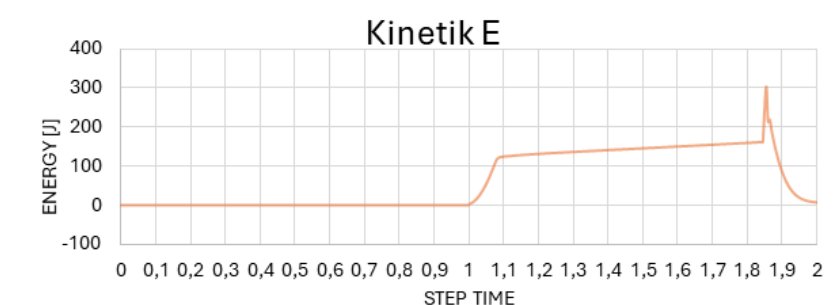
In dynamic explicit analyses, the introduction of load as a ramp serves crucial purposes. First, it significantly enhances convergence by allowing the model to gradually adapt to the applied load. This gradual adjustment mitigates the risk of numerical instabilities or convergence issues commonly associated

with sudden changes or shocks in load. By preventing oscillations during the solution process (as seen in the total energy graph in **Part A** section 2.2.3), ramp loading ensures a more stable and reliable analysis. Moreover, ramp loading improves the accuracy of predictions regarding structural response.

Dynamic events in real-world scenarios rarely involve instantaneous load application; instead, loads typically build up gradually over time. By simulating this gradual progression through ramp loading, the analysis captures the realistic behaviour of the structure. This approach is particularly crucial when dealing with highly nonlinear responses or significant inertial effects. Hence when a ramped load, such as settlement in this case, is applied, the Kinetic Energy remains constant throughout the duration of the ramp period if it does not lead to collapse. However, if the applied load is sufficient to cause collapse, the Kinetic Energy will exhibit a sudden jump followed by a decrease to zero, indicating a loss of equilibrium followed by collapse.

Indeed, this anticipated behaviour was confirmed when the settlement was gradually applied via a ramp. Initially, the value of 0.4 was ramped over a certain period of the duration of the step time, followed by ramping 0.5m, 0.6m, and 0.7m. Throughout all these stages, all graphs displayed a plateau in the kinetic energy, signifying equilibrium.

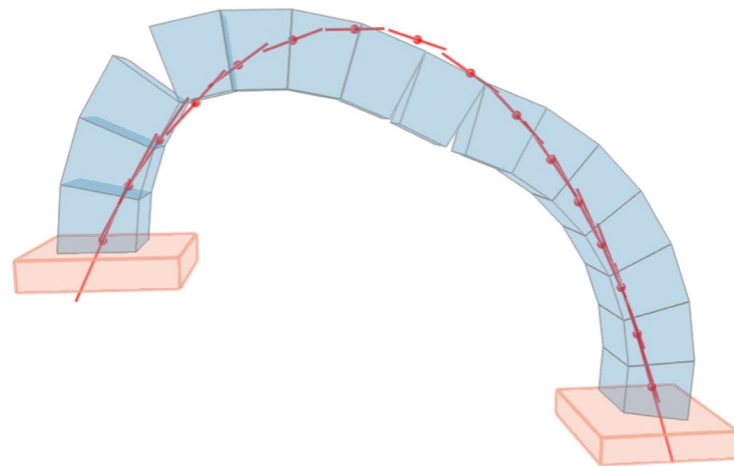
However, upon ramping the value $\delta=0.85\text{m}$ over the step time period, the thrust line observed is identical to the thrust line obtained through the PRD method (see Figure 105). Additionally a distinct jump in the kinetic energy was evident followed by a drop to zero, as can be seen in the following graph (Figure 104). Hence, the results found using the MC model in ABAQUS present an 85% reduction compared to the collapse settlement equal to 1m found through PRD. The latter is due to the rigid body behaviour and Mohr-Coulomb model which exhibits elasticity in compression. This matter has been addressed in the section “Elastic vs. Rigid Compression Models in Masonry”.



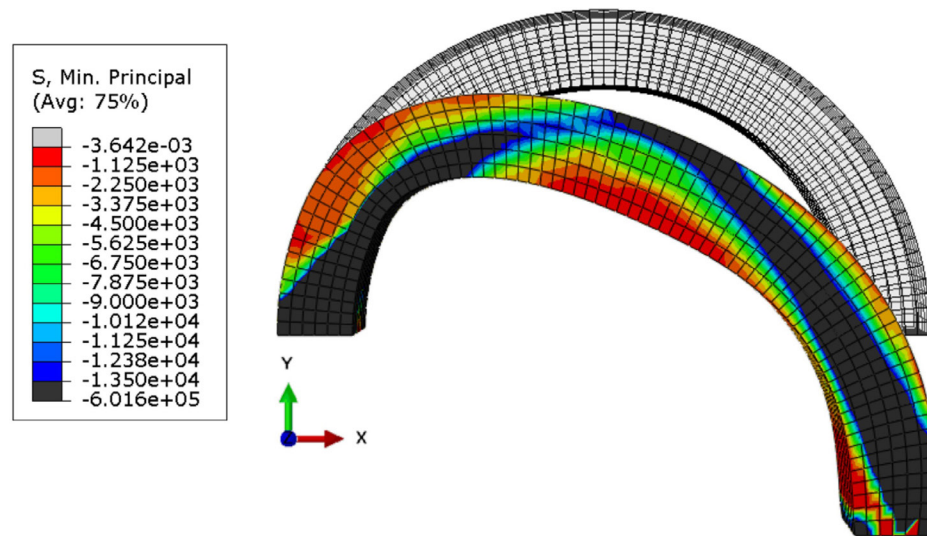
[Figure 104]
Graph showing the Kinetic Energy of the MC model in function of the step time, which is equivalent to the displacement.

Thrust Line Comparison

PRD results
at $\delta = 0.85 \text{ m}$



MC model - $S_{\min MC}$ results
at $\delta = 0.85 \text{ m}$

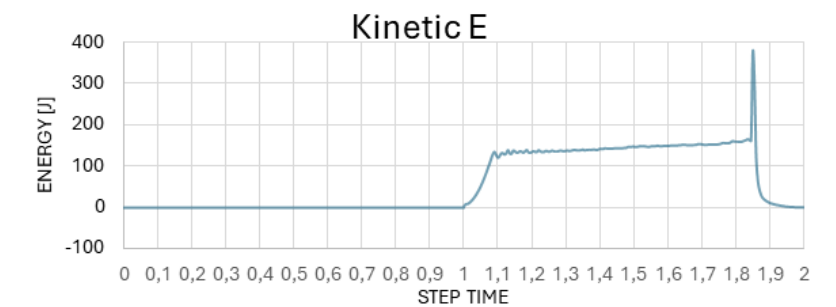


[Figure 105]

Comparison between the obtained thrust line and deformation visualized through the PRD method and the minimum principal stress via the MC model in ABAQUS at the limit value of the vertical settlement.

MC compared to CDP

Just like the prior analysis, we investigate the vertical settlement behaviour using an explicit dynamic analysis, which consists of two steps. The first step solely considers the gravity load, while the second step incorporates both gravity load and vertical settlement. This approach mirrors the earlier discussion in addition to the load ramping. It can be seen from the following graph (Figure 106) that just as the MC model, the CDP model was able to capture the limit vertical settlement responsible for the collapse mechanism initiation and it can be identified at $\delta=0.85\text{m}$.

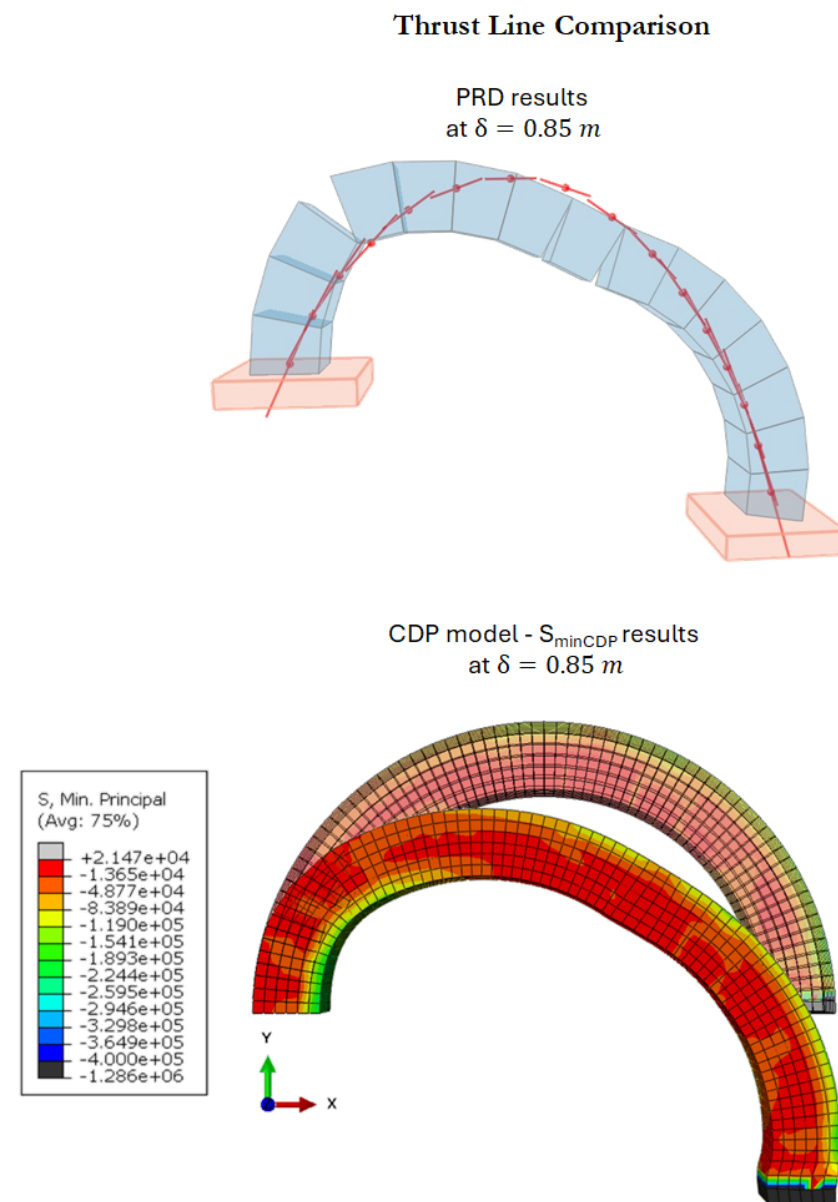


[Figure 106]

Graph showing the Kinetic Energy of the CDP model in function of the step time, which is equivalent to the displacement.

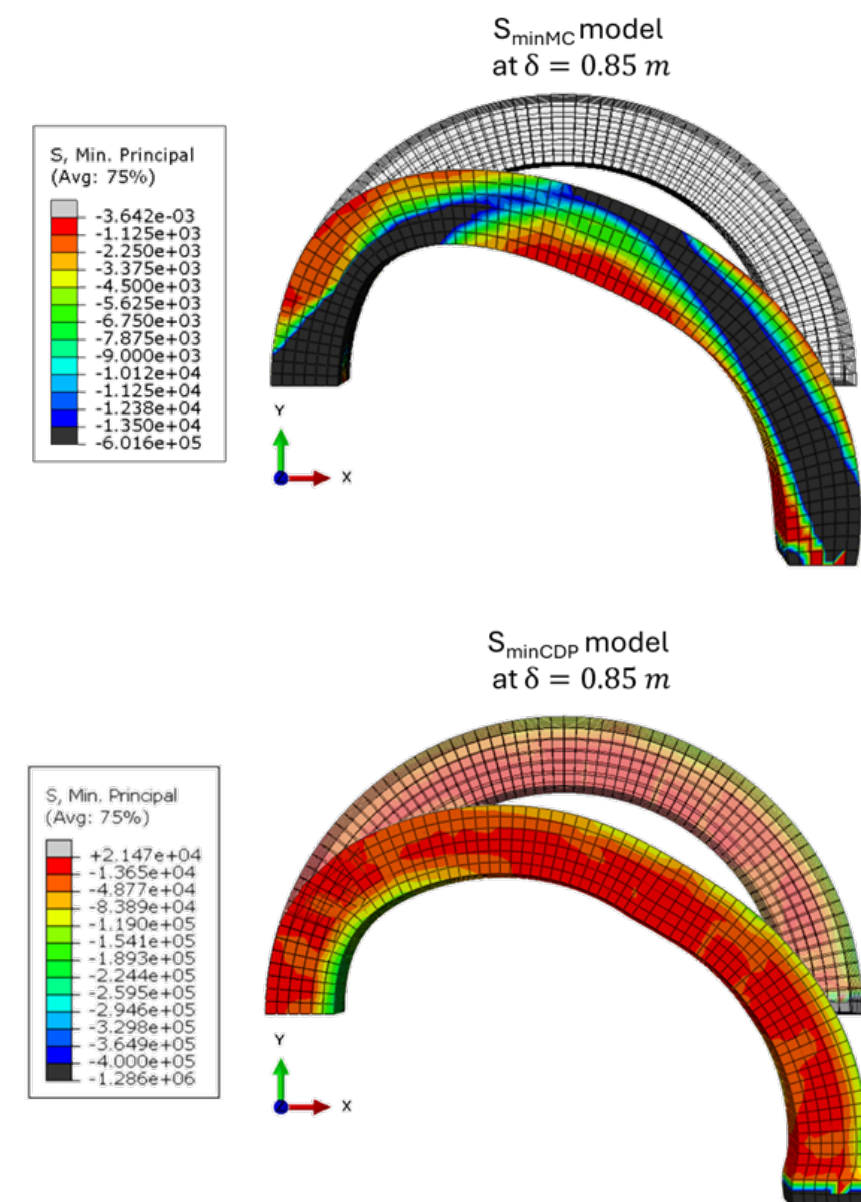
Examining Figure 107, it is evident that the settlement value which induces collapse is $\delta=0.85\text{m}$. This results found using the CDP model in ABAQUS presents an 85% reduction compared to the collapse settlement equal to 1.00m found through PRD. The latter is due to the rigid body behaviour and CDP model which exhibits elasticity in compression. This matter has been addressed in the section “Elastic vs. Rigid Compression Models in Masonry”. However, in Figure 108 the line of thrust is unidentifiable through the visualization of the minimum principal stress in the CDP model due its smeariness, as opposed to the visualization of the minimum principal stress of the MC model. In the latter, the black band clearly portrays the expected line of thrust at $\delta=0.85\text{m}$.

Furthermore, a comparison of the minimum principal plastic strain PE is presented in Figure 109. Both models successfully identify four hinge openings as expected. However, the location and concentration of the strain in the MC model more closely aligns with those observed through PRD.



[Figure 107]

Comparison between the thrust line and the deformation visualized through the PRD method and the minimum principal stress via the CDP model in ABAQUS, at the limit value of the vertical settlement.



[Figure 108]

Comparison of the results of the minimum principal stress, the obtained thrust line and deformation between the MC model and the CDP model in ABAQUS at the limit value of the vertical settlement.

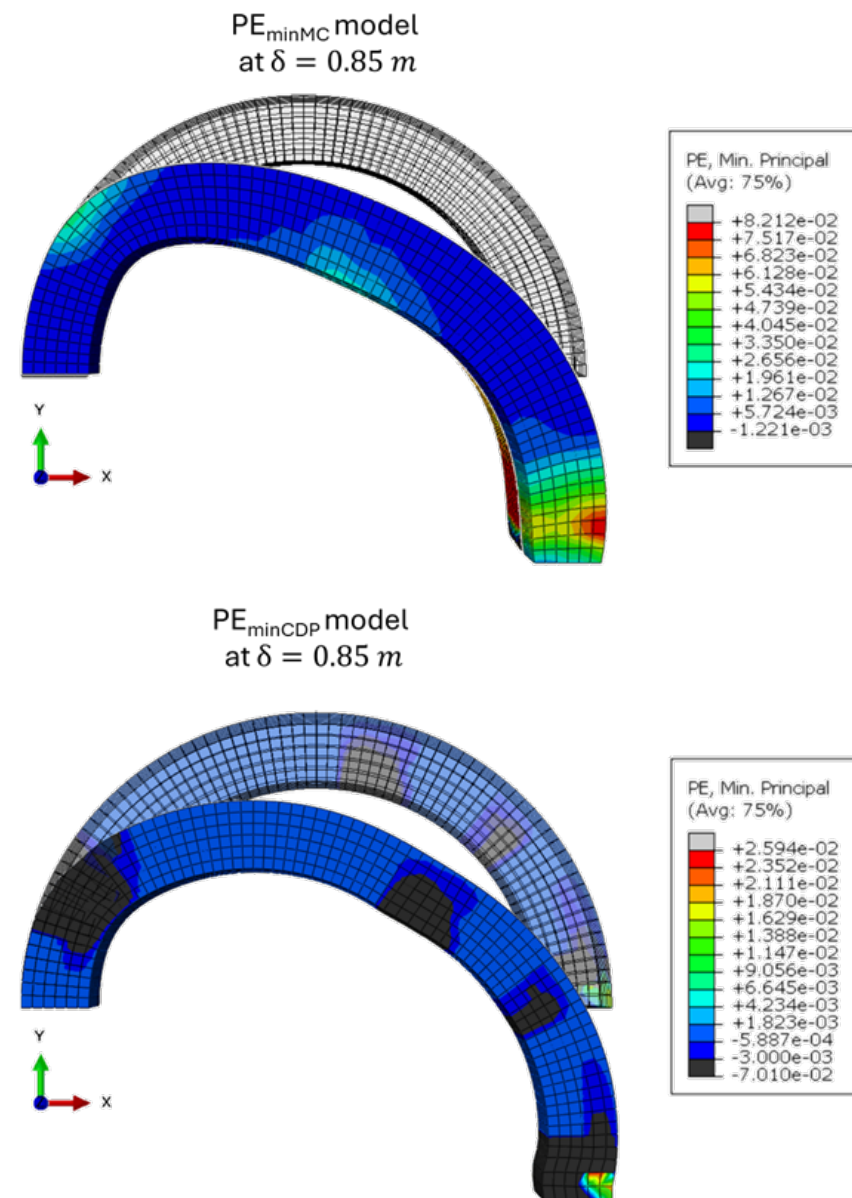
3. CASE STUDIES

In this section we apply a FEA on case studies using the MC model previously validated and compare it to the reference approaches used in literature, which are PRD and FE CDP model.

3.1. CASE STUDY 1: TORRE DEL PIANO

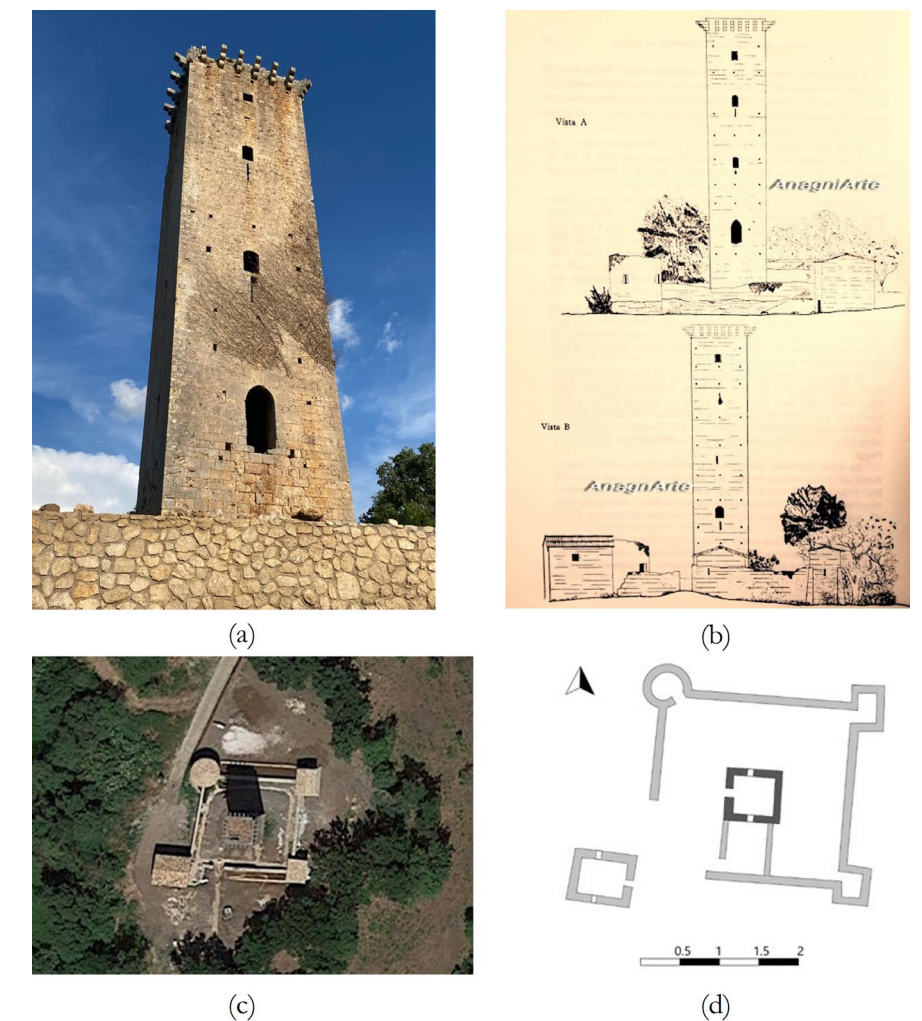
3.1.1. HISTORICAL BACKGROUND AND GEOMETRY

Piglio is considered the heir of the ancient "Capitulum Hernicum", associated with the Hernicis who were allies of Rome during the Iron Age (Garrod et al., 2012; Nibby, 1848; Terrinoni, 1888). As an indication of its historical significance, Roman artifacts such as entablatures, epigraphs, and ruins can still be found in the terrains around Piglio. It is in 1080, near Piglio, Anagni and Acuto, that the "Torre Del Piano" was constructed. The construction was done using coralline stones and featuring merlons (Nibby, 1848), as can be seen in Figure 110a and Figure 110b.



[Figure 109]

Comparison of the results of the minimum principal plastic strain between the MC model and the CDP model in ABAQUS at the limit value of the vertical settlement.



[Figure 110]

Torre del Piano: (a) Picture of the tower, (b) drawing by Mario Anzelotti according to Anagni Scuola futura (Genitoriscuola, 2017), (c) arial view of the tower and (d) plan of the tower.

Positioned strategically on a fifteen-meter-high tufa rock, the tower played a crucial role in early warning systems against enemy attacks. During the 15th century, the tower was reinforced with a bastion and four turrets made of stone, alongside three towers (two squared structures and one cylindrical structure) and a contemporary church (see Figure 110c and Figure 110d) (Nibby, 1848). This structure was part of a network of towers that served as a communication system used for surveillance, defence and for enemy detection across Ciociaria. Subsequent events, such as a lightning strike in 1922 and an interior fire, caused notable damage (Nibby, 1848).

During the preparation for the International Summer School on Historic Masonry Structures (HIMASS) a thorough geometrical and material survey was conducted so that this Tower can serve as a case study for structural analysis (Bohler & Heinz, 1999). Laser scanning and photogrammetry captured precise geometric data that were used in (Gerges et al., 2023a) and in this thesis. The tower, with a 5.6-meter-square base and standing 25.5 meters tall, displayed varying wall thicknesses and stone sizes along its height (Bohler & Heinz, 1999). Visual inspections revealed damage from lightning strikes, abandonment signs, and cracks, which were meticulously analysed using photogrammetry and expert assessment. Detailed drawings and renders of the tower's facades and crack patterns were produced (AutoCAD, 2022) and (Rhino 7, 2022) can be found in (Gerges et al., 2023a).

3.1.2. MATERIAL PROPERTIES

As previously mentioned, the tower was constructed using coralina stone, a type of limestone sedimentary rock, on a 15-meter-high tufa rock layer. With reference to the Italian Code NTC 2018, this study adopts an LC1 level due to the absence of in-situ testing data (NTC, 2018). Table 21 presents the assumed material properties based on this classification based on Table 1.

Material Properties			
Mass Density ρ	Young modulus E	Poisson ratio ν	Compression resistance f_m
2100Kg/m ³	2400Mpa	0.1	6MPa

[Table 21]
Material properties adopted for Torre
del Piano.

3.1.3. COMPARATIVE ANALYSES

In this section the critical leaning and fracture propagation of Torre del Piano is examined following the methodologies adopted in (Gerges et al., 2023a) using three distinct approaches:

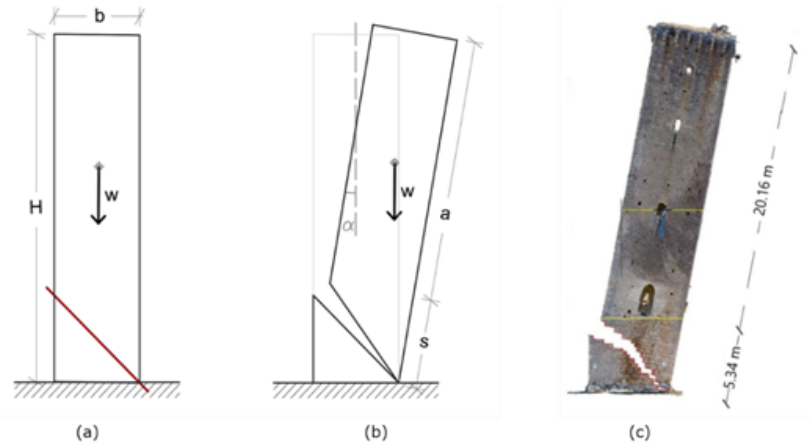
- A simplified approach: based on Heyman's theoretical strategy to assess elongation failure of the tilted tower (Heyman, 1992).
- Pseudo-static seismic capacity analysis: using the PRD method to evaluate the tower's lateral capacity against horizontal seismic actions and determine collapse mechanisms, Horizontal capacity analysis through an FEA using both a CDP model and an MC model.

A comparison among these approaches aim to provide insights into the validity of the MC model in detecting the potential fracture pattern associated with tower tilting and seismic events and the horizontal multiplier associated with the initiation of the collapse mechanism.

3.1.3.1. SIMPLIFIED APPROACH

Theoretical calculations

The results of the simplified approach are based on Heyman's paper (Heyman, 1992) which focuses on leaning towers, offering a practical method to analyse crack formation and collapse. It starts with equations for rectangular masonry blocks and extends these principles to tower structures. The results of this theoretical approach are presented in (Gerges et al., 2023a) for Torre del Piano which has H , the height, equal to 25.5m and b , the width, equal to 5.6 m. It was found that the critical angle of inclination $\alpha = 10.47^\circ$ and the distance from the tope of the tower of the fracture initiation $a = 20.16$ m Figure 111 illustrates the linear fracture propagation accordingly.



[Figure 111]
Figure from (Gerges et al., 2023a)
representing the geometric features
and tilting analysis of a hollow tower
with simplified theoretical calculations.
(a) Main features including a potential
linear crack (in red). (b) Visual
representation of the deformed tilted
configuration. (c) Solution of the tilting
test conducted on Torre del Piano.

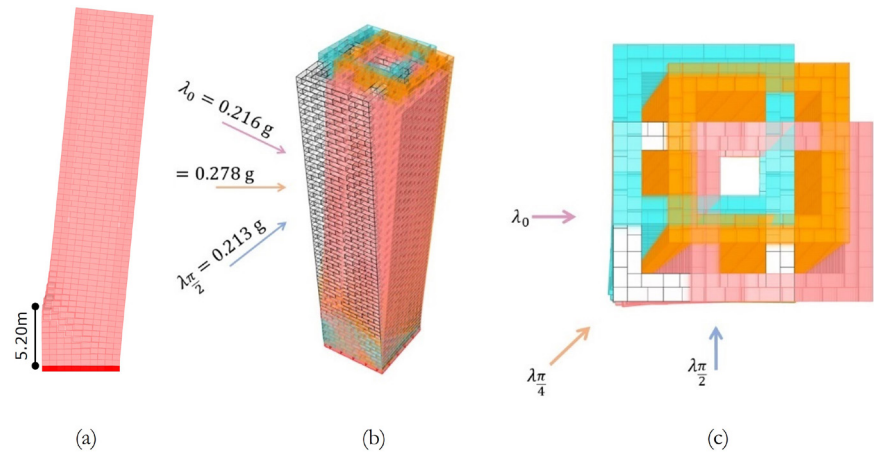
3.1.3.2 PSEUDO-STATIC SEISMIC CAPACITY ANALYSIS

PRD method results

A detailed analysis is provided in (Gerges et al., 2023a), where eventually the solution was reached by incrementally increasing horizontal forces while maintaining zero displacement at the boundary supports.

The solution converges when the horizontal forces become unbounded. Three azimuth angles are examined, and the corresponding solutions, showing the λ which initiate the mechanism, are illustrated in Figure 112.

[Figure 112]
PRD analysis of the tower considering three azimuth angles showing the initiation and propagation of fractures: (a) fracture initiation point, (b) 3D view, and (c) fracture propagation plan.



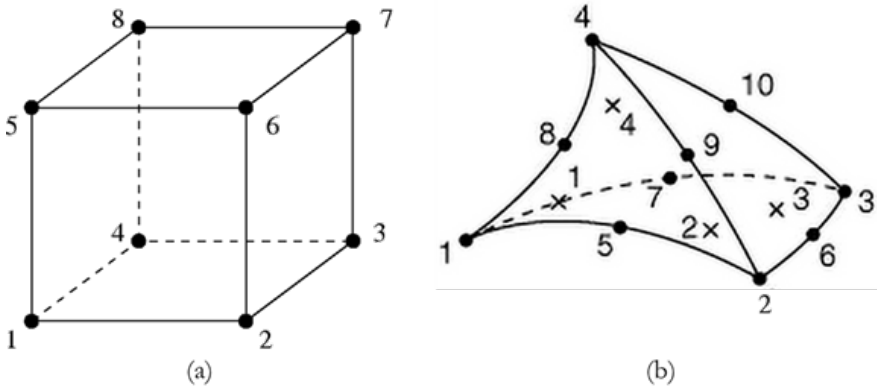
FEA results – CDP model

In this analysis the parameters implemented for the CDP model in ABAQUS (Abaqus, 2023) are presented in Table 22 and

Table 23 and are identical to those adopted in (Gerges et al., 2023a).

Two types of mesh were considered for this analysis. The first mesh is a tetrahedral mesh of global mesh size equal to 1 and a type C3D10M element are adopted for this model generating 13330 number of elements. The second mesh is a bottom-up mesh extrusion of a global size of 0.2 and an element type C3D8R generating 33420 elements. Figure 113 shows the elements corresponding to each mesh.

[Figure 113]
Two types of mesh elements in ABAQUS: (a) quadratic tetrahedral elements of type C3D10M, (b) linear hexahedral elements of type C3D8R.



[Table 22]
Assumed values for some parameters of the CDP model in ABAQUS.

Dilatation angle	Eccentricity	f_{b0}/f_{c0}	K_c	Viscosity
10°	0.1	1.16	0.667	$5 \cdot 10^{-5}$

[Table 23]
Values for the stress-strain behavior of the material in the plastic range.

