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Anti-Gauss Laguerre polynomials: Some properties and a new interpolation process

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ABSTRACT

Anti-Gauss Laguerre quadrature formulas are based on the zeros of polynomials, we call them Anti-Gauss polynomials, defined in terms of Laguerre orthogonal polynomials. This paper establishes new properties of the Anti-Gauss Laguerre polynomials, and analyzes some "truncated" interpolation processes essentially based on their zeros. Estimates of the corresponding Lebesgue constants are proved, and error bounds in spaces of locally continuous functions, equipped with uniform weighted norms are given. Finally, some numerical tests are presented about the behavior of the Lebesgue functions and Lebesgue constants, and numerical experiments dealing with the approximation of functions having different smoothness are proposed. Comparisons with the results achieved by truncated Lagrange interpolation processes based on Laguerre zeros are shown.

1. Introduction

In 1996 Laurie [1] introduced a new $(m+1)$-point interpolatory quadrature rule, called "anti-Gauss" formula, with a high degree of exactness, based on distinct real nodes and having always positive coefficients. Such rules have been extensively studied by many authors, especially with the aim of constructing efficient quadrature rules, in turn, the main tool in approximating the solutions of integral equations by Nyström methods [2–5]. As proved by Laurie, the $(m+1)$ anti-Gauss nodes are the zeros of a certain polynomial $\tilde{p}_{m+1}$, defined as a linear combination of orthogonal polynomials w.r.t. a weight function w. Several useful properties of $\{\tilde{p}_{m+1}\}_m$ and their zeros have been proved in [1,6]. In particular, we mention the internality of such zeros for a large class of weight functions, and the interlacing properties with the zeros of the orthogonal polynomial $p_m(w,\cdot)$ to which are related.

In this paper, we consider the polynomial sequence $\{\tilde{p}_{m+1}\}_m$ associated to the Laguerre weight $w(x) = x^\alpha e^{-x}$, $\alpha > -1$, hereafter referred to as the *Anti-Gauss Laguerre polynomials*. We will provide some new estimates of the Anti-Gauss polynomial $\tilde{p}_{m+1}$, such as the distance between the zeros and relationships with the zeros of $p_m(w,\cdot)$, estimates of the polynomial $\tilde{p}_{m+1}$ and its derivatives. All this information is useful to investigate whether or not the matrix of zeros of the sequence $\{\tilde{p}_{m+1}\}_m$ can be a candidate for the construction of "optimal" interpolation processes in some weighted spaces of continuous functions, i.e. such that the corresponding Lebesgue constants diverge logarithmically. Lagrange interpolating polynomials share many interesting properties and fair qualities, as well as the ease of computation, elementary manipulation, and, for careful choices of the interpolation set of nodes, the improvement of the accuracy as the smoothness of the function increases. Based on these, Lagrange interpolation processes have been successfully employed in the simultaneous approximation of functions, for the construction of efficient quadrature formulae, and in numerical

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methods for solving functional equations. However, as just mentioned, one of the main difficulties is the choice of the set of interpolation nodes, since the so-called Lebesgue constants, which depend on them, appear both in the theoretical convergence estimate of the error and as amplification factor of the round-off errors.

Here, we focus on the approximation of functions in weighted spaces C_{u}, which include functions with possible algebraic singularities at the origin, and growing exponentially for $x \to \infty$. In C_{u} the Lagrange interpolation polynomial based on the zeros of generalized Laguerre polynomials is a "bad" process since the corresponding Lebesgue constants behave as m^{τ} (see e.g. [7–9]). To overcome this problem, several modifications have been proposed [7,8,10–12], all essentially based on the zeros of the generalized Laguerre polynomials.

In this paper we study the behavior of the Lagrange polynomial based on the zeros of the Anti-Gauss Laguerre polynomial. As usual, the study of the Lebesgue constants involves, among others, the estimate of the weighted fundamental Lagrange polynomial related to the node "closest" to the evaluation point (see, for example, the estimate (56)). Unfortunately, we prove that in the case we select the Anti-Gauss zeros, the divergence of this term behaves like m^{τ}, with $\tau > 1$. The worst behavior is that of the first node, since it is very close to the origin. To improve the process, we introduced a modified interpolating polynomial, which is basically obtained by replacing the first zero of the Anti-Gauss polynomial $\tilde{p}_{m+1}(\cdot)$ with the first zero of the Laguerre polynomial $p_m(w,\cdot)$. Although the process is still not optimal, we have convergence in suitable subspaces of C_{u}.

The paper is structured as follows. In Section 2 we fix some notations, define the function spaces, and collect some basic known facts about orthogonal polynomials and the Anti-Gauss Laguerre polynomial. Section 3 contains useful estimates for the Anti-Gauss polynomial, its derivatives, and new properties on the zeros. All these results are essential for the investigation of the interpolation processes presented in Section 4. Section 5 contains some numerical tests and Section 6 collects the proofs of our theoretical results. Finally, Section 7 deals with some conclusions and outlines possible research perspectives.

2. Notations and preliminary results

Throughout the paper C will denote a positive constant, which may have different values at different occurrences, and the notation $C \neq C(m, f, \ldots)$ will be used to underline that the constant C is independent of $m, f, \ldots$. Moreover, if $A, B > 0$ are quantities depending on some parameters, the writing $A \sim B$, has to be understood as there exists a constant $0 < C \neq C(A, B)$ such that $\frac{B}{C} \leq A \leq CB$.

Furthermore, $\mathbb{P}_m$ will denote the space of the algebraic polynomials of degree at most m.

2.1. Function spaces

Let u be the following weight function

$$\mathrm{u}(x) = x^{\gamma} e^{-\frac{x}{2}}, \qquad x \geq 0, \quad \gamma \geq 0, \tag{1}$$

and let C_{u} be the space defined as

$$C_{\mathrm{u}} = \begin{cases} \left\{ f \in C^0((0,\infty)) : \lim\limits_{\substack{x\to+\infty \\ x\to 0^+}} (f\mathrm{u})(x) = 0 \right\}, & \text{if} \quad \gamma > 0, \\[2ex] \left\{ f \in C^0([0,\infty)) : \lim\limits_{x\to+\infty} (f\mathrm{u})(x) = 0 \right\}, & \text{if} \quad \gamma = 0, \end{cases}$$

equipped with the norm

$$\|f\|_{C_{\mathrm{u}}} = \|f\mathrm{u}\|_{\infty} = \sup_{x\geq 0} |(f\mathrm{u})(x)| .$$

For smoother functions, the Sobolev-type space of order $r \in \mathbb{N}$ is defined as

$$W_r(\mathrm{u}) = \left\{ f \in C_{\mathrm{u}} \ : \ f^{(r-1)} \in AC(0,+\infty) \text{ and } \ \|f^{(r)}\varphi^r \mathrm{u}\| < +\infty \right\},$$

where $\varphi(x) = \sqrt{x}$ and $AC((0,+\infty))$ is the set of the absolutely continuous functions on every closed subset of $(0,+\infty)$. The space $W_r(\mathrm{u})$ is endowed by the norm

$$\|f\|_{W_r(\mathrm{u})} := \|f\mathrm{u}\|_{\infty} + \|f^{(r)}\varphi^r \mathrm{u}\|_{\infty}.$$

In order to deal with more refined subspaces of C_{u}, for any $\lambda \in \mathbb{R}^+$, let $Z_{\lambda}(\mathrm{u})$ be the Zygmund-type space of parameter $0 < \lambda < r$, $r \in \mathbb{N}$,

$$Z_{\lambda}(\mathrm{u}) = \left\{ f \in C_{\mathrm{u}} \ : \ \sup_{t>0} \frac{\Omega_{\varphi}^r(f,t)_{\mathrm{u}}}{t^{\lambda}} < +\infty \right\},$$

where

$$\Omega_{\varphi}^r(f,t)_{\mathrm{u}} = \sup_{0<h\leq t} \|\Delta_{h\varphi}^r f\mathrm{u}\|_{I_{rh}}, \quad t > 0,$$

is the main part of the rth φ–modulus of smoothness, $I_{rh} = \left[4r^2h^2, \frac{C}{h^2}\right]$, being C a fixed positive constant, and

$$\Delta^r_{h\varphi} f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + h\varphi(x)(r-k)\right).$$

The space $Z_\lambda(\mathrm{u})$ is equipped with the norm

$$\|f\|_{Z_\lambda(\mathrm{u})} = \|f\|_{C_\mathrm{u}} + \sup_{t>0} \frac{\Omega^r_\varphi(f,t)_\mathrm{u}}{t^\lambda}.$$

Finally, we recall the definition of the error of best approximation by polynomials of $f \in C_\mathrm{u}$

$$E_m(f)_\mathrm{u} := \inf_{p \in \mathbb{P}_m} \|(f-p)\mathrm{u}\|_\infty.$$

It can be estimated in Zygmund and Sobolev subspaces as [13,14]

$$E_m(f)_\mathrm{u} \le \frac{C}{\sqrt{m^r}} \|f\|_{W_r(\mathrm{u})}, \quad \forall f \in W_r(\mathrm{u}), \tag{2}$$

and

$$E_m(f)_\mathrm{u} \le \frac{C}{\sqrt{m^\lambda}} \|f\|_{Z_\lambda(\mathrm{u})}, \quad \forall f \in Z_\lambda(\mathrm{u}), \tag{3}$$

where in both the cases $C \neq C(m, f)$.

