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Three-way focusing Zakharov-Shabat systems with zero diagonal entry

Francesco Demontis[1] · **Cornelis van der Mee**[1]

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Abstract
In this article we develop the direct and inverse scattering theory of the AKNS system $\boldsymbol{V}_x = (ik\Sigma + \mathcal{Q}(x))\boldsymbol{V}$, where $\Sigma = \text{diag}(1, 0, -1)$ and $\mathcal{Q}(x)$ is an off-diagonal 3×3 $\boldsymbol{\sigma}$-focusing potential with entries in $L^1(\mathbb{R})$. We derive the time evolution of the scattering data which, through the inverse scattering transform, lead to the solution of the initial-value problem for a generalized long-wave-short-wave equation. We determine the essential spectrum of the AKNS system rigorously.

Keywords Singular Zakharov-Shabat system · Inverse scattering · Long-wave-short-wave equation

Mathematics Subject Classification 34A55 · 34L25

1 Introduction

The direct and inverse scattering theory of the AKNS system

$$v_x = (ik\Sigma + \mathcal{Q})v, \tag{1.1}$$

where Σ is a diagonal matrix of order n with nonzero real entries, $\mathcal{Q}(x)$ is an $n \times n$ matrix potential anticommuting with Σ with its entries belonging to $L^1(\mathbb{R})$ and $x \in \mathbb{R}$ is position, has been studied at great length [1–4, 10, 15, 32, 39]. Its direct scattering theory relies primarily on the interplay between n linearly independent column-vector Jost solutions continuous in $k \in \mathbb{C}^+ \cup \mathbb{R}$ and analytic in $k \in \mathbb{C}^+$ and n linearly independent Jost solutions continuous in $k \in \mathbb{C}^- \cup \mathbb{R}$ and analytic in $k \in \mathbb{C}^-$. The scattering data resulting from the direct scattering theory consist of two sets of (matrix) reflection coefficients and as many (matrix) norming constants as there are

✉ Francesco Demontis
fdemontis@unica.it

Cornelis van der Mee
cornelis110553@gmail.com

1 Dip. Matematica e Informatica, Università di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy

discrete eigenvalues including multiplicities in $\mathbb{C}^+$ or in $\mathbb{C}^-$. Its inverse scattering theory consists of converting the scattering data into a Marchenko integral equation whose solution leads to the potential $\mathcal{Q}(x)$. An alternative way of carrying out inverse scattering consists of converting the scattering data into a Riemann-Hilbert problem whose solution leads to the potential $\mathcal{Q}(x)$. The direct and inverse scattering theory of the AKNS system (1.1) with nonsingular real diagonal matrix Σ has been worked out in detail in various textbooks [3, 10, 15, 29, 32]. A general theory of developing the direct and inverse scattering for an arbitrary nonsingular complex diagonal matrix Σ has been presented in [5–7].

Few publications are devoted to AKNS systems (1.1), where Σ is a real diagonal matrix with at least one of its diagonal entries vanishing. Among the very few papers devoted to the subject matter is that by Newell [31], where $\Sigma = \text{diag}(1, 0, -1)$. In [31] Newell studies a solvable model for long-wave-short-wave interaction first presented by Benney [8, 9] and defines only two Jost solutions analytic in $k \in \mathbb{C}^+$ and only two Jost solutions analytic in $k \in \mathbb{C}^-$ instead of three. In the present article we define a third Jost solution analytic in $k \in \mathbb{C}^+$ and a third Jost solution analytic in $k \in \mathbb{C}^+$, thus arriving at two fundamental systems of Jost solutions. We mention that Kaup [23] has studied 3×3 AKNS systems, where Σ is a nonsingular diagonal matrix with distinct diagonal entries.