Compression		Tension	
ϵ	σ [MPa]	ϵ	σ [MPa]
0.000	6.000	0.00	0.200
0.010	5.400	0.007	0.005
0.040	5.400	0.100	0.005
0.100	4.600		

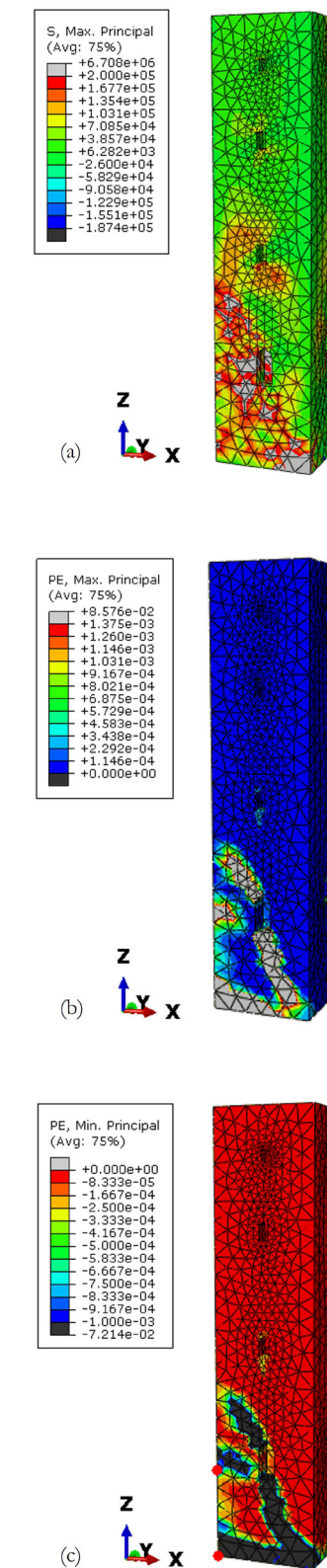
Firstly, the result of the analysis using the CDP model will be discussed for the tetrahedral meshing. Considering that the highest tensile stress is 0.2 MPa, it is selected as highest upper limit for the maximum principal stress.

It can be seen in Figure 114a that the grey areas indicate where the tensile stress surpasses 0.2 MPa. This visualization corresponds to $\lambda = 0.5$ where the grey areas first join and mark the line of fracture that separates the tower's base from the collapsing section. The orientation and propagation of the fracture line aligns with Heyman's analysis (Heyman, 1992) and are consistent with the findings of the PRD approach. However, the λ identified with CDP is higher than that identified through the PRD method.

As for the max principal strain PE, when the maximum principal strain is positive it signifies tensile deformation. The grey areas that can be seen in Figure 114b highlight the highest the tensile stretching experienced by the material. This aligns with the fracture line identified in the maximum principal stress visualization and corroborating Heyman's findings. Conversely for the minimum principal strain PE, usually the negative values of minimum plastic strain indicate that the material is experiencing tensile strain and separation.

In Figure 114c all the values of the min plastic strain are negative, however, to analyse the results the highest negative values are considered and as depicted the black band represents the highest region experiencing tensile strain. The distance from the fracture initiation to the tower's base is indicated by the two red dots in Figure 114c. It was identified as equal to 4.96m, consequently the distance a from the top is in this case 20.54m, as opposed to the 20.16m found through theoretical calculations for the Torre del Piano based on (Heyman, 1992).

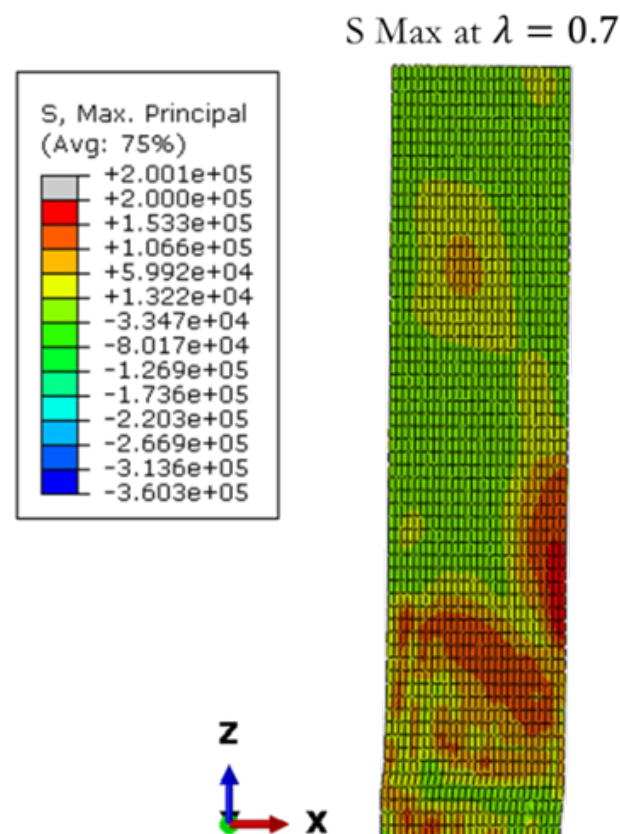
All fracture patterns in the visualization module for the maximum principal stress, the maximum and minimum principal strains are aligned, and the measured distance for the fracture initiation closely match theoretical calculations.



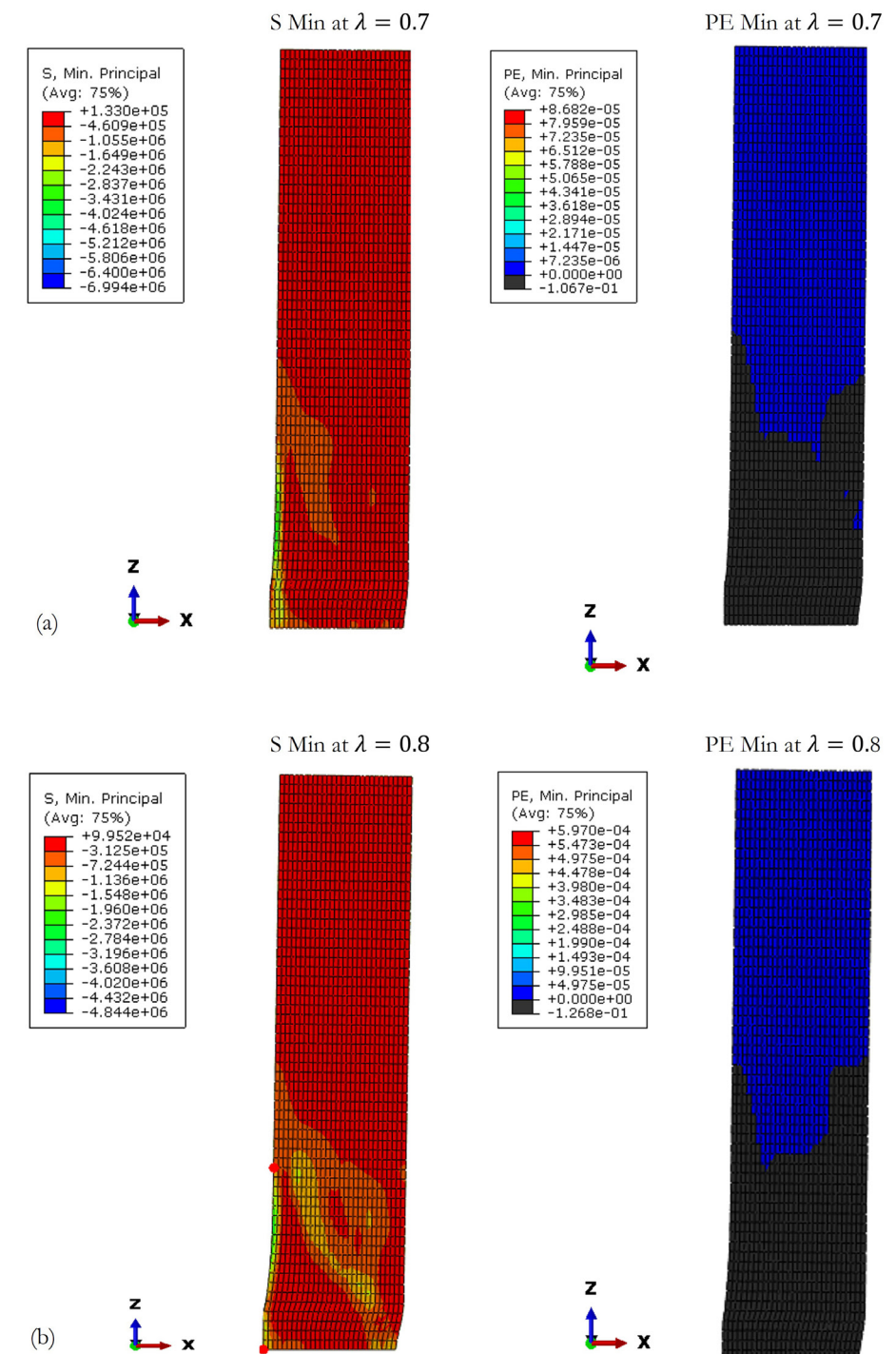
[Figure 114]
Horizontal capacity analysis results via CDP model: (a) maximum principal stress, (b) maximum principal plastic strain and (c) minimum principal plastic strain.

Moving forward, the results obtained from the previously described bottom-up mesh will be presented. Figure 115 illustrates that the maximum principal stress matches the ultimate 0.2 MPa stress at $\lambda=0.7$, highlighting by the red smeared stress where the separation might occur. In line with this observation, when investigating the minimum principal stress in Figure 116a, the initial formation of the smeared thrust line was observed. However, the fracture line becomes evident only when the collapse multiplier reaches a 0.8 value as can be seen in Figure 116b.

Figure 116 also presents a side-by-side comparison of the minimum principal stress against the minimum principal strain for each horizontal multiplier value. To differentiate between positive and negative values of the plastic strain, the upper limit is set equal to 0, revealing compressive strain (black areas) and tensile strain (blue areas) where detachment might take place. It is evident that the visualization of the plastic strain is not as desired and does not accurately represent the fracture line nor differentiate tower's intact base from the collapsing part. Despite these limitations and the fact that the multiplier value greatly exceeds that found through the PRD approach the stress analyses give valuable insights into fracture initiation and propagation. Nevertheless, the identified initiation point of the fracture is inaccurate as it is 7.8m from the tower's base. This value differs significantly from the theoretically calculated location.



[Figure 115]
Maximum principal stress results obtained by adopting a CDP model for the horizontal capacity analysis of the Torre del Piano and attributing to it a bottom-up mesh.

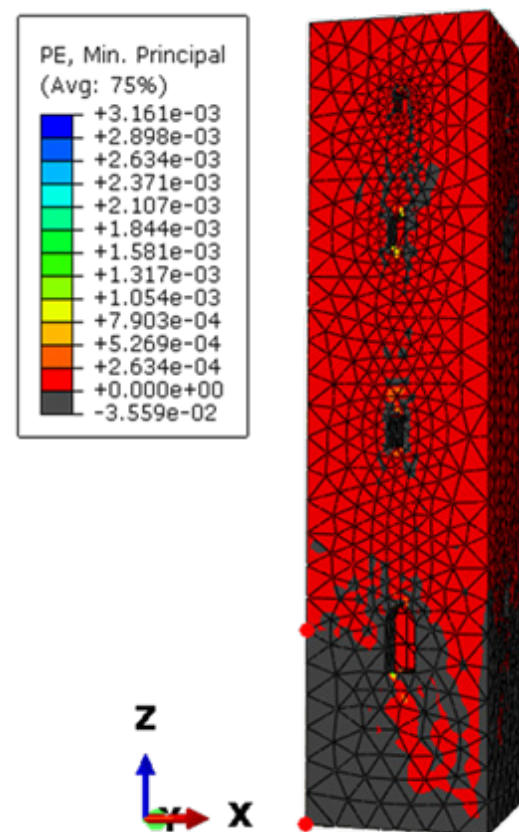


[Figure 116]
Visualization of results of minimum principal stress and plastic strain for the horizontal capacity analysis of the Torre del Piano, attributed a CDP model and a bottom-up mesh: (a) at $\lambda=0.7$ and (b) at $\lambda=0.8$.

FEA results – MC model

In line with prior analyses using the CDP model, the tower was also analysed using the MC model and considering two mesh types. Therefore, the same material characteristics such as mesh properties, element number, and elastic properties are maintained in these analyses. However, the MC model was specified with an associated flow rule characterized by a friction angle of 89.9° and a cohesion of 0.5 Pa.

The initial discussion will focus on the results obtained from tetrahedral meshing of Torre del Piano. In Figure 117, the minimum presented plastic strain PE with an upper bound of 0 that serves to distinguish between negative and positive values, highlighting tensile and compressive strains accordingly. It is important to note that at $\lambda=0.2$, a value identical to the one found by PRD, clear fracture line can be visualized, serving as a boundary between the intact tower's base and the collapsing part (see Figure 117). The measured distance from the point of initiation to the tower's base is 5.9 meters, which compares to the theoretical value of 5.34 meters and the 5.2 meters determined by the PRD method.



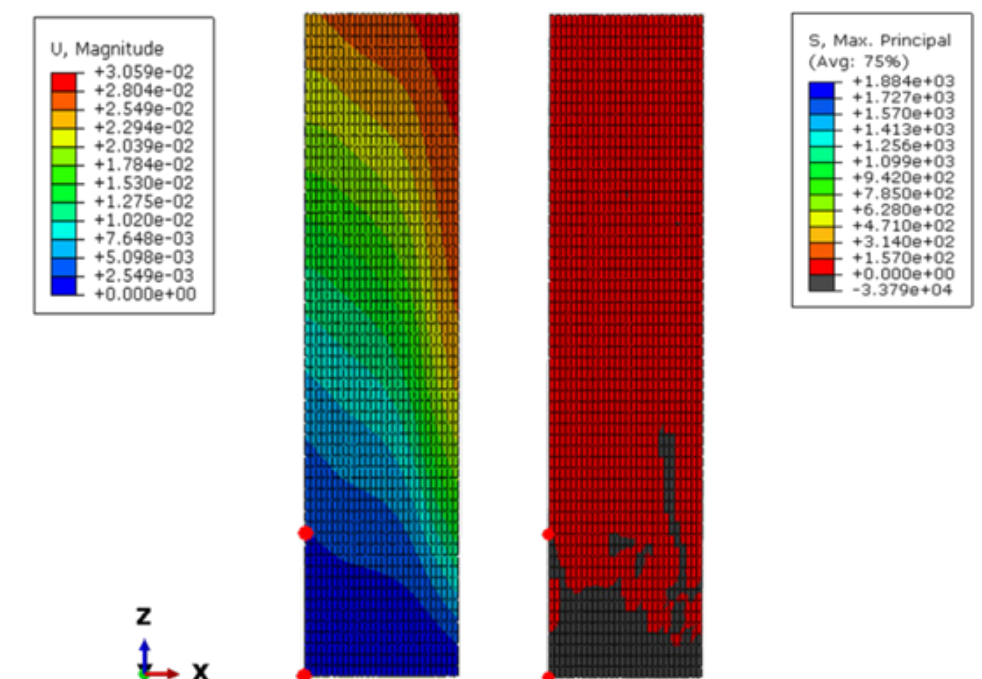
[Figure 117]
Maximum principal plastic strain results obtained by adopting a MC model for the horizontal capacity analysis of the Torre del Piano and attributing to it a tetrahedral mesh.

Figure 117 presents the results obtained for the bottom-up mesh following an extrusion having a global size of 0.2 and an element type C3D8R, generating 33420 elements. Upon inspecting the displacement magnitude U results in the ABAQUS visualization module, it is observed at $\lambda = 0.15$, a distinct 45-degree line that shows the displacement of the parts of the tower indicating that at collapse the base will separate from the collapsing part following this form of crack propagation.

To pinpoint the exact moment of mechanism initiation, attention should be given to when a complete joint stress line forms as a boundary between two areas, initiating thus the collapse mechanism. From that point, a comparison between the maximum principal stress S Max and displacement magnitude is conducted. The lower bound of S Max was set at 0 to differentiate positive (indicating tensile stress) and negative (compression) stress values.

At $\lambda=0.3$ the boundary line is clearly visualized in S Max, as depicted in Figure 118. A direct comparison between displacement magnitude and maximum principal stress reveals synchronization in the boundary line separation, pinpointing the mechanism of collapse initiation at $\lambda = 0.3$, a value slightly greater than that found by the PRD method.

Furthermore, when calculating the distance from the fracture initiation point to the tower's base, the resulting value of 5.37 meters closely matches the theoretical value of 5.34 meters calculated according to (Heyman, 1992) making it the closest among all previous analyses conducted.



[Figure 118]
Displacement magnitude and maximum principal stress results obtained by adopting a MC model for the horizontal capacity analysis of the Torre del Piano and attributing to it a bottom-up mesh.

3.1.4. CONCLUSION

A comprehensive comparison of the computational time required for each analysis is presented in Table 24. Based on earlier findings, and taking into account computational efficiency, the most effective way to analyse the horizontal capacity of Torre del Piano is by using the MC model with a bottom-up mesh configuration. This approach portrayed a good balance between quality and accuracy while also being efficient in terms of time. This configuration presented a $\lambda = 0.3$, pinpointing the crack initiation at the tower's base around 5.37 meters from the ground. These outcomes closely align with the calculations based on Heyman calculations and the PRD method results, demonstrating a strong balance between accuracy and computational efficiency.

[Table 24]
Comparison of the finite element analysis computational time required for the horizontal capacity analysis of the Torre del Piano between the CDP and MC models.

	Computational Time [s]	
	Tetrahedral mesh	Bottom-up mesh
CDP model	1118	204
MC model	35000	6648

3.2. CASE STUDY 2: ACUEDUCTO DE LOS CINCO OJOS

In this section the “Acueducto de los Cinco Ojos” is studied. The background information on the history and geometry of the structure as well as on the seismic hazard of the zone in which the Aqueduct is located have been taken from the Master’s thesis developed by Simona Ena at the University of Cagliari (Ena, 2021), under the supervision of Professor Cristina Porcu of the University of Cagliari and prof. Salvador Ivorra of the University of Alicante (Spain), who also provided the data of the aqueduct and carried out the modal tests on it. A summary and a concise overview of this information serve as an introductory part preceding the comparative analyses conducted for the purpose of the present thesis.

This section will present the results of the analyses conducted in ABAQUS (Abaqus, 2023) employing the two masonry models MC and CDP. The comparison between the two approaches aims to provide insights into the validity of the MC model in detecting the response of the structure under dynamic loadings representative of a real earthquake.

3.2.1. HISTORICAL BACKGROUND AND GEOMETRY

The Aqueduct de los Cinco Ojos is a remarkable hydraulic engineering achievement dating back to the 18th century, situated near the Elche marshes and the Vinalopó River in Aspe, Spain (Español & López, 2015, 2020). This aqueduct forms an integral part of Alicante Province’s historical hydraulic infrastructure network, designed to convey drinking water from Aspe’s Barrenas Source to Elche. It is part of a larger system that includes other aqueducts such as the Aqueduct de los Cuatro Ojos, Aqueduct del Rabosero, Aqueduct del Barranco de la Coca, and Aqueduct del Barranco de Carboneras (Español & López, 2015, 2020).

During the 18th century, Elche experienced significant population growth, leading to a rising demand for potable water that exceeded local resources. This necessitated the construction of an aqueduct, with the Fuente de Barrenas identified as the most suitable water source. Construction of the aqueduct began in 1785 under the patronage and partial funding of José Tormo, the Bishop of the Diocese of Orihuela. The architectural design and canalization project were led by architect Miguel Francia and José González de Coniedo, an accomplished architect, sculptor, and painter from Aspe (1735 – 1820).

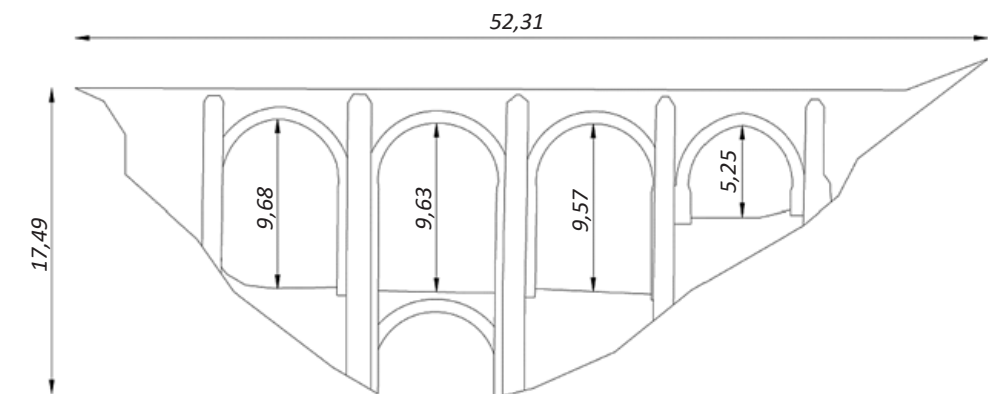
The detailed project plan, submitted by Francia and González de Coniedo in October 1783 and approved by the Council of Castile in February 1784, incorporated comprehensive considerations of terrain properties, geographic characteristics, material sourcing, and construction methods. Water from the Fuente de Barrenas located just 1 km from Aspe on the left bank of the Tarafa River beneath the current Monforte Road, was channelled through a central gallery and transverse galleries to the main conduit. From there, the water flowed into an open channel before being directed through a siphon to the Tarafa riverbed control cabin, eventually traveling underground across orchards and agricultural lands, crossing the Alicante Highway, and culminating at the Los Barrancos site, where the most impressive sections of the aqueduct can be observed.

The Aqueduct de los Cinco Ojos, named for its five arches arranged across two levels, represents the pinnacle of architectural and hydraulic engineering excellence within this canalization network. The aqueduct comprises 37 structures, including aqueducts and sewage networks, and is distinguished by its elegant proportions, harmonious architectural design, and structural stability (Español & López, 2015, 2020).

The aqueduct was originally designed to support its own weight and the weight of transported water through an enclosed section, later replaced by an external canal to facilitate water flow. Key structural elements include load-bearing arches that efficiently transfer loads to sturdy brick masonry pillars constructed with lime mortar (Español & López, 2015, 2020). These pillars primarily work under compression, transmitting the loads to the underlying foundations. To enhance lateral stability, the pillars are reinforced transversely with a masonry composed of various rock fragments bound with mortar, providing essential additional support.

Decorative elements such as cornices and keystones in the upper arches not only contribute to the aqueduct's stability but also enhance its aesthetic appeal. The aqueduct's dimensions, meticulously obtained through laser scanner surveys, reveal its monumental scale, with a maximum height of 16.8 (Español & López, 2015, 2020). The thesis student Simona Ena used the AutoCAD (AutoCAD, 2021) drawing document provided by Jose Antonio Huesca Tortosa to derive the geometric dimensions for the model. This document contained an elevation profile obtained via laser scanning before any interventions, which included areas of damage at the base of pillars

and within the arch intrados. However, in Ena's thesis damaged areas were digitally removed from the profile using AutoCAD software. Furthermore, non-structural elements were excluded from the FEM model to prevent convergence issues (see Figure 119).



[Figure 119]
Elevation view of the Aqueduct de los
Cinco Ojos.

3.2.2. MATERIAL PROPERTIES

The modelling process in ABAQUS (*Abaqus*, 2023) involved two key aspects: geometric model development and assignment of material properties based on detailed analyses and adaptations. Simona Ena's thesis extensively detailed these steps to accurately represent the Los Cinco Ojos Aqueduct's behaviour under various conditions.

In ABAQUS, the geometric model was imported using standard ACIS text files (.sat) to create parts suitable for detailed analysis. The model employed was meshed (see Figure 120a) using C3D10 quadratic tetrahedral elements with 10 nodes (see Figure 120b).

The identification and assignment of material properties were critical for the accurate simulation of the aqueduct's components. Simona Ena's thesis detailed an extensive and iterative process, based on matching numerical and experimental first frequencies, to identify the values of elastic moduli and mass densities of the two masonry materials the aqueduct is made of. Due to the absence of experimental data specific to the aqueduct's materials, reference was made to maximum and minimum values of mechanical parameters for various types of masonry structures outlined in Circular 21/01/2019 (CNT, 2019). The initial modelling used assumptions based on average and maximum values of Young's modulus and specific weight for solid brick masonry

and finely textured stone masonry. However, these models did not match experimental frequencies, leading to adjustments. Subsequent trials refined material parameters by considering lower specific weights and higher Young's modulus values based on revised classifications. The iterative identification process eventually led to the material properties listed in Table 25, that were assumed in the model of the Aqueduct in ABAQUS.

[Table 25]
Identified values of the Young's modulus, Poisson ratio and mass density of the two masonry materials the aqueduct is made of.

Solid brick masonry with lime mortar		Rubble stone masonry with good texture	
E [MPa]	ρ [kN/m ³]	E [MPa]	ρ [kN/m ³]
1500	18	1740	21
$\nu = 0,25$		$\nu = 0,3$	

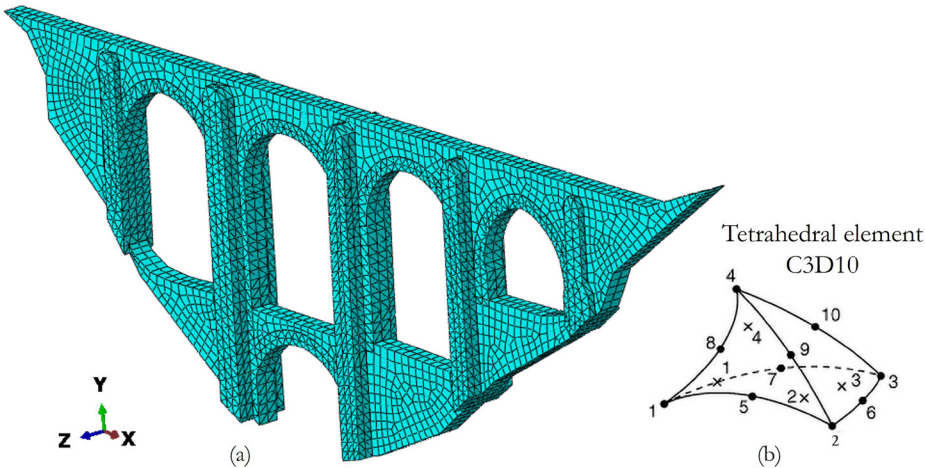
3.2.3. SEISMIC ACTION

To carry out the dynamic analyses, the strongest of the seven spectrum-consistent earthquakes found in (Ena, 2021) was considered. The selected earthquake exhibits 7.23 m/s² and 8.21 m/s² maximum peak accelerations in the longitudinal and transversal direction, respectively. The data concerning the earthquake are presented in Table 26 and Figure 121a.

In the analyses, the longitudinal component of the earthquake was applied in the X direction of the structure, while the transverse component was applied along the Z direction, hereby considering the self weight acts in the Y direction (see Figure 120a). No vertical component of the earthquake is considered.

To address the worst-case scenario, the decision to apply the transverse component in the Z direction was based on selecting the highest peak amplitude (8.21 m/s²) in the out-of-plane direction of the structure.

[Figure 120]
Numerical model of the Aqueduct de los Cinco Ojos: (a) the meshed assembly and (b) the tetrahedral element forming the mesh.



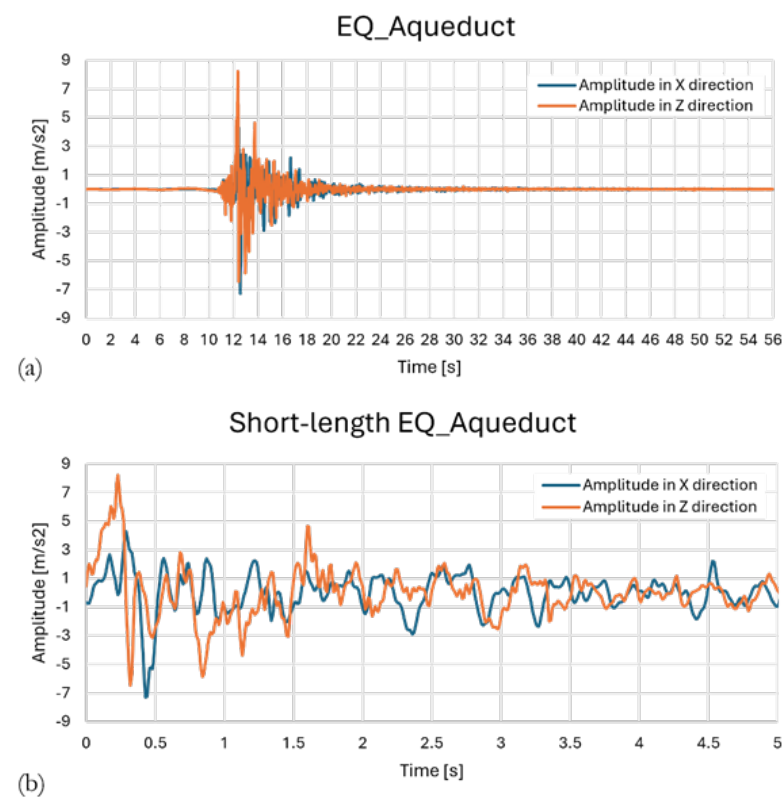
and within the arch intrados. However, in Ena' thesis damaged areas were digitally removed from the profile using AutoCAD software. Furthermore, non-structural elements were excluded from the FEM model to prevent convergence issues (see Figure 119).

[Table 26]
Characteristics of the strongest earthquake from the combination found with REXEL (SET30) (Ena, 2021).

	Waveform ID	Earthquake ID	Name	Date	M _{wc}	Epicentral Distance [km]
EQ_Aqueduct	6349	2142	South Iceland (aftershock)	21/06/2000	6.4	5

In order to reduce the computational time, the original accelerograms plotted in Figure 121a were suitably cut by applying a procedure based on the Arias Intensity percentage content, which is described in (Gerges et al., 2023b) and recalled in Appendix C. This procedure assure to save 95 % of the energy content of the earthquake, while drastically reducing its duration. Through this procedure, the short-length accelerograms presented in Figure 121b were obtained.

The two short-length accelerograms were applied to the aqueduct model to carry out dynamic analyses with the MC approach and the CDP approach. The comparison between the two approaches aims to provide insights into the validity of the MC model in detecting the response of the structure under dynamic loadings.



[Figure 121]
Earthquake retrieved from (Ena, 2021):
(a) Original full length earthquake, (b):
short length earthquake cut as per the
procedure in Appendix C.

3.2.3.1. THNLA computational method

To assess the Aqueduct's dynamic response to a given earthquake, a time history non-linear analysis (THNLA) can be performed in Abaqus. THNLA is an advanced method used in seismic engineering to assess the dynamic response of structures under spectrum-consistent earthquakes (SCEs) (M. C. Porcu et al., 2018, 2019; Vielma-Perez et al., 2020). Unlike linear analyses that assume material behaviour to be either linear-elastic or elastic-plastic, THNLA integrates the nonlinear differential equations of motion under SCEs, considering the complex behaviour of materials and connections within the structure. In THNLA, the structural response to seismic loading is analysed over time, accounting for material nonlinearities such as yielding, stiffness changes, and large deformations. This method provides detailed insights into a structure's behaviour under realistic seismic conditions, offering valuable results applicable to various types of structures and materials.

Compared to simpler linear methods like Modal Response Spectrum Analysis (MRSA), which combines the contribution of the most significant modes taken from design response spectra, THNLA offers a more comprehensive understanding of structural performance during seismic events (Vielma-Perez et al., 2020). THNLA is considered a powerful and code-compliant tool for seismic assessment, enabling engineers to design earthquake-resistant

structures capable of withstanding complex dynamic loads with greater accuracy and reliability (Gerges et al., 2023c; Krawinkler & Seneviratna, 1998; Paglietti et al., 2011).

In the context of structural dynamics, selecting an appropriate time integration procedure for THNLA is crucial. The demands of such analyses necessitate stable and accurate methods due to the fundamental challenges with explicit approaches (Dahlquist, 1963). Notably, explicit methods lack stability within the class of linear multistep methods (Dahlquist, 1963).

Implicit unconditionally stable algorithms are favoured in structural dynamics, particularly for scenarios where low-frequency mode responses are critical, such as seismic analysis. These algorithms allow for the selection of time steps independently of stability considerations in linear analysis, resulting in significant computational savings (Dahlquist, 1963).