2.2. Laguerre orthogonal polynomials

Let $w(x) = x^\alpha e^{-x}$, with $\alpha > -1$ and let $\{p^w_j(x)\}_j$ be the corresponding sequence of orthogonal Laguerre monic polynomials, that is

$$\langle p^w_j, p^w_i \rangle_w = \int_0^\infty p^w_j(x) p^w_i(x) w(x) dx = \begin{cases} 0, & i \neq j \\ \dfrac{\Gamma(i+\alpha+1)}{i!}, & i = j, \end{cases} \tag{4}$$

where Γ is the Gamma function. The orthogonal sequence (see, for instance, [15]) satisfies the following three-term recurrence relation

$$\begin{cases} p^w_{-1}(x) = 0, \\ p^w_0(x) = 1, \\ p^w_{i+1}(x) = (x - a_i) p^w_i(x) - b_i \, p^w_{i-1}(x), \quad i = 0, 1, 2, \dots \end{cases} \tag{5}$$

where the coefficients are given by

$$\begin{aligned} a_i &= 2i + \alpha + 1, \quad i = 0, 1, 2, \dots \\ b_0 &= \Gamma(1+\alpha), \\ b_i &= i(i+\alpha), \quad i = 1, 2, \dots . \end{aligned} \tag{6}$$

Denoting by $\{x_{m,k}\}_{k=1}^m$ the zeros of $p^w_m(\cdot)$ in increasing order, that is $x_{m,k} < x_{m,k+1}$ for each $k = 1, \dots, m-1$, we recall that they are the eigenvalues of the following tridiagonal matrix [16]

$$J_m = \begin{bmatrix} a_0 & \sqrt{b_1} & & & \\ \sqrt{b_1} & a_1 & \sqrt{b_2} & & \\ & \sqrt{b_2} & a_2 & \ddots & \\ & & \ddots & \ddots & \sqrt{b_{m-1}} \\ & & & \sqrt{b_{m-1}} & a_{m-1} \end{bmatrix},$$

and

$$\frac{C}{m} < x_{m,1} < x_{m,2} < \cdots < x_{m,m} < 4m, \qquad C \neq C(m).$$

In what follows, the following asymptotic formulae for the Laguerre zeros (see [17, Th. 10]) will be useful

$$x_{m,m-k+1} = \nu + 2^{\frac{2}{3}} c_k \nu^{\frac{1}{3}} + d, \qquad k = 1, 2, \dots, m,$$

where $\nu = 4m + 2\alpha + 2$,

$$\begin{aligned} d \; = \; & \frac{2^{\frac{4}{3}}}{5} c_k^2 \nu^{-\frac{1}{3}} + \left(\frac{11}{35} - \alpha^2 - \frac{12}{175} c_k^3\right) \nu^{-1} + \left(\frac{16}{1575} c_k + \frac{92}{7875} c_k^4\right) \frac{2}{3} \nu^{-\frac{5}{3}} \\ & - \left(\frac{15152}{3031875} c_k^5 + \frac{1088}{121275} c_k^2\right) 2^{1/3} \nu^{-\frac{7}{3}} + \mathcal{O}\left(\nu^{-3}\right), \end{aligned}$$

and $\{c_k\}_{k=1,2,\dots}$ denote the negative zeros in decreasing order of the Airy function $\mathrm{Ai}(x)$. In particular, recalling that $c_1 \sim -2.33811$ and $c_2 \sim -4.08795$, we deduce that for $m \geq 6$ it is $0 < d < 1$, and hence for m sufficiently large, and $-1 < \alpha \leq 1$

$$x_{m,m} < 4m + 2\alpha + 3 - (2.3)2^{\frac{2}{3}}\nu^{\frac{1}{3}}. \tag{7}$$

Moreover, uniformly in k and m [18] (see also [7])

$$x_{m,k} \sim \frac{k^2}{m}, \quad k = 1, 2, \dots, m, \tag{8}$$

and then

$$x_{m,k+1} \sim x_{m,k}. \tag{9}$$

For any fixed $0 < \theta < 1$, let $x_{m,j}$ $(j = j(m,\theta))$ be the node defined as

$$x_{m,j} = \min\left\{x_{m,k} : x_{m,k} \geq 4m\theta, \quad k = 1, 2, \dots, m\right\}. \tag{10}$$

About the distance between two consecutive zeros of $p_m^w(\cdot)$ in the interval $[0, x_{m,j}]$ the following estimate holds [19]

$$\Delta x_{m,k} = x_{m,k+1} - x_{m,k} \sim \sqrt{\frac{x_{m,k}}{m}}, \quad k = 1, 2, \dots, j, \tag{11}$$

and, by (8), one has

$$\Delta x_{m,k} \sim \Delta x_{m,k+1}. \tag{12}$$

In what follows, $\{p_m(w,\cdot)\}_m$ will denote the sequence of orthonormal Laguerre polynomials with positive leading coefficients, that is, setting

$$\zeta_m := \sqrt{\langle p_m^w(\cdot), p_m^w(\cdot)\rangle_w} = \sqrt{\frac{\Gamma(m+\alpha+1)}{m!}}, \tag{13}$$

the sequence is defined as

$$\begin{aligned} p_m(w,x) &= \frac{p_m^w(x)}{\zeta_m}, \\ p_m(w,x) &= \gamma_m(w)x^m + \text{ terms of lower degree }, \qquad \gamma_m(w) > 0. \end{aligned} \tag{14}$$

Denoting by d the index of a zero of $p_m(w,\cdot)$ closest to $x \in [0, 4m]$, i.e. $|x - x_{m,d}| = \min_{1\leq k\leq m}|x - x_{m,k}|$, for some positive constant $C \neq C(m,x,d)$, the following inequalities hold true [7]

$$\begin{aligned} \frac{1}{C}\left(\frac{x - x_{m,d}}{x_d - x_{m,d\pm 1}}\right)^2 &\leq p_m^2(w,x)e^{-x}\left(x + \frac{1}{m}\right)^{\alpha+\frac{1}{2}}\sqrt{|4m - x| + m^{\frac{1}{3}}} \\ &\leq C\left(\frac{x - x_{m,d}}{x_{m,d} - x_{m,d\pm 1}}\right)^2. \end{aligned} \tag{15}$$

Therefore, by (15), and introducing the interval

$$\mathcal{A}_m := \mathcal{A}_m(c) = \left(\frac{c}{m}, \ \nu - \nu^{\frac{1}{3}}\right), \quad \nu = 4m + 2\alpha + 2, \quad 0 < c \neq c(m), \tag{16}$$

one deduces

$$|p_m(w,x)|\sqrt{w(x)} \leq \frac{C}{\sqrt[4]{x(4m - x)}}, \quad x \in \mathcal{A}_m, \quad C \neq C(m). \tag{17}$$

Finally, we recall that the Christoffel numbers defined as

$$\lambda_{m,k} := \lambda_{m,k}(w) = \left[\sum_{k=0}^{m-1} p_k^2(w, x_{m,k})\right]^{-1}, \tag{18}$$

can be computed through the matrix J_m, i.e. [16]

$$\lambda_{m,k} = b_0 v_{k,1}^2,$$

where $v_{k,1}$ is the first component of the eigenvector corresponding to the eigenvalue $x_{m,k}$.

2.3. Anti-Gauss Laguerre polynomials

Let $\{\tilde{p}_k(\cdot)\}_k$ be the sequence of the monic *Anti-Gauss Laguerre polynomials*, which can be defined in terms of the Laguerre polynomials $p_k^w(\cdot)$ by [1]

$$\tilde{p}_{k+1}(x) = p_{k+1}^w(x) - b_k p_{k-1}^w(x), \tag{19}$$

where the coefficients b_k are defined in (6). Such polynomials play a crucial role in the anti-Gauss quadrature formula, introduced for the first time by Laurie [1].

The mth anti-Gauss Laguerre rule has the form

$$\int_0^\infty f(x)w(x)dx = \sum_{k=1}^{m+1} \tilde{\lambda}_{m+1,k} f(\tilde{y}_{m+1,k}) + R_m^{AG}(f) := G_{m+1}^{AG}(f) + R_m^{AG}(f)$$

$$G_{m+1}^{AG} = \sum_{k=1}^{m+1} \tilde{\lambda}_{m+1,k} f(\tilde{y}_{m+1,k}), \quad R_m^{AG}(f) = \int_0^\infty f(x)w(x)dx - G_{m+1}^{AG}(f),$$

where $\{\tilde{y}_{m+1,k}\}_{k=1}^{m+1}$ are the zeros of the polynomial $\tilde{p}_{m+1}(\cdot)$ given in (19) and $\{\tilde{\lambda}_{m+1,k}\}_{k=1}^{m+1}$ are positive coefficients. $G_{m+1}^{AG}(f)$ is an interpolatory rule having degree of exactness $2m-1$, and

$$R_m^{AG}(f) = 0, \ \forall f \in \mathbb{P}_{2m-1},$$

and the zeros and weights are chosen such that

$$R_m^{AG}(p) = -R_m^G(p), \quad p \in \mathbb{P}_{2m+1},$$

where $R_m^G(f)$ is the error of the Gauss–Laguerre quadrature formula $G_m(f)$

$$\int_0^\infty f(x)w(x)dx = G_m(f) + R_m^G(f) := \sum_{k=1}^{m} \lambda_{m,k} f(x_{m,k}) + R_m^G(f),$$

with $x_{m,k}$ the zeros of $p_m(w,\cdot)$ and $\lambda_{m,k}$ the Christoffel numbers in (18). Recently, such formulae have attracted some attention for their utility (see, for instance, [2,5,20–22]) also in the multivariate case [3,4,23]. Indeed, they allow one to build new rules, namely, averaged or stratified quadrature formulae, which are characterized by a higher accuracy and smaller computational cost. In addition, they also provide numerical estimates for the error of the Gauss rule for a fixed number of points.

Coming back to the Anti-Gauss Laguerre polynomials in (19), it is known that the zeros $\{x_{m,k}\}_{k=1}^{m}$ of $p_m(w,\cdot)$ interlace the zeros $\{\tilde{y}_{m+1,k}\}_{k=1}^{m+1}$ of $\tilde{p}_{m+1}(\cdot)$, i.e.

$$\tilde{y}_{m+1,1} < x_{m,1} < \tilde{y}_{m+1,2} < x_{m,2} < \cdots < x_{m,m} < \tilde{y}_{m+1,m+1}, \tag{20}$$

and $\{\tilde{y}_{m+1,k}\}_{k=1}^{m+1} \in (0,\infty)$ for each $\alpha > -1$. Similarly to the zeros and weights of the Gauss–Laguerre rule, the computation of $\{\tilde{y}_{m+1,k}, \tilde{\lambda}_{m+1,k}\}_{k=1}^{m+1}$ is not a difficult task, since they are related to the eigenpairs of the matrix

$$\widetilde{J}_{m+1} = \begin{bmatrix} J_m & \sqrt{2b_m}\mathbf{e}_m \\ \sqrt{2b_m}\mathbf{e}_m^T & a_m \end{bmatrix}, \tag{21}$$

with $\mathbf{e}_m = (0,0,\dots,1)^T \in \mathbb{R}^m$, and a_m, b_m defined in (6). Indeed, the zeros $\{\tilde{y}_{m+1,k}\}_{k=1}^{m+1}$ are the eigenvalues of $\widetilde{J}_{m+1}$, and the coefficients $\{\tilde{\lambda}_{m+1,k}\}_{k=1}^{m+1}$ are given by

$$\tilde{\lambda}_{m+1,k} = b_0 \tilde{v}_{k,1}^2,$$

where $\tilde{v}_{k,1}$ is the first component of the eigenvector corresponding to $\tilde{y}_{m+1,k}$.