In this article we develop a direct and inverse scattering theory of the three-way AKNS system (1.1), where $\Sigma = \text{diag}(1, 0, -1)$. To define the third linearly independent Jost solutions analytic in either $k \in \mathbb{C}^+$ or in $k \in \mathbb{C}^-$, we take the vector products of the complex conjugates of the dual Jost solutions satisfying $\breve{v}_x = -\breve{v}(ik\Sigma + \mathcal{Q})$, where we adopt an interesting idea applied to the Manakov system [i.e., (1.1) with $\Sigma = \text{diag}(1, -1, -1)$] with nonvanishing boundary conditions [33]. Similar constructions of a third Jost-like solution have been given before by Deift et al. [13], Kaup [24], and McKean [30]. Once we have defined three linearly independent Jost solutions analytic in $k \in \mathbb{C}^+$ written as the columns of a 3×3 matrix and three linearly independent Jost solutions analytic in $k \in \mathbb{C}^-$ written as the columns of another 3×3 matrix, we write the first 3×3 matrix as the other multiplied from the right by a 3×3 matrix depending only on $k \in \mathbb{R}$, where the diagonal entries are the transmission coefficients and, apart from minus signs, the off-diagonal entries are the reflection coefficients. As our scattering data, we have thus found the reflection coefficients and, corresponding to each pole of the transmission coefficients in $\mathbb{C}^+$ or in $\mathbb{C}^-$, the so-called norming constants. Using these scattering data, the inverse scattering problem can then be solved alternatively by either the Marchenko method or the Riemann-Hilbert method. In this article we have decided not to furnish explicit solutions of the inverse scattering problem.

The study of one-dimensional wave propagation poses several challenging mathematical problems [27, 36]. The most elementary mathematical model for studying the interaction between long waves and short waves is the AKNS system studied by Newell [31]. Similar integrable equations were formulated by Yajima and Oikawa [38]. More recently, Caso-Huerta et al. [11, 12] have studied more general long-wave-short-wave interaction models. These so-called YON equations (named after Yajima, Oikawa, and Newell) can be derived from the zero curvature condition

$$\boldsymbol{X}_t - \boldsymbol{T}_x + \boldsymbol{X}\boldsymbol{T} - \boldsymbol{T}\boldsymbol{X} = 0_{3\times 3}, \tag{1.2}$$

where the Lax pair $(\boldsymbol{X}, \boldsymbol{T})$ is given by [12, Eqs. (2.2), (3.1), and (3.2)] as well as by [11, Eqs. (7)]. Generalizing this Lax pair to $\boldsymbol{X} = ik\Sigma + \mathcal{Q}$ and $\boldsymbol{T} = (ik)^2 A + ikB + C$, where $A = \frac{i}{3}\text{diag}(1, -2, 1)$,

$$\mathcal{Q} = \begin{bmatrix} 0 & S & iL \\ T^* & 0 & S^* \\ iK & T & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & iS & 0 \\ iT^* & 0 & -iS^* \\ 0 & -iT & 0 \end{bmatrix}, \tag{1.3a}$$

$$C = \begin{bmatrix} -iST^* & iS_x - LT & i|S|^2 \\ -iT_x^* - KS^* & iST^* + iTS^* & -iS_x^* - LT^* \\ i|T|^2 & iT_x - KS & -iTS^* \end{bmatrix}, \tag{1.3b}$$

and L and K are real functions and S and T are complex functions, we get from the zero curvature condition (1.2) the generalized long-wave-short-wave equations

$$\begin{cases} L_t - 2(|S|^2)_x + 2iL(ST^* - TS^*) = 0, \\ K_t - 2(|T|^2)_x + 2iK(TS^* - ST^*) = 0, \\ iS_t + S_{xx} + iL_xT - 2S^2T^* + KLS = 0, \\ iT_t + T_{xx} + iK_xS - 2T^2S^* + KLT = 0, \end{cases} \tag{1.4}$$

which, apart from a rescaling of (x, t) and the constants f_j, have been studied before by Wright [37, Eqs. (52)-(53)]. The YON equations are the special case, where $K = \alpha^2 L - \beta$ and $T = \alpha S$ for the real constants α and β. Recently, Geng and Li [18, 28, 35] have formulated matrix extensions of the Yajima-Oikawa equations which can be derived from a $(n+2)\times(n+2)$ matrix Lax pair $(\boldsymbol{X}, \boldsymbol{T})$, where $\boldsymbol{X} = i\lambda\Sigma + \mathcal{Q}$ and Σ is a singular matrix of rank one or two.