Newmark's valuable work on implicit methods introduced step-by-step time integration (SSTI) methods tailored for solving dynamic problems under seismic and blast loading conditions (Newmark, 1959). Since then, researchers have enhanced and adapted Newmark's techniques to address specific structural challenges, leading to the development of efficient SSTI algorithms (Argyris et al., 1974; Belytschko, 1976). For dynamic nonlinear analysis, the HHT α method remains one of the most popular SSTI algorithms (Hilber et al., 1977; Miranda et al., 1989). This method, derived from the Newmark method, incorporates a dissipation parameter (α) crucial for improving numerical damping, especially effective for modelling nonlinear oscillations (Hilber et al., 1977; Miranda et al., 1989).

The stability and efficiency offered by implicit unconditionally stable algorithms, exemplified by the HHT α method, make them particularly suitable for THNLA, where robust integration of nonlinear differential equations under spectrum-consistent earthquakes (SCEs) is essential (Dahlquist, 1963). This approach ensures accurate simulation of structural response under realistic loading conditions, aligning well with the demands of THNLA.

Hence, according to the previously stated, since it is preferable the use of implicit dynamic integration for the THNLA, when applying the CDP model an implicit dynamic step will be adopted. However, due to the MC model's limitation it will be attributed an explicit dynamic integration step. This aligns

with the findings of (Němec et al., 2017) which indicate that both implicit and explicit methods are effective and comparable for seismic analysis in structural dynamic studies. Despite the implicit method being slower than the explicit method, it achieved practically identical results, demonstrating only a 1% difference in extreme displacement values.

Both implicit and explicit steps have a step duration of 6 seconds. The initial 5 seconds include the combined effects of the structure's self-weight and the earthquake, illustrating the dynamic response. The subsequent 1 second interval allows for post-earthquake visualization, focusing on the structure's behaviour once the dissipates and it remains under its own self weight.

3.2.3.2. Results of the finite element analyses

FEA results – CDP model

After implementation, in ABAQUS, the model was meshed with a global size equal to 0.5, generating 32508 elements: 2016 are C3D8R elements and the rest are C3D10M elements. A representation of both meshes has been portrayed in Figure 113.

Due to the lack of experimental data on the mechanical characteristics of the two types of masonry that make up the Aqueduct of Los Cinco Ojos, defining the constitutive relationships for tension and compression is particularly challenging. The initial elastic portion of the stress-strain curves under compression was derived considering the Young’s modulus and the compressive strength previously calibrated. The compressive strength of the solid brick and lime mortar masonry was assumed to be 6.3 MPa, and that of the irregular soft stone masonry was assumed to be 2.3 MPa. Three ratios between compressive and tensile strength, say $R_f = f_t / f_c$, were considered in the study: 1/30, 1/60, and 1/100. $R_f = 1/100$ is actually a very low value, which however led to explore scenarios with significantly reduced tensile strength, reflecting Heyman's assumption of negligible tensile capacity.

The values given in Table 3 (Section 2.2.1.b) are assumed for the parameters of the CDP model in ABAQUS. Table 27 to Table 29 provide the values of strength and strain at the yield limit, for the three values of R_f here considered and for the two types of masonry.

[Table 27]

Tensile and compressive strengths and elastic deformation limits for the CDP model with $f_t = f_c/30$.

[Table 28]

Tensile and compressive strengths and elastic deformation limits for the CDP model with $f_t = f_c/60$.

[Table 29]

Tensile and compressive strengths and elastic deformation limits for the CDP model with $f_t = f_c/100$.

CDP model_ $R_f = 1/30$							
Solid brick masonry with lime mortar				Rubble stone masonry with good texture			
f_c	$f_t = f_c/30$	$\epsilon_{el,c}$	$\epsilon_{el,t}$	f_c	$f_t = f_c/30$	$\epsilon_{el,c}$	$\epsilon_{el,t}$
[MPa]	[MPa]	-	-	[MPa]	[MPa]	-	-
6.3	0.21	0.0024	8.0E-05	2.3	0.076	0.0017	5.68E-05

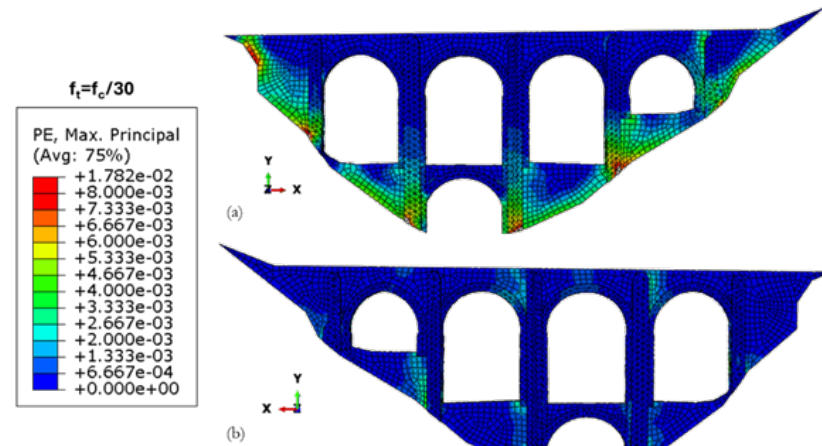
CDP model_ $R_f = 1/60$							
Solid brick masonry with lime mortar				Rubble stone masonry with good texture			
f_c	$f_t = f_c/60$	$\epsilon_{el,c}$	$\epsilon_{el,t}$	f_c	$f_t = f_c/60$	$\epsilon_{el,c}$	$\epsilon_{el,t}$
[MPa]	[MPa]	-	-	[MPa]	[MPa]	-	-
6.3	0.105	0.0024	8.0E-05	2.3	0.038	0.0017	5.68E-05

CDP model_ $R_f = 1/100$							
Solid brick masonry with lime mortar				Rubble stone masonry with good texture			
f_c	$f_t = f_c/100$	$\epsilon_{el,c}$	$\epsilon_{el,t}$	f_c	$f_t = f_c/100$	$\epsilon_{el,c}$	$\epsilon_{el,t}$
[MPa]	[MPa]	-	-	[MPa]	[MPa]	-	-
6.3	0.063	0.0024	8.0E-05	2.3	0.023	0.0017	5.68E-05

The specific time chosen for result visualization is immediately after the conclusion of the peak seismic activity during the short-length earthquake in

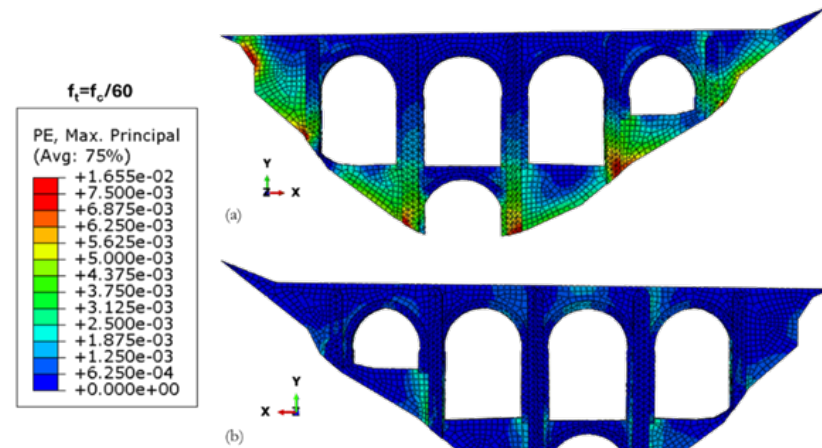
Figure 121b, at $t = 0.35$. The maximum plastic strain contours for the three CDP models are presented from see Figure 122 to Figure 124. In these figures, colors from green to red indicate the zones where plastic strain occurs. The blue areas remain instead in the elastic range. However, the predominant areas are characterized by smeared green, suggesting lower but dominant levels of strain.

It can be clearly seen in Figure 122 that for a compressive to tensile stress ratio of 1/30 indicates a 0.53% maximum plastic strain in these green areas. At a ratio of 1/60 (Figure 123) the percentage of maximum plastic strain is reduced to 0.5% and remains quasi-equal at a 0.49% value at a ratio of 1/100 in Figure 124. These results portray an exceptionally low decrease of the maximum plastic strain influenced by the stress ratio reduction.



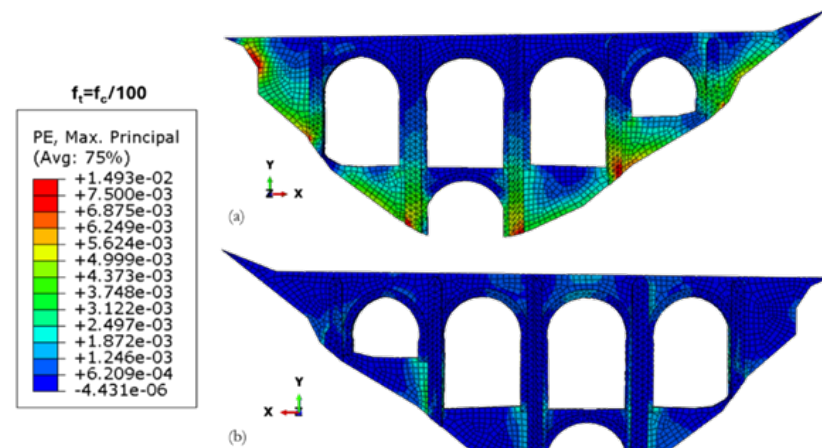
[Figure 122]

Maximum principal plastic strain results with CDP model for $f_t = f_c/30$: (a) Front view of the Aqueduct de los Cinco Ojos, (b) Back view of the Aqueduct de los Cinco Ojos.



[Figure 123]

Maximum principal plastic strain results with CDP model for $f_t = f_c/60$: (a) Front view of the Aqueduct de los Cinco Ojos, (b) Back view of the Aqueduct de los Cinco Ojos.



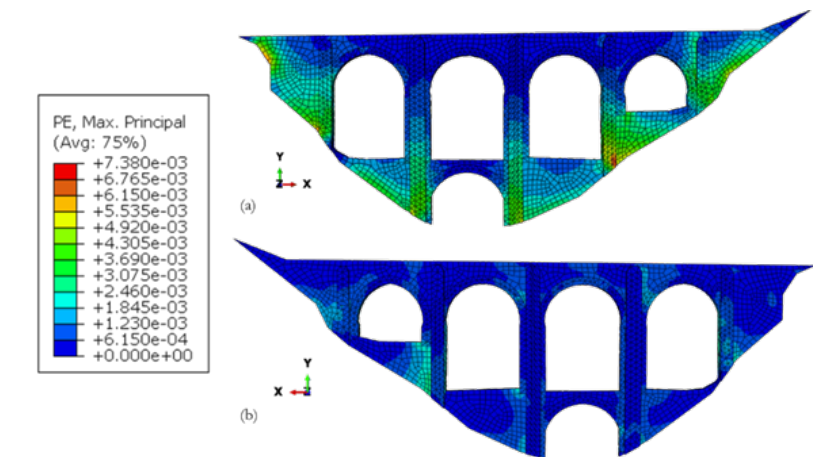
[Figure 124]

Maximum principal plastic strain results with CDP model for $f_t = f_c/100$: (a) Front view of the Aqueduct de los Cinco Ojos, (b) Back view of the Aqueduct de Los Cinco Ojos.

FEA results – MC model

The same numerical model, having same material properties, same mesh characteristics and under same loading and boundary conditions, has been analysed with the MC failure criteria in ABAQUS. An associative flow rule was assumed with an angle of friction equal to 89.9° and a cohesion of 0.5 Pa, values that have been previously determined in section 2 and adopted throughout the thesis. The maximum plastic strain contours obtained in this case, are provided in Figure 125 for the front and back of the aqueduct. They are taken at the same instant considered in Figure 122 to Figure 124.

The comparison between the CDP model and the MC model shows that very similar results in terms of plastic strain are found, although more red zones are detected with the CDP model.



[Figure 125]

Maximum principal plastic strain results with MC model: (a) Front view of the Aqueduct de los Cinco Ojos, (b) Back view of the Aqueduct de los Cinco Ojos.

3.2.4. CONCLUSION

Analyzing the results shown from Figure 122 to Figure 125, it is evident that the concentration of maximum plastic strain is observed at the bottom of the aqueduct and on the front face of the aqueduct, where it is greatly more pronounced than on the back face.

It appears that the concentration of maximum (positive) plastic strains occurs at the base of the aqueduct (along the boundaries) and in correspondence of the front face of the structure, where such a kind of “tensile induced deformations” are greatly more pronounced than those detected on the back face. High positive values of principal plastic strain denote the effect of tensile stresses above the strength limits, producing detachment failure. This detachment is typically associated to in-plane material disarrangements or to in-plane boundary condition adjustments and is characterized by visible separations between adjacent parts -passing through the masonry blocks or along the joints- which are essentially uniform across the thickness, previously described in section 1.1.1.a.

In the present case, this kind of smeared fracture pattern at the base and at the front surface of the structure, with a non-uniform through thickness pattern which dies out going from front to back, indicates out-of-plane failure of the structure in the front to back direction.

This out-of-plane rocking of the aqueduct structure is the most probable mechanism under the severe seismic action which was applied out-of-plane.

To assess whether it is solely a local detachment failure mechanism, which according to section 1.1.1.a does not generally raise concern, or it is rather a global out-of-plane rocking mechanism, which could lead to partial or global collapse, an interpretation of the structure’s status through and specifically after the effect of the earthquake is needed. This will help determine whether the aqueduct is likely to maintain its structural integrity or not. To this purpose, the time-histories of the Kinetic Energy (KE) and the Total Energy (TE) are analysed for both MC and CDP models.

First of all, it should be noted that the three different CDP models (with different tension/compression strength ratio R_f) led to the same kinetic

energy and total energy graphs, as can be inferred from the diagrams depicted respectively in Figure 126 and Figure 128. In the same diagrams also the graphs of KE and TE obtained for the MC model are given.

Figure 126a shows that the KE exhibits a significant rise at the beginning of the earthquake, followed by a gradual decrease to a relatively stable plateau from the third to the fifth second of seismic activity, for both MC and CDP models.

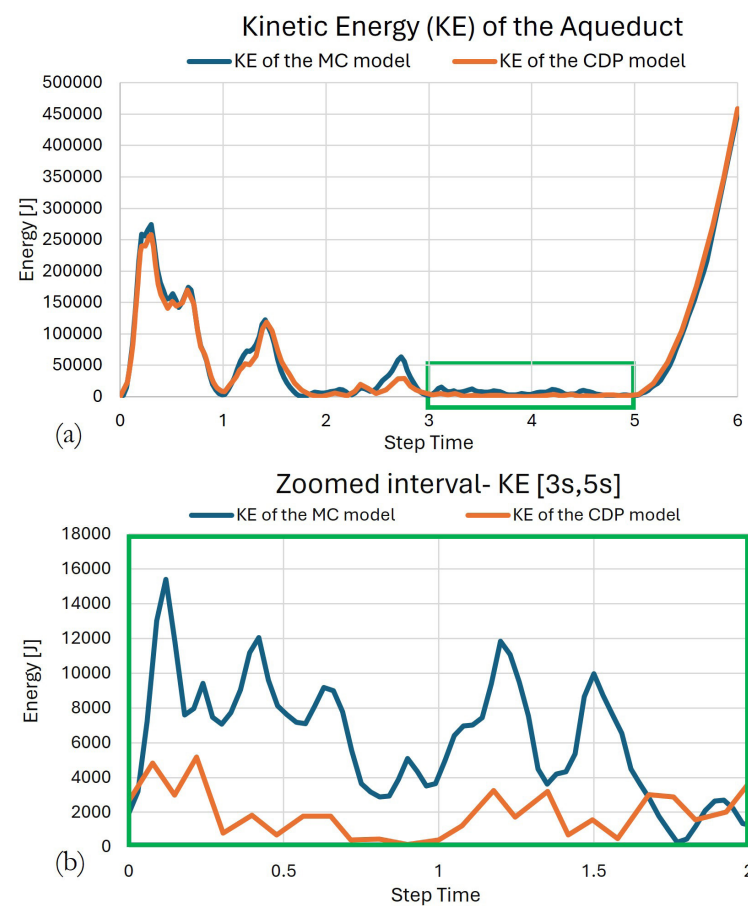
This kind of response is natural under the effect of an earthquake which has a pronounced (impulse type) peak in the first few seconds, followed by a kind of harmonic oscillatory peaceful phase which is prone to produce a stabilizing effect on the motion.

Upon a closer inspection of Figure 126b, it is found that the structure vibrates with diminishing intensity with the decrease in the earthquake’s energy. Hence, during the interval $3 \text{ s} < t < 5 \text{ s}$ following the initial dynamic response to the earthquake is drastically reduced, this reduction in the structure's movement creates the impression of a state of equilibrium.

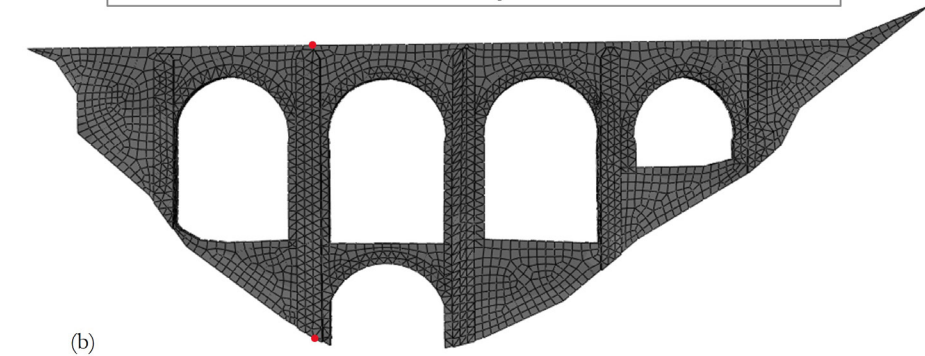
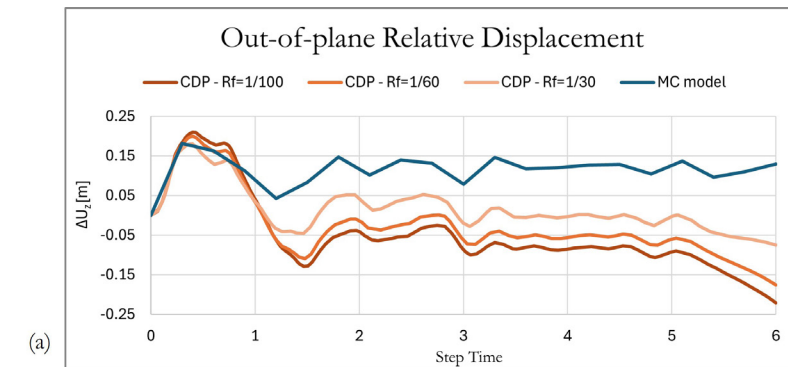
This kind of failure deferral is interrupted after the earthquake subsides (say after $t = 5\text{s}$) and the structure is left solely under its self-weight. As it is shown in Figure 126a, a sudden jump in the KE occurs in this phase. This circumstance suggests that as the earthquake ended and the structure remained under its own weight, it did not find an equilibrium state, as signalled by the spike in KE observed at the end.

This loss of equilibrium and subsequent out-of-plane movement are further emphasized in Figure 127a, showcasing the time history of relative out-of-plane displacement between nodes highlighted in Figure 127b. During the interval $3 \text{ s} < t < 5 \text{ s}$, all four finite element models exhibit substantial relative displacement at the seismic peak, reaching 20 cm for CDP models and 18 cm for the MC model. The residual relative displacement for $t > 5 \text{ s}$ is neither constant nor returns to zero, indicating the absence of equilibrium under self-weight conditions.

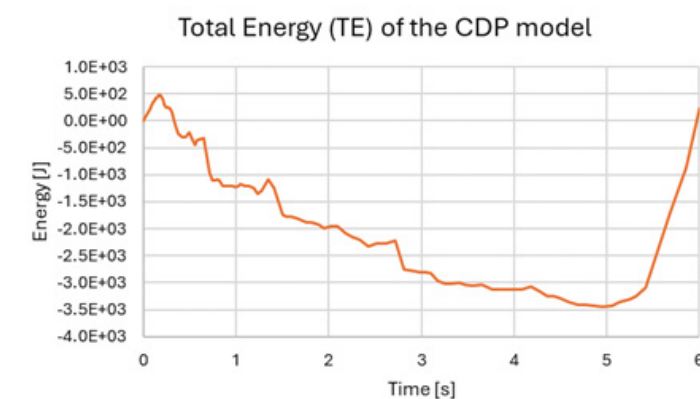
Additionally, in Figure 128 and Figure 129, it can be seen that during the same interval of time ($t > 5$ s), the total energy (TE) of both CDP and MC models does not become constant, which would be expected under static loads and stable conditions. Ideally, the total energy of a system is conserved if there is a state of mechanical balance. Continuous changes in total energy can signify energy dissipation, work done by extra external forces, or shifts in system behaviour, all of which are critical considerations in maintaining structural integrity and equilibrium. Hence, the continuous change in the TE energy graphs for both models, may indicate significant structural changes such as partial or total collapse.



[Figure 126]
Kinetic Energy graphs of the MC and CDP models of the structure: (a) KE over the complete step time, (b) lower scale evidencing the energy between 3 to 5 seconds of the earthquake.

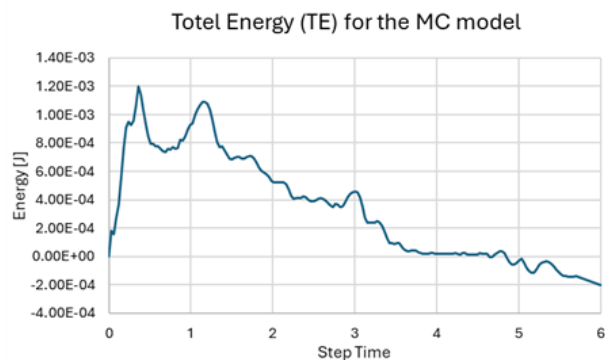


[Figure 127]
Relative out-of-plane displacement: (a) depicts the time histories for the four finite element models, and (b) highlights the bottom and lower nodes (shown in red) associated with the relative displacements.



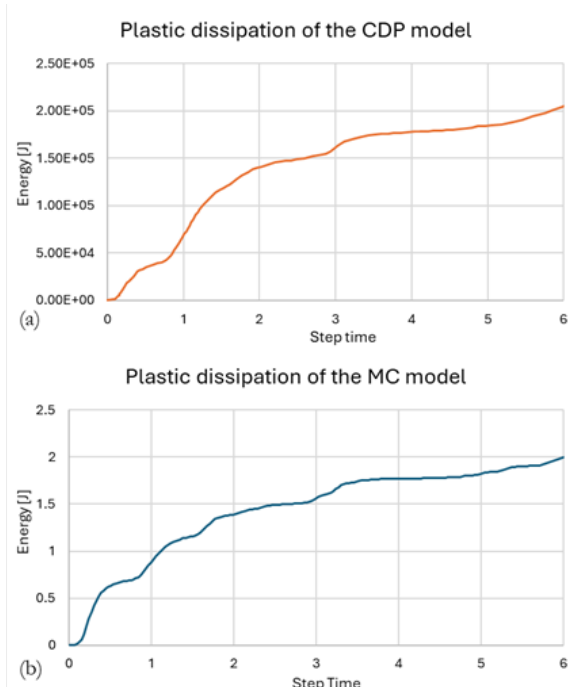
[Figure 128]
Total Energy graph of the CDP models of the structure.

[Figure 129]
Total Energy graph of the MC model of
the structure.



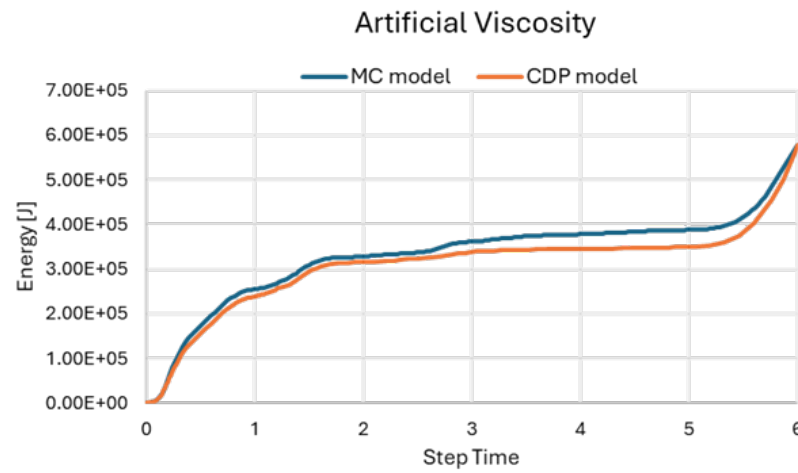
However, it is important to highlight the difference in magnitude of the TE between the CDP and MC models. This difference stems from the ability of CDP to account for phenomena such as strain softening, damage evolution, and fracture energy dissipation, which are fundamental in materials like brick, mortar, stone, or concrete. Softening involves a gradual decrease in mechanical resistance under continuous deformation, leading to progressive internal crack growth and eventual failure. For a detailed mathematical framework on how CDP incorporates these phenomena, the reader is referred to sub-section 2.2.1.2. “CDP methodology”. This complexity of CDP results in a significantly higher total energy requirement and dissipation compared to the simpler MC model. Any cohesionless MC model under the associative flow rule does not exhibit plastic dissipation, the reader is referred to **Part A**, sub-section 2. In fact, when comparing the plastic dissipation energy between these two models in Figure 130, the magnitude of this difference becomes apparent.

[Figure 130]
Plastic dissipation of the Aqueduct de
los Cinco Ojos: (a) plastic dissipation
graph of the MC model and (b) plastic
dissipation graph of CDP model.



We have to recall that the dynamical analyses performed with ABAQUS, in order to stabilize the results and reduce the computation time, apply an added damping obtained by considering an artificial viscosity. The presence of a consistent level of such a dissipated energy is then a signal of the structure being out of equilibrium. Additionally, the continuous increase observed in Figure 131 of such an energy at $t > 5$ s for both models further suggest a total loss of equilibrium and potential failure.

[Figure 131]
Artificial Viscosity considered in
ABAQUS for stabilization for the MC
and CDP models.



Such energy behaviours, rapid increases in KE, the TE graphs experience continuous change, the presence of artificial viscosity in addition to the maximum PE concentration discussed previously, highlight potential collapse scenarios. It can be concluded that the structure, under the worst-case scenario we adopted, will exhibit a collapse scenario due to both detachment failure and out-of-plane rocking.

PART B

DYNAMIC ANALYSIS OF A HISTORIC
LEBANESE HAMMAM USING CDP MODEL

1. BATHS OF
THE LEVANT

1.1. Historical Overview

The process of amalgamation between Eastern and Western traditions, though initiated earlier, notably accelerated during two pivotal epochs corresponding to the dissolution of the Levant border separating the East and West. These significant junctures coincide with the reigns of Alexander the Great and the Umayyads, during which the convergence of cultural influences became notably pronounced(Sibley, 2006).

Numerous Islamic dynasties emerged subsequent to the advent of Islam. Each of these empires exhibited distinctive characteristics in their art, architecture, and construction techniques. These architectural distinctions were influenced by factors such as climate conditions, the availability of construction materials, and the prevailing regional cultures. Consequently, the architectural marvels of these dynasties served as poignant reflections of the respective epochs in which they were constructed, encapsulating the prevailing religious faith of their respective regions (Creswell & Allan, 1958).

The emergence of Islamic architecture was significantly shaped by the assimilation of pre-Islamic ancient civilizations, beginning with the transition from the late naturalistic tradition, which was prevalent in the art of the eastern Mediterranean shores. This evolution extended to incorporate the refined artistic contributions of the Byzantines and Sassanids (Hillenbrand, 2000). In light of the absence of a native architectural tradition amenable to the requirements of a significant empire, the Arab civilization chose to assimilate the architectural methodologies of the vanquished Sassanian and Byzantine empires. Given their administrative centre in Syria, the Byzantine architectural influence held a more pronounced sway, although the incorporation of Sassanian elements steadily gained prominence. In numerous instances, Byzantine or even Roman architectural structures were directly appropriated with minimal or no alterations. Nonetheless, these conquests did yield innovations, both in the realm of architectural typology and in the elevated

significance accorded to decorative aspects (Petersen, 1996). It is notable that artists who embraced Islam as their faith, along with skilled artisans from the region, pioneered innovative techniques and effectively translated Islamic philosophies into artistic expressions. This synthesis was a transformative process, imbuing the architectural heritage with a distinctly Islamic character that drew upon the rich heritage of previous civilizations (Hillenbrand, 1994).

Historians have identified the eighth century during the Umayyad era as the initial phase of the development of Islamic architecture. The Islamic religion's customs and rituals required distinct spatial requirements hence Mosques, desert castles, and Hammams (bathhouses) are regarded as the earliest architectural manifestations within the historical trajectory of Islamic architecture (Petersen, 1996).

Dating back to the Hellenistic era, during the Romans and Byzantine times, baths existed and flourished. The Roman baths were massive structures, generally occupying the centre of a city, they masonry structures with vaults and domes. The Romans were not satisfied with only a single bath of cold or warm water. They did however go through a series of baths, where vapor and water were its main agents. Baths were built around three principal rooms: frigidarium, tepidarium and caldarium. The exact order of the course followed is not clear. There is no evidence of a general practice but rather it all depended on the individual’s whim to either take a cold or a warm bath. First, In the apodyterium, bathers were expected to undress as it is not permitted to enter unless naked. Bathers enter to the frigidarium which contained the cold bath. Those who desired to go through the process of warm bathing and sudation, entered into the tepidarium. The later did not contain water, it was heated with warm air of a tepid temperature, in order to prepare the bather’s body for the high heat of the warm baths and the vapor in the caldarium; and, upon exit, to obviate the danger of an abrupt transition to the open air (Smith, 1842).

With the conquest of Muslim Arabs of the levant, bathing habits continued in the region. The Umayyads conquered Bilad al-Sham in the 7th century with Mu’awiya as governor. There is Archaeological evidence in Tabariyyah and its region, supporting the creation of the statehood infrastructure for the Umayyads. Monumental architectural construction and development took place in Greater Syria including mosques, palaces, baths and fortresses. Mu’awiya established an Umayyad base of operations between Tiberias and the road to Damascus in the region of the Byzantine capital of Palaestina Secunda. (Muslims conquered Roman-Byzantine Tiberias (Tabariyya) the capital of

Palaestina Secunda in 635 and made it the capital of Jund al-Urdunn). His pattern for construction in Bilad al -Sham merges Syrian cultural precedents with traditions from Arabia. There were no grand monuments portraying Islam, rather construction over previously existing ruins in the site of the Roman-Byzantine precedents such as baths, infrastructure and others were repurposed in an Islamic frame (St. Laurent, 2020).

The history of the Islamic baths evolved and developed up until the 20th century. Their history witnessed a rapid development and adaptation to the needs of the religion by retaining the habits that were appropriate to the needs of Islam and by disregarding others that failed to do so. Although pools existed in the baths of Greater Syria. In most of the bath’s feature like the cold plunge pool disappeared to be replaced by running water to avoid their pollution by an unwashed and impure person. In addition, washing by pouring water over the bather replaced Hot pools at the end of the bathing ritual (Sibley, 2007).

In contrast to Roman bathhouses, Islamic bathhouses are seldom freestanding structures; they are typically integrated within the surrounding urban context, thereby limiting the extent of their exterior walls to the front entrance façade. As a fundamental component of the urban landscape in Islamic cities, baths are frequently found as part of a triad alongside the mosque and the suq (marketplace). Their proximity to mosques is attributed to their role in facilitating the essential ablution process which is a mandatory cleansing ritual to ensure purity before prayer. Additionally, these bathhouses serve a vital social function by providing a communal gathering space for both genders within society. It is noteworthy that Islamic bathhouses do not maintain separate sections for males and females; rather, they operate on distinct schedules to accommodate both.