Let us note that, by applying to the last term in (19) the three term recurrence relation (5), we have

$$\tilde{p}_{m+1}(x) = 2p_{m+1}^w(x) + (a_m - x)p_m^w(x), \quad a_m \sim 2m.$$

Then, by (13) and (14), we can also write

$$\begin{aligned} \tilde{p}_{m+1}(x) &= \zeta_m \left[2p_{m+1}(w,x)\sqrt{\frac{m+\alpha+1}{m+1}} + (a_m - x)p_m(w,x) \right] \\ &\sim \zeta_m \left[2p_{m+1}(w,x) + (a_m - x)p_m(w,x) \right], \end{aligned} \tag{22}$$

where the constants in $\sim$ do not depend on m. Moreover, since $p_m^w(0) = 1$, we easily get

$$\tilde{p}_{m+1}(0) = 2(m+1) + \alpha + 1. \tag{23}$$

Finally, we recall that in [6] it was proved

$$\tilde{\lambda}_{m+1,k} = 2\frac{\langle p_m^w(\cdot), p_m^w(\cdot)\rangle_w}{p_m^w(\tilde{y}_k)\tilde{p}'_{m+1}(\tilde{y}_k)}, \quad k = 1,2,\dots,m+1,$$

or equivalently

$$\tilde{\lambda}_{m+1,k} = 2\frac{\zeta_m}{p_m(w,\tilde{y}_k)\tilde{p}'_{m+1}(\tilde{y}_k)}, \quad k = 1,2,\dots,m+1. \tag{24}$$

3. Some estimates for Anti-Gauss Laguerre polynomials

In this section we collect some new properties concerning the Anti-Gauss Laguerre polynomials $\{\tilde{p}_{m+1}(\cdot)\}_m$, and their zeros. From now on we set

$$\mathcal{D}_m := \left(\frac{C}{m}, 3m\right). \tag{25}$$

The following Proposition holds true.

Proposition 3.1. *For any $x \in \mathcal{A}_m$, with $\mathcal{A}_m$ defined in (16), the following estimate holds*

$$\tilde{p}_{m+1}(w,x)|\sqrt{w(x)} \le C\,\zeta_m \begin{cases} \dfrac{(4m-x)^{\frac{3}{4}}}{\sqrt[4]{x}}, & x \le 3m, \\ \dfrac{m}{\sqrt[4]{x(4m-x)}} & x \ge 3m. \end{cases} \tag{26}$$

Moreover, for $x \in \mathcal{D}_m$, with $\mathcal{D}_m$ defined in (25), we have

$$|\tilde{p}'_{m+1}(x)|\sqrt{w(x)} \le C\,\zeta_m \sqrt{\frac{m}{x}}\,\frac{(4m-x)^{\frac{3}{4}}}{\sqrt[4]{x}}, \tag{27}$$

and

$$|\tilde{p}''_{m+1}(x)|\sqrt{w(x)} \le C\,\zeta_m\,\frac{m}{x}\,\frac{(4m-x)^{\frac{3}{4}}}{\sqrt[4]{x}}, \tag{28}$$

where in all the estimates $0 < C \neq C(m)$ and ζ_m is defined in (13).

From now on, to simplify the notation, we set $\tilde{y}_k := \tilde{y}_{m+1,k}$, $x_k := x_{m,k}$, $\tilde{\lambda}_k := \tilde{\lambda}_{m+1,k}$, and $\lambda_k := \lambda_{m,k}$. Similarly to the Laguerre case, we define the special node $\tilde{y}_j$. With x_j given in (10) for any fixed $\theta \le \frac{3}{4}$, here we define

$$\tilde{y}_j = \max_k \left\{\tilde{y}_k \,:\, \tilde{y}_k < x_j\right\}. \tag{29}$$

About some distances between the zeros, the following proposition holds.

Proposition 3.2. *Let $\tilde{y}_j$ as defined in (29) and set $\Delta\tilde{y}_k = \tilde{y}_{k+1} - \tilde{y}_k$. Then*

(a) For $k = 1, 2, \dots, m-1$, one has

$$\Delta\tilde{y}_k \le C\Delta x_k, \qquad 0 < C \neq C(m) \tag{30}$$

(b) For $k = 1, 2, \dots, j$, one has

$$x_k - \tilde{y}_k \ge C\frac{\Delta x_k}{m}, \qquad 0 < C \neq C(m). \tag{31}$$

(c) For $k = 1, 2, \dots, j$, one deduces

$$\Delta\tilde{y}_k \ge C\frac{\Delta x_k}{m}, \qquad 0 < C \neq C(m) \tag{32}$$

(d) The first and last zeros of $\tilde{p}_{m+1}(\cdot)$ satisfy the following bounds

$$\frac{C}{m^{\frac{\alpha}{2}+\frac{3}{2}}} \le \tilde{y}_1 \le \frac{C}{m}, \quad 0 < C \neq C(m). \tag{33}$$

and

$$\tilde{y}_{m+1} \le 4.5m + 9.5 - (2.3)2^{\frac{2}{3}}\nu_1^{\frac{1}{3}}, \tag{34}$$

where $\nu_1 = 4(m+1) + 2\alpha + 2$ with $-1 < \alpha \le 1$.

Proposition 3.3. *Let $w(x) = x^\alpha e^{-x}$, $\alpha > -1$ and u be the weight defined in (1). Then, the following estimates hold true*

$$\tilde{\lambda}_k \le C\Delta\tilde{y}_k\, w(\tilde{y}_k), \tag{35}$$

and

$$\frac{1}{|\tilde{p}'_{m+1}(\tilde{y}_k)|\mathrm{u}(\tilde{y}_k)} \le C\,\frac{\Delta\tilde{y}_k}{\zeta_m}\,\frac{\tilde{y}_k^{\frac{\alpha}{2}-\gamma-\frac{1}{4}}}{(4m-\tilde{y}_k)^{\frac{1}{4}}}, \quad k = 1, \dots, j, \tag{36}$$

where $C \neq C(m)$.

4. A new interpolation process

In this section, we want to discuss the construction of some interpolation processes, essentially based on the Anti-Gauss Laguerre zeros $\{\tilde{y}_{m+1,k}\}_{k=1}^{m+1}$.

Let us first consider the truncated Lagrange polynomial $\widetilde{\mathcal{L}}_{m+1}(f,\cdot)$, interpolating a given function $f \in C_{\mathrm{u}}$ at the first j zeros of $\tilde{p}_{m+1}(\cdot)$, and vanishing at $\{\tilde{y}_k\}_{k=j+1}^{m+1}$, where the index j is defined in (29). Then, the polynomial is defined as

$$\widetilde{\mathcal{L}}_{m+1}(f,x) = \sum_{k=1}^{j} \widetilde{\ell}_{m+1,k}(x) f(\tilde{y}_k), \qquad \widetilde{\ell}_{m+1,k}(x) = \frac{\tilde{p}_{m+1}(x)}{(x-\tilde{y}_k)\tilde{p}'_{m+1}(\tilde{y}_k)}.$$

Of course, the main question is the convergence $\widetilde{\mathcal{L}}_{m+1}(f) \to f$ when $m \to \infty$, strictly related to the behavior of the Lebesgue constant associated to the truncated Anti-Gauss Laguerre process and defined as

$$\|\tilde{\mathcal{L}}_{m+1}\|_{C_{\mathrm{u}}\to C_{\mathrm{u}}} = \sup_{x\geq 0} \mathrm{u}(x) \sum_{k=1}^{j} \frac{|\tilde{\ell}_{m+1,k}(x)|}{\mathrm{u}(\tilde{y}_k)}.$$

Unfortunately, as the next lemma infers, they seem to have a "bad" behavior, especially "around" the first zero $\tilde{y}_1$.

Lemma 4.1. *Denoting by d the index of the zero of $\tilde{p}_{m+1}$ which is closest to $x \in \mathcal{D}_m$, then*

$$\frac{\sqrt{w(x)}}{\sqrt{w(x_d)}} |\widetilde{\ell}_{m+1,d}(x)| \leq C \begin{cases} m, & d \geq 2, \\ m^{\frac{\alpha}{2}+\frac{3}{2}}, & d = 1, \end{cases}$$

where $C \neq C(m)$.

Thus, in order to obtain a little bit improvement, let us introduce a modified interpolation process, based on the zeros of the polynomial $q_{m+1}(\cdot)$ obtained by replacing the first zero $\tilde{y}_1$ of $\tilde{p}_{m+1}$ with the first zero x_1 of $p_m(w,\cdot)$. To be more precise, let $q_{m+1} \in \mathbb{P}_{m+1}$ defined as

$$q_{m+1}(x) = \frac{\tilde{p}_{m+1}(x)}{x-\tilde{y}_1}(x-z_1), \quad z_1 = x_{m,1} =: x_1, \tag{37}$$

whose zeros $\{z_k\}_{k=1}^{m+1}$ are

$$z_k = \begin{cases} x_1, & k = 1 \\ \tilde{y}_k, & k = 2,\dots,m+1. \end{cases} \tag{38}$$

Hence, the Lagrange polynomial interpolating a function f at the first j zeros z_k and vanishing at $\{z_k\}_{k=j+1}^{m+1}$, where the index j is defined in (29), is given by

$$\tilde{L}_{m+1}(f,x) = \sum_{k=1}^{j} \tilde{l}_{m+1,k}(x) f(z_k), \qquad \tilde{l}_{m+1,k}(x) = \frac{q_{m+1}(x)}{(x-z_k)q'_{m+1}(z_k)}.$$

Note that $\tilde{L}_{m+1}(f,x) \in \mathbb{P}_m$ but $\tilde{L}_{m+1} : C_{\mathrm{u}} \to \mathbb{P}_m$ is not a projector. However, letting

$$\mathcal{P}_m^* = \left\{ p \in \mathbb{P}_m : p(z_i) = 0, \quad z_i > z_j \right\} \subset \mathbb{P}_m,$$

with z_j defined in (29), $\tilde{L}_{m+1}$ projects C_{u} on $\mathcal{P}_m^*$. Moreover, $\bigcup_m \mathcal{P}_m^*$ is dense in C_{u}. In fact, setting

$$\tilde{E}_m(f)_{\mathrm{u}} := \inf_{Q\in\mathcal{P}_m^*} \|(f-Q)\mathrm{u}\|_\infty,$$

from a result in [24] we can deduce the following.

Lemma 4.2. *For any function $f \in C_{\mathrm{u}}$,*

$$\tilde{E}_m(f)_{\mathrm{u}} \leq C\left\{E_M(f)_{\mathrm{u}} + e^{-Am}\|f\mathrm{u}\|_\infty\right\}, \tag{39}$$

where for $0 < \theta \leq \frac{3}{4}$, $M = \left\lfloor m\frac{\theta}{1+\theta} \right\rfloor$ and A and C are positive constants such that $A \neq A(m,f)$ and $C \neq C(m,f)$.

In view of (39), $\tilde{E}_m(f)_{\mathrm{u}}$ can be estimated by the best approximation error $E_M(f)_{\mathrm{u}}$, where M is a proper fraction of m depending on θ.

The following Proposition is essential to estimate the Lebesgue constants associated to the Lagrange interpolating process.