The singularity of the matrix $\Sigma = \text{diag}(1, 0, -1)$ means that the AKNS system (1.1) with potential $\mathcal{Q}$ cannot be viewed as a compact perturbation of the "free" AKNS system $v_x = ik\Sigma v$ in the standard way. By denoting by $\mathbf{1}$ stands the identity operator on a suitable function space, instead of viewing the "Hamiltonian" $\mathbb{L} = -i\Sigma^{-1}(\partial\mathbf{1} - \mathcal{Q})$ as a resolvent compact perturbation of the "free Hamiltonian $\mathbb{L}_0 = -i\Sigma^{-1}\partial\mathbf{1}$, we now consider the operator polynomial $L(\lambda) = -\partial_x I_3 + i\lambda\Sigma + \mathcal{Q}$ as a perturbation of the operator polynomial $L_0(\lambda) = -\partial_x I_3 + i\lambda\Sigma$, where, for nonreal λ, $L(\lambda)L_0(\lambda)^{-1}$ and $L_0(\lambda)^{-1}L(\lambda)$ are power compact perturbations of the identity on suitable Hilbert spaces. In this way we prove [cf. Appendix C] that the nonreal spectrum consists of isolated eigenvalues of finite algebraic multiplicity which can only accumulate at points of the real λ-line.

Let us discuss the contents of the various sections. In Sects. 2–4 we discuss the scattering solutions of the AKNS system $v_x = (ik\Sigma + \mathcal{Q})v$, those of the dual AKNS system $\breve{v}_x = -\breve{v}(ik\Sigma + \mathcal{Q})$, and the reflection and transmission coefficients. Section 5 is devoted to deriving the Marchenko integral equations. In Sect. 6 we derive the time evolution of the scattering data which lead to the solution of the initial-value problem for the generalized long-wave-short-wave equations (1.4) by means of the inverse scattering transform. Three appendices have been added. These appendices regard linear equations for wedge products, Wiener algebra properties of scattering solutions and scattering coefficients, and determining the essential spectrum of the AKNS system rigorously.

2 Jost solutions

Let us define the *Jost matrices* $\Psi(x,k)$ and $\Phi(x,k)$ as those 3×3 matrix solutions of the AKNS system (1.1) satisfying the asymptotic conditions

$$\Psi(x,k)=e^{ikx\Sigma}[I_3+o(1)],\qquad x\to+\infty,\tag{2.1a}$$

$$\Phi(x,k)=e^{ikx\Sigma}[I_3+o(1)],\qquad x\to-\infty,\tag{2.1b}$$

where $\det\Psi(x,k)=\det\Phi(x,k)=1$ for $(x,k)\in\mathbb{R}^2$. Since the entries of $\mathcal{Q}(x)$ belong to $L^1(\mathbb{R})$, we easily arrive at the Volterra integral equations

$$\Psi(x,k)=e^{ikx\Sigma}-\int_x^\infty dy\, e^{-ik(y-x)\Sigma}\mathcal{Q}(y)\Psi(y,k),\tag{2.2a}$$

$$\Phi(x,k)=e^{ikx\Sigma}+\int_{-\infty}^x dy\, e^{ik(x-y)\Sigma}\mathcal{Q}(y)\Phi(y,k).\tag{2.2b}$$

Letting $M(x,k)=\Psi(x,k)e^{-ikx\Sigma}$ and $N(x,k)=\Phi(x,k)e^{-ikx\Sigma}$ stand for the so-called *Faddeev functions*, we obtain the Volterra integral equations

$$M(x,k)=I_3-\int_x^\infty dy\, e^{-ik(y-x)\Sigma}\mathcal{Q}(y)M(y,k)e^{ik(y-x)\Sigma},\tag{2.3a}$$

$$N(x,k)=I_3+\int_{-\infty}^x dy\, e^{ik(x-y)\Sigma}\mathcal{Q}(y)N(y,k)e^{-ik(x-y)\Sigma}.\tag{2.3b}$$