1.2. Evolution of Architectural Layout and Features

An architectural evolution between the 12th and 18th centuries of the historic public baths took place. The functional layout of the Islamic hammam was established during the Umayyad dynasty were early examples can be found in the Syrian desert bathhouses and the majority are in the early Islamic districts of Filastin and Urdun (Hamilton, 1978) . The most known early example of an Islamic bath house is located in the Umayyad desert palace “Qusayr Amra” in today’s northeastern Jordanian desert (Creswell & Allan, 1958) .

Damascus, the oldest city in the world, was the center of power and the capital of the first Islamic dynasty, the Umayyad Dynasty, so the evolution of its public bathhouses illustrates early developments of the hammâm typology in general in the area. It was a chosen site for a number of public baths throughout many centuries and it provides a variety of examples from different historical periods (Kayyal, 1964).

According to the last survey conducted in 2004 by M. Sibley, It was apparent that 14 out of 20 existing hammams were still operating due to the fact that some were victims of modernization, change of function or natural decay. Efforts to restore and preserve this heritage were due pre-war period. Of those that still existed they date back to the period between the 12th and 14th centuries, with the vast majority predating the Ottoman period (Sibley, 2007). This indicates the high heritage value of these buildings and the necessity to save them from possible future closure and decay. During the war and post-war period, the baths in Damascus revived as costumer number increased due to the number of people that fled North-Syria and the power outages.

However, the evolutions of the Hammams of DamsCUS shed a light about the transition that occurred between 12th and 18th century. Based on surveys, Ecochard and Le Coeur (Écochard & Le Coeur, 1942, 1943) produced a comprehensive record of the baths of Damascus, carried out detailed drawings and careful analysis of the layout of 29 hammams, 12 before 14th century, five from the 14th century three from the 15th and 5 from the 16th and 17th century and four from the 18th and 19th century. The evolution of the internal organization from 12th to 18th century is done by comparing 6 hammams of different periods: Al-bzuriye (1180), Ez-Zen (1384), Tayruzi (1444), Malike (16th century second half), Fethi (1745) and Mezze (1933).

Importance and proportions varied throughout the periods, in addition to the spatial organization. Pre 14th century two organization existed: a sequential linear organization where bathing space are organized along an axis sequentially, and a central organization where the spaces are organized around a main octagonal room (Sibley & Jackson, 2012). This simultaneous organization remained only until the 14th century, this coexistence of both linear and central organizations within the same city between the 12th and 14th centuries can be explained by the existence of two sources of influences in antiquity. The central organization is reminiscent of Byzantine architecture, whereas the sequential linear organization is reminiscent of the early Umayyad baths (Sibley, 2007; Sibley & Jackson, 2012).

In addition, the main spaces of the hammam are usually covered in barrel vaults or domes. Traditionally, the domes were typically supported by pendentives, bridging the space between the corners of the room's walls and the circular dome above with a triangular portion of a sphere. However, squinches with arches at each corner, designed to support the dome, became more prevalent during the Mamluk period and gradually waned during the Ottoman period.

Towards the end of the Ottoman era, a noticeable shift occurred in the construction of bathing rooms. These spaces departed from the previous square units capped with circular domes, instead adopting rectangular rooms in which the dome arose directly from the room's walls, eliminating the need for pendentives. The transition from the walls to the curved surface of the dome occurred within the dome structure itself, above the room's walls (Kayyal, 1964).

A distinctive architectural hallmark of hammams is the presence of perforations in the domes and vaults above their bathing areas, adorned Figure 132. This design evolution marks a significant departure from the Byzantine era, during which the bathing rooms were illuminated by suspended lanterns positioned at the apex of the dome and windows located at the lower periphery of the dome. In contrast, Islamic bathhouses are renowned for their employment of numerous circular or star-shaped apertures scattered across the entire surface of the dome, each sealed with glass covers Figure 132. These openings, known locally as "qamariyyat", are constructed using pottery tubes integrated into the dome structure, and they are meticulously arranged in various ornate configurations. The “qamariyyat” allow daylight beams cutting through the thick steam, creating suggestive atmospheres at different times of the day, while excess steam escape from them. This architectural element can be traced back as early as the 7th century, as exemplified by Qusayr Amra in Jordan in

Figure 133. Other examples among many, can be found in the hammams of Bab-Al-Bahr, Cairo, Egypt (Sibley and Jackson, 2012); Fethi, Damasco, Siria (Sibley, 2006); Elbasan, Albania (Castiglia & Bevilacqua, 2008).

As for the heating system’s evolution, up till the 10th century, baths had a hypocaust system of heating with heating pipes integrated in the walls. All of the heated rooms are organized along the same axis, a system of heating inherited from Roman and Byzantine times. From the 10th to the 18th century, smoke ducts travelling under the floor of the washing rooms replaced the hypocaust system and central organization of baths dominated. After the 18th century, the internal organization reverted back to a linear organization along an axis with a loss of complex architectural forms (Sibley, 2007).



[Figure 132]
Example of the “qamariyyat” present in some of Lebanon’s historic monuments.



[Figure 133]
Tracing the “qamariyyat” back to Qusayr Amra in Jordan.

2. BEIT EL-DIN: HISTORY, CONSTRUCTION AND DYNAMIC STABILITY

2.1. Data Collection

The case study is the Beit El-Din palace in Beit El-Din, Chouf region, Lebanon (see Figure 134) the town’s name translates to "House of Faith". The boundaries of the region expand from Damour to the mountain peaks of Niha to the east. It is in the Chouf region where the roots of regional Lebanese architecture can be found, hence it presents a fertile environment for research. The abundancy of white limestone quarries from earlier geological formations in the area was a contributing factor in the common use of limestone in the are as well as the presence of the best masonry craftsmanship.



[Figure 134]
Map of Lebanon.

The period from the 17th century till the middle of the 19th century is called the “Emirate”. Most historians characterize it as being the period of conception of the Lebanese identity, which manifested in the 20th century by the French establishment of the nation-state of Lebanon (Salibi, 1965). Nominal independence was given to the territory under the emirate, that covered mainly the Mount Lebanon chain, by the Ottoman Empire.

In the first half of the 19th century Emir Bashir II of the Shihab Dynasty, ruled Mount Lebanon Emirates, started building the Beit Ed-Din palace in 1788 at the site of the Druze hermitage, 850 m above sea level, naming it after the town's name (Figure 135). Emir Bashir ordered the construction of this palace after he prevailed in battles against other competing lords of Mount Lebanon as means to impress both friends and foes (Salibi, 1965). The palace is said to be designed with the help of Italian architects and executed by masons from Mount Lebanon and Damascus. Yet no evidence of any radical deviation from regional building traditions has manifested in this palace, it remained “oriental” in style (Haddad, 2007). Many travellers, like John Murray, were aware of these aspects that have not gone unnoticed (Haddad, 2007; Murray, 1875).

It is worth noting that the only proof of the involvement of Italian architects is the word of mouth as there is no official historical reference nor publication that asserts this as fact. This claim is since Emir Bashir had relations with foreign dignitaries and rulers from Italy and France as his predecessor Fakhreddine II two centuries before him (Haddad, 2007). In this territory Emir Fakhreddin Maan II had direct relations with the court of the Medicis. He requested after his return from Florence to Mount Lebanon, the Medicis' assistance in the art of modern fortifications. Around 1631 Tuscan experts arrived to Lebanon who included an architect by the name of Francesco Cioli and a master builder Francesco Fagni (Tritton, 1938). Some of the works attributed to Fagni are the supervision of infrastructural and urban works like construction of bridges at Nahr-el-Kalb, Beirut, Sidon and in Deir el Kamar, the capital of the emirate. On the other hand, no specific buildings were attributed to Cioli. Presumably the latter have been in charge of the construction of the palace of Beirut, and the one in Sidon (Tritton, 1938).



[Figure 135]
Overview of the Beit El-Din monument.

The palace was used by the Ottomans as a government building when Emir Bashir II was sent into exile after 1840. Its role was preserved during the French Mandate, and it served as a local administrative office at the time.

In 1934, it was declared a national monument by an official decree during the Mandate, (Nb 7242 dated 1930), specifying the use as an archaeological museum. An additional law, ref 208 dated 5/6/1940, extend the protection to the Palace of Beit El-Din again. In the same year, Bechara el Khoury, the first Lebanese President, declared it the official president's summer residence (Maher et al., 1984).

One of the adversities faced in this research is the scarcity of published material on the subject a problem faced by many colleagues before me. I feel compelled and challenged to contribute to the literature about structural integrity, conservation, and history of the edifices of Lebanon. Since only a concise description of monuments similar to Beit El-Din can be found in Friedrich Ragette’s book “Architecture in Lebanon” (Seabrook, 1926).

Since 1999, the Palace is under the authority of the Ministry of Culture and managed by the Directorate of Antiquities (DGA). Parts of the palace are today open to the public while the rest remains inaccessible as a presidential security protocol. During my inspection of their archives to find information useful for my research, although a lot of documents have been inspected, information about the architecture, the material constitution, the reconstruction, and restoration process were scarce. This can be attributed to the absence of bureaucracies during war time and to the lack of documentation of detailed interventions.

As it is known, Lebanon is a Mediterranean country that has witnessed the perils of civil war from 1975 till 1990. During this period Beit El-Din Palace was heavily damaged. Many significant structures in Chouf area were shattered and bombed. Individuals took the initiative of the reconstruction process and restoration, which has proven to be more effective contrary to urban Beirut. And the main reason of the individuals’ initiative which boosted the rehabilitation/restoration process is the benefits of fewer restrictions at the time due to the absence of bureaucracies (even with little financial support) and, of course, the absence of written conservation legislation (Maher et al., 1984).

After 1984, when fighting in the area receded, and as an example of individual initiative, the Beit El-Din palace was managed by local ruler, Minister Walid Jumblat who ordered its restoration and initiated the actual mosaic museum with a collection from Jiyeh.



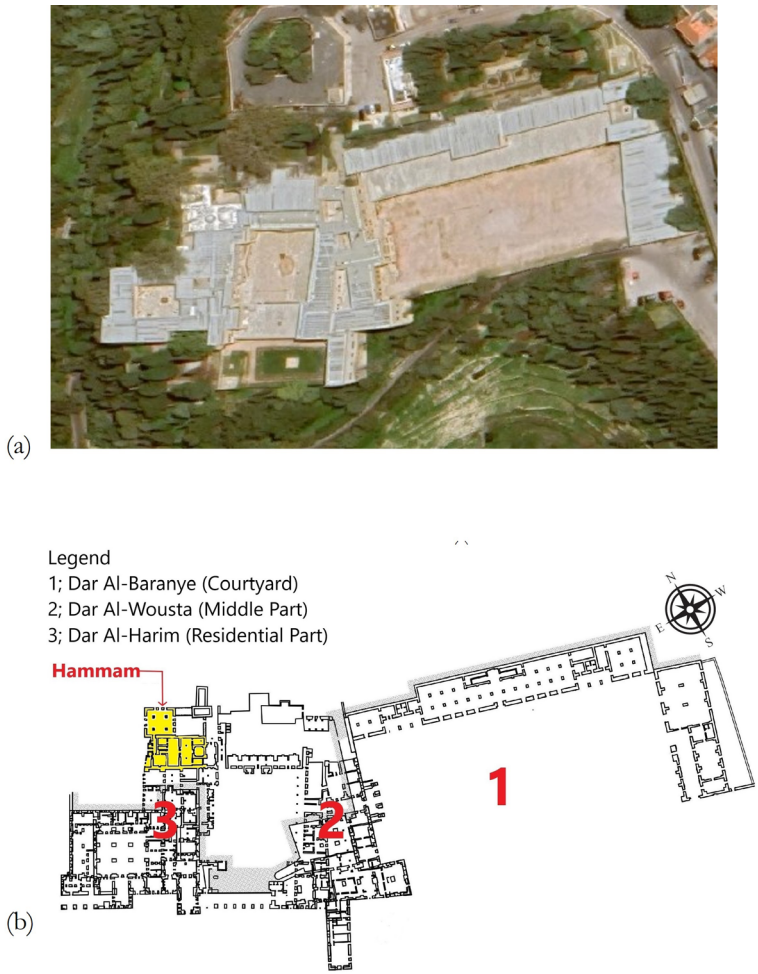
[Figure 136]
The mosaic museum of Beit El-Din.

2.2. Geometry and Construction

The Chouf has a distinctive landscape characterized by terraced topography. An important architectural component is the integration of the complex typology to fit well into the terrain of the authentic landscape pattern. An aerial view of the Beit-El-Din palace in Lebanon, is provided in Figure 137a, respectively. A plan of the palace is displayed in Figure 137b.

Beit El-Din encompasses multiple arcaded galleries, outer and inner courtyards with beauteous marble and alabaster fountains in their centre (Murray, 1875). As can be seen in Figure 135 the palace’s main entrance welcomes you to a very large courtyard, a grand open 'maidan, (Dar-Al-Baranye). Above the maidan was the antiquarian museum in an exterior open gallery which displayed the Emir's personal artifacts and memorabilia collection. Then there is an ascent by means of a staircase leading to the middle public reception wing with its aesthetic courtyard (Dar-Al-Wousta) and the residential part (Dar-Al-Harim) of the building complex (Figure 137b).

Dar Al-Harim, at the far end of the palace, is divided into an upper and a lower part. The Upper Harim consists of private apartments where Emir Bashir, his entourage, and his women (Harim) resided. Currently, such part is the summer residence of the Lebanese President. The Damascus style furnished apartments were light with high gold and glitter arabesque ceilings, tessellated pavements of marble and inlaid walls. The third more private inner courtyard is reached by an open Iwan arched way situated in the Lower Harim where the thermal baths (Hammam) are in the northern edge of Dar Al-Harim.



[Figure 137]
The Beit El-Din palace: (a) Aerial view and (b) Plan view of the palace.

In addition to the perfect proportions of the volumetric spaces the 'ablaq' masonry's beauty is noteworthy in some sections of Beit El-Din as can be seen in Figure 138.

Ablaq masonry originated from southern Syria, during the Mamluk period during the 14th and 15th century. It consists of combining two kinds of stones of light and darker colours. Mainly limestone and basalt or dolomite are used in alteration for their abundancy in the area for decorative and structural merits for the basalt and dolomite stone are said to be effective in resisting of earthquake (Cytryn-Silverman, 2010). On the other hand, in other sections of Beit El-Din, the stones used are only white limestone from the quarries of Chouf (Figure 139).



[Figure 138]
Pictures showing the Ablaq masonry in
some sections of Beit El-Din.



[Figure 139]
Example of stones used in the Beit
El-Din Palace.

As has been mentioned, many historical buildings typical of the Arabic culture, such as mosques and hammams, were built during the Ottoman Empire (1445–1912). They represent the heritage product of cultural traditions, construction techniques, and aesthetic tastes of a historical era. Most of these monuments are masonry structures standing in earthquake-prone countries, those bordering the Mediterranean Sea, which makes their seismic assessment a concern to preserve historical architectural heritage for future generations.

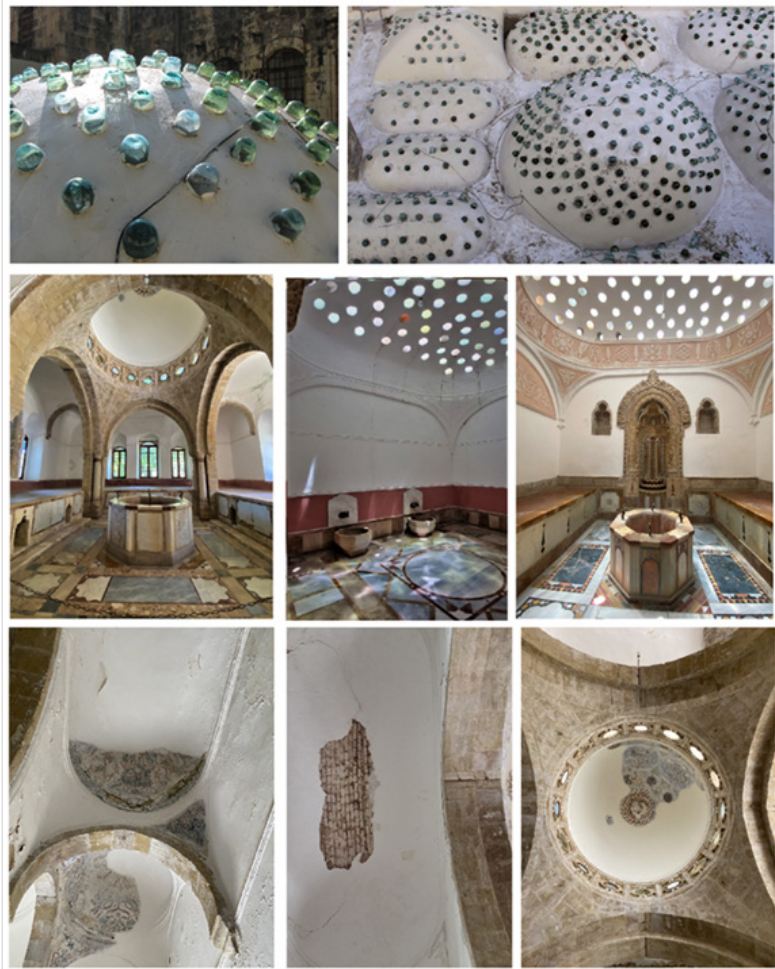
Of particular interest to us within this complex is the hammam, which is a masonry structure characterized by domes and barrel vaults pierced by holes. The considerable number and the different shapes of pierced vaults (16 over a total of 25), the attractive decorative patterns formed by the holes and the light effects shaped by glass semi-globes over them (formerly polychrome but now monochrome only), make the Beit-El-Din Hammam the most superb Hammam of Lebanon and one of the most attractive of the Arabic world (see Figure 140).

The study of the structural behaviour of historical monuments involves both a preliminary survey on their complex geometry and the choice of the more suitable method for analysis out of those described in the state-of-the-art section to achieve the planned goal.

Suitably modelling its structural behaviour and assessing its performance under exceptional dynamic actions is a challenging goal.

In this study, the macro-modelling approach is adopted, which involves using continuous models where masonry components are represented as a homogeneous material with averaged properties. This strategy, widely used for modelling large and complex monuments, is particularly common in analysing full-scale 3D unreinforced masonry buildings as previously described in the “State of the Art” section.

Since Historical masonry monuments are in fact massive structures, typically able to carry loads in compression but vulnerable to horizontal loads, due to their very low resistance to tensile actions. Implementing an appropriate numerical model of such structures, and identifying it, is a pivotal prerequisite for a reliable earthquake resistant assessment (Pierdicca et al., 2016; M. C. Porcu et al., 2021).



[Figure 140]
The holed domes and vaults of the
Hammam of Beit El-Din.

Based on a survey of archival data and in-situ measurements, a detailed three dimensional geometrical and numerical model of the Beit-El-Din Hamman was built for the present investigation. The holes in the vaults were reproduced in the numerical model, while a twin model (without holes) was also considered to assess the influence of the holes on the stress pattern and damage distribution in the vaults.

An isotropic linear behaviour of the homogenized masonry material was first considered to identify the numerical model. In-situ modal tests under environmental excitations were carried out to find the first natural frequencies of the structure. An iterative procedure, based on comparing numerical and experimental eigenfrequencies, was then performed to find the best-fitting values of the mass density and the elastic modulus.

Subsequently, THNLA is considered to assess the performance of the structure under real earthquakes scaled to the Lebanese design target spectrum. Pulse-like and multi-peak records are considered to assess the susceptibility of the structure to different kinds of ground motions. The damage pattern under the two considered ground motions was finally investigated.

The study aims to give some preliminary indications on the seismic vulnerability of the Beit-El-Din Hammam, understanding the structural response of this monument. For this analysis we adopt the CDP model and not the MC, previous analyses conducted on the Aqueduct de los Cinco Ojos and the conclusions drawn, considering the computational time factor.

2.3. Modelling the Hammam of Beit El-Din

An initial investigation was conducted to gather comprehensive data related to the historical background, geometric attributes, and construction materials of the monument under consideration. Subsequently, a meticulous geometrical model was created in AutoCAD (AutoCAD, 2022), wherein buildings B1 and B2 were initially treated as separate entities for the sake of simplicity. Focusing on building B1, the internal structure, characterized by a centrally positioned main dome supported by four columns, was prioritized in the modelling process due to its inherent structural self-sufficiency. Following its integration with the remaining portion of the building. A brief description of the modelling of multiple essential elements is presented in the following.

2.3.1. THE ARCHES OF THE MAIN DOME

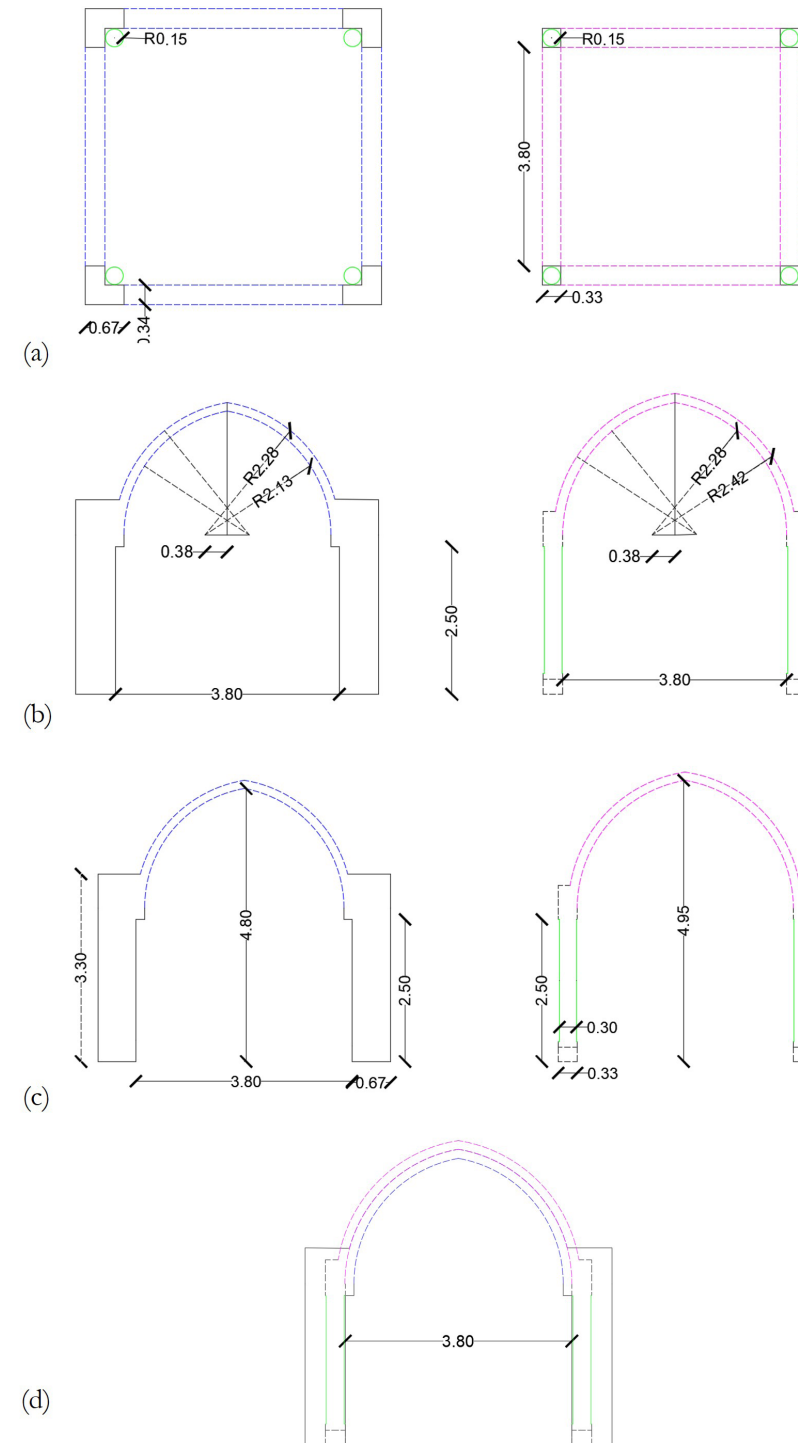
The Arch of the B1 domes is pointed arches as can be seen in Figure 141, which were typical of the Islamic architecture in the 19th century (insert reference). On site measurements were retrieved to draw the model. A plan was drawn in Figure 142a.

The exterior arches of 15 cm thickness lie on a column of 2.6 m height. The light of the arch is 3.8 m, knowing the height as well of the arch, which is equal to 4.8 m, it was deduced that the centre of each part of the arch lies on the 1/10 of the light from the centre of the horizontal line Figure 142b. Thus, the curvature of each side of the pointed arch was drawn tangent to the vertical on the springing (Willis, 1842).

The interior arches over which lies the pendentive have a similar thickness of 15 cm, with a 4.1 m light and a 4.95m height. The arches are as well tangent to the vertical on the springing which lie on a 2.6 m circular column with a square base. The centre of their curvature is as well located at 1/10 of the span Figure 142b. Taking all the above measurements into account, a 2D model was produced as can be seen in Figure 142c and Figure 142d.



[Figure 141]
Interior view of the main dome of B1, note the pointed arches, the dome with circular openings at the base and the common square base for the rectangular and L-shaped columns.



[Figure 142]

Phases of the 2D modelling of the arches of the main dome: (a) Plan view. (b) Centre of the pointed arches, (c) The Exterior and interior arches with dimensions and (d) The overlapping of the 2D sketches.

The 2D model of the arches was then transformed into a 3D model by several steps of extruding and overlapping the structure. First the triangular columns with the exterior arches were extruded and dispositioned to form the square shape of the plan. Then the circular columns were extruded and placed over their common base with the rectangular columns. Afterwards the interior arches were extruded and placed above the circular columns, thus creating the entirety of the 3D structure (see Figure 143).



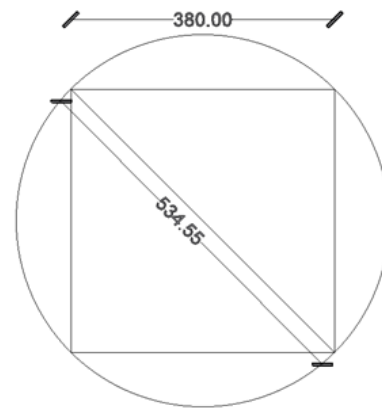
[Figure 143]

Phases of the 3D modelling of the arches of the main dome of B1.

2.3.2. THE PENDENTIVES

The procedure to create the pendentives lies at the beginning to specify the area covered by the dome which is the distance between the columns of the arches and to specify the height of the pendentives that is equal to the distance from the springing of the arches to the base of the dome.

In this specific case the area covered by the dome is a 3.8m*3.8m, a square shape with these dimensions is drawn in the 2D phase of the modelling followed by a circle with the diameter equal to the diagonal of the previously drawn square as can be seen in Figure 144.

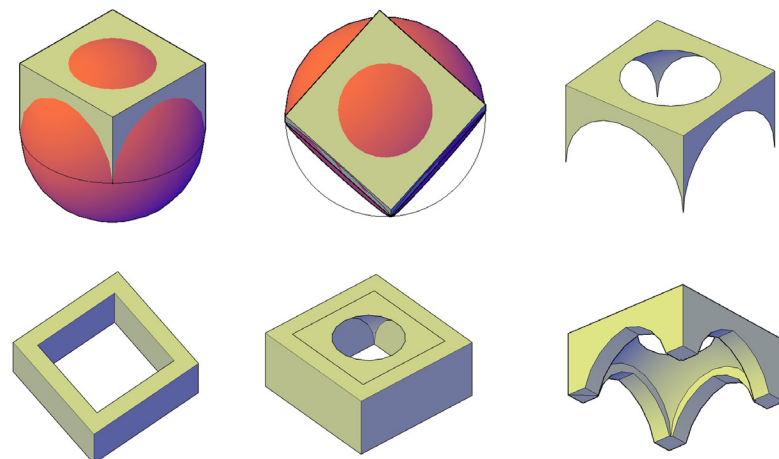


[Figure 144]
2D Phase of the modelling of the
pendentives.

According to a survey during waterproofing works on Beit El-Din, found in the archives of the Directorate General of Antiquities (DGA) in Lebanon, the height of the roof is equal to 5.65m thus in modelling the pendentives, the square should be extruded 2.35 m to cover the distance between the top of the arches' springing and the roof.

The circle is then transformed into a sphere. The latter is then sliced following the sides of the cube as can be seen in Figure 145, and then subtracted from the cube producing thus the interior face of the pendentives.

Another hollow cube is created with the larger sides equal to the distance between the exterior faces of the column and the smaller sides equal to the distance between the interior faces of the columns. Thus, the hollow cube and the interior face of the pensive are then merged into one solid as can be seen in Figure 145. Finally, the latter's sides are sliced according to the 2D shape of the arches, resulting eventually in the shape of the pensive.

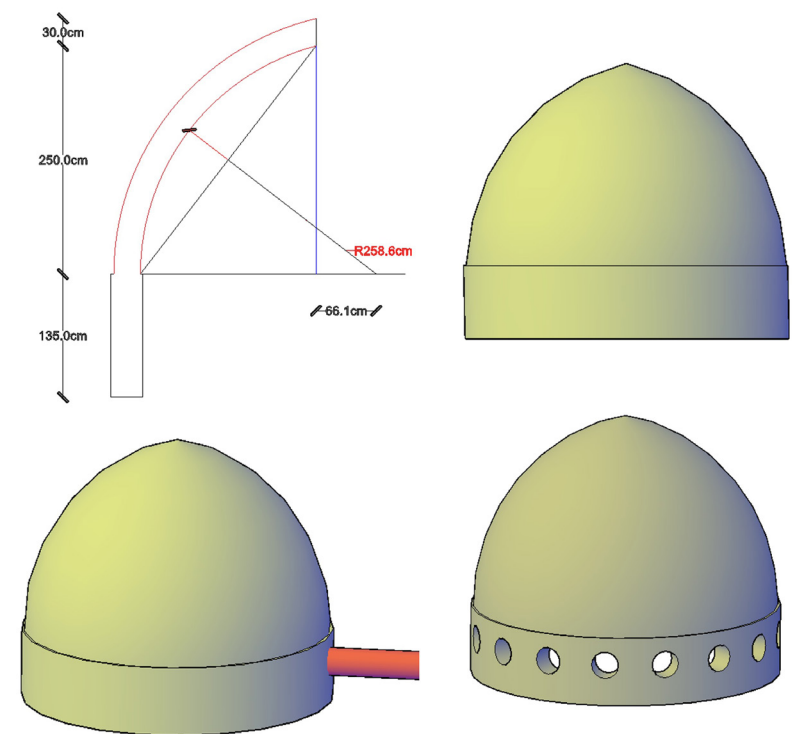


[Figure 145]
Phases of modelling the pensive of
the main dome in B1.

2.3.3. THE PENDENTIVES

The central dome, which is a pointed one, is standing at a height of 2.5 meters with a radius of 5.8 meters (which is 1/10 of the span). Beneath it rests the drum, standing at a height of 1.35 meters, adorned with 16 circular windows.