Proposition 4.3. *For any $x \in \mathcal{A}_m$, with $\mathcal{A}_m$ defined in (16), and ζ_m defined in (13), one has*

$$|q_{m+1}(x)|\sqrt{w(x)} \leq C\zeta_m \begin{cases} \frac{(4m-x)^{\frac{3}{4}}}{\sqrt[4]{x}}, & x \leq 3m, \\ \frac{m}{\sqrt[4]{x(4m-x)}} & x \geq 3m, \end{cases} \quad C \neq C(m). \tag{40}$$

Moreover,

$$\frac{1}{|q'_{m+1}(z_1)|\sqrt{w(z_1)}} \le C\frac{\Delta x_1}{\zeta_m}\sqrt[4]{(4m-z_1)z_1}, \quad C \ne C(m), \tag{41}$$

and for $k = 2, 3, \dots, j$

$$\frac{1}{|q'_{m+1}(\tilde{y}_k)|\sqrt{w(\tilde{y}_k)}} \le C\frac{\Delta \tilde{y}_k}{\zeta_m}\frac{1}{\sqrt[4]{\tilde{y}_k(4m-\tilde{y}_k)}}, \quad C \ne C(m). \tag{42}$$

The next theorem proves that, under a suitable assumption on the parameters γ and α, the Lebesgue constants associated to the modified process, i.e.

$$\|\tilde{L}_{m+1}\|_{C_{\mathrm{u}}\to C_{\mathrm{u}}} = \sup_{x\ge 0} \mathrm{u}(x)\sum_{k=1}^{j}\frac{|\tilde{l}_{m+1,k}(x)|}{\mathrm{u}(z_k)},$$

diverge with order $m\log m$.

Theorem 4.4. *Let* $w(x) = x^\alpha e^{-x}$ *and* u *as in* (1). *Assuming that*

$$\max\left(0, \frac{\alpha}{2}+\frac{1}{4}\right) \le \gamma \le \frac{\alpha}{2}+\frac{5}{4}, \tag{43}$$

then, for any $f \in C_{\mathrm{u}}$, *we have*

$$\|\tilde{L}_{m+1}(f)\|_{C_{\mathrm{u}}} \le C\, m\, \log m\, \|f\|_{C_{\mathrm{u}}}$$

with $C \ne C(m, f)$.

Remark 4.1. Recalling that P. Vértesi proved, in a more general context [9], that for any matrix of interpolation knots in $[0, 4m]$, the Lebesgue constants in C_{u} have an order greater than or equal to $\log m$, the Lagrange polynomial introduced above is not an optimal process. On the other hand, this is the first result in the study of Lagrange interpolation processes on Anti-Gauss Laguerre zeros.

About the convergence, the following theorem holds.

Theorem 4.5. *Let* $w(x) = x^\alpha e^{-x}$ *and* $\mathrm{u}(x) = x^\gamma e^{-\frac{x}{2}}$ *where* γ, α *satisfy* (43). *Then, for any* $f \in C_{\mathrm{u}}$

$$\|[f - \tilde{L}_{m+1}(f)]\mathrm{u}\|_\infty \le Cm\log m\left(E_M(f)_{\mathrm{u}} + e^{-Am}\|f\|_{C_{\mathrm{u}}}\right),$$

where $M = \left\lfloor m\frac{\theta}{(1+\theta)}\right\rfloor$ *with* θ *fixed with* $0 < \theta \le \frac{3}{4}$, *and* C, A *are positive constants independent of* m, f.

We remark, that by Theorem 4.5 and using (2) and (3) we immediately get the following estimates.

Corollary 4.6. *Under the assumption of Theorem 4.5, for any* $f \in Z_\lambda(\mathrm{u})$, $\lambda > 2$, *we have*

$$\|[f - \tilde{L}_{m+1}(f)]\mathrm{u}\|_\infty \le C\frac{\log m}{m^{\frac{\lambda}{2}-1}}\|f\|_{Z_\lambda(\mathrm{u})},$$

and for any $f \in W_r(\mathrm{u})$, $r \ge 3$, *we have*

$$\|[f - \tilde{L}_{m+1}(f)]\mathrm{u}\|_\infty \le C\frac{\log m}{m^{\frac{r}{2}-1}}\|f\|_{W_r(\mathrm{u})},$$

where $C \ne C(m, f)$ *in both estimates.*

Remark 4.2. Based on Remark 4.1, our result is not sharp, since the norm of the operator $\tilde{L}_{m+1} : C_u \to C_u$ behaves as $m\log m$ instead of $\log m$. However, as with other interpolation processes based on suitable matrices of nodes, the error in weighted uniform norm decreases according to the smoothness of the function f. Moreover, by comparing estimates of Corollary 4.6 with the analogous results for the truncated Lagrange polynomial $\mathcal{L}_{m+1}(w, f)$ in (44), under the same hypotheses, it can be seen that only one order of m is lost, since

$$\|[f - \mathcal{L}_{m+1}(w, f)]\mathrm{u}\|_\infty \le C\frac{\log m}{m^{\frac{\lambda}{2}}}\|f\|_{Z_\lambda(\mathrm{u})}, \quad f \in Z_\lambda(\mathrm{u}),$$

$$\|[f - \mathcal{L}_{m+1}(w, f)]\mathrm{u}\|_\infty \le C\frac{\log m}{m^{\frac{r}{2}}}\|f\|_{W_r(\mathrm{u})}, \quad f \in W_r(\mathrm{u}).$$

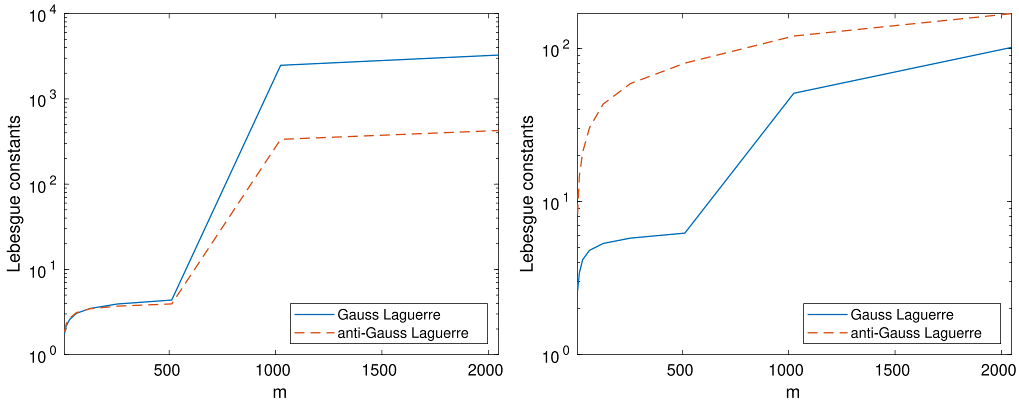

Fig. 1. The Lebesgue constants for $\alpha = 0$ and $\gamma = 0.25$ to the left, and for $\alpha = 0$ and $\gamma = 1.25$ to the right.

5. Numerical experiments

In this section, we give some tests to assess the performance of the truncated Anti-Gauss interpolation process above introduced. In details, we propose tests about

1. the behavior of the Lebesgue constants and Lebesgue functions, for different choices of the parameters α and γ satisfying (43) (see Example 5.1)
2. the approximation of functions having different regularity (see Example 5.2).

In all the cases, we compare the results with those achieved by the optimal truncated interpolation process introduced in [10] (see also [25]) based on the first j zeros of the Laguerre polynomial $p_m(w, \cdot)$, where j is the index of the node defined in (10)

$$\mathcal{L}_{m+1}(w, f, x) = \sum_{k=1}^{j} \ell_{m+1,k}(w, x) f(x_k), \tag{44}$$

with

$$\begin{aligned} \ell_{m+1,k}(w, x) &= \frac{p_m(w, x)}{p'_m(w, x_k)(x - x_k)} \frac{4m - x}{4m - x_k}, \quad k = 1, 2, \dots, m, \\ \ell_{m+1,m+1}(w, x) &= \frac{p_m(w, x)}{p_m(w, 4m)}. \end{aligned}$$

We remark that, as proved in [25], under the same assumptions (43), the associated Lebesgue constants diverge as $\log m$ in C_{u}.

All the computations are performed on an Intel Xeon E-2244G system with 16 Gb RAM, running Matlab R2024b. The Matlab software is available from GitHub at https://github.com/CaNAgroup/AGLagrange and at the web page https://bugs.unica.it/cana/software.html.

Example 5.1. In Fig. 1 the red dashed line represents the weighted Lebesgue constants values associated to $\tilde{L}_{m+1}$, as m increases in the case $w(x) = e^{-x}$, and fixing in (29) $\theta = 0.15$. In this case, we consider two weights functions u, both satisfying the conditions (43). Specifically, we consider the smallest and the largest values that γ can take for this choice of $\alpha = 0$. For increasing values of m, in both cases we compare their values with the Lebesgue constants associated to $\mathcal{L}_{m+1}(w)$ in (44), represented by the continuous blue line. First, we note that for lower values of γ there are no major differences between the two interpolation methods. On the other hand, as γ approaches the right limit, we observe that the Lebesgue constants of $\tilde{L}_{m+1}$ are larger than the optimal ones of $\mathcal{L}_{m+1}(w)$, although not excessively so.

In Fig. 2, we have plotted the Lebesgue functions associated to the weight $w(x) = x^{0.6}e^{-x}$ and $\mathrm{u}(x) = x^{0.55}e^{-x/2}$ for two fixed values $n = 32$ and $n = 128$. Compared to those associated with (44), we do not observe large variations between the two functions.

Example 5.2. Let us consider the functions

$$f_1(x) = \sin(x), \qquad f_2(x) = \left|x - \frac{1}{2}\right|^{9/2}, \qquad f_3(x) = \frac{e^{x/4}}{1 + x^2}. \tag{45}$$

and let us approximate them with the interpolation polynomials involving the weight functions $w(x) = e^{-x}$ for f_1; $w(x) = x^{0.6}e^{-x}$ for f_2, and $w(x) = x^{-1/2}e^{-x}$ for f_3.

For increasing values of m, in Tables 1 and 2, we plot the following errors

$$\epsilon_m^{AG}(f_i) = \max_{x \in X} |[f_i(x) - L_{m+1}(f_i, x)]\mathrm{u}(x)|, \qquad i = 1, 2, 3,$$

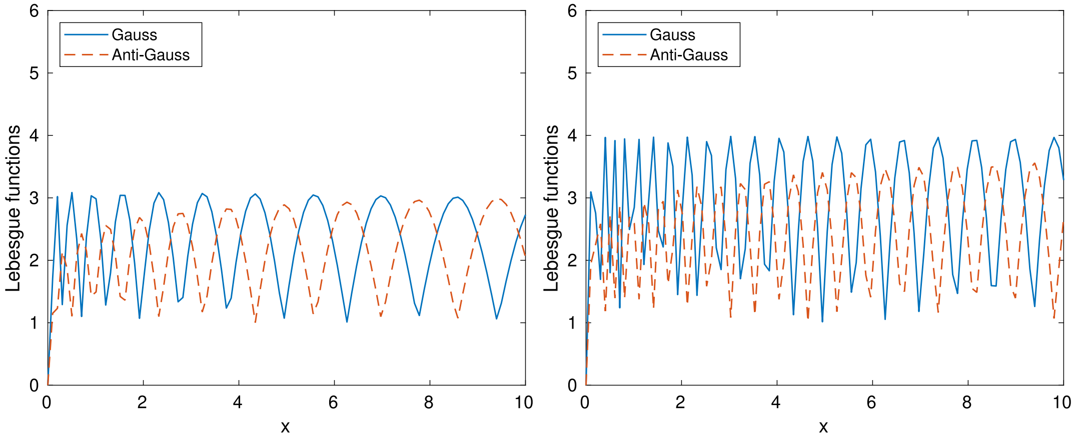

Fig. 2. Lebesgue functions for $\alpha = 0.6$, $\gamma = 0.55$ and $n = 32$ to the left and $n = 128$ to the right.