Letting $\boldsymbol{\sigma}$ stand for a hermitian and unitary 3×3 matrix, we call the potential $\boldsymbol{\sigma}$*-focusing* if

$$\mathcal{Q}(x)^\dagger=-\boldsymbol{\sigma}\,\mathcal{Q}(x)\boldsymbol{\sigma},\tag{2.4}$$

where the dagger stands for the matrix conjugate transpose. If $\boldsymbol{\sigma}=I_3$, the potential is called *focusing*. There are four types of $\boldsymbol{\sigma}$-focusing potentials where $\boldsymbol{\sigma}$ is a diagonal matrix with diagonal entries ± 1 (see the four expressions of $\mathcal{Q}$ below corresponding to the possible choices of 1 or -1 as an entry in the diagonal matrix $\boldsymbol{\sigma}$) and hence commutes with Σ, namely-1

$$\mathcal{Q}=\underbrace{\begin{bmatrix}0&q_{12}&q_{13}\\-q_{12}^*&0&q_{23}\\-q_{13}^*&-q_{23}^*&0\end{bmatrix}}_{\text{for }\boldsymbol{\sigma}=I_3};\qquad \mathcal{Q}=\underbrace{\begin{bmatrix}0&q_{12}&q_{13}\\q_{12}^*&0&q_{23}\\-q_{13}^*&q_{23}^*&0\end{bmatrix}}_{\text{for }\boldsymbol{\sigma}=\mathrm{diag}(1,-1,1)},$$

$$\underbrace{\mathcal{Q}=\begin{bmatrix}0&q_{12}&q_{13}\\-q_{12}^*&0&q_{23}\\q_{13}^*&q_{23}^*&0\end{bmatrix}}_{\text{for }\boldsymbol{\sigma}=\text{diag}(1,1,-1)};\qquad \underbrace{\mathcal{Q}=\begin{bmatrix}0&q_{12}&q_{13}\\q_{12}^*&0&q_{23}\\q_{13}^*&-q_{23}^*&0\end{bmatrix}}_{\text{for }\boldsymbol{\sigma}=\text{diag}(1,-1,-1)}.$$

Other important choices of $\boldsymbol{\sigma}$ are the matrices $\tilde{\sigma}_1=\left[\begin{smallmatrix}0&0&1\\0&-1&0\\1&0&0\end{smallmatrix}\right]$ and $\tilde{\sigma}_2=\left[\begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix}\right]$, anti-commuting with Σ, where the corresponding $\tilde{\sigma}_1$-focusing and $\tilde{\sigma}_2$-focusing matrices have the form

$$\underbrace{\mathcal{Q}=\begin{bmatrix}0&q_{12}&q_{13}=-q_{13}^*\\q_{32}^*&0&q_{12}^*\\q_{31}=-q_{31}^*&q_{32}&0\end{bmatrix}}_{\text{for }\boldsymbol{\sigma}=\tilde{\sigma}_1};\quad \underbrace{\mathcal{Q}=\begin{bmatrix}0&q_{12}&q_{13}=-q_{13}^*\\-q_{32}^*&0&-q_{12}^*\\q_{31}=-q_{31}^*&q_{32}&0\end{bmatrix}}_{\text{for }\boldsymbol{\sigma}=\tilde{\sigma}_2}.$$

For $\boldsymbol{\sigma}=\tilde{\sigma}_1$ the potential $\mathcal{Q}$ has the general form of the potential in (1.3a).

For hermitian and unitary matrices $\boldsymbol{\sigma}$ commuting with Σ, we get for $(x,k)\in\mathbb{R}^2$

$$\Psi(x,k)^\dagger=\boldsymbol{\sigma}\Psi(x,k)^{-1}\boldsymbol{\sigma},\qquad \Phi(x,k)^\dagger=\boldsymbol{\sigma}\Phi(x,k)^{-1}\boldsymbol{\sigma},\tag{2.5a}$$

$$M(x,k)^\dagger=\boldsymbol{\sigma}M(x,k)^{-1}\boldsymbol{\sigma},\qquad N(x,k)^\dagger=\boldsymbol{\sigma}N(x,k)^{-1}\boldsymbol{\sigma},\tag{2.5b}$$

so that the Jost matrices are $\boldsymbol{\sigma}$-unitary for $(x,k)\in\mathbb{R}^2$. For hermitian and unitary matrices $\boldsymbol{\sigma}$ anticommuting with Σ (such as $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$), we get for $(x,k)\in\mathbb{R}^2$

$$\Psi(x,k)^\dagger=\boldsymbol{\sigma}\Psi(x,-k)^{-1}\boldsymbol{\sigma},\qquad \Phi(x,k)^\dagger=\boldsymbol{\sigma}\Phi(x,-k)^{-1}\boldsymbol{\sigma},\tag{2.6a}$$

$$M(x,k)^\dagger=\boldsymbol{\sigma}M(x,-k)^{-1}\boldsymbol{\sigma},\qquad N(x,k)^\dagger=\boldsymbol{\sigma}N(x,-k)^{-1}\boldsymbol{\sigma}.\tag{2.6b}$$