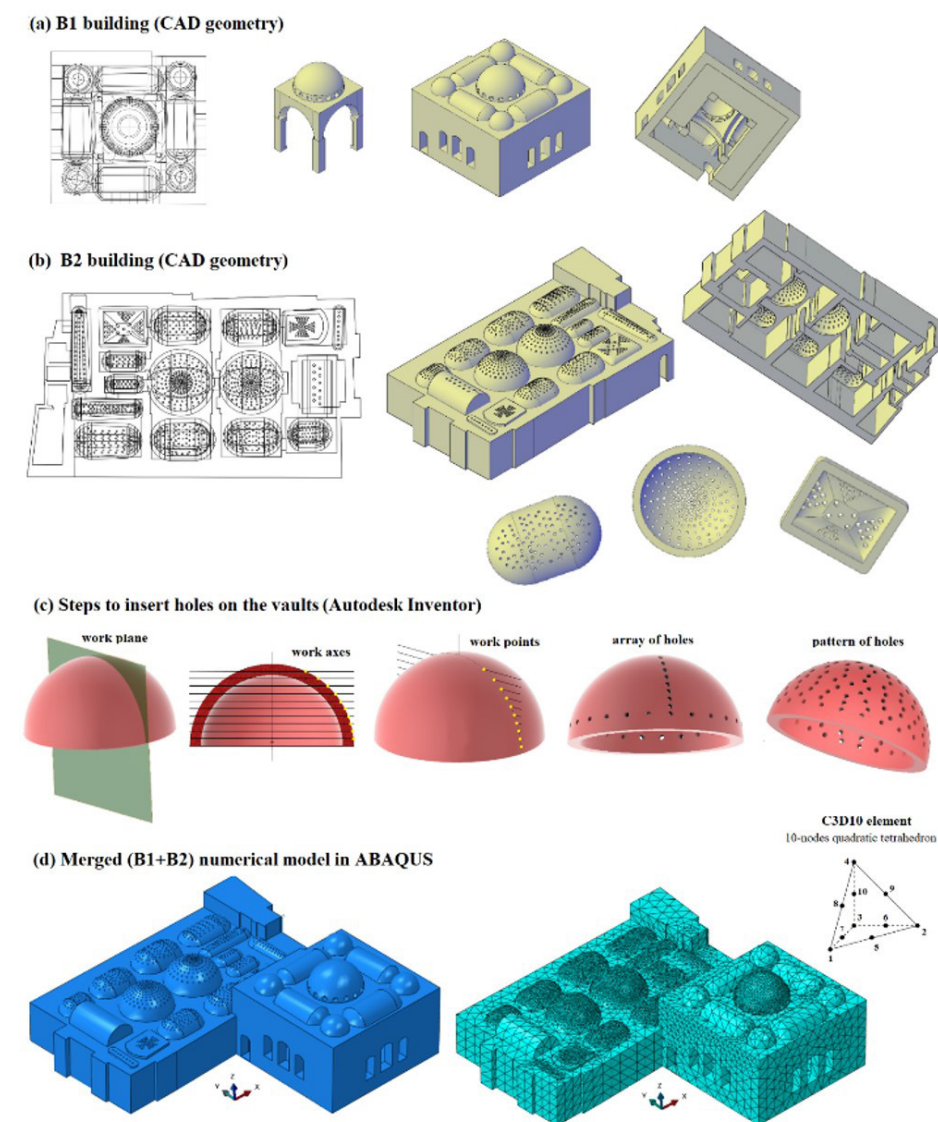
The process of creating a 3D model of the central dome involved tracing a section of the dome and the drum, then rotating it around the central axis of rotation as depicted in blue in Figure 146. Subsequently, a cylinder matching the desired diameter of the circular windows was generated and rotated at an angle of 22.5° with the blue axis as axis of rotation, gradually subtracting from the solid cylinder from the drum until all windows were formed accordingly.



[Figure 146]
The Process of modelling the central dome
and its drum with circular windows.

Two distinct models of B2 were generated. In the initial model of B2, no perforations in the vaults were considered. This model served as a basis for the comparative analyses detailed in a later section. Subsequently, a second model was developed, incorporating perforations of 12 cm diameter in alignment with their actual positions in the authentic structure. Figure 147 illustrates the final model of B2, featuring vaults with perforations.

Accurately modelling the holes on a curved surface posed challenges, necessitating a dedicated procedure facilitated by Autodesk Inventor 3DCAD software (Inventor, 2022). The procedure starts with the creation of specific work planes vertically intersecting the vaults. Subsequently, an array of horizontal lines, corresponding to the hole levels, functioned as work axes for intersecting the surface and establishing work points. Ultimately, a hole was located at each work point and arranged in a circular pattern around the vertical axis. Notably, only the essential work axes and corresponding work points required for replicating the authentic pattern of each vault were considered. Figure 147c shows the process of modelling perforations in one of the domes of B2 through Autodesk Inventor (Inventor, 2022).



[Figure 147]
Modelling the Beit-El-Din hammam.
(a-b) Geometrical 3D model of B1 and
B2 buildings in CAD environment; (c)
main steps to create the holes in one
of the domes of B2 with Autodesk
Inventor; (d) fully merged FEM 3D
model in ABAQUS.

The CAD geometrical representations of the B1 and B2 buildings were integrated into ABAQUS (Abaqus, 2022), and merged to form the numerical model of the hammam. This involved merging the geometry and mesh of all nodes with a tolerance of 1E-06. The resultant merged model of the hammam in ABAQUS is illustrated in Figure 147d.

Notably, B1 and B2 share a common wall, which was consolidated as a singular entity in the final comprehensive model. The mesh employed a 10-node quadratic tetrahedron C3D10 element, with a characteristic element length of 2E-01, incorporating localized refinements. Following a convergence analysis, the entire model comprised 258,181 elements.

The isolation assumption of the hammam from the remainder of the palace encouraged the exclusion of the building section along the South-East side of B2 (Dar Al-Harim portion numbered as 3 in Figure 137b) from the model. While this segment serves as a lateral constraint for the hammam, its omission may render the model more flexible than the authentic structure. To address this, potential inclusion of the building bordering B2 could be achieved by introducing springs in the model, necessitating dedicated studies for assigning suitable values to the stiffness constants. It is noteworthy, however, that the numerical model of historical masonry monuments often exhibits greater stiffness than the actual structure, particularly in the elastic range. This discrepancy arises from factors such as nonlinear effects, stiffness degradation, the presence of cracks, voids, defects, and ineffective constraints between orthogonal walls, as well as soil-structure interaction all challenging to accurately account for. Moreover, it is observed that the modal behaviour of the hammam is predominantly governed by the more flexible B1, with B2 playing a marginal role, especially in lower frequency modes. Consequently, the exclusion of the palace portion bordering B2 and its associated constraints is not anticipated to significantly impact the results of this study.

For modal identification, a homogeneous isotropic linear elastic material was initially employed. Subsequently, the CDP model was introduced for nonlinear dynamic analyses under actual earthquake conditions.

The analysis comprised a dynamic implicit step, accounting for the gravitational weight of the model in the 'loads' module. Regarding boundary conditions, the base was fixed in the Z-direction, while the remaining boundaries were defined as 'accelerations,' with input amplitudes for the considered accelerograms in the X and Y directions.

No consideration for soil-structure interaction was made in this study. All analyses were conducted in ABAQUS/Standard, leveraging an implicit solver to address the dynamic equilibrium between external forces and internal reactions through an incremental and iterative procedure at each solution step.

2.4. ENVIRONMENTAL MODAL TESTS

To facilitate the refinement of the numerical model of the Beit-El-Din Hammam, an ambient vibration test was conducted, aiming to capture and analyse the structure's response to ambient vibrations. This approach provides valuable insights into certain dynamic characteristics of the hammam, particularly when subjected to small deformations within the linear elastic range.

Ambient vibration tests offer a straightforward, rapid, cost-effective, and non-intrusive means of acquiring modal data for the examined structure (Kibboua et al., 2008). This methodological advantage aligns with the pursuit of comprehending the dynamic behaviour of the Beit-El-Din Hammam.

The ambient vibration test on the hammam buildings took place on February 10, 2022, following the necessary authorizations. Weather conditions during the test were characterized by instability, featuring light winds, intermittent rain, and an ambient temperature of approximately 4°C. The instrumentation employed comprised a Lennartz LE-3D-5 s seismometer (the sensor) linked to a CityShark-II™ recorder (Chatelain & Guillier, 2013). The sensor was strategically positioned and levelled at five distinct locations (labelled 001 to 005) on the roofs of buildings B1 (001, 002, 003) and B2 (004 and 005), seeking proximity to the roof centres. Images captured during the experimental campaign are presented in Figure 148 and the spatial arrangement of the measurement points is depicted in Figure 149.

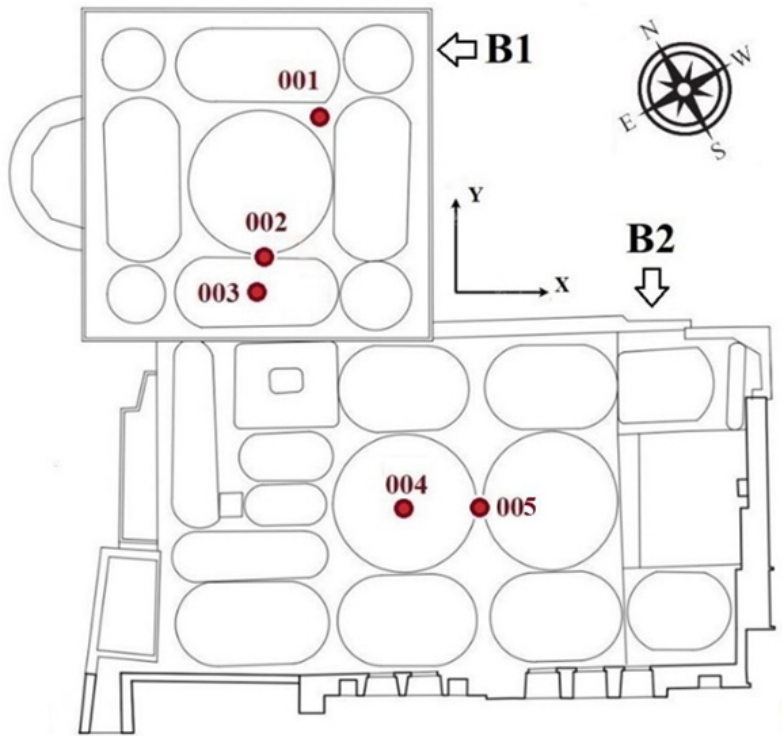
However, practical constraints, arising from the height and inaccessibility of the central dome, precluded direct access to the centre of B1. Despite these limitations, the chosen locations offered a representative sampling of the structure's response to ambient vibrations during the test campaign.



[Figure 148]
Environmental modal tests and sensor
installation on the roof.

Placing sensors on rooftops is ideal for discerning fundamental frequencies of a structure, as emphasized by (Salameh et al., 2016). This strategic placement capitalizes on the principle that the higher the sensor is situated, the greater the amplitude of the recorded signal. However, the application of this approach encountered challenges in the case of the Beit-El-Din Hammam due to the presence of numerous curved surfaces and considerations of safety and inaccessibility. These factors complicated the task of distributing sensors across a more extensive array of points on the rooftops.

The selection of sensor positions is crucial for experimental dynamic tests involving modal shapes (Loi et al., 2022), influencing the precision of mode shape identification. The number and location of sensors wield influence over the precision of mode shape identification, while having a lesser impact on the determined frequency values, as demonstrated by (Diaferio et al., 2022). Although the present study does not delve into the identification of mode shapes for the hammam, the placement of sensors at five distinct positions was sufficient for extracting the natural frequencies' values, essential for delineating the uncertain parameters of the numerical model.



[Figure 149]
Position of sensors on the
Hammam Roof.

For each designated sensor location, the response to ambient vibrations was recorded bidirectionally for a duration of 30 minutes at a sampling rate of 100 samples per second. In the case of measurement 001, recordings were obtained in the North and East directions, while for measurements 002 to 005, the sensor components were oriented to capture motion in the longitudinal and transverse directions of the structure. Signal processing was carried out using the GEOPSY open-source software (www.geopsy.org, 2020 version) (Wathelet et al., 2020) within the frequency range of 0.1–50 Hz. A processing procedure akin to that employed by Salameh et al. (2016) was applied, involving the calculation of smoothed Power Spectral Density (PSD) functions across multiple signal windows, and subsequent statistical analysis.

The automatic Anti-triggering algorithm Short-Time Average over Long-Time Average (STA/LTA) in GEOPSY (with settings of STA over 1 s, STA over 30 s, and lower and upper limits of STA/LTA ratio equal to 0.2 and 2.5, respectively) was selected to maintain stable signal windows for individual spectra, thereby eliminating transient effects from ambient vibration recordings. A 10% cosine taper at both ends of the windows was implemented to minimize Fast Fourier Transform distortion, and the Konno-Ohmachi smoothing function (Konno & Ohmachi, 1998) was applied to each spectrum with a constant value of 80

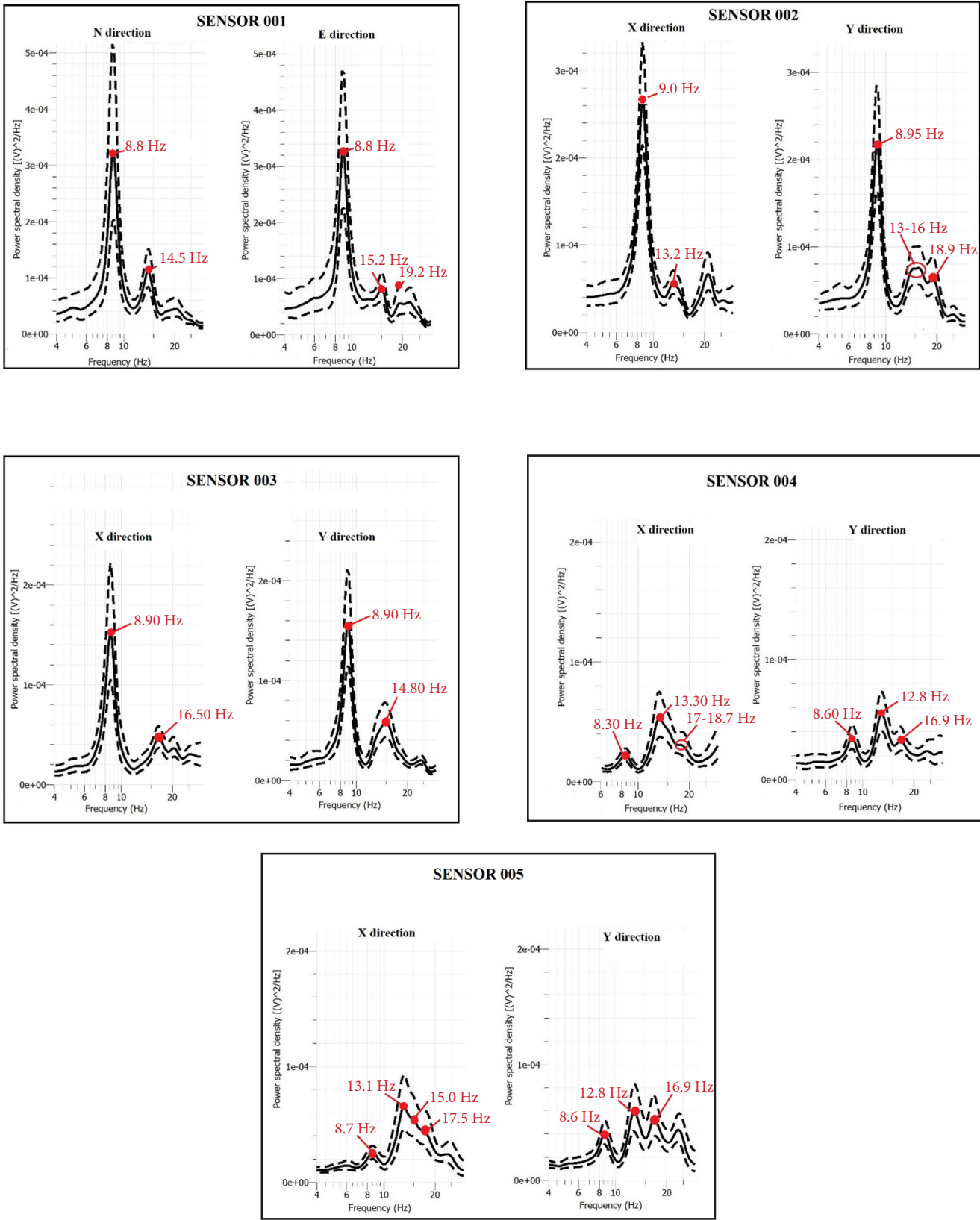
for discerning slightly different frequencies. The mean spectra were obtained by averaging individual smoothed spectra. Figure 150 displays the PSD spectra obtained through this procedure for the two components of each of the five sensor placements, with continuous lines representing mean spectra and dashed lines indicating mean plus and minus one standard deviation curves.

Modal frequencies, corresponding to peaks in each component, were identified using the GEOPSY automatic picking method, which is based on locating the value at the boundary between the standard deviation domains of two peak frequencies. The procedure underwent systematic validation through visual inspection, with manual removal of signal sections containing high-frequency noise.

The precision and reliability of the obtained experimental frequencies are indicative of the effectiveness of the adopted methodology in capturing the dynamic behaviour of the Beit-El-Din Hammam. The identified fundamental frequencies in the longitudinal and transverse directions play a crucial role in characterizing the structural response under ambient vibrations. It's noteworthy that the resonance frequencies observed in these experiments provide essential information for validating and calibrating the numerical model, enhancing its accuracy in simulating the actual structural behaviour. Moreover, the clarity in identifying the fundamental frequencies despite the shared common wall between buildings B1 and B2 underscores the accuracy of the conducted ambient vibration tests. This robustness is particularly significant in scenarios where neighbouring structures might introduce complexity in frequency identification. The equivalence between numerical predictions and experimental outcomes, as presented in Table 30, is essential in validating the numerical model.

MODE	EXPERIMENTAL Frequency ranges (Hz)	NUMERICAL Frequency (Hz)
1		8.58
2	8.3 ÷ 9.0	9.40
3	12.8 ÷ 13.3	11.71
4	14.5 ÷ 16.0	15.36
5		17.10
6	16.9 ÷ 18.0	17.45
7		19.25
8	18.7 ÷ 20.5	19.60

[Table 30]
Comparison between experimental
and numerical frequencies.



[Figure 150]
Diagrams of the power spectral density detected at the five sensor positions in the two horizontal directions.

A more comprehensive investigation involving simultaneous recordings at multiple locations could provide additional modal information and offer a detailed representation of mode shapes. However, for the current study's objectives, an extensive modal investigation is deemed unnecessary and impractical. Several key reasons support this decision.

First and foremost, the primary focus of this study lies in assessing the post-elastic behaviour of the structure. Consequently, the significance of evaluating modal properties is somewhat diminished in the context of the study's objectives. Notably, when nonlinear material behaviour is considered, the influence of dynamic effects within the elastic range diminishes as the system enters the plastic range (Paglietti & Porcu, 2001). This rationale justifies assigning lesser importance to the detection of modal shapes and the identification of higher modes.

The results obtained from the present experimental campaign have facilitated a reliable identification of the first frequencies in the two main directions. This level of information is generally adequate for many dynamic studies, given that higher modes' contributions tend to become increasingly negligible in elastic analyses under seismic forces. Seismic codes, recognizing this phenomenon, often permit the neglect of higher mode contributions.

Masonry, as a material, inherently exhibits inelastic behaviour, particularly when subjected to robust and alternating forces. In instances where a homogeneous elastic-plastic model is assumed for masonry, as in the present study, and nonlinear dynamic analyses are conducted under intense seismic events, calibrating the elastic modulus and mass density values of the model may suffice. For this purpose, the fundamental frequencies in the two main directions are deemed adequate.

In conclusion, the comprehensive ambient vibration testing, along with the subsequent processing and analysis, not only provided the fundamental frequencies of the Beit-El-Din Hammam but also ensured a strong basis for further refinement and validation of the numerical model. This work improves the model's predictive accuracy for the structure's dynamic behaviour under various loading conditions. (Salameh et al., 2016; Wathelet et al., 2020).

2.5. NUMERICAL MODEL IDENTIFICATION

Building upon the outcomes of the extensive ambient vibration testing detailed previously, this section engages in a dynamic identification of the numerical model for the Beit-El-Din Hammam, a crucial aspect in seismic structural assessments. However, this process is inherently fraught with uncertainties, arising from deficiencies in the structural survey of historical buildings concerning materials, geometry, and mechanical characteristics (M. C. Porcu et al., 2021). Additionally, experimental errors, as highlighted by (Rizzo et al., 2020), may propagate onto the estimation of natural frequencies. Although a well-conducted study can significantly mitigate such uncertainties, complete elimination remains elusive, especially when dealing with historical masonry structures.

[Table 31]
Identified properties of the Beit-El-Din masonry from (CNT, 2019; NTC, 2018).

Material Properties	
Mass density	1800 kg/m³
Young modulus	1400 MPa
Poisson ratio	0.125

[Table 32]
Numerical frequencies and participating masses for the first 8 modes (in bold higher than 5% values).

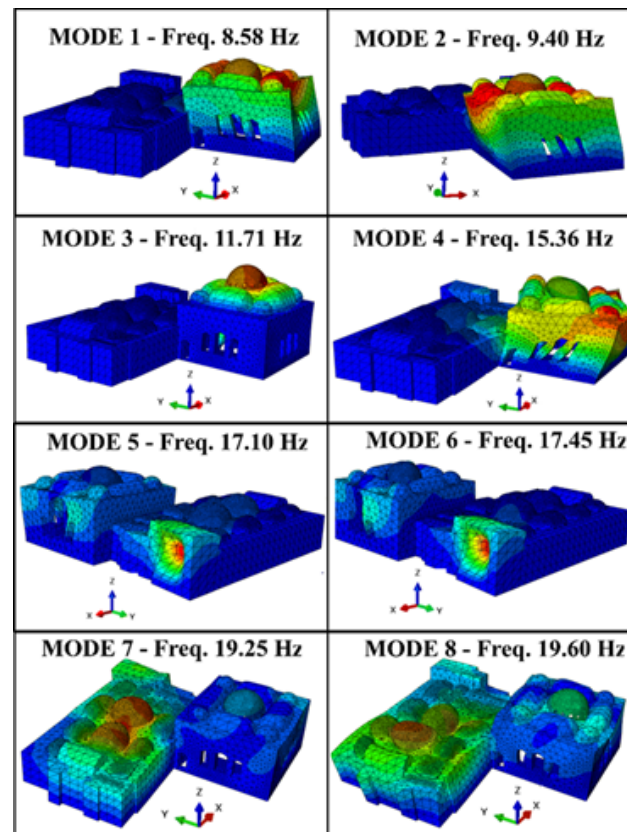
Mode		Freq. (Hz)	Participating masses (%)					
n.	type		UX	UY	UZ	ROT-X	ROT-Y	ROT-Z
1	Flexural Y dir. &	8.58	1.75	37.68	0.12	2.77	0.33	20.25
2	Flexural X dir. &	9.40	46.50	1.55	0.01	0.16	3.61	9.42
3	Mainly vertical Z	11.71	0.01	0.00	97.32	30.71	38.08	0.01
4	Torsional	15.36	0.67	2.12	0.02	0.23	0.09	13.12
5	Local	17.10	19.55	0.22	0.40	0.68	12.13	20.01
6	Local	17.46	8.20	0.17	0.08	0.09	7.43	2.27
7	Mixed	19.25	16.09	15.20	0.14	16.97	22.46	3.40
8	Mixed	19.60	5.68	42.40	0.00	42.64	7.66	0.00

Initially assuming a linear elastic isotropic behaviour of the material, an iterative procedure comparing numerical and experimental frequency values was employed to identify the numerical model of the Beit-El-Din Hammam.

The selection of mass density γ and Young’s modulus E as updating unknown parameters was refined through multiple iterations. For this study, a Poisson’s ratio ν of 0.1, a realistic value for masonry (Angelillo, 2014; Angelillo et al., 2010), was adopted. The iterative modification of Young’s modulus and mass density values commenced with initial values falling within the typical ranges provided by Italian Regulations in Table C8A.2.1 of Circolare No. 617 del 2 febbraio (2009) for unreinforced masonry. Material properties of the identified model are detailed in Table 31 while Table 32 elucidates the participating masses relevant to the first eight modes.

Analysis of Table 32 and Figure 151 reveals that the first two modes are flexural and torsional mixed modes, a common occurrence in buildings with irregular plans, as evidenced in Vielma et al. (2021). The irregular plan of the Beit-el-Din Hammam (B1+B2 buildings) justifies the prevalence of mixed modes. It's noteworthy that even the plan-regular building B1 exhibits flexural/torsional mixed modes in its first two modes. The dominance of dynamic behaviour by B1, the most flexible part of the structure, is evident in the first modes of the hammam. The presence of mixed flexural/torsional modes is a characteristic feature in masonry tower-like structures, owing to their comparatively low torsional stiffness resulting from the absence of horizontal rigid floors. This integration of dynamic insights, along with the acknowledgment of uncertainties, contributes to a nuanced understanding of the seismic response of historical masonry structures like the Beit-El-Din Hammam.

The frequencies of the first two modes exhibit a proximity (8.58 Hz and 9.40 Hz), falling within the experimental ranges extracted from the diagrams in Figure 150 and detailed in Table 30. The third mode (11.71 Hz) primarily manifests as a local translational mode, predominantly in the vertical direction, implicating the main dome of building B1 and the surrounding barrel vaults. This mode suggests a potential structural vulnerability. The fourth mode (15.36 Hz) is predominantly a Z-torsional mode. Modes 5 and 6 are local modes with closely aligned frequencies (17.10 Hz and 17.46 Hz), activating a flexural out-of-plane deformation in a lateral room of B2, characterized by a slenderer wall than the rest of the structure. Moving on, modes 7 and 8,



[Figure 151]
Numerical mode shapes and
corresponding frequencies for the
first identified eight modes.

again closely aligned in frequency (19.25 Hz and 19.60 Hz), are identified as mixed modes predominantly involving building B2. A generally satisfactory agreement between the experimental ranges and numerical frequencies is observed (refer to Table 30), validating the numerical model.

In the elastic range, a damping ratio (ξ) of 0.08 (8%) was adopted for all modes, a value typical for stone masonry (Pierdicca et al., 2016). The numerical model incorporates a Rayleigh damping matrix, resulting from a linear combination of mass and stiffness matrices.

The coefficients α and β for this combination were determined by solving a system of two linear equations:

$$\xi = \frac{\alpha}{2f_1} + \frac{\beta f_1}{2} \quad (88)$$

$$\xi = \frac{\alpha}{2f_2} + \frac{\beta f_2}{2} \quad (89)$$

2.6. SPECTRUM CONSISTENT LEBANESE EARTHQUAKES

The scarcity of recorded earthquakes in the Lebanese region presents challenges in obtaining suitable sets of spectrum-consistent accelerograms for dynamic analyses. Nevertheless spectrum consistent earthquakes for the Lebanese zone were identified through a rigorous and detailed procedure in (Gerges et al., 2023b). Appendix B provides detailed information on this procedure, including details of the essential earthquakes relevant to this part of the thesis. It is essential to reiterate that these records underwent comprehensive elaboration, involving amplitude scaling and frequency content modification, utilizing SeismoMatch (SeismoMatch, 2016), a commercial software proficient in aligning earthquake accelerograms with a predefined target response spectrum. Adhering to ASCE/SEI 7-16 (ASCE 7-16, 2016) guidelines, a 10% lower tolerance and a 10% upper tolerance were established, with the matching procedure iterated 250 times. Following the matching in SeismoMatch (SeismoMatch, 2016), subsequent corrections were implemented using SeismoSignal (SeismoSignal, 2016).

Beit El-Din is situated west of the Yammounch fault branch, the principal strike-slip component of the Dead Sea fault system in Lebanon Figure 152. The design response spectrum specific to the Beit El-Din zone corresponds to Soil Type C it is represented in black as the target spectrum in Figure 153.

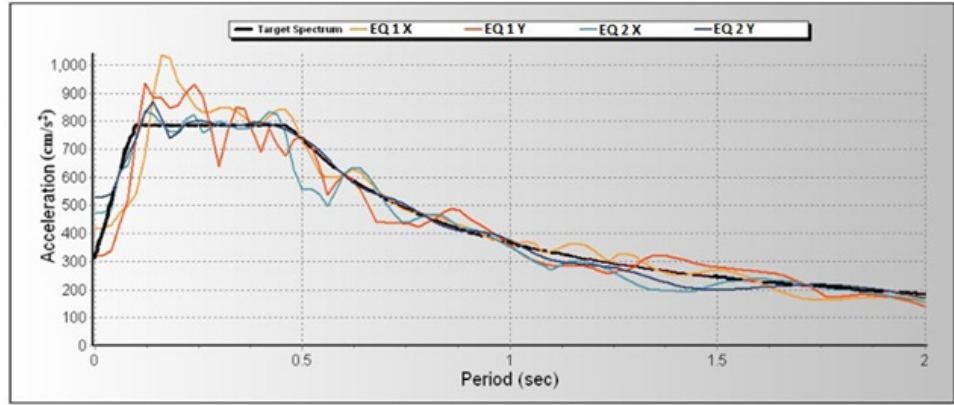
From the numerous earthquakes matched to Soil Type C spectrum in Appendix B, two specific events detailed in Table 33 were deliberately selected for their distinctive characteristics. In Appendix B, the two earthquakes are designated as “IBC-C-E2” and “IBC-C-E5”; however, for the remainder of this study, we will refer to them as EQ 1 and EQ 2, respectively. The first earthquake exhibits multiple peaks, with the strongest exceeding 4 m/s², while the second earthquake is characterized as pulse-like, featuring a primary peak reaching almost 6 m/s². Pulse-like earthquakes are conventionally anticipated to be more destructive than non-pulse-like events. Consequently, these two earthquakes were intentionally chosen to conduct a comprehensive assessment of the seismic vulnerability of the Beit-El-Din Hammam.

[Figure 152]
Faults of Lebanon.