Table 1
Numerical results for the functions f_1 and f_2 in Example 5.2.

m	j	$\epsilon_m^{AG}(f_1)$	$\epsilon_m^{G}(f_1)$
4	3	7.98e−02	1.30e−01
8	7	2.36e−02	2.80e−02
16	13	1.37e−03	7.97e−04
32	26	4.16e−06	2.76e−06
64	52	3.92e−11	3.65e−11
128	105	1.44e−15	1.22e−15

m	j	$\epsilon_m^{AG}(f_2)$	$\epsilon_m^{G}(f_2)$
16	13	5.08e−01	3.95e−01
32	26	4.09e−04	3.29e−04
64	52	6.26e−05	3.01e−05
128	105	1.45e−05	1.51e−05
256	209	2.53e−06	2.75e−06
512	419	4.88e−07	3.12e−07

Table 2
Numerical results for the function f_3 in Example 5.2.

m	j	$\epsilon_m^{AG}(f_3)$	$\epsilon_m^{G}(f_3)$
4	3	5.29e−02	4.38e−02
8	7	3.06e−02	2.36e−02
16	13	2.76e−03	1.67e−03
32	26	4.34e−04	3.79e−04
64	52	1.32e−05	1.26e−05
128	105	1.89e−07	1.28e−07
256	210	2.79e−10	1.90e−10
512	419	2.50e−14	1.69e−14

and

$$\epsilon_m^G(f_i) = \max_{x\in X} |[f_i(x) - \mathcal{L}_{m+1}(f_i, x)]\mathrm{u}(x)|, \qquad i = 1, 2, 3,$$

where X is a sufficiently large uniform mesh of points in the finite interval $(0, 10)$, and

$$\mathrm{u}(x) = \begin{cases} e^{-x/2}, & i = 1, \\ xe^{-x/2}, & i = 2, \\ \sqrt{x}e^{-x/2}, & i = 3. \end{cases}$$

As we can see, in all cases the new interpolation gives comparable errors to the optimal one (44). In the case of the function f_1, the convergence is fast due to the smoothness of f_1. For f_2, since it belongs to $Z_{9/2}(\mathrm{u})$, according to Corollary 4.6, the theoretical error of our process is $\mathcal{O}(m^{-5/4} \log m)$, but the numerical results are better than the theoretical estimate. Finally, f_3 is also analytic, and by fixing $m = 512$, we get almost machine precision.

6. Proofs

First, let us recall that for any $P_m \in \mathbb{P}_m$ one has [8, Theorem 2.1] (see also [25, (4.1)])

$$\|P_m\mathrm{u}\|_\infty \le \max_{x\in\mathcal{A}_m} |P_m(x)\mathrm{u}(x)|, \qquad C \ne C(m). \tag{46}$$

Moreover, we also mention the following lemma which is crucial for some of our results.

Lemma 6.1 ([13, Lemma 8]). *For $\theta \in (0,1)$, and for any $x, y \in \left(\frac{C}{m}, 4m\theta\right)$ s.t. $|x-y| \le C\sqrt{\frac{x}{m}}$, it is*

$$w(x) \sim w(y), \quad C \neq C(m).$$

Proof of Proposition 3.1. In view of estimate (17), we can deduce

$$|p_{m+1}(w,x)|\sqrt{w(x)} \le \frac{C}{\sqrt[4]{x(4m-x)}}, \qquad x \in \mathcal{A}_m, \tag{47}$$

and

$$|(a_m - x)p_m(w,x)|\sqrt{w(x)} \le C \begin{cases} \dfrac{(4m-x)^{\frac{3}{4}}}{\sqrt[4]{x}}, & \dfrac{C}{m} \le x \le 3m, \\ \dfrac{m}{\sqrt[4]{x(4m-x)}} & 3m \le x \le 4m - Cm^{\frac{1}{3}}. \end{cases}$$

Hence, (26) follows by combining the last two estimates with (22).

Let us now prove (27). To this end, let us recall [26, page 128]

$$\frac{d}{dx}p_m^w(x) = -p_{m-1}^{\bar{w}}(x), \quad \bar{w}(x) = xw(x),$$

and thus, by (4), we have

$$p_m'(w,x) = -\sqrt{m}p_{m-1}(\bar{w},x). \tag{48}$$

So, by (22), it follows

$$\tilde{p}_{m+1}'(x) = -\zeta_m \left[2\sqrt{m+1}p_m(\bar{w},x) + (a_m - x)\sqrt{m}p_{m-1}(\bar{w},x) + p_m(w,x)\right], \tag{49}$$

so that, by using estimate (17), for any $x \in \mathcal{D}_m$, we have

$$\begin{aligned} |\tilde{p}_{m+1}'(x)|\sqrt{\bar{w}(x)} &\le C\,\zeta_m \Big[2\sqrt{m+1}|p_m(\bar{w},x)|\sqrt{\bar{w}(x)} \\ &\quad + |a_m - x|\sqrt{m}|p_{m-1}(\bar{w},x)|\sqrt{\bar{w}(x)} + |p_m(w,x)|\sqrt{\bar{w}(x)}\Big] \\ &\le C\,\zeta_m \left(\frac{\sqrt{m}}{\sqrt[4]{x(4m-x)}} + \frac{\sqrt{m}|a_m - x|}{\sqrt[4]{x(4m-x)}} + \frac{\sqrt{m}}{\sqrt[4]{x(4m-x)}}\right) \\ &\le C\,\zeta_m \frac{\sqrt{m}}{\sqrt[4]{x}}\left((4m-x)^{\frac{3}{4}} + \frac{1}{\sqrt[4]{4m-x}}\right) \\ &\le C\,\zeta_m \sqrt{m}\,\frac{(4m-x)^{\frac{3}{4}}}{\sqrt[4]{x}}, \end{aligned}$$

and then (27) follows. To prove (28), differentiate twice (22)

$$\tilde{p}_{m+1}''(x) = \zeta_m \left[2p_{m+1}''(w,x)\sqrt{\frac{m+\alpha+1}{m+1}} + (a_m - x)p_m''(w,x) - 2p_m'(w,x)\right].$$

By using twice (48), we deduce

$$\begin{aligned} \tilde{p}_{m+1}''(x) = \zeta_m \Big[&2\,p_{m-1}(\bar{\bar{w}},x)\sqrt{(m+\alpha+1)m} + (a_m - x)\sqrt{m(m-1)}p_{m-2}(\bar{\bar{w}},x) \\ &+2\sqrt{m}\,p_{m-1}(\bar{w},x)\Big] \end{aligned}$$

where $\bar{\bar{w}}(x) = x^2\ w(x)$, and by (17) we have for $x \in \mathcal{D}_m$

$$|\tilde{p}_{m+1}''(x)|\sqrt{w(x)}x \le C\zeta_m\,m\,\frac{(4m-x)^{\frac{3}{4}}}{\sqrt[4]{x}},$$

by which (28) follows. □

Proof of Proposition 3.2.

(a) Setting $\Delta x_{m,0} = x_{m,1}$, by (20) one has

$$\tilde{y}_{k+1} - \tilde{y}_k \le x_{m,k+1} - x_{m,k-1} \le \Delta x_{m,k} + \Delta x_{m,k-1}, \quad k = 1, 2, \dots, m-1.$$

Then, using (11) and (12), (30) follows.

(b) Now we prove (31). Setting

$$Q_{2m+1}(x) = \tilde{p}_{m+1}(x) p_m(w, x),$$

in view of the interlacing property, we have $Q'_{2m+1}(x_k) > 0$, $Q'_{2m+1}(\tilde{y}_k) < 0$, and hence

$$0 < Q'_{2m+1}(x_k) - Q'_{2m+1}(\tilde{y}_k) = (x_k - \tilde{y}_k) Q''_{2m+1}(\xi_k),$$

where $\xi_k \in (\tilde{y}_k, x_k)$. Therefore

$$\begin{aligned} \frac{1}{(x_k - \tilde{y}_k)} &\leq \frac{Q''_{2m+1}(\xi_k)}{Q'_{2m+1}(x_k)} \\ &= \frac{\tilde{p}''_{m+1}(\xi_k) p_m(w, \xi_k) + 2\tilde{p}'_{m+1}(\xi_k) p'_m(w, \xi_k) + \tilde{p}_{m+1}(\xi_k) p''_m(w, \xi_k)}{\tilde{p}_{m+1}(x_k)\, p'_m(w, x_k)}. \end{aligned} \tag{50}$$

Let us estimate the denominator. We recall that (see e.g. [24, estimate (29)])

$$\frac{1}{|p'_m(w, x_k)| \sqrt{w(x_k)}} \sim \Delta x_k \sqrt[4]{4m x_k}, \qquad k = 1, 2, \dots, j$$

and by (15) and (22) we get

$$|\tilde{p}_{m+1}(x_k)| \sqrt{w(x_k)} \sim \zeta_m |p_{m+1}(w, x_k)| \sqrt{w(x_k)} \geq C \frac{\zeta_m}{\sqrt[4]{x_k(4m - x_k)}}.$$

Then, we deduce

$$\frac{1}{Q'_{2m+1}(x_k) w(x_k)} \leq C \Delta x_k \frac{\sqrt{x_k} \sqrt[4]{4m(4m - x_k)}}{\zeta_m} \leq C \Delta x_k \frac{\sqrt{x_k m}}{\zeta_m}. \tag{51}$$

Let us now estimate the numerator $Q''_{2m+1}(\xi_k)$. By (28), (27) and (47) we have

$$|Q''_{2m+1}(\xi_k)| w(\xi_k) \leq C\, m\, \zeta_m \frac{\sqrt{4m - \xi_k}}{\xi_k \sqrt{\xi_k}}. \tag{52}$$

Hence, combining (51) and (52) in (50), and by virtue of Lemma 6.1, we conclude for $k = 1, 2, \dots, j$

$$\begin{aligned} \frac{1}{(x_k - y_k)} &\leq \frac{Q''_{2m+1}(\xi_k) w(\xi_k)}{Q'_{2m+1}(x_k) w(x_k)} \frac{w(x_k)}{w(\xi_k)} \\ &\leq C m \Delta x_k \sqrt{x_k m} \frac{\sqrt{4m - \xi_k}}{\xi_k \sqrt{\xi_k}} \\ &\leq C \frac{m}{\Delta x_k} \end{aligned}$$

namely (31).

(c) Estimate (32) easily follows by $\tilde{y}_{k+1} - \tilde{y}_k > x_k - \tilde{y}_k$ combined with (31).

(d) The upper bound in (33) is a consequence of the interlacing property and (8), by which $\tilde{y}_1 < x_1 \sim \frac{1}{m}$. To prove the reverse inequality (33), consider

$$|\tilde{p}_{m+1}(0)| = |\tilde{p}_{m+1}(\tilde{y}_1) - \tilde{p}_{m+1}(0)| = |\tilde{p}'_{m+1}(\xi)|\ \tilde{y}_1, \quad \xi \in (0, \tilde{y}_1).$$

Now, by (49) and estimate (15), it follows for $\xi \in (0, \tilde{y}_1)$

$$|\tilde{p}'_{m+1}(\xi)| \left(\xi + \frac{1}{m} \right)^{\frac{\alpha}{2} + \frac{3}{4}} \leq\ \zeta_m m^{\frac{5}{4}}.$$

Then, taking into account (23), we have at least

$$\tilde{y}_1 = \frac{|\tilde{p}_{m+1}(0)|}{|\tilde{p}'_{m+1}(\xi)|} \geq C \frac{m}{\zeta_m\ m^{\frac{\alpha}{2} + 2}} \geq \frac{C}{m^{\frac{\alpha}{2} + \frac{3}{2}}},$$

where in the last bound we used $\zeta_m > \sqrt{m+1}$.

In order to prove (34) let us rewrite the matrix $\widetilde{J}_{m+1}$ defined in (21) as

$$\begin{aligned} \widetilde{J}_{m+1} &= \begin{bmatrix} J_m & \sqrt{b_m} \mathbf{e}_m \\ \sqrt{b_m} \mathbf{e}_m^T & a_m \end{bmatrix} + \begin{bmatrix} \mathbf{0} & (\sqrt{2} - 1)\sqrt{b_m} \mathbf{e}_m \\ (\sqrt{2} - 1)\sqrt{b_m} \mathbf{e}_m^T & 0 \end{bmatrix} \\ &= J_{m+1} + \hat{J}_{m+1}, \end{aligned}$$

where $\mathbf{0}$ is the null matrix of order m. Note that $J_{m+1}, \widetilde{J}_{m+1}, \hat{J}_{m+1}$ are Hermitian matrices such that the eigenvalues of J_{m+1} are the zeros $x_{m+1,1} < x_{m+1,2} < \cdots < x_{m+1,m+1}$ of $p_{m+1}(w, \cdot)$, those of $\widetilde{J}_{m+1}$ are the zeros $\tilde{y}_1 < \tilde{y}_2 < \cdots < \tilde{y}_{m+1}$ of $\tilde{p}_{m+1}$, and the eigenvalues $\hat{\sigma}_1 < \hat{\sigma}_2 < \cdots < \hat{\sigma}_{m+1}$, of $\hat{J}_{m+1}$ are such that $\hat{\sigma}_{m+1} = (\sqrt{2}-1)b_m$ with $b_m = \sqrt{m(m+\alpha)}$. Therefore, by [27, p.315, Th. 5.10] according to which

$$\tilde{y}_i \le x_{m+1,i} + \hat{\sigma}_{m+1},$$

we can immediately deduce

$$\tilde{y}_{m+1} \le x_{m+1,m+1} + (\sqrt{2}-1)b_m. \tag{53}$$

Hence, in view of (7) with $m+1$ in place of m

$$x_{m+1,m+1} < 4(m+1) + 2\alpha + 3 - (2.3)2^{\frac{2}{3}} \nu_1^{\frac{1}{3}}, \quad \nu_1 = 4(m+1) + 2\alpha + 2,$$

and thus, by (53) we can conclude

$$\tilde{y}_{m+1} \le 4.5m + 9.5 - (2.3)2^{\frac{2}{3}} \nu_1^{\frac{1}{3}}. \quad \square$$

Proof of Proposition 3.3. By [23, Corollary 1] we can write

$$\tilde{\lambda}_k\, e^{\tilde{y}_k} \le \int_{\tilde{y}_{k-2}}^{\tilde{y}_{k+2}} x^\alpha dx,$$

and hence

$$\frac{\tilde{\lambda}_k}{w(\tilde{y}_k)} \le \frac{\int_{\tilde{y}_{k-2}}^{\tilde{y}_{k+2}} x^\alpha dx}{\tilde{y}_k^\alpha} \le \frac{\xi^\alpha}{\tilde{y}_k^\alpha} 2\Delta\tilde{y}_k,$$

where $\xi \in (\tilde{y}_{k-2}, \tilde{y}_{k+2})$.

Let us now observe that using (9) and (20) we can deduce that $\tilde{y}_{k-2} \sim \tilde{y}_k$, since $\tilde{y}_{k-2} > x_{k-1} \sim x_k \ge \tilde{y}_k$, and $\tilde{y}_{k+2} \sim \tilde{y}_k$, since $\tilde{y}_{k+2} < x_{k+2} \sim x_{k-1} < \tilde{y}_k$. Therefore,

$$\frac{\tilde{\lambda}_k}{w(\tilde{y}_k)} \le C\Delta\tilde{y}_k,$$

and the (35) follows.

Now we prove (36). By (24), we can write

$$\frac{1}{\tilde{p}'_{m+1}(\tilde{y}_k)} = \frac{\tilde{\lambda}_{m+1,k}\, p_m(w, \tilde{y}_k)}{2\zeta_m},$$

and in view of (35)

$$\frac{1}{|\tilde{p}'_{m+1}(\tilde{y}_k)|\mathrm{u}(\tilde{y}_k)} \le C\, \frac{\Delta\tilde{y}_k}{\zeta_m}\, \frac{\sqrt{w(\tilde{y}_k)}}{\mathrm{u}(\tilde{y}_k)} |p_m(w, \tilde{y}_k)| \sqrt{w(\tilde{y}_k)}.$$

Thus, by (17), it follows for $k = 1, 2, \ldots, j$,

$$\frac{1}{|\tilde{p}'_{m+1}(\tilde{y}_k)|\mathrm{u}(\tilde{y}_k)} \le C\, \frac{\Delta\tilde{y}_k}{\zeta_m}\, \frac{\tilde{y}_k^{\frac{\alpha}{2}-\gamma-\frac{1}{4}}}{(4m-\tilde{y}_k)^{\frac{1}{4}}}. \quad \square$$

Proof of Lemma 4.1. By definition,

$$\frac{\sqrt{w(x)}}{\sqrt{w(x_d)}} |\tilde{\ell}_{m+1,d}(x)| = \frac{\sqrt{w(x)}}{\sqrt{w(x_d)}} \left| \frac{\tilde{p}_{m+1}(x)}{(x-\tilde{y}_d)\tilde{p}'_{m+1}(\tilde{y}_d)} \right|,$$

so that using (27), for $x \in \mathcal{D}_m$, we deduce

$$\left| \frac{\tilde{p}_{m+1}(x)}{x-\tilde{y}_d} \right| \sqrt{w(x)} = |\tilde{p}'_{m+1}(\xi)| \sqrt{w(x)} \le C\zeta_m \sqrt{m}\, \frac{(4m-x)^{\frac{3}{4}}}{x^{\frac{3}{4}}} \frac{\sqrt{w(x)}}{\sqrt{w(\xi)}}.$$

Moreover, taking into account that $|\xi - \tilde{y}_d| < |\tilde{y}_{d+1} - \tilde{y}_{d-1}| \le C\sqrt{\frac{x_d}{m}}$, by Lemma 6.1, we have

$$\left| \frac{\tilde{p}_{m+1}(x)}{x-\tilde{y}_d} \right| \sqrt{w(x)} \le C\zeta_m \sqrt{m}\, \frac{(4m-x)^{\frac{3}{4}}}{x^{\frac{3}{4}}},$$

and using (36) we get

$$\frac{\sqrt{w(x)}}{\sqrt{w(\tilde{y}_d)}} |\tilde{\ell}_{m+1,d}(x)| \le C\sqrt{m}\, \frac{\Delta\tilde{y}_d}{\tilde{y}_d^{\frac{1}{4}} x^{\frac{3}{4}}}\, \frac{(4m-x)^{\frac{3}{4}}}{(4m-\tilde{y}_d)^{\frac{1}{4}}} \le C\sqrt{m}\frac{\Delta\tilde{y}_d}{\tilde{y}_d^{\frac{1}{2}}}\, \frac{(4m-\tilde{y}_d)^{\frac{1}{2}}}{\tilde{y}_d^{\frac{1}{2}}}. \tag{54}$$

Now, for $d \geq 2$, taking into account $\frac{(4m-\tilde{y}_d)^{\frac{1}{2}}}{\tilde{y}_d^{\frac{1}{2}}} \leq \frac{C}{\Delta x_{d-1}}$, $\Delta \tilde{y}_d \leq C\Delta x_d$ and $\tilde{y}_d > \frac{C}{m}$, it follows

$$\frac{\sqrt{w(x)}}{\sqrt{w(x_d)}}|\widetilde{\ell}_{m+1,d}(x)| \leq Cm.$$

For $d = 1$, by (54), since $\Delta y_1 \leq \frac{C}{m}$, $y_1 \geq \frac{C}{m^{\frac{\alpha}{2}+\frac{3}{2}}}$, it follows

$$\frac{\sqrt{w(x)}}{\sqrt{w(\tilde{y}_d)}}|\widetilde{\ell}_{m+1,d}(x)| \leq Cm^{\frac{\alpha}{2}+\frac{3}{2}},$$

and the lemma is completely proved. □

Proof of Proposition 4.3. Assume first $x \in \mathcal{A}_m \cap \left\{|x - z_1| \leq \frac{C}{m}\right\}$. By definition (37)

$$\begin{aligned}\left||q_{m+1}(x)|\sqrt{w(x)}\right| &= \left|\frac{\tilde{p}_{m+1}(x)}{x-\tilde{y}_1}\right| |x - z_1|\sqrt{w(x)} \\ &= \left|\tilde{p}'_{m+1}(\xi)\right| |x - z_1|\sqrt{w(\xi)}\frac{\sqrt{w(x)}}{\sqrt{w(\xi)}}\end{aligned} \tag{55}$$

with $|\xi - z_1| < |x - z_1|$. By Lemma 6.1 and (27), and using $|x - z_1| \leq \frac{C}{m}$ one deduces

$$|q_{m+1}(x)|\sqrt{w(x)}| \leq C\frac{|\tilde{p}'_{m+1}(\xi)|\sqrt{\bar{w}(\xi)}}{m\sqrt{\xi}} \leq C\frac{\zeta_m}{\sqrt{m}}\frac{(4m-\xi)^{\frac{3}{4}}}{\sqrt[4]{\xi}\sqrt{\xi}} \leq C\zeta_m\frac{(4m-x)^{\frac{3}{4}}}{\sqrt[4]{x}},$$

since $\frac{1}{\sqrt{\xi}} \leq C\sqrt{m}$, and $\xi \sim x$.