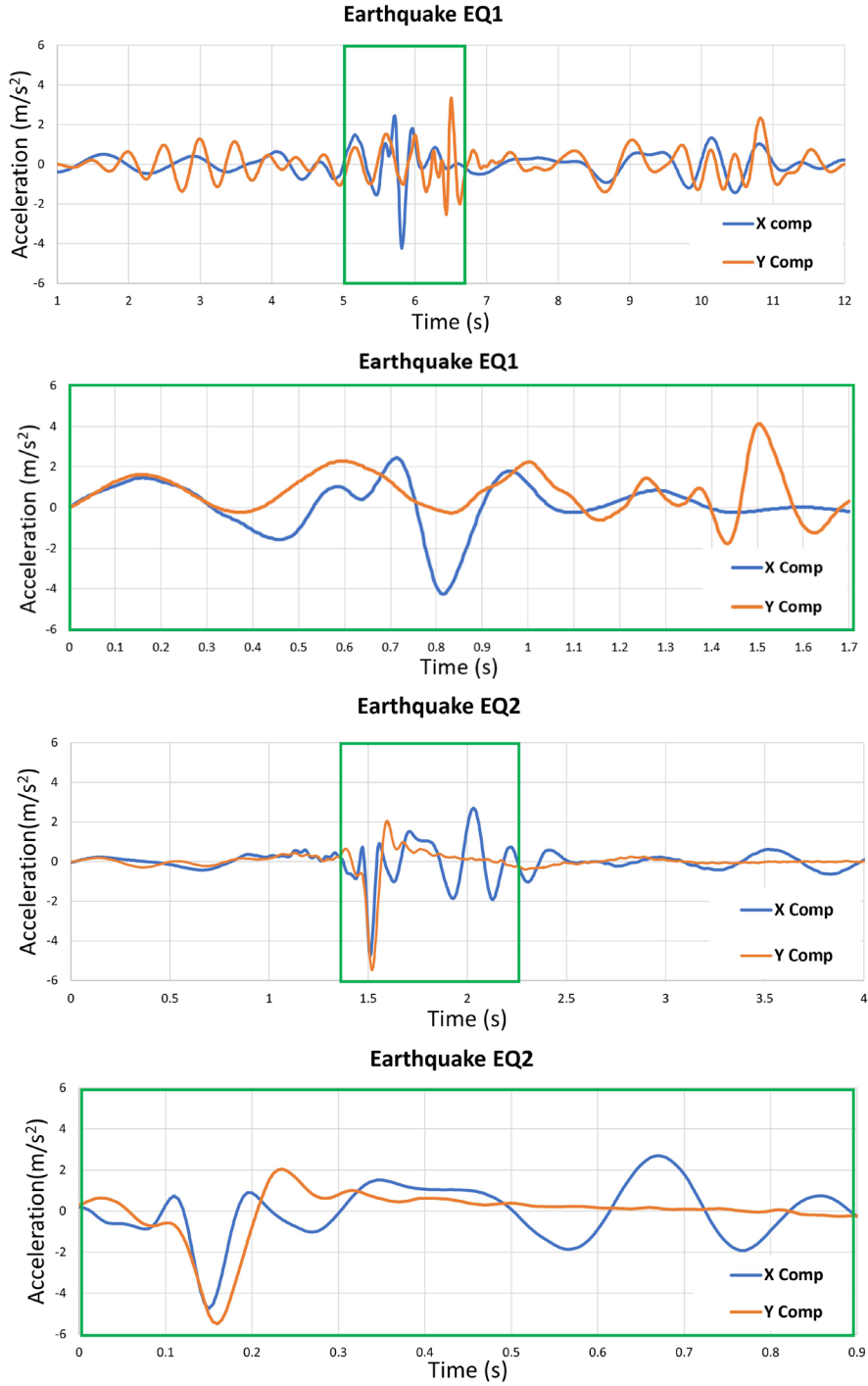
Figure 153 exclusively illustrates the spectra of the two selected earthquakes alongside the designated target spectrum in both the X and Y directions. It is noteworthy that, in accordance with Lebanese code specifications, the same design response spectrum was applied to both directions. Accelerograms in the X and Y directions for the two chosen earthquakes are presented in Figure 154.

[Figure 153]
Target design spectrum and records'
spectra in the two horizontal directions.



To reduce the computational burden the time windows highlighted in green in Figure 154 considered, and the accelerograms referred to as EQ1 and EQ2 were applied in the nonlinear analyses. This choice follows the Arias Intensity procedure presented in Appendix C, which allows to drastically decrease the earthquake length and preserve the effectiveness of the results.

In this study, we have not considered the vertical component of earthquakes. According to seismic code standards (EC8, 2004)), only the horizontal components are relevant for this type of structure.



[Figure 154]
The two spectrum-consistent earthquakes with the time-windows selected
for the analyses and a zoom of the time-windows considered in the analyses.

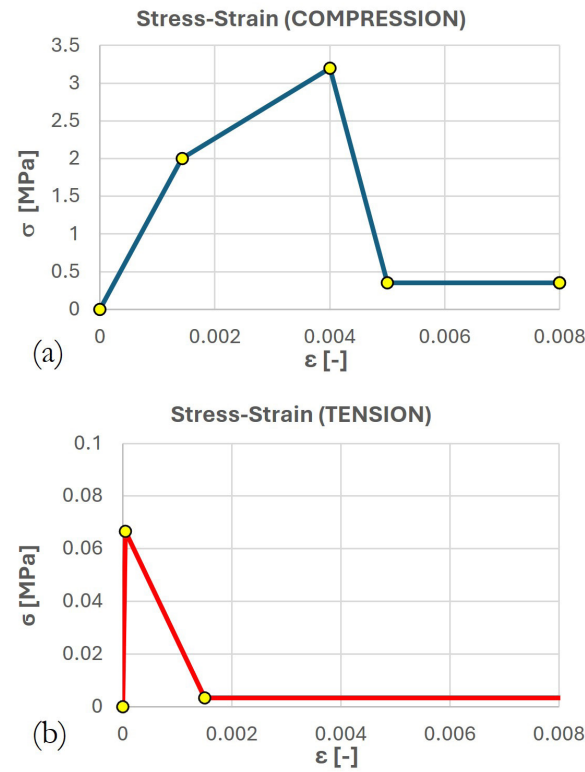
[Table 33]
Selected earthquakes.

Earthquake	Event ID	Location	Station Name	Date	Mw	Epicentral distance (km)
EQ1	EMSC.20180704_0000036	Dead Sea East	Ghor Haditha Jordan	2018/04/07	4.1	159
EQ2	IL - 1987-0002	Dead Sea East	Izrael	1984/08/24	5.3	34.8

Based on the findings from the time-history nonlinear analysis (THNLA) of the Aqueduct de los Cinco Ojos outlined in sub-section 3.2 of Part A, the selected finite element model is CDP. This decision is driven by the analysis objective of identifying potentially affected areas under seismic loading, while also considering computational efficiency. The CDP model is deemed more suitable for this task compared to MC, given its ability to efficiently handle the required computations.

The stress-strain diagrams displayed in Figure 155 were assumed to model the CDP behaviour of the masonry of the Beit-El-Din Hammam in ABAQUS. The compressive uniaxial diagram is made by four linear branches (an initial linear elastic one, then a hardening and a softening branch, and a final constant-stress branch). The tensile diagram is made instead by three branches only, no hardening being considered in this case according to the CDP model. The linearisation of the CDP diagrams has been made for the sake of simplicity, in agreement with existing references, e.g., (Bertolesi et al., 2017; M. C. Porcu et al., 2021). Because of the lack of experimental data on the masonry of the Beit-El-Din Hammam, some literature data have been exploited to assign empirical values to parameters so that the values listed in Table 34 were finally assumed. Reference to the experimental tests performed by Di Ludovico (Di Ludovico et al., 2020) was first made. They involved column-like specimens built with limestone blocks (130 mm × 130 mm × 400 mm), hydraulic mortar with poor mechanical properties and 5–10 mm thick joints to simulate the behaviour of masonry of historical buildings. The directions provided by the Italian Building Regulations ((NTC, 2018) were also considered to get the typical value ranges of strength and elastic parameters of different kinds of historical masonry. The tensile strength was assumed to be 1/30 of the compressive strength as per the findings concerning stone masonry in Appendix A.

The values assumed for the CDP model in ABAQUS are given in Table 35. It should be noted that the choice of these values has been discussed in sub-section 2.2.1.b.of the “State of the Art” section.



[Figure 155]
(a) Compression and tension uniaxial stress-strain in the CDP theoretical model,
(b) Compression and tension uniaxial stress-strain curves assumed for the Beit-El-Din Hammam limestone masonry.

[Table 34]
Values modelling the anisotropic stress-strain behaviour of masonry for the CDP model.

Compression		Tension ($f_t=f_c/30$)	
σ	ϵ	σ	ϵ
[MPa]	[%]	[MPa]	[%]
0	0	0	0
2	0.143	0.066	0.004762
3.2	0.4	0.00333	0.15
0.35	0.5	0.00333	0.4
0.35	0.6		

[Table 35]
Assumed values for parameters of the CDP model in ABAQUS.

Dilatation angle	Eccentricity	f_{b0}/f_{c0}	K_c	Viscosity
ψ	ϵ			χ
10°	0.1	1.16	0.667	5.00E-05

2.7. RESULTS OF THE NONLINEAR DYNAMIC ANALYSES

THNLA were carried out under the two spectrum-consistent earthquakes. The X and Y components of each earthquake were applied simultaneously. The analyses were performed using the reduced earthquake in Figure 154, with time-windows of 1.7 seconds for EQ1 and 0.9 seconds for EQ2. The simulations required a run time of about 250,978 sec in wall clock time (2.9 days) for EQ1 and of 126,788 sec (1.47 days) for EQ2 using parallelization across 6 processors.

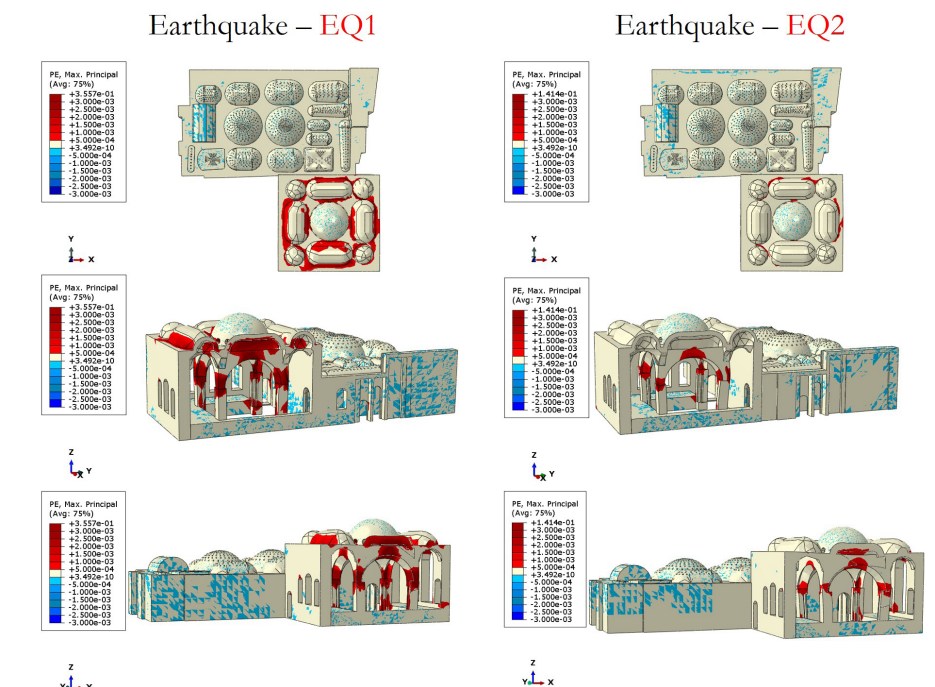
The contours of the maximum principal strains in the deformed structure are given in Figure 156 the right and the left columns referring to EQ1 and EQ2 respectively. The tensile strains are displayed in red to better highlight the zones where crack damage can occur in the structure under severe seismic actions. The comparison shows that EQ1 is the most demanding earthquake for the structure, despite it being characterized by a peak ground acceleration (PGA) lower than EQ2, with PGA values of 4.53 in the x direction and 4.13 in the y direction for EQ1, compared to 5.04 in the x direction and 5.12 in the y direction for EQ2.. It can be also inferred that building B1 is the most affected part of the structure. All vaults of B1 are affected by tensile yield under EQ1, the barrel vaults being the most damaged. The pattern of tensile plastic strain on the vaulted roof of B1 is instead less critical under EQ2.

The sections displayed in Figure 156 also show that, under both the considered earthquakes, three plastic hinges (two at the springings and one at the crown) formed in the arches around the main dome of B1 building. Three hinges are, however, insufficient to form a kinematically admissible collapse mechanism, which should consist instead of at least four hinges (Heyman, 1982).

The two plots in Figure 157 provide the total energy dissipated by plastic yielding during the action of earthquakes EQ1 and EQ2, respectively. From the comparison of the two graphs, it can be inferred that plastic dissipation under EQ1 is about twice than that under EQ2, this remarking again that the structure is affected by much higher inelastic deformation under the multi-peak earthquake. It is well known that dissipation of seismic energy through plastic deformation is regarded as a beneficial effect in earthquake-resistant structures, whatever material they are made of, see, e.g., (Liel et al.,

2011; M. Porcu, 2017). In masonry, the opening and closing of cracks leads to a dissipation of energy, which is evidence of a ductile behaviour of the material. A widespread crack pattern, in fact, does not necessarily indicate a critical unsafe condition, although monitoring its evolution would be a prudent choice (Ottoni et al., 2010).

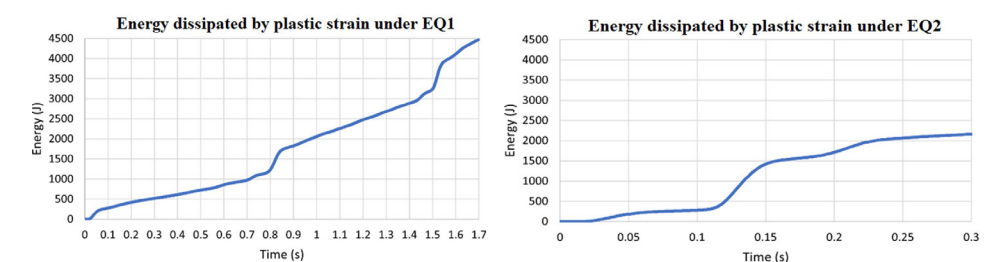
[Figure 156]
Contours of the maximum principal plastic strain (PE) under EQ1 and EQ2.



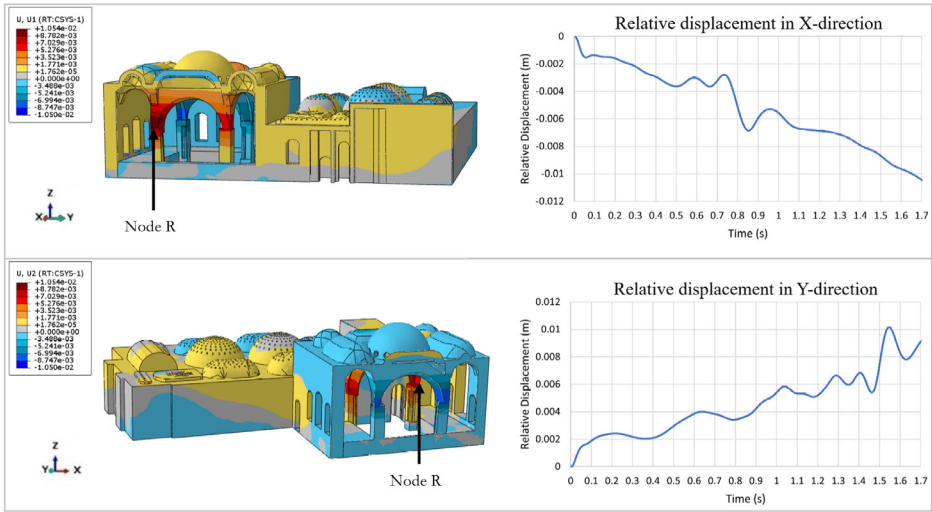
With reference to two sections of the structure, Figure 158 presents the contours and graphs of the maximum displacements in the X and Y directions under EQ1. The maximum values are reached in both directions at the base of the arch, which is significantly affected by the earthquake, as also shown by Figure 156. The time-histories of the relative displacements of node R, where the maximum displacements occur, are plotted in the same figure. These plots indicate that a maximum value of approximately 1 cm in both directions is reached at this node by the end of the ground motion from EQ1.

EQ1 has already proven to be the most demanding, hence the relative displacement was considered for it. Given that the relative displacement was low, equal to 1 cm, the relative displacement under EQ2, the less demanding earthquake, was not explored.

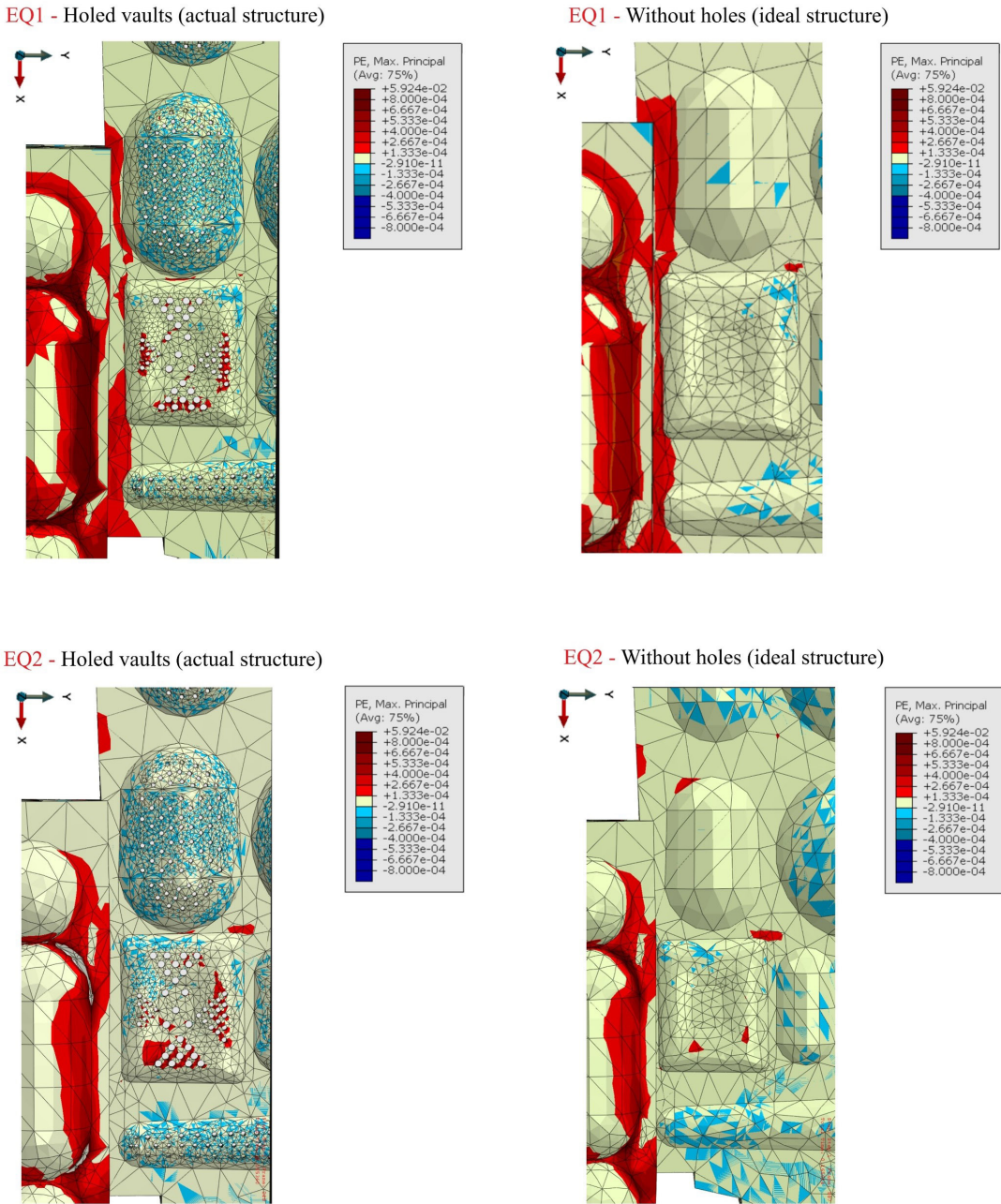
[Figure 157]
Plastic energy dissipated under EQ1 and EQ2.



[Figure 158]
Contours of the maximum relative
displacements under EQ1 and
time-history of the X and Y relative
displacements of node R at the springing
of the arch of B1 main dome.



To investigate the impact of holes on the seismic behaviour of the structure's vaults, we included a twin model without holes in this study. Nonlinear time-history analyses were conducted under EQ1 and EQ2 for both the model with holes and the model without holes. While overall differences in global behaviour between the two models were minimal, distinct local effects were observed. Specifically, one of the vaults exhibited more pronounced tensile damage in the model with holes compared to the model without holes. This is shown in Figure 159, which presents a comparison of the plastic strain (PE) contours obtained for the model with holes and the model without holes under EQ1 and EQ2. To highlight the differences in the plastic strain patterns more clearly, a zoomed-in top view is included, and a proper magnified scale has been adopted.



[Figure 159]
Comparison between maximum principal plastic strain (PE) in the holed and
not-holed models under EQ1 and EQ2.

2.8. CONCLUSIONS

The Hammam of Beit-El-Din stands as one of Lebanon's most exquisite masonry buildings, characterised by its holed domes and barrel vaults. Preservation of cultural heritage against the adverse effects of natural disasters aligns with the objectives outlined in Goal 11 of the United Nations (UN) Agenda 2030 for sustainable development, emphasizing the importance of safeguarding historical structures like the Hammam of Beit-El-Din.

This study aims to possible areas of damage where cracks might occur and give an insight to the Hammam’s vulnerability against seismic events. To achieve this goal, a comprehensive three-dimensional finite-element model was developed for conducting numerical dynamic analyses. In-situ modal tests were performed to determine the natural frequencies of the structure. The numerical model was calibrated based on these experimental modal results, demonstrating a satisfactory correlation between experimental and numerical frequencies.

In accordance with Lebanese design response standards, two seismic events were considered: a pulse-like earthquake and a multi-peak earthquake, each with a peak ground acceleration (PGA) of 0.6 g and 0.4 g, respectively. Both horizontal components of these earthquakes were applied simultaneously in time-history nonlinear analyses, incorporating the anisotropic nonlinear behaviour of the masonry using the CDP model.

The latter was chosen due to considerations of computational efficiency and the specific goals of the analysis. The monument posed significant computational demands, requiring substantial time even with parallelization across 6 processors. This underscores that selecting the MC model would have resulted in excessively long and large computational burden.

The analysis revealed that the multi-peak earthquake causes greater damage compared to the pulse-like earthquake among the considered ground motions. Under both earthquakes, the tallest and most flexible building (B1) within the

hammam complex experienced the most significant effects. In the latter a distinct pattern of tensile damage was observed along the contours of the vaults, with the barrel vaults exhibiting the most pronounced damage. The four arches supporting the main dome of B1 were found to exhibit three plastic hinges (two at the springings and one at the crown) under both seismic events. However, the presence of three hinges alone is insufficient to initiate a collapse mechanism.

To evaluate the influence of holes on masonry vault damage patterns, a comparative analysis was conducted using a twin model without holes. Overall, the seismic behaviour of the hammam did not exhibit significant differences between the model with holes and the model without holes. However, certain vaults with holes displayed a more extensive pattern of local tensile damage compared to those without holes.

The findings of this study demonstrate the Beit-El-Din Hammam's resilience against severe earthquakes. Throughout its existence, this monument has endured significant seismic events, including the destructive 1956 Chouf earthquake (magnitude 5.3). More recently, it withstood the catastrophic 2023 Kahramanmaras earthquake near the Turkish-Syrian border, which registered a magnitude of 5.8 in Lebanon. Following this event, the author of this thesis inspected the monument and found it to be in a very sound and unaffected condition.

It is worth noting that many ancient monuments are still standing after being hit by exceptional natural events during their life. The role of masonry ductility in the seismic vulnerability assessment as well as the full understanding of the post-elastic behaviour of masonry curved structures is essential when analysing such structures under seismic action.

CONCLUSIVE
REMARKS

Modelling masonry is a challenging task, specifically when considering the assessment of a 3D large scale structure under large displacements and deformations. Besides the obvious complexity of modelling the compound geometry of a church, a palace or a fortress, there is an inherent difficulty in modelling the material behaviour of masonry. Masonry exhibits fracture and the ability of masonry structures to stand depend crucially on the friction forces triggered by the vertical weights. Modelling fracture and friction represents a tremendously difficult job even for modern industrial materials whose local properties are controlled and reproducible. Macro-modelling and the use of discrete approaches comes in handy as it is a highly less complex and time-consuming modelling approach when compared to the more classical approaches of the Theory of Structures, but suffers from the arbitrary nature of the partition we choose and from the inherent limitation in the repertoire of possible kinematical mechanisms.

The standard way to approach the Theory of Structures is to assume that the structure is an assembly of connected and continuous parts for which equilibrium of forces, deformation, and constitutive relations can be introduced and exploited.

In the case of masonry, the feeble tensile resistance of masonry put at a stake the satisfaction of equilibrium in many cases that could seem common for modernly conceived structures, and the assumption of a standard deformable continuum is challenged by the evidence of concentrated fractures which, to be legitimate, requires that the problem be formulated in a much more sophisticated mathematical environment. Finally we have to make a choice about the degree of complexity we wish to provide the constitutive restrictions we may use to represent the material behaviour.

In this last respect, going for a sophisticated, “closer to reality”, type of model requiring the setting of several material parameters may not be the wisest option. The more a physical problem appears complicated the more we should use parsimony in the introduction of material constants, the main challenge being the use of the least number of parameters which still allows a sound mechanical assessment of the structure.

We may say that continuum damage models belong to the class of sophisticated continuum based approaches, since they consider average stresses and strains whose varying elastic and multi-parameter inelastic properties along material axes require the setting of several parameters. These models, such as the concrete damage plasticity (CDP) model, that were first introduced for the modelling of concrete in reinforced concrete structures, have been subsequently applied to masonry. CDP is a plastic material in compression and in tension which softens and accumulates damage at its inside, damage being the way CDP models fracture.

On the other side of sophistication are the continuum approaches based on the assumption that the material is unilateral in compression (then it can develop fractures at zero energy cost) and does not allow anelastic sliding. These assumptions, that are sufficient to adopt the theorems of Limit Analysis, maybe complemented by different assumptions about the deformation in compression. The Normal Rigid No-Tension (NRNT) material considers that the material is rigid in compression and the Normal Elastic No-Tension (NENT) material assumes a linear elastic stress strain relation when it shortens.

In the present thesis these two different material assumptions are implemented into the commercial program for structural analysis ABAQUS, and the results obtained in the analysis of many benchmark problems and of some case studies are analysed and compared.

ABAQUS, as all the displacement based numerical approximations of the balance laws of Mechanics, cannot handle zero energy modes and then is inherently incapable of managing materials which are exactly unilateral. Therefore in the present study the closest possible approximation to NENT materials that ABAQUS can allow is considered. This approximation is actually obtained by pushing to its limits a standard associated plastic Mohr-Coulomb material, linear elastic inside the elastic domain, by adopting friction angles of values up to 89.9° and values of cohesion as small as 10^{-9} of the Young modulus.

In Part A of this thesis the (CDP) model, implemented in ABAQUS, has been applied to the analysis of masonry arches, giving unsatisfactory results

when compared to other methods and known experimental and analytical results. For these elementary small scale masonry structures, CDP appears as inherently inadequate to catch the unilateral blocky behaviour of the assembly. Specifically CDP was able to locate roughly the possible places at which hinges might form, although without the ability to predict any real localization, and was unable to accurately predict the collapse load in most of the cases.

The introduction of the idealized Mohr-Coulomb model for masonry as a continuum to mirror NENT and NRNT masonry like materials, has been calibrated in Part A. Following this process, the model was benchmarked against two reference approaches which are the Piecewise rigid displacement (PRD) approach and the CDP model. The MC model was able to reproduce the thrust lines, their evolution, the hinge locations, the limit horizontal multiplier and the limit vertical settlement at collapse mechanism initiation with satisfactory accuracy, both in the geometrically linear and nonlinear range.

In section 3 of Part A, when applied to case studies concerning large-scale masonry structures, both CDP and MC models demonstrated a satisfactory overall performance when compared to PRD and to each other.

For instance, when applied to the Torre del Piano model, with a tetrahedral mesh, CDP was able to provide the right fracture propagation pattern and determining the right location of fracture initiation. However, it was unable to adequately predict the right collapse load horizontal multiplier. When considering the CDP discretization with the bottom-up mesh of the tower the obtained results were greatly different than those obtained on adopting the tetrahedral mesh, being highly inaccurate. Only the possible fracture pattern could be correctly identified after a thorough analysis of the results.

When analysing the Torre del Piano using the MC model with both discretization configurations, the results were accurate and satisfactory. The MC model predicted the exact collapse multiplier value in addition to the correct fracture initiation point. Additionally, when considering a bottom-up mesh discretization, the MC model exhibited a large decrease of the amount of computational time required. Even though the MC model was computationally slower than CDP, it has proved to yield still accurate results by adopting a simplified mesh which allowed faster analyses.

This places the MC model at an advantage with regards to CDP in the frame of small-scale macro-models and large-scale macro-models under pseudo-static analyses. It excels in accurately representing structural behaviour and critical collapse characteristics.

It is important to acknowledge that the extreme MC model that is here adopted, imposes significantly higher computational demands compared to CDP. Then, when considering large-scale models under pseudo-static loads choosing CDP with an adequate mesh, depending on the desired outcome of the analysis -especially if the precise level of loss of equilibrium is not a primary concern-, may be preferable due to its computational efficiency.

Another large-scale macro-model of masonry was analysed, the Aqueduct de los Cinco Ojos, this time under a dynamic time-history non-linear analysis (THNLA). Using again the two finite element ABAQUS models, both models in this case were able to similarly predict the behaviour of the structure, as they captured similar results from the point of view of minimum and maximum plastic strain concentration and with respect to the behaviour portrayed through energy graphs. However, in this case, the CDP model gave a larger advantage over the MC model as it required a definitely lesser computational burden. For instance, the longest analysis conducted with the CDP model was completed within 12 hours, whereas the MC model necessitated a substantially longer time-frame of 7 days.

Hence, depending on the desired goal of the analyses one can use CDP and analyse the results with a discerning approach, understanding them not as absolute truths, but rather as nuanced signals. Basically we cannot ask to CDP what it cannot do, namely predict correctly strain concentration and macro-block partition, keeping in mind that its results must be interpreted as gross and average indications of failure, to serve as a guide and to be complemented by simpler and more reliable equilibrium analyses. This underscores the necessity for expertise in masonry behaviour to effectively analyse CDP results.

To prove the validity of this statement, in Part B of this thesis, a complex case study from Lebanon, my home country, was chosen. Due to the large number of generated elements and to the significant demand posed by THNLA, it was clear that the choice of our extreme MC model was out of the question

due to its inherent high computational burden. Hence the CDP model was applied and the results were adequately analysed.

The conclusions drawn from this research provide valuable insights for practitioners involved in the preservation of historical structures addressing the complex challenge of modelling masonry, particularly for large-scale 3D structures under significant displacements and deformations. By comparing different modelling approaches, it offers a nuanced understanding of their respective strengths and limitations. The research demonstrates the effectiveness of the MC model in accurately predicting structural behaviour and collapse characteristics in masonry, making it a powerful tool for future applications in both academic and professional settings with the least number of required parameters. Additionally, the critical analysis and original implementation of this model in the nonlinear dynamic analysis software ABAQUS open new avenues for future research. These contributions include a detailed synthesis of the evolution of masonry calculation methods and it underscores the necessity of tailoring models to specific analytical goals and structural characteristics.

Furthermore, this thesis has made considerable progress in reconciling two seemingly incompatible worlds: the discrete blocks of masonry and continuous material theories. It bridges the gap between theories by applying a limit analysis based solution for masonry in the finite element world. This work may serve as guidance for future scholars who wish to apply finite element methods for masonry structures.

Finally, one can conclude that the MC model proves to be an adequate tool to analyse masonry macro-models within the finite element environment whether they are of small or large scale. Nevertheless, the model still needs refining and limitation testing specifically determining factors that can optimize computational efficiency. As demonstrated in the thesis, the computational time required for the MC model is still reasonably acceptable for models composed of tens of thousands of elements. However, for models containing hundreds of thousands of elements, it becomes significantly more challenging, indicating a need for further research to enhance computational efficiency. Meanwhile the existing damage plasticity model remains suitable for seismic dynamic analyses of large-scale masonry structures when properly calibrated and tuned, and cross checked with more speedy and sound equilibrium based methods.