Assume now $x \in \mathcal{A}_m \cap \{|x - z_1| > \frac{C}{m}\}$. Using $|x - \tilde{y}_1| = x - \tilde{y}_1 > x - x_1$ and by applying in (55), estimate (26), the assertion (40) follows.

To prove (41), start from $q'_{m+1}(z_1) = \frac{\tilde{p}_{m+1}(z_1)}{z_1-\tilde{y}_1} = \zeta_m 2\frac{p_{m+1}(w,z_1)}{z_1-\tilde{y}_1}$, and hence

$$\left|\frac{1}{q'_{m+1}(z_1)\sqrt{w(z_1)}}\right| = \frac{z_1-\tilde{y}_1}{2\zeta_m|p_{m+1}(w,z_1)|\sqrt{w(z_1)}}.$$

By (15), taking into account $\frac{z_1-\tilde{y}_1}{\sqrt{z_1}} \leq \frac{C}{\sqrt{m}}$

$$\left|\frac{1}{q'_{m+1}(z_1)\sqrt{w(z_1)}\sqrt{z_1}}\right| \leq C\frac{\sqrt[4]{z_1(4m-z_1)}}{\zeta_m\sqrt{m}} \leq C\Delta x_1\frac{\sqrt[4]{4m-z_1}}{\zeta_m\sqrt[4]{z_1}},$$

and also

$$\left|\frac{1}{q'_{m+1}(z_1)\sqrt{w(z_1)}}\right| \leq C\frac{\Delta x_1}{\zeta_m}\sqrt[4]{(4m-z_1)z_1}.$$

To prove (42), we observe that for $k \geq 2$ it is $q'_{m+1}(\tilde{y}_k) = \frac{\tilde{p}'_{m+1}(\tilde{y}_k)}{\tilde{y}_k-\tilde{y}_1}(\tilde{y}_k - x_1)$ and hence

$$\frac{1}{|q'_{m+1}(\tilde{y}_k)|\sqrt{w(\tilde{y}_k)}} = \frac{1}{|\tilde{p}'_{m+1}(\tilde{y}_k)|\sqrt{w(\tilde{y}_k)}}\frac{\tilde{y}_k-\tilde{y}_1}{\tilde{y}_k-x_1}.$$

First we show that $\tilde{y}_2 - x_1 \geq \frac{C}{m}$. Starting from

$$p_m(w,\tilde{y}_2) = p'_m(w,\xi)(\tilde{y}_2 - x_1), \quad \xi \in (x_1, \tilde{y}_2)$$

we get

$$\frac{1}{\tilde{y}_2 - x_1} \leq C\sqrt{m}\frac{|p_m(\bar{w},\xi)|}{|p_m(w,\tilde{y}_2)|}\frac{\sqrt{w(\xi)}}{\sqrt{w(\tilde{y}_2)}}\frac{\sqrt{w(\tilde{y}_2)}}{\sqrt{w(\xi)}}$$

and by Lemma 6.1 and estimate (15) we have at least

$$\frac{1}{\tilde{y}_2 - x_1} \leq C\sqrt{m}\frac{\tilde{y}_2^{\frac{1}{4}}}{\xi^{\frac{3}{4}}}\frac{\sqrt[4]{4m-\tilde{y}_2}}{\sqrt[4]{4m-\xi}} \leq Cm,$$

and hence

$$\tilde{y}_2 - x_1 \geq \frac{C}{m}.$$

As a consequence, since $x_1 - \tilde{y}_1 \le \frac{C}{m}$, we have

$$\frac{\tilde{y}_k - \tilde{y}_1}{\tilde{y}_k - x_1} = 1 + \frac{x_1 - \tilde{y}_1}{\tilde{y}_k - x_1} \le \frac{x_1 - \tilde{y}_1}{\tilde{y}_2 - x_1} \le C.$$

Then, in view of (36)

$$\frac{1}{|q'_{m+1}(\tilde{y}_k)|\sqrt{w(\tilde{y}_k)}} \le C \frac{\Delta x_k}{\zeta_m \sqrt[4]{\tilde{y}_k(4m - \tilde{y}_k)}}, \quad k = 2, \dots, j. \quad \square$$

The following three lemmas are useful for the proof of Theorem 4.4.

Lemma 6.2. *For $x \in \mathcal{A}_m$, the following estimate holds true*

$$\max_{x \in \mathcal{A}_m} |q'_{m+1}(x)|\sqrt{w(x)}\sqrt{x} \le C\,\zeta_m\, m^{\frac{3}{2}}, \quad C \ne C(m).$$

Proof. By the Bernstein inequality [13]

$$\max_{x \ge 0} |P'_m(x)\sqrt{w(x)x}| \le C\sqrt{m} \max_{x \ge 0} |P_m(x)\sqrt{w(x)}|$$

and by (46) we have

$$|q'_{m+1}(x)\sqrt{w(x)x}| \le C\sqrt{m} \max_{x \in \mathcal{A}_m} |q_{m+1}(x)\sqrt{w(x)}|.$$

Since by Proposition 4.3

$$\max_{x \in \mathcal{A}_m} |q_{m+1}(x)\sqrt{w(x)}| \le C\zeta_m\, m,$$

the lemma follows. $\square$

Lemma 6.3. *Let d be the index of the zero of $\tilde{p}_{m+1}$ that is closest to $x \in \mathcal{D}_m$. Then, the following estimate holds*

$$\frac{\sqrt{w(x)}}{\sqrt{w(x_d)}} |\tilde{l}_{m+1,d}(x)| \le Cm, \quad C \ne C(m).$$

Proof. For $d = 1$, by (55)

$$\tilde{l}_{m+1,1}(x) \frac{\sqrt{w(x)}}{\sqrt{w(z_1)}} = \left| \frac{\tilde{p}_{m+1}(\xi)}{q'_{m+1}(z_1)} \right| \frac{\sqrt{w(x)}}{\sqrt{w(z_1)}}.$$

Using (36) and Lemma 6.1, we have

$$\frac{\sqrt{w(x)}}{\sqrt{w(z_1)}} |\tilde{l}_{m+1,1}(x)| \le C\sqrt{m}\Delta z_1 \sqrt[4]{(4m - z_1)z_1} \frac{(4m - x)^{\frac{3}{4}}}{x^{\frac{3}{4}}} \le Cm$$

since $\sqrt{m}\Delta z_1 \le \frac{C}{\sqrt{m}}$, $4m - x \le Cm$, $x \ge \frac{C}{m}$, and $\sqrt[4]{(4m - z_1)z_1} \le C$.

Now assume $2 \le d \le j$. By

$$\left| \frac{q_{m+1}(x)}{x - z_d} \right| = |q'_{m+1}(\xi_d)|, \qquad |x - \xi_d| < |x - z_d|,$$

by Lemma 6.2, (42) and Lemma 6.1, we have

$$\frac{\sqrt{w(x)}}{\sqrt{w(z_d)}} |\tilde{l}_{m+1,d}(x)| \le Cm^{\frac{3}{2}} \frac{\Delta z_d}{\sqrt[4]{(4m - z_d)z_d}}$$

and by $\sqrt{m}\Delta z_k \le \frac{C}{\sqrt{m}}$, $4m - x \le Cm$, $x \ge \frac{C}{m}$, and $\sqrt[4]{(4m - z_1)z_1} \le C$ we get

$$\frac{\sqrt{w(x)}}{\sqrt{w(z_d)}} |\tilde{l}_{m+1,d}(x)| \le Cm. \quad \square$$

Lemma 6.4. *Let $0 \le \rho \le 1$, with j defined in (29). Denoting by d the index of a zero of q_{m+1} closest to $x \in \mathcal{D}_m$, it is*

$$\sum_{\substack{k=1 \\ k \ne d}}^{j} \left(\frac{x}{z_k} \right)^{\rho} \frac{\Delta z_k}{|x - z_k|} \le C \log m,$$

$C \ne C(m, x)$.

Proof. First note that by definition (30) and (38), one has

$$\left(\frac{x}{\tilde{z}_1}\right)^{\rho}\frac{\Delta z_1}{|x-z_1|}=\left(\frac{x}{x_1}\right)^{\rho}\frac{\tilde{y}_2-x_1}{|x-x_1|}<\left(\frac{x}{x_1}\right)^{\rho}\frac{\tilde{y}_2-\tilde{y}_1}{|x-x_1|}\le C\left(\frac{x}{x_1}\right)^{\rho}\frac{\Delta x_1}{|x-x_1|}.$$

Moreover, by the interlacing property (20), estimate (9), and (30), for $k\ge 2$, we have

$$\left(\frac{x}{z_k}\right)^{\rho}\frac{\Delta z_k}{|x-z_k|}=\left(\frac{x}{\tilde{y}_k}\right)^{\rho}\frac{\Delta \tilde{y}_k}{|x-\tilde{y}_k|}\le C\left(\frac{x}{x_{k-1}}\right)^{\rho}\frac{\Delta x_k}{|x-x_k|}\le C\left(\frac{x}{x_k}\right)^{\rho}\frac{\Delta x_k}{|x-x_k|}.$$

Therefore

$$\sum_{\substack{k=1\\k\neq d}}^{j}\left(\frac{x}{z_k}\right)^{\rho}\frac{\Delta z_k}{|x-z_k|}\le C\sum_{\substack{k=1\\k\neq d}}^{j}\left(\frac{x}{x_k}\right)^{\rho}\frac{x_k}{|x-x_k|},$$

the assertion follows by extending the last sum up to m and applying [26, Lemma 4.1.3 page 245]. □

Proof of Theorem 4.4. By (46)

$$\begin{aligned}\|\tilde{L}_{m+1}(f)\mathrm{u}\|_\infty &\le C\max_{x\in\mathcal{A}_m}|\tilde{L}_{m+1}(f,x)\mathrm{u}(x)|\\ &\le \|f\mathrm{u}\|_\infty\max_{x\in\mathcal{A}_m}\sum_{k=1}^{j}\frac{|q_{m+1}(x)|}{|q'_{m+1}(z_k)(x-z_k)|}\frac{\mathrm{u}(x)}{\mathrm{u}(z_k)}\end{aligned}$$