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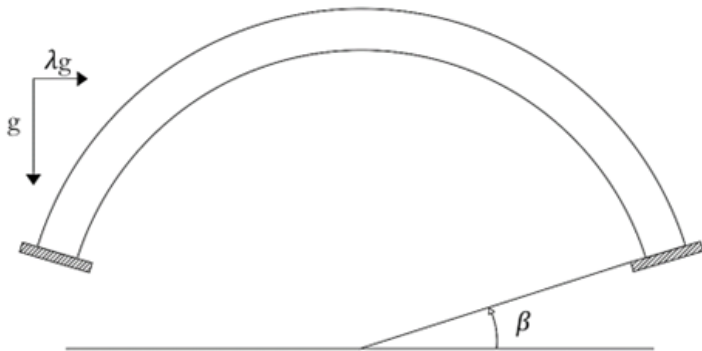
APPENDIX A

COMPARATIVE STUDIES ON THE CDP MODEL

In this section, aspects of analysis using the CDP model are investigated. In the first subsection, a comparison between the use of an incremental dynamic explicit step and an incremental dynamic implicit step is conducted to evaluate λ and inspect hinge formation through of minimum plastic strain visualization.

In the second subsection, the influence of the variations in yield tensile strength on stone masonry arches of different thicknesses is examined, and the results of the analyses are compared.

It is important to note that all the analyses conducted focus on horizontal capacity, where all analysed arches are subjected to self-weight and a horizontal incremental action directly proportional to the mass, which increases through the loading parameter λ . An example illustrating the loading conditions on an arch, which has been meshed with an element length of 0.5 (the mesh size used for all analyses), is presented in Figure A1.



[Figure A1]
Example of an arch subjected to vertical and horizontal loads applied to the centroids of each finite element.

The parameters used in the CDP model are referenced from sub-section 2.2.1.4. of the 'State of the Art' section (see Table A1). The values for Young's modulus, mass density, and Poisson's ratio are consistent with those employed in the Benchmark section and are recalled in Table A2.

[Table A1]
Assumed values for parameters of the CDP model in ABAQUS.

Dilatation angle ψ	Eccentricity ϵ	f_{b0}/f_{c0}	K_c	Viscosity χ
10°	0.1	1.16	0.667	5.00E-05

[Table A2]
Material properties adopted for the limestone arches from Table 1.

Material Properties			
Mass Density ρ	Young modulus E	Poisson ratio ν	Compression resistance f_m
1800Kg/m ³	1400Mpa	0.1	2MPa

A.1. Explicit vs Implicit dynamic approach

The aim of this comparison is to study and compare the horizontal collapse multiplier, hinge location, and minimum plastic strain resulting from implicit dynamic incremental analysis versus explicit dynamic incremental analysis for two circular arches, one with springing angle $\beta=0^\circ$ and the other $\beta=20^\circ$. Both arches are subjected to their self-weight and a horizontal incremental action proportional to their mass. Finally, a comparison of the required computational time is presented, and the optimal choice is determined.

a. Circular arch with springing $\beta=0^\circ$

The structural analysis focused on a circular arch with a thickness of 0.5 meters and a radius of 1 meter. The aim was to study plastic strain development and hinge formation within the arch using two different analysis methods: explicit and implicit dynamics.

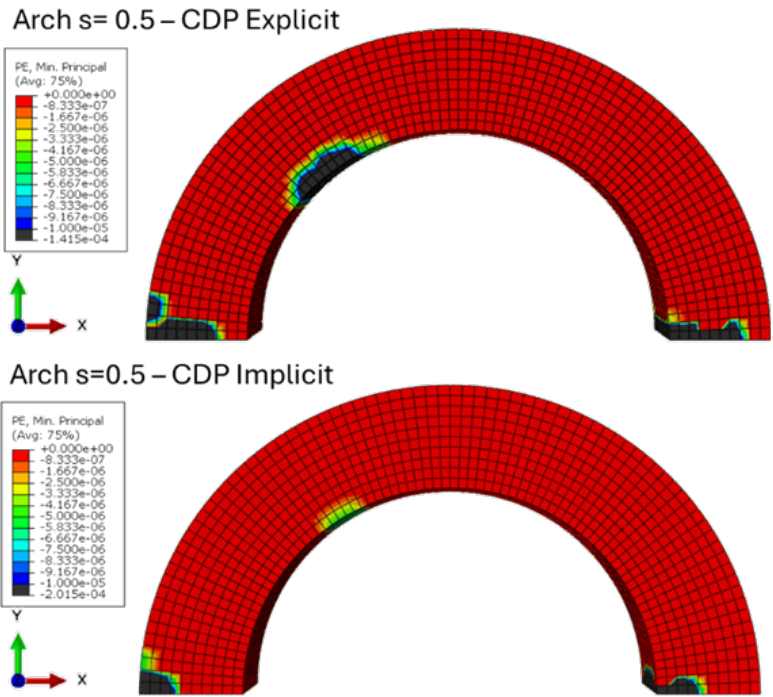
The visualization of the results was based on the horizontal multiplier value λ found in Part A sub-section 2.4.1.a. for the arch with a thickness of 0.5 meters, where $\lambda=1$.

Figure A 2 presents a visualization highlighting the distribution of minimum plastic strain. In this representation, black regions indicate areas with the highest negative values of minimum plastic strain, indicating tensile strain hence the location of separation of the material which pinpoints the openings

of hinges. The analysis compared results obtained through both explicit and implicit methods. In the explicit incremental analysis, the highest recorded negative plastic strain value was $1.4 \text{ E-}04$ (-0.014%), while the implicit analysis showed a slightly higher value of $2 \text{ E-}04$ (-0.02%).

Despite this numerical difference, the comparison suggests that the results are quasi-equal. This implies, in addition to having the same value of λ , that the variation in plastic strain values between the two methods is relatively minor and does not significantly affect the overall interpretation of the structural behavior.

Moreover, the computational time was compared in Table A3, where the explicit analysis, as expected, exhibits faster processing time compared to the implicit dynamic analysis.



[Figure A2]
Minimum principal plastic strain results for the analysis of the semi-circular arch with springing $\beta=0^\circ$, of radius $r = 1\text{m}$ and thickness $s = 0.5\text{m}$, using the CDP model under both explicit and implicit dynamic steps.

[Table A3]
Comparison of computational time between MC Explicit, CDP Explicit, CDP Implicit for a horizontal capacity analysis, for a round arch (i.e. $\beta = 0^\circ$) with internal radius $r = 1\text{m}$, of thickness $s = 0.5\text{m}$.

s [m]	Nb. Elements	Time [s] CDP-Explicit	Time [s] CDP-Implicit
0.5	7800	64	509

b. Circular arch with springing $\beta=20^\circ$

Another analysis conducted on a circular arch with springing angle equal to 20° instead of 0° and a radius of 1.5 meters, the same investigative approach applied to assess plastic strain development and hinge formation using both explicit and implicit dynamics.

It is worth noting that the visualizations in Figure A3 are according to the results found in Part A sub-section 2.3.1.2 for $\lambda = 0.95$.

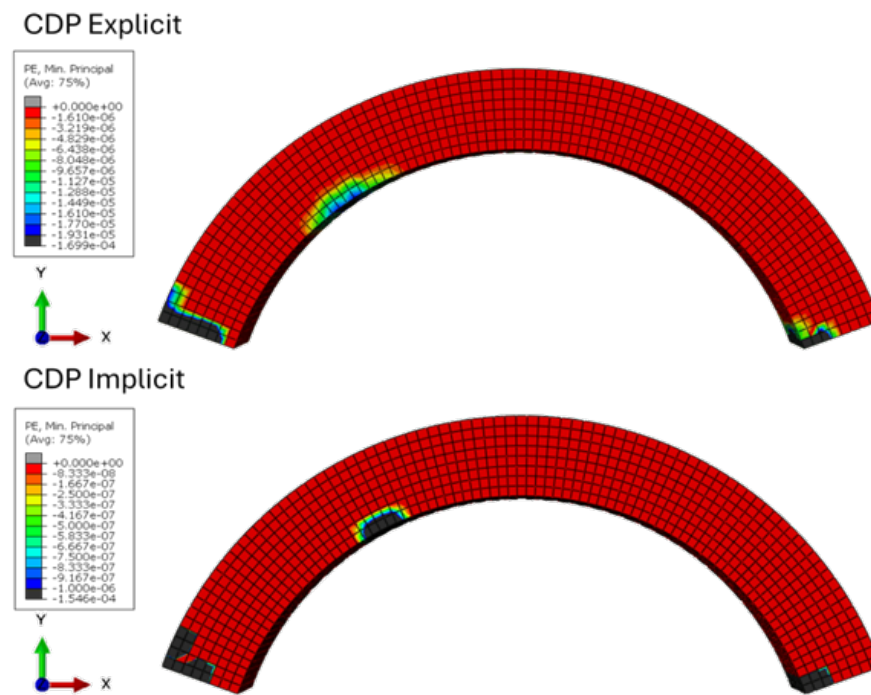
Like the previous study, the visualization of the results focused on identifying regions of high negative plastic strain, denoting tensile stress and potential hinge openings. Despite variations in arch geometry, the interpretation drawn here is consistent with the previous findings. In the explicit analysis, the highest recorded negative plastic strain value was $-1.69 \text{ E-}04$ (-0.0169%), while the implicit analysis yielded a slightly lower value of $-1.54 \text{ E-}04$ (-0.0154%). The latter found values indicate minor discrepancies between the two analysis methods.

The consistency in results, as well as the equivalent horizontal multiplier value λ , highlights the reliability of the explicit analytical approach across different arch configurations.

Furthermore, a comparison of the computational time is provided in Table A4 for this arch configuration. It is evident that the explicit analysis method has a higher processing speed compared to the implicit dynamic analysis.

[Table A4]
Comparison of computational time between MC Explicit, CDP Explicit, CDP Implicit for a horizontal capacity analysis, for a circular arch having a springing angle. $\beta = 20^\circ$, with internal radius $r = 1.5\text{m}$, of thickness $s = 0.4\text{m}$.

Nb. Elements	MC-Explicit	CDP-Explicit	CDP-Implicit
5312	851s	40s	649s



[Figure A3]
Minimum principal plastic strain results for the analysis of the semi-circular arch with springing $\beta=20^\circ$, of radius $r = 1.5\text{m}$ and thickness $s = 0.4\text{m}$, using the CDP model under both explicit and implicit dynamic steps.

In conclusion, the analyses conducted on different arch geometries demonstrate the alignment of results between explicit dynamic and implicit dynamic approaches. Moreover, computational comparison clearly places the explicit analysis method as the most efficient.

This supports the decision to use explicit analysis in the Benchmark section for the CDP model, allowing for a consistent and adequate comparison with the MC model, which also utilizes explicit dynamic incremental steps. Overall, this choice ensures a valid and effective comparison between these models.

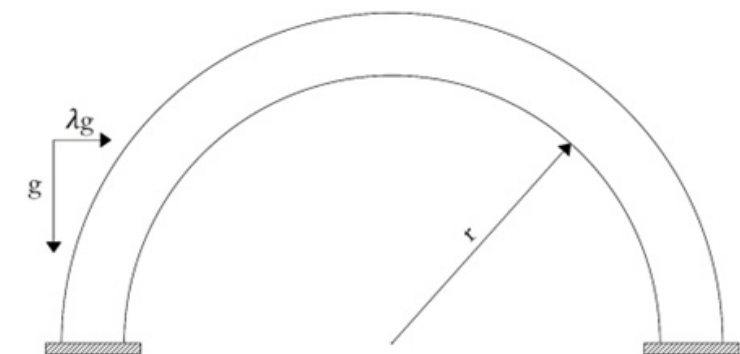
A.2. Impact of tensile yield strength variation on stone arches

In literature when adopting the CDP model for masonry analysis in ABAQUS the tensile stress of masonry is calculated to consider all possible variations as being equal to several ratios of the compressive stress. Indeed, values ranging from 1/60 (Valente & Milani, 2016) to 1/10 (Bertolesi et al., 2017; M. C. Porcu et al., 2021) of the compressive strength are adopted in the literature for the tensile strength of historical unreinforced masonry, the intermediate ratio 1/30 has been adopted as well in (Jain et al., 2020). This choice is generally based on the masonry type and the unit and mortar which forms it.

For limestone, a tensile strength of about 1/30 of the compressive strength is also experimentally found by (Magenes et al., 2010) on stone masonry walls under cyclic tests. The experimental ratio calculated from the data provided is approximately 23.94. While the ratio calculated according to the Italian building code (NTC, 2018) is 32, considering that the suggested mean compressive strength is 3.2 MPa and the suggested mean tensile strength is 0.1.

To verify if the best choice to adopt throughout the thesis is the 1/30 ratio a parametric study was conducted on four circular arches of different thicknesses (0.2m, 0.3m, 0.4m, and 0.5m). These arches were analysed using the same material properties, CDP parameters, and compressive strength as those specified in section 2.3.1.1.

The arches are subjected to self-weight and horizontal incremental loading, where the forces are directly proportional to mass and increase with the loading parameter λ (see Figure A4).



[Figure A4]
Loading conditions applied to a round arch (i.e. $\beta = 0^\circ$) of varying thickness s , with internal fixed radius $r = 1\text{m}$.

This study serves to investigate the impact of varying ratios of tensile strength to compressive strength in masonry. Three ratios were examined: 1/30, 1/60, and 1/100, with the latter ratio chosen to explore scenarios where the tensile strength is exceptionally low, aligning with Heyman's assumption of zero tensile strength.

The choice of ratios allowed for a comprehensive analysis, ranging from more typical values used in literature to extreme cases where tensile strength is significantly diminished. The exploration of these ratios is done through a comparison of the visualization of stress results and minimum plastic strain.

It can be observed from Figure A5 to Figure A8, that optimal visualization of stress results, for the value of horizontal multiplier identified in section 2.3.1.1., specifically Mises and minimum principal stresses, occurs when the scale is set to 1/30. When the scale is set to 1/60 or 1/100, adjusting the limits only yields suboptimal visualizations. In the event that the set limits surpass the adjusted ones, in hopes for better results, the resulting visualization of stresses becomes increasingly smeared, deviating further from the intended depiction of a thrust line. For the various thicknesses the best parameter to identify the probable hinge locations in the minimum principal stress (compressive stress).

It's essential to highlight that when using a 1/100 scale ratio, there's a notable limitation in clearly illustrating the opening of hinges for minimum potential energy (PE). Therefore, the discussion will focus solely on the 1/30 and 1/60 ratios. For thicknesses of $s=0.2$ m and 0.3 m, a clearer depiction of hinge openings is observed with the 1/30 ratio compared to 1/60. Conversely, for thicker arches like $s=0.4$ m and $s=0.5$ m, the 1/60 ratio illustrates all four hinges (though PE is much more smeared than desired), while the 1/30 ratio only reveals three. However, this discrepancy with the 1/30 ratio is mitigated when visualizing minimum principal stress or von Mises, as it enables precise identification of all four hinges. Thus, compensating for this flaw in PE visualization at 1/30 is feasible.

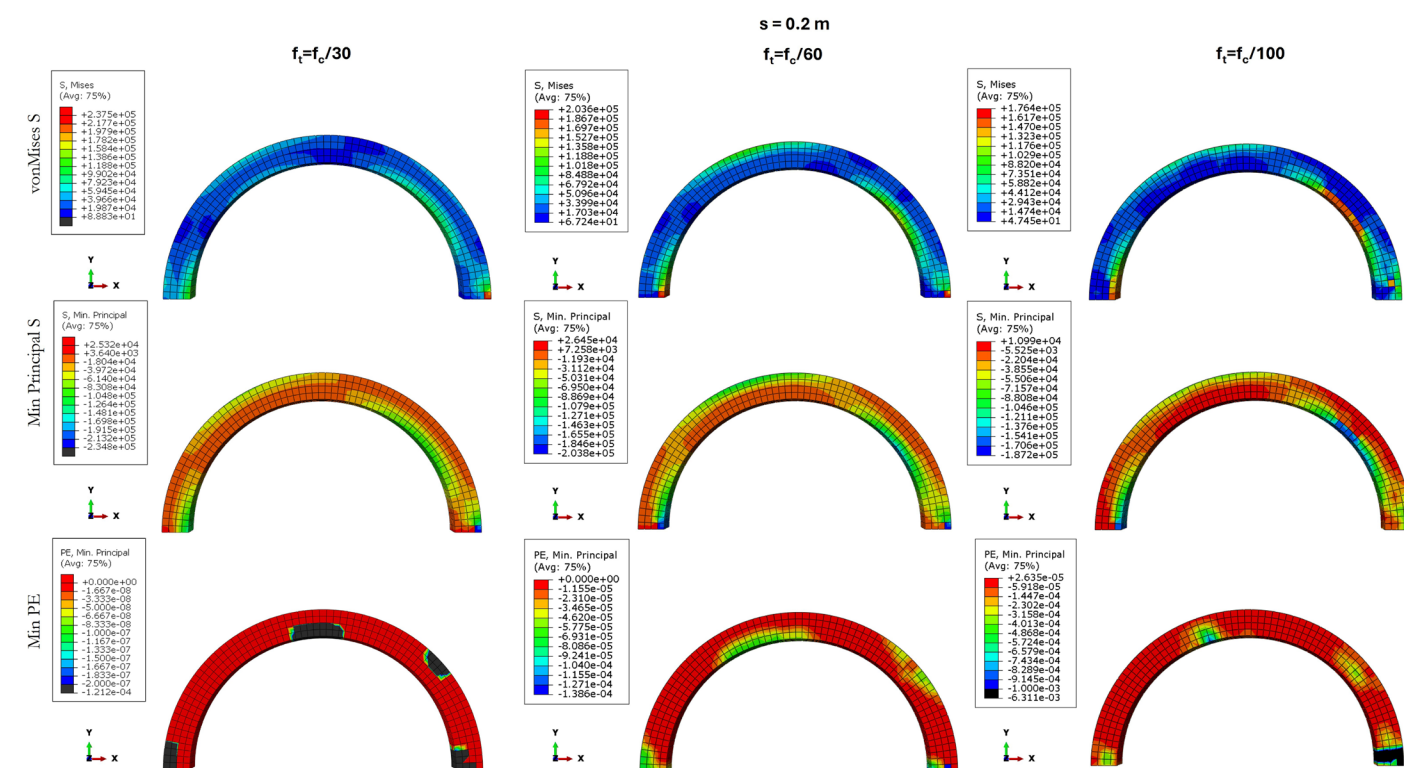
Finally, when comparing computational time in Table A5, it can be seen that for all the various thicknesses 1/30 ratio is the most efficient. In addition, it can be affirmed that adopting the 1/30 value is the most adequate for stone masonry despite occasional inaccuracies in the minimum PE. The latter mentioned limitation can be surpassed since any discrepancies in PE can be

mitigated through a side-by-side comparison with the accurately depicted stresses (S Min and Von Mises stress). Therefore, the overall benefits of adopting the 1/30 value remain compelling.

In conclusion, considering both computational efficiency and overall visualization quality of the results derived from the parametric study, it is evident that adopting the 1/30 for stone masonry is most adequate.

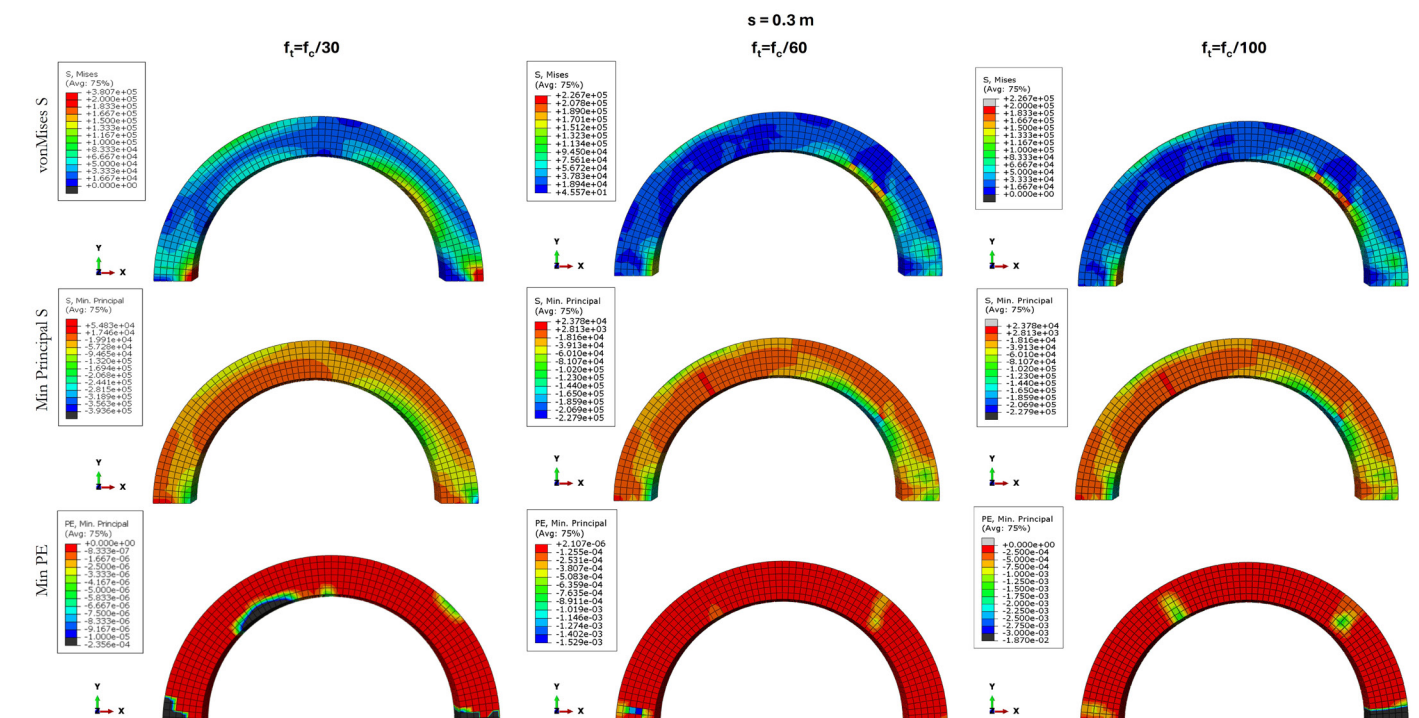
[Table A5]
Computational time in [s] for the different ratios of the tensile stress studied over the various thicknesses of a stone arch.

Thickness	Computational time [s]		
s	$R_f = \frac{f_t}{f_c} = 1/30$	$R_f = \frac{f_t}{f_c} = 1/60$	$R_f = \frac{f_t}{f_c} = 1/30$
0.2	9	11	11
0.3	19	29	30
0.4	37	61	66
0.5	64	114	127



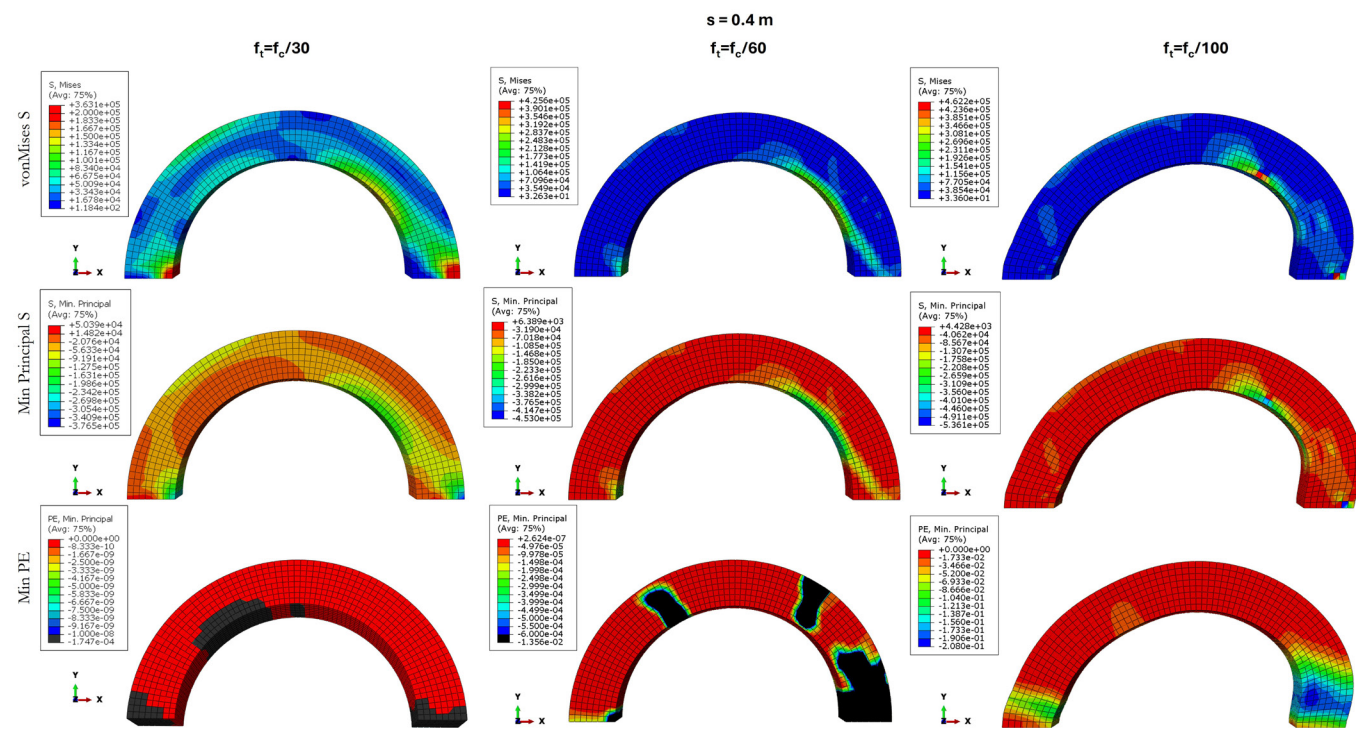
[Figure A5]

A side-by-side comparison of Von Mises stress, minimum principal stress, and minimum plastic strain (PE) across different ratios of tensile stress, for arches with a thickness of $s=0.2\text{m}$.



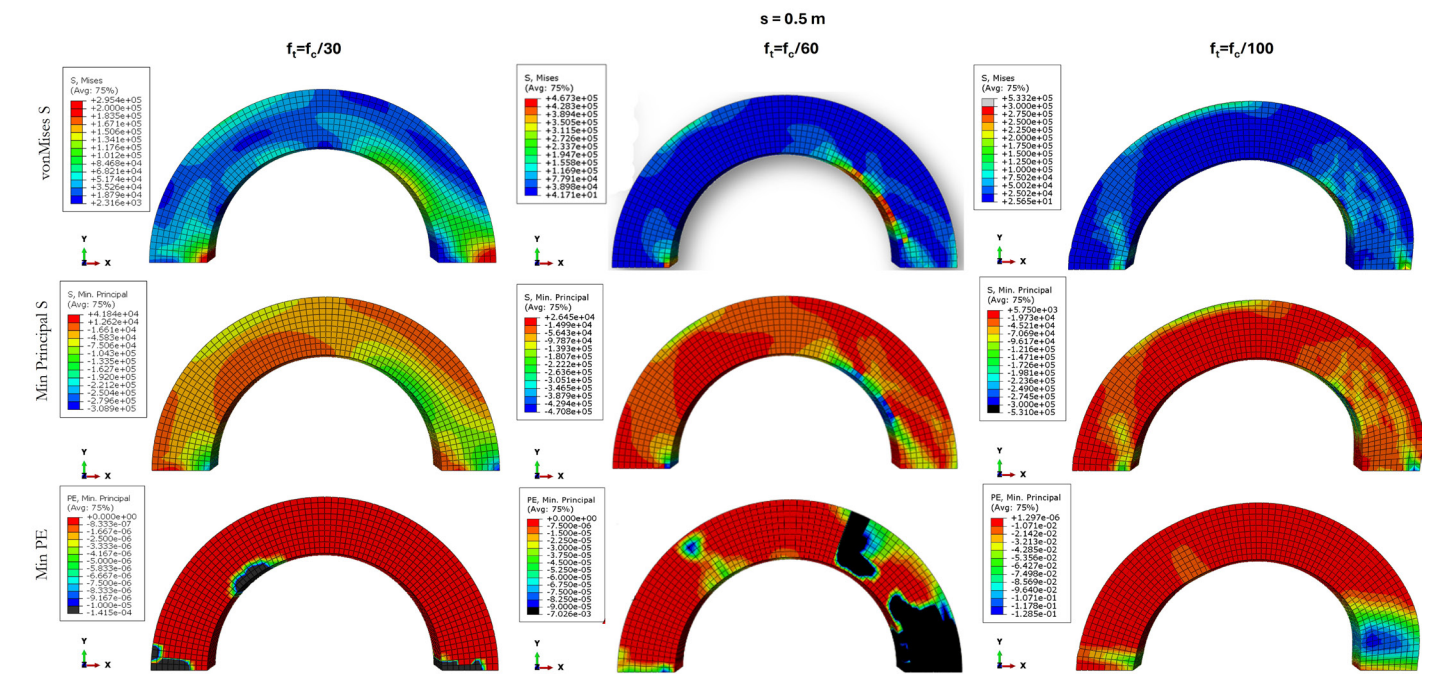
[Figure A6]

A side-by-side comparison of Von Mises stress, minimum principal stress, and minimum plastic strain (PE) across different ratios of tensile stress, for arches with a thickness of $s=0.3\text{m}$.



[Figure A7]

A side-by-side comparison of Von Mises stress, minimum principal stress, and minimum plastic strain (PE) across different ratios of tensile stress, for arches with a thickness of $s=0.4\text{m}$.



[Figure A8]

A side-by-side comparison of Von Mises stress, minimum principal stress, and minimum plastic strain (PE) across different ratios of tensile stress, for arches with a thickness of $s=0.5\text{m}$.

APPENDIX B

In this section we present a specific part of a study detailed in (Gerges et al., 2023b), which addresses the complexity and potential inconsistencies in Lebanese regulations, which reference multiple international codes. Hence, the study compares Lebanese design response spectra and provides spectrum-consistent accelerograms to aid in the seismic design amid challenges posed by limited earthquake data.

We focus in this section on the part concerned with obtaining suitable sets of spectrum-consistent accelerograms that can be applied for dynamic analyses of Lebanese structures according to the local regulations, based on the International Building Code (IBC).

SPECTRUM CONSISTENT EARTHQUAKES

B.1. Seismicity of Lebanon

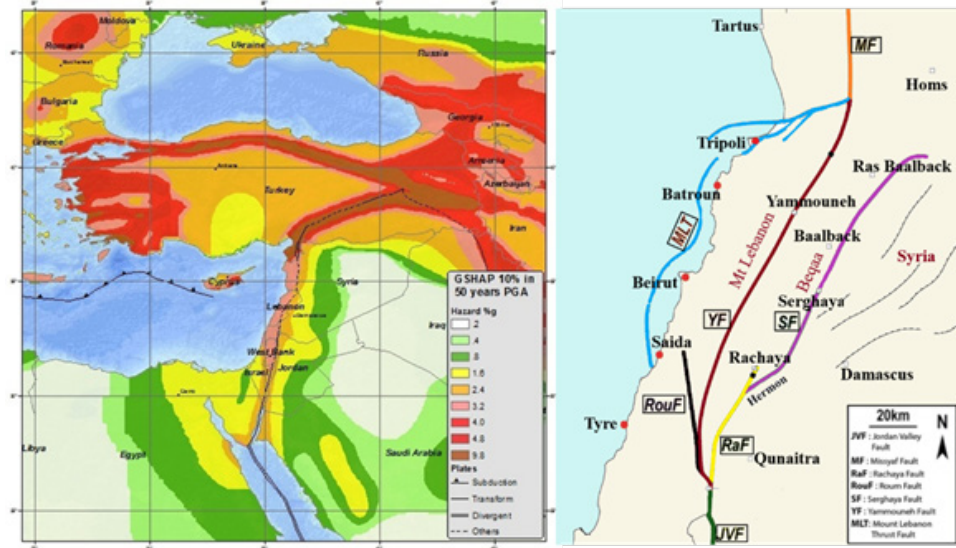
The tectonic complexity of the Eastern Mediterranean area is well known, Lebanon lies astride the Dead Sea Transform Fault (DSTF) in a region of complex tectonic plate interaction (see Figure B1a), due to the presence of three major plates (the Arabian, African and Eurasian Plates) and two sub-plates (the Anatolian and Sinai subplates) (Demir et al., 2024). The DSTF is a strike-slip fault between the African and the Arabian plates (Brax et al., 2016), which extends South-North approximately 1000 km from the north of the Gulf of Aqaba through Wadi Araba, the Dead Sea, the Jordan valley, the Beqaa valley, and merges eventually at the east of the Anatolian fault (Arango & Lubkowski, 2012). It can be divided into two main sections joined by a restraining bend along Lebanon, where it splits into five main fault branches: the Yammouneh, Roum, Serghaya, Rachaiya and Hasbaya faults, see Figure B1b. The activity of these faults has been the topic of recent paleoseismic (Demir et al., 2020; Palanci et al., 2023; Trifunac & Brady, 1975) and geodetic (Zhong & Zareian, 2014) investigations which indicated that such faults are still active.

[Table B1]
Some of the strongest Lebanese earthquakes.

	Date	Location	Magnitude
1	9-Jul-551 A.D.	Phoenicia (Roman Province), Byzantine Empire (now Lebanon)	7.5
2	20-May-1202	Southwestern Syria	7.6
3	30-Oct-1759	Ottoman Syria part known today as Lebanon	6.7
4	25-Nov-1759	Ottoman Syria part known today as Lebanon	7.4
5	29-Sep-1918	Northwest of Tartus, Syria	6.6
6	16-Mar-1956	Southeast of Baabda, Mohafazat Mont-Liban, Lebanon	5.3
	16-Mar-1956	Southwest of Zahle, Mohafazat Béqaa, Lebanon	5.5
7	26-Mar-1997	Nabatiyé et Tahta, Mohafazat Nabatiyé, Lebanon	5.0
8	15-Feb-2008	Tyros, South Governorate, Lebanon	5.1

Major earthquakes have hit Lebanon, A list of the strongest Lebanese earthquakes is provided in Table B1. The most destructive of the modern era may be the 1759 earthquake, which hit Beirut and Damascus (6.7 Richter degrees). Recently, the 1956 earthquake, which consists of twin shocks of $M_w=5.3$ and $M_w=5.5$, affected the regions of El-Chouf, Jezzine, Saida and parts of the Beqa'a, causing huge structural damage both to buildings and infrastructure.

[Figure B1]
(a) Seismic map of the Eastern Mediterranean area; (b) active faults in Lebanon. Figure from (Gerges et al., 2023b).



B.2. Design Response Spectra for Lebanon

The initial seismic design regulations in Lebanon were developed in the early 1990s. However, governmental decrees N. 646, N. 14293, and N. 7964, issued in 2004, 2005, and 2012 respectively, established the current seismic design rules in the country. These regulations are outlined in the Lebanese Standards Institution (LIBNOR) Standard NL135 (NL135, 2012). NL135 applies a uniform seismic hazard classification (2c) across Lebanon and directs designers to consult either the French Norm PS92 (PS92, 1995), the Uniform Building Code (UBC, 1997), or the International Building Code (IBC, 1997) for seismic design parameters.

Additionally, NL135 (NL135, 2012) states that the Eurocode 8 (EC8, 2004) will be adopted as the sole seismic code in Lebanon within the next five years. It is worth mentioning that some of the codes referenced by NL135 are no longer used in their countries of origin. For instance, PS92 is no longer in effect in France, where Eurocode 8 is now the standard seismic code. As for IBC is a comprehensive code incorporating earlier U.S. standards, including the UBC, which remains widely used in countries like Lebanon.

To get a complete overview of the seismic design codes applicable in Lebanon, (Gerges et al., 2023b) presents the design response spectra (DRS) relevant to PS92, UBC, IBC and EC8.

However, in this section the focus will be solely on IBC which was the reference code used in Part B of this thesis. The Lebanese design spectra based on IBC (IBC, 1997) have been built based on the following 5 assumptions:

1. Only the *elastic* design response spectra are considered.
2. The reference peak ground acceleration (PGA) is taken equal to $2.5 \frac{m^2}{s}$.
3. A period range **0s – 4s** will be here considered in the spectra.
4. A damping ratio of 5% is considered.
5. The vertical seismic action is neglected.

The first and second assumptions are mandatory. To obtain spectrum-consistent earthquakes for transient dynamic analyses, the acceleration records must in fact be obtained by referring to the elastic (and not to the reduced) design response spectra.

On the other hand, the second assumption meets the Lebanese code recommendations which assume a seismic zone factor $Z=0.25$.

In relation to Assumption 3, it should be noted that certain codes provide formulas and parameters for calculating design spectra for fundamental periods (T) exceeding 4 seconds. However, buildings typically have fundamental periods falling within the 0 to 4-second range, leading codes like EC8 to confine the use of design response spectra to this interval.

Regarding Assumption 4, a 5% damping ratio is commonly used in most codes for reinforced concrete and steel buildings, which is realistic for these materials but may result in conservative outcomes for more dissipative materials like masonry or timber.

As for Assumption 5, seismic codes generally suggest accounting for the vertical ground motion in specific scenarios (e.g., long-span bays, floor beams supporting columns, bridges) due to its slight impact on buildings. However, it's important to recognize that the vertical component of earthquakes can influence the post-elastic behaviour of certain types of buildings (Valdés-Vázquez et al., 2021).

To build the DRS, in terms of horizontal absolute acceleration $S_e(T)$, according to IBC (IBC, 1997) the following formulas to build the DRS

$$S_{e,c}(T) = S_{DS} * [0.4 + 0.6 * (T/T_0)] , \text{ for } T \leq T_0 \quad (90)$$

$$S_e(T) = S_{DS} * \left[0.4 + 0.6 * \left(\frac{T}{T_0} \right) \right] , \text{ for } T \leq T_0 \quad (90)$$

$$S_e(T) = S_{DS} , \text{ for } T_0 < T \leq T_s \quad (91)$$

$$S_e(T) = \frac{S_{D1}}{T} , \text{ for } T \geq T_s \quad (92)$$

IBC provides a fourth branch of the spectrum for values of the period exceeding 4s. However, this branch will not be considered here, according to the previously stated third assumption.

S_{DS} , S_{D1} , T_0 and T_s can be obtained through:

$$S_{DS} = \frac{2}{3} * S_{MS} \quad (93)$$

$$S_{MS} = F_a * S_s \quad (94)$$

$$S_{D1} = \frac{2}{3} * S_{M1} \quad (95)$$

$$S_{M1} = F_v * S_1 \quad (96)$$

$$T_0 = 0.2 \frac{S_{D1}}{S_{DS}} \quad (97)$$

In accordance with NL135, Table B2 provides the values of the parameters S_1 , S_s , F_a and F_v for the five site classes. The spectra obtained through equations (93) to (98) and Table B2 are displayed in Figure B2, which are the same for the two main horizontal directions. It can be observed that on the third branch of the IBC DRS the design acceleration increases as the mechanical properties of the soil decreases.

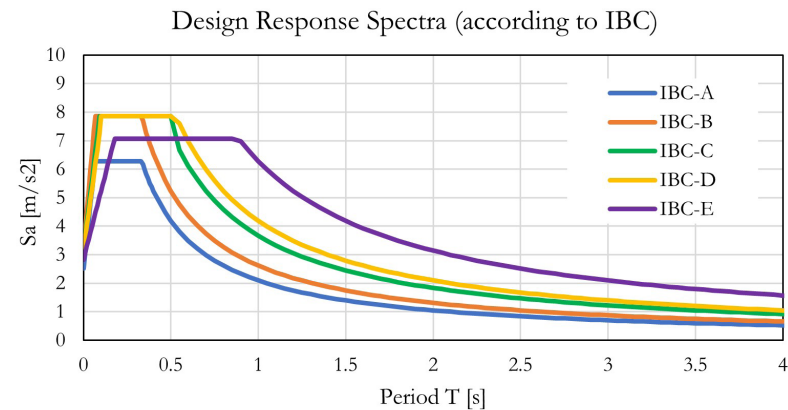
[Table B2]

Values of the parameters for different site classes (according to UBC, IBC and NL135).

Site class	IBC			
	S_1	S_s	F_a	F_l
A	0.4	1.2	0.8	0.8
B	0.4	1.2	1.0	1.0
C	0.4	1.2	1.0	1.4
D	0.4	1.2	1.0	1.6
E	0.4	1.2	0.9	2.4

[Figure B2]

Lebanese horizontal design response spectra as obtained according to American IBC code for the five soil types (A, B, C, D, E) and 5% damping ratio.



B.3. Suites of Earthquakes Consistent with the Lebanese DRS

To conduct seismic time-history analyses for designing or evaluating new or existing buildings, appropriate earthquakes are required. Codes provide guidelines for obtaining accelerograms that match the reference design spectrum specific to the site's seismicity. This can be achieved using artificial or real seismic records. In this study, real records were referenced to develop suites of spectrum-consistent accelerograms specific for Lebanese seismic design.

In this dissertation only the suites obtained referring to the IBC elastic design response spectra will be discussed.

Due to the scarcity of records found for Lebanon, the research was extended to a wider area of the Middle East region, which included the Dead Sea, the Mediterranean Sea, Syria, and Turkey. Two main steps characterize the procedure that has been followed to obtain spectrum consistent earthquakes.

First, thorough research of suitable records through available databases ESM and COSMOS (COSMOS, 1994; ESM, 1997). Among the several earthquakes retrieved, thirteen have been finally selected.

Table B3 lists the selected earthquakes, which are here labelled from E1 to E13. Each earthquake is decomposed in the two main horizontal directions (the vertical component is neglected as stated in the previous section in the fifth assumption).

The second steps, revolves around obtaining earthquakes that are consistent to the DRS as generally recommend by codes when carrying out time-history analyses. A minimum number of three earthquakes is suggested by IBC with reference to ASCE/SEI 7-16 (ASCE 7-16, 2016). If considering a suite of seven earthquakes, both codes permit averaging the outcomes from these analyses. Various methods exist for selecting the most suitable set of ground

motions, ranging from simple selection based on seismological characteristics to more sophisticated frequency domain procedures involving wavelet adjustments to match one or multiple target spectra (Hancock et al., 2006). The number of records required to obtain a stable and unbiased estimate of the inelastic response of a structure was found to significantly depend on the kind of method adopted (Hancock et al., 2008). A large number of accelerograms is required when the scaling to the target spectral acceleration adopted. However, the Spectral Matching (SM) method offers an alternative that can reduce this requirement. The scarcity of recorded ground motions in a country may justify adopting the SM method (Léger et al., 1993). SM adjusts both the amplitude and frequency content of earthquakes to align their response spectrum with the target design spectrum, often resulting in more stable outcomes (Hancock et al., 2008). Adding wavelets offers similar advantages to Fourier adjustment, with the added benefit of introducing less energy into the ground motion and preserving the non-stationary characteristics of the original ground motion (Hancock et al., 2006). An effective SM algorithm was developed by (Abrahamson, 1992), and subsequently refined by (Hancock et al., 2006), which is based on (Kaul, 1978; Lilhanand & Tseng, 1987, 1988).

SM method is preferred with respect to other methods given the scarcity of strong recorded earthquakes in the Lebanese area. SeismoMatch (SeismoMatch, 2016) application, which is based on the Abrahamson-Hancock algorithm, was used to obtain suites of earthquakes consistent with the IBC design spectra (Figure B 2). The values of the matching parameters are listed in Table B4 and the earthquakes listed in Table B3 were considered in SeismoMatch (SeismoSignal, 2016).

In Table B4, the mismatch tolerance denotes the maximum allowable discrepancy between the matched earthquake spectrum and the target one. SeismoMatch allows to set one single value for the mismatch tolerance, which means giving the same value to the upper and lower bounds. EC8 suggests a 10% lower bound, but it does not give an upper bound. On the other hand, IBC does not provide specific values for the mismatch tolerance. The matter of ground motion selection and modification for new building projects in the United States was recently addressed by the Building Seismic Safety Council BSSC (FEMA P-1050, 2015) as also discussed in (Baker et al., 2023). Although a specific value of 10% lower bound is suggested by BSSC for

scaled earthquakes, no specific values are given for matched earthquakes, the spectra of which are only required not to fall below the target spectrum. It is also worth noting that some codes provide a lower and an upper tolerance bound such as the Italian Code (NTC, 2018) that provides a 10% lower tolerance limit and a 30% upper tolerance limit.

A 10% upper and lower bound (i.e. a mismatch tolerance equal to 0.1) was adopted in (Gerges et al., 2023b). This choice is also in accordance with what was assumed in (Baker et al., 2023). The number of maximum iterations was set to 200 for a scale factor of 1. The default value was adopted for the other parameters appearing in Table B4, which are related to the Abrahamson-Hancock algorithm (Hancock et al., 2006). Typically, the higher the maximum number of waves for the wavelet model the better the matching outcome, although considering more than 10 waves has no significant effect on the results. On the other hand, 20 additional waves (or sub-iterations) are enough to prevent divergence in the Abrahamson-Hancock algorithm, as shown in (Hancock et al., 2008). When the off-diagonal terms of the response-to-time matrix do not exceed the 0.7 value recommended by SeismoMatch, better convergence can be reached. In addition the design spectrum is split into sub-groups, the recommended number of which being 250 (Hancock et al., 2008).

Id.	Station Code	Event ID	Location	Date	Magnitude
E1	UJAP	EMSC- 20140524_0000020	DEAD SEA	2014- 05-24	4.3 (M _L)
E2	GHAJ	EMSC- 20180704_0000036	DEAD SEA	2018- 04-07	4.1 (M _W)
E3	MSBI UJAP	EMSC- 20180704_0000512	DEAD SEA	2018- 04-07	4.7 (M _W)
E4	GHAJ MSBI	EMSC- 20190626_0000058	DEAD SEA	2019- 06-26	4.0 (M _W)
E5	IZR	IL-1984-0002	DEAD SEA	1984- 08-24	5.3 (M _W)
E6	KRKR SUAN	SY-1996-0001	JORDAN - SYRIA	1996- 12-24	5.5 (M _W)
E7	3101 3103	TK-2006-0033	JORDAN - SYRIA	2006- 03-29	4.1 (M _W)
E8	EIL	EMSC- 20110220_0000203	RED_SEA	2011- 02-20	3.7 (M _L)
E9	CMRD SLFK	EMSC- 20200415_0000051	SYRIAN COAST	2020- 04-15	4.7 (m _b)
E10	KRTS	EMSC- 20200403_0000177	SYRIAN COAST	2020- 04-03	4.7 (M _W)
E11	KRTD	EMSC- 20130726_0000002	CENTRAL TURKEY	2013- 07-26	4.3 (M _W)
E12	BOZY KRTD	EMSC- 20120511_0000032	CYPRUS REGION	2012- 05-11	5.3 (M _W)
E13	OSC1 OSC2	EMSC- 20170108_0000026	CYPRUS REGION	2017- 01-08	4.1 (M _L)

[Table B3]

Suite of all the earthquakes consistent with the Lebanese response spectra
(according to IBC).

Mismatch tolerance	Max iterations	Scale factor	Min eigenvalue	Max number of waves	Additional waves	Off diagonal reduction	Group size
0.1	200	1	0.1	10	20	0.7	250

[Table B4]

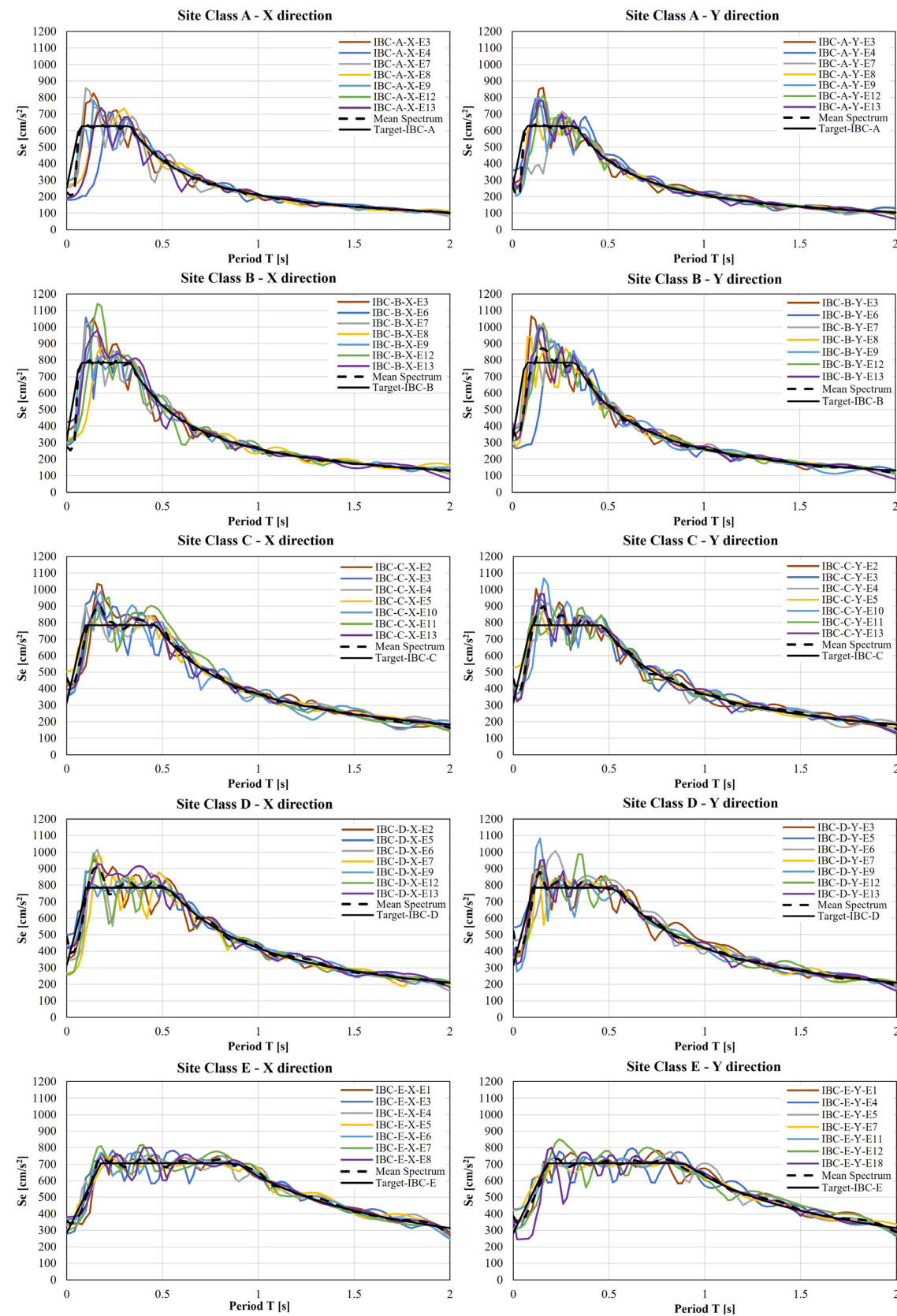
Values of the matching parameters used in SeismoMatch.

B.4. IBC Spectrum Consistent Earthquakes

By following the previously detailed two-step procedure, five suites of earthquakes consistent with the IBC DRS were derived. The spectra of the IBC-spectrum-consistent earthquakes, which have been baseline corrected, are compared in Figure B3 with the target design spectrum of each site class in the two main directions. The accelerograms of the previously matched earthquakes are displayed in Figure B4.

In (Gerges et al., 2023b) a repository is present from which the accelerograms are freely downloadable for the currently employed seismic design codes in Lebanon, namely EC8 and IBC. They can be used for time-history (linear or non-linear) analyses of Lebanese structures.

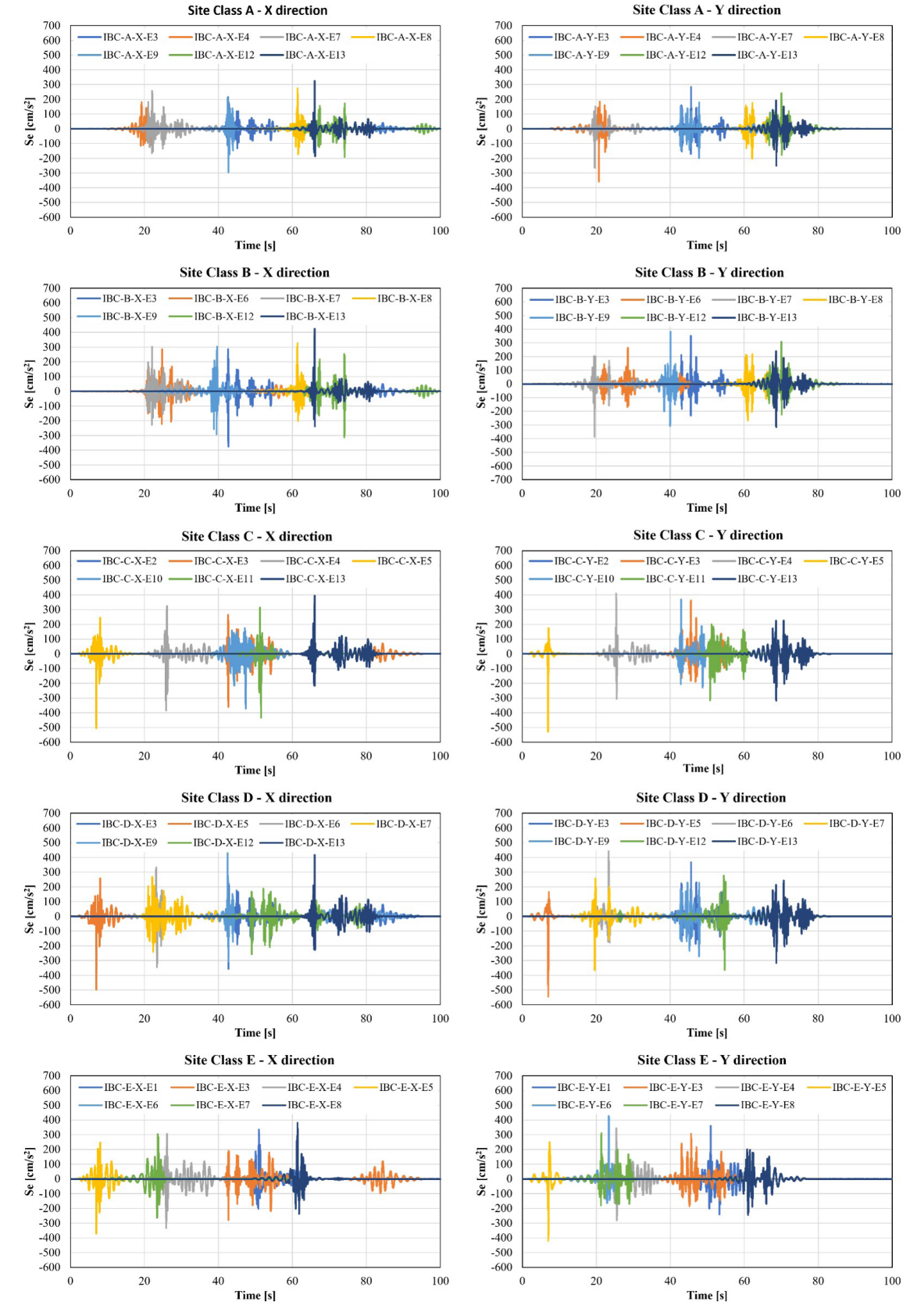
Spectra of matched earthquakes (according to IBC design spectra)



[Figure B3]

Target design spectra and response spectra of the in the X and Y direction (left and right diagrams) for the five soil types.

Accelerograms of matched earthquakes (according to IBC design spectra)



[Figure B4]

Accelerograms of the suites of IBC-spectrum-consistent earthquakes referred to in Figure 5.

APPENDIX C

In this section another part of the study detailed in (Gerges et al., 2023b), which addresses the computational burden of time-history non-linear analyses. A procedure was presented to reduce the duration of accelerograms while maintaining earthquake energy content and efficiency, hence ensuring the accuracy and reliability of results.

An overview of this procedure is presented as it has been used to conduct the study in Part B of this thesis.

C.1. Arias Intensity for Efficient Accelerograms

Given the typically intensive computational demand of nonlinear time-history analyses, it is advantageous to use shorter ground motions in numerical models. However, when cutting a full recorded accelerogram, it is crucial to preserve the most significant peaks and retain a substantial percentage of the seismic energy from the entire event. In this context, the Arias intensity I_A can be a valuable metric for such a requirement since it is an indicator of the destructive potential of an earthquake. The damage experienced by a structure is proportional to the energy dissipated during the seismic event, the Arias intensity is defined as the integral of the dissipated energy $E(\omega)$ over the frequency domain (Arias, 1970):

$$I_A = \int_0^\omega E(\omega) d\omega \quad (99)$$

In the time domain, I_A is expressed as the integral of the square acceleration over the complete duration t of the earthquake (Lilhanand & Tseng, 1987):

$$I_A = \frac{\pi}{2g} \int_0^{t_d} a^2(t) dt \quad (100)$$

where $a(t)$ is the ground acceleration amplitude, g the gravity acceleration and t_d is the total duration of the record.

The Arias intensity defined by Equation (100) is like the formulation used by Trifunac and Brady (Trifunac & Brady, 1975) when introducing the limits of the significant duration of an earthquake.

Hence, the Arias intensity is a good parameter to be adopted to specify the most efficient and energy-conserving time-window of the earthquake. To this end, the SeismoSignal (SeismoSignal, 2016) program can be used. This program calculates the effective duration of an earthquake for a filtered and baseline-corrected ground motion, corresponding to the interval from 5% to 95% of the I_A . An example will be presented to demonstrate how SeismoSignal can be used to determine an appropriate time-window for a specific earthquake.

The example will refer to the components of the IBC-spectrum-consistent earthquakes displayed in Figure B4, namely IBC-C-X-E5 and IBC-C-Y-E5, which is referred to as EQ2 in Part B 2.2.6. of this thesis. Once the accelerograms (see Figure C1a) are uploaded in SeismoSignal and the option “effective duration” is set, the values given in Table C1 can be extracted. Following the time-windows relevant to the 5-95% range of I_A from Table C1, the diagrams in Figure C1b can be extracted. The suitable time windows are bordered in green.

It's important to recognize that the starting and ending times of the time-windows differ for the two components, as evident from Figure C1c. However, to ensure that equivalent-duration components of the earthquake are used for analysis, a single time-window must be set. This approach involves selecting the earliest start time and latest end time from the two time-windows identified using the Arias intensity method, see Figure C1d. Subsequently the full-length accelerograms of Figure C1a are reduced to the those in Figure C1e, which retain 95% of the earthquake energy.

To validate the procedure, a four-story reinforced concrete building model with typical residential loads was used in (Gerges et al., 2023b). A nonlinear time-history analyses using EC8 consistent accelerograms using both the full-length and the short-length earthquakes. Results showed small differences in roof displacement and shear forces between the two scenarios, validating the procedure's effectiveness, the reader is thus referred to (Gerges et al., 2023b) for further details.

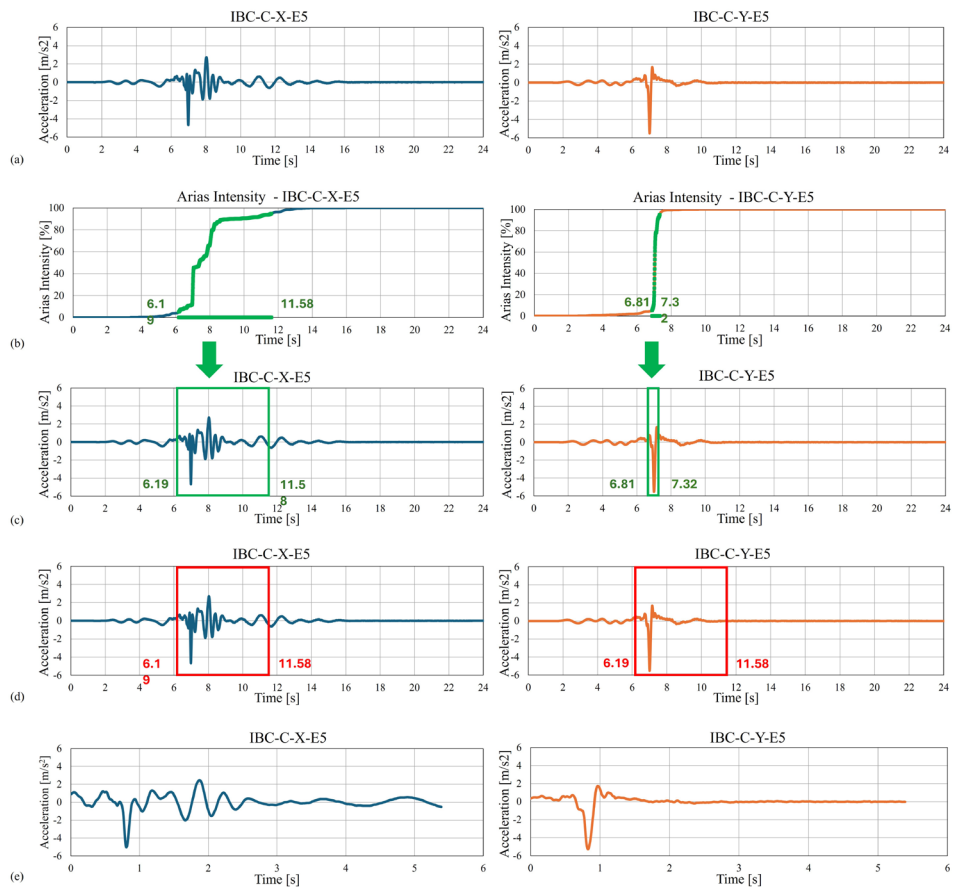
[Table C1]
Duration and Arias Intensity for
IBC-C-X-E5 (from SeismoSignal).

Earthquake component	Duration [s]	Cumulative I_A [m/s]	Percentage of I_A [%]
IBC-C-X-E5	6.19	0.3	5
	11.58	6.2	95
IBC-C-Y-E5	6.81	0.2	5
	7.32	4.1	95

For each earthquake, two accelerograms to be applied in the main horizontal directions, are made available and freely downloadable from a repository.

Since the time length of such accelerograms is typically rather long, which implies a large computational burden, the matter of shortening the spectrum-consistent accelerograms was also addressed in the paper. Based on the Arias intensity of a given earthquake, a procedure is proposed, which can be implemented with the help of commercial matching tools. This procedure was shown to drastically reduce the duration of the horizontal components of a given earthquake while saving 95% of its energy content. It can be used as a general tool to obtain suitable short-length accelerograms from given spectrum-consistent full-length ones. As obtained from this procedure, the corresponding ten suites of short length accelerograms are also made freely downloadable from the repository. They can be profitably exploited to reduce the burden of linear and non-linear time-history analyses of Lebanese structures, while saving the effectiveness of the results.

[Figure C2]
Procedure to obtain a suitable time-window for time-history analyses. (a) Full-length X and Y accelerograms; (b) Arias Intensity of the two components of the earthquake, where the range from 5% to 95% is highlighted in green; (c) time windows corresponding to 5-95% of ξ in green; (d) final chosen time-window for the two components, in red; (e) cut earthquake components for the numerical analyses.



To address the matter of lack of proper accelerograms to be used for the seismic assessment of the Lebanese structures through time-history analyses, wide research in many seismic databases was carried out which eventually led to find thirteen strong earthquakes recorded in the Lebanese area. By means of elaborations of such records, both in the time and in the frequency domain, suitable sets of accelerograms that are consistent with the design response spectra of IBC and of EC8 are found, respectively, for the different considered soil categories. IBC is considered because it is the only code still in force in its country among the three referred by the Lebanese code. EC8 is considered since it is announced to be the future reference code in Lebanon.