Denote by d the index of the zero of q_{m+1}, closest to x, and recalling Lemma 6.3, we can write

$$\begin{aligned}\|\tilde{L}_{m+1}(f)\mathrm{u}\|_\infty &\le \|f\mathrm{u}\|_\infty\max_{x\in\mathcal{A}_m}\sum_{\substack{k=1\\k\neq d}}^{j}|\tilde{l}_{m+1,k}(x)|\frac{\mathrm{u}(x)}{\mathrm{u}(z_k)}+|\tilde{l}_{m+1,d}(x)|\frac{\mathrm{u}(x)}{\mathrm{u}(z_d)}\\ &=: \|f\mathrm{u}\|_\infty\max_{x\in\mathcal{A}_m}\left\{\Sigma(x)+|\tilde{l}_{m+1,d}(x)|\frac{\mathrm{u}(x)}{\mathrm{u}(z_d)}\right\}.\end{aligned}\tag{56}$$

To estimate $\Sigma(x)$ assume first $x\in\mathcal{A}_m\cap\{x\le 3m\}$. For $k=1$, $k\neq d$, by Proposition 4.3 and estimate (41) we have

$$\begin{aligned}|\tilde{l}_{m+1,1}(x)|\frac{\mathrm{u}(x)}{\mathrm{u}(x_1)} &\le C\frac{\Delta x_1}{|x-x_1|}\left(\frac{x}{x_1}\right)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}(4m-x)^{\frac{3}{4}}\sqrt[4]{4m-x_1}\\ &\le Cm\frac{\Delta x_1}{|x-x_1|}\left(\frac{x}{x_1}\right)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}},\end{aligned}$$

and for $k>1$, $k\neq d$, by Proposition 4.3 and estimate (42)

$$\begin{aligned}|\tilde{l}_{m+1,k}(x)|\frac{\mathrm{u}(x)}{\mathrm{u}(z_k)} &\le Cm\frac{\Delta x_k}{|x-z_k|}\left(\frac{x}{z_k}\right)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}\left(\frac{4m-x}{4m-z_k}\right)^{\frac{1}{4}}\\ &\le Cm\frac{\Delta x_k}{|x-z_k|}\left(\frac{x}{z_k}\right)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}},\end{aligned}$$

since $\left(\frac{4m-x}{4m-z_k}\right)^{\frac{1}{4}}\le C$. Hence we have at least

$$\Sigma(x)\le Cm\sum_{\substack{k=1\\k\neq d}}^{j}\frac{\Delta x_k}{|x-z_k|}\left(\frac{x}{z_k}\right)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}.$$

Assume now $x\in\mathcal{A}_m\cap\{x>3m\}$. For $k=1$, $k\neq d$, by Proposition 4.3 and (41), we have

$$\begin{aligned}|\tilde{l}_{m+1,1}(x)|\frac{\mathrm{u}(x)}{\mathrm{u}(x_1)} &\le Cm\frac{\Delta x_1}{|x-x_1|}\left(\frac{x}{x_1}\right)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}\frac{\sqrt[4]{4m-x_1}}{\sqrt[4]{4m-x}}\\ &\le Cm\frac{\Delta x_1}{|x-x_1|}\left(\frac{x}{x_1}\right)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}},\end{aligned}$$

and for $k>1$, $k\neq d$, by Proposition 4.3 and (42), we obtain

$$\begin{aligned}|\tilde{l}_{m+1,k}(x)|\frac{\mathrm{u}(x)}{\mathrm{u}(z_k)} &\le Cm\frac{\Delta x_k}{|x-z_k|}\left(\frac{x}{z_k}\right)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}\frac{1}{\sqrt{z_k}\sqrt[4]{(4m-x)(4m-z_k)}}\\ &\le Cm\frac{\Delta x_k}{|x-z_k|}\left(\frac{x}{z_k}\right)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}},\end{aligned}$$

since $\sqrt[4]{(4m-x)z_k} \geq C$ and $\sqrt[4]{(4m-z_k)z_k} \geq C$. Hence, we have at least

$$\Sigma(x) \leq Cm \sum_{\substack{k=1 \\ k\neq d}}^{j} \frac{\Delta x_k}{|x-z_k|} \left(\frac{x}{z_k}\right)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}.$$

The thesis follows by Lemmas 6.3 and 6.4, under the assumption $0 \leq \gamma - \frac{\alpha}{2} + \frac{1}{4} \leq 1$. □

Proof of Theorem 4.5. Let $P^* \in \mathcal{P}_m^*$. Then, we can write

$$\|(f - \tilde{L}_{m+1}(f))\mathrm{u}\|_\infty \leq \|[f - P^*]\mathrm{u}\|_\infty + \|\tilde{L}_{m+1}(f - P^*)\mathrm{u}\|_\infty$$

from which by applying Theorem 4.4 to the second term we have

$$\|(f - \tilde{L}_{m+1}(f))\mathrm{u}\|_\infty \leq C\|[f - P^*]\mathrm{u}\|_\infty(1 + m\log m)$$

and in view of (39) it follows at least

$$\|(f - \tilde{L}_{m+1}(f))\mathrm{u}\|_\infty \leq Cm\log m\; E_M(f)_{\mathrm{u}}, \quad C \neq C(m, f). \quad \square$$

7. Conclusions

Anti-Gaussian integration formulas have proven to be very valuable for the construction of other quadrature schemes characterized by high accuracy and low computational cost. To our knowledge, the present work is the first introducing an interpolation process based on the anti-Gaussian nodes in the Laguerre case. It is not "optimal" in the space C_{u}, since the associated Lebesgue constants diverge as $m \log m$, and the convergence is ensured in subspaces of C_{u}, smaller than those in which the truncated interpolation process defined in (44) converges. However, we recall that the research of "good" matrices of interpolation knots is still an open problem, especially on unbounded intervals, since interpolating polynomials are an useful tool in many applications of numerical analysis. Hence, it is our intention to investigate about the employment of this new interpolating sequence, both alone and in combination with the truncated Lagrange polynomial $\mathcal{L}_{m+1}(w)$, for the simultaneous approximation of functions in Sobolev subspaces of C_{u}, the approximation of singular/hypersingular integral transforms, and also in the numerical treatment of integral equations of different types.

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Data availability

No data was used for the research described in the article.

References

[1] D.P. Laurie, Anti-Gaussian quadrature formulas, Math. Comp. 65 (214) (1996) 739–747.
[2] P. Díaz de Alba, L. Fermo, G. Rodriguez, Solution of second kind Fredholm integral equations by means of Gauss and anti-Gauss quadrature rules, Numer. Math. 146 (4) (2020) 699–728.
[3] P. Díaz de Alba, L. Fermo, G. Rodriguez, Anti-Gauss cubature rules with applications to Fredholm integral equations on the square, SIAM J. Sci. Comput. 47 (2) (2025) A689–A712.
[4] D.L. Djukić, L. Fermo, R.M. Mutavdžić Djukić, Averaged Nyström interpolants for bivariate Fredholm integral equations on the real positive semi-axes, Electron. Trans. Numer. Anal. 61 (2024) 51–65.
[5] L. Fermo, L. Reichel, G. Rodriguez, M. Spalević, Averaged Nyström interpolants for the solution of Fredholm integral equations of the second kind, Appl. Math. Comput. 467 (2024) 128482.
[6] S.E. Notaris, Anti-Gaussian quadrature formulae based on the zeros of stieltjes polynomials, BIT Numer. Math. 58 (1) (2018) 179–198.
[7] G. Mastroianni, D. Occorsio, Lagrange interpolation at laguerre zeros in some weighted uniform spaces, Acta Math. Hung. 90 (1–2) (2001) 27–52.
[8] G. Mastroianni, J. Szabados, Polynomial approximation on the real semiaxis with generalized laguerre weights, Stud. Univ. Babes-Bolyai Math. 52 (4) (2007) 105–128.
[9] P. Vértesi, On the Lebesgue function of weighted Lagrange interpolation. II, J. Aust. Math. Soc. A-Pure Math. Stat. 65 (2) (1998) 145–162.
[10] C. Laurita, G. Mastroianni, L_p–Convergence of Lagrange interpolation on the semiaxis, Acta Math. Hung. 120 (3) (2008) 249–273.
[11] D. Occorsio, Extended Lagrange interpolation in weighted uniform norm, Appl. Math. Comput. 211 (2009) 10–22.
[12] D. Occorsio, A method to evaluate the Hilbert transform on $(0, +\infty)$, Appl. Math. Comput. 217 (12) (2011) 5667–5679.

[13] M.C. De Bonis, G. Mastroianni, M. Viggiano, K-functionals, moduli of smoothness and weighted best approximation on the semiaxis, in: L. Leindler, F. Schipp, J. Szabados (Eds.), Functions, Series, Operators, Alexits Memorial Conference, Janos Bolyai Mathematical Society, Budapest, Hungary, 2002, pp. 181–211.
[14] G. Mastroianni, Polynomial inequalities, functional spaces and best approximation on the real semiaxis with Laguerre weights, Electron. Trans. Numer. Anal. 14 (2002) 142–151.
[15] W. Gautschi, Orthogonal polynomials. Computation and approximation, in: Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford, 2004.
[16] G. Golub, J.H. Welsch, Calculation of Gauss quadrature rules, Math. Comp. 23 (1969) 221–230.
[17] L. Gatteschi, Asymptotics and bounds for the zeros of Laguerre polynomials: a survey, J. Comput. Appl. Math. 144 (2002) 7–27.
[18] G. Szegö, Orthogonal polynomials, Amer. Math. Soc., vol. 23, Amer. Math. Soc. Colloq. Publ., Providence, 1975.
[19] M. De Bonis, B. Della Vecchia, G. Mastroianni, Approximation of the Hilbert transform on the real semiaxis using Laguerre zeros, J. Comput. Appl. Math. 140 (1–2) (2002) 209–229.
[20] L. Reichel, M.M. Spalević, A new representation of generalized averaged Gauss quadrature rules, Appl. Numer. Math. 165 (2021) 614–619.
[21] L. Reichel, M.M. Spalević, Averaged Gauss quadrature formulas: Properties and applications, J. Comput. Appl. Math. 410 (2022).
[22] M.M. Spalević, On generalized averaged Gaussian formulas, Math. Comp. 76 (2007) 1483–1492.
[23] D.L. Djukić, L. Fermo, R.M. Mutavdžić Djukić, Averaged cubature schemes on the real positive semiaxis, Numer. Algorithms 92 (2023) 545–569.
[24] G. Mastroianni, D. Occorsio, Some quadrature formulae with non standard weights, J. Comput. Appl. Math. 235 (3) (2010) 602–614.
[25] G. Mastroianni, G.V. Milovanović, Some numerical methods for second-kind Fredholm integral equations on the real semiaxis, IMA J. Numer. Anal. 29 (4) (2009) 1046–1066.
[26] G. Mastroianni, G. Milovanovic, Interpolation Processes: Basic Theory and Applications, in: Springer Monographs in Mathematics, 2008.
[27] G.W. Stewart, Matrix Algorithms: Volume 1: Basic Decompositions, SIAM, 1998.