To derive (2.5) and (2.6), we observe that either side of the four identities in (2.5a) and (2.6a) satisfies the same homogeneous linear first order ODE system with **the same asymptotic conditions as** $x\to\pm\infty$ boundary condition.

To derive the triangular representations of the first and third columns of the Faddeev functions, we write the columns of I_3 as $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$. We get

Theorem 2.1 *There exist* 3×3 *matrices* $K(x,y)$ *and* $J(x,y)$ *such that the triangular representations*

$$M(x,k)\boldsymbol{e}_1=\boldsymbol{e}_1+\int_x^\infty dy\, e^{ik(y-x)}K(x,y)\boldsymbol{e}_1,\tag{2.7a}$$

$$M(x,k)\boldsymbol{e}_3=\boldsymbol{e}_3+\int_x^\infty dy\, e^{-ik(y-x)}K(x,y)\boldsymbol{e}_3,\tag{2.7b}$$

$$N(x,k)\boldsymbol{e}_1=\boldsymbol{e}_1+\int_{-\infty}^x dy\, e^{-ik(x-y)}J(x,y)\boldsymbol{e}_1,\tag{2.7c}$$

$$N(x,k)\boldsymbol{e}_3=\boldsymbol{e}_3+\int_{-\infty}^x dy\, e^{ik(x-y)}J(x,y)\boldsymbol{e}_3,\tag{2.7d}$$

are true, where

$$\sum_{s=1,3}\left(\int_x^\infty dy\,\|K(x,y)\boldsymbol{e}_s\|+\int_{-\infty}^x dy\,\|J(x,y)\boldsymbol{e}_s\|\right)<+\infty. \tag{2.8}$$

As a result, $M(x,k)\boldsymbol{e}_1$ and $N(x,k)\boldsymbol{e}_3$ are analytic in $k\in\mathbb{C}^+$, are continuous in $k\in\mathbb{C}^+\cup\mathbb{R}$, and tend to $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$ as $k\to\infty$ from within $\mathbb{C}^+\cup\mathbb{R}$, whereas $N(x,k)\boldsymbol{e}_1$ and $M(x,k)\boldsymbol{e}_3$ are analytic in $k\in\mathbb{C}^-$, are continuous in $k\in\mathbb{C}^-\cup\mathbb{R}$, and tend to $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$ as $k\to\infty$ from within $\mathbb{C}^-\cup\mathbb{R}$.

Proof Postmultiplying (2.3a) and (2.3b) by $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$ we obtain

$$M(x,k)\boldsymbol{e}_1=\boldsymbol{e}_1-\int_x^\infty dy\,e^{ik(y-x)\boldsymbol{N}_-}\mathcal{Q}(y)M(y,k)\boldsymbol{e}_1, \tag{2.9a}$$

$$M(x,k)\boldsymbol{e}_3=\boldsymbol{e}_3-\int_x^\infty dy\,e^{-ik(y-x)\boldsymbol{N}_+}\mathcal{Q}(y)M(y,k)\boldsymbol{e}_3, \tag{2.9b}$$

$$N(x,k)\boldsymbol{e}_1=\boldsymbol{e}_1+\int_{-\infty}^x dy\,e^{-ik(x-y)\boldsymbol{N}_-}\mathcal{Q}(y)N(y,k)\boldsymbol{e}_1, \tag{2.9c}$$

$$N(x,k)\boldsymbol{e}_3=\boldsymbol{e}_3+\int_{-\infty}^x dy\,e^{ik(x-y)\boldsymbol{N}_+}\mathcal{Q}(y)N(y,k)\boldsymbol{e}_3, \tag{2.9d}$$

where $\boldsymbol{N}_+=\operatorname{diag}(2,1,0)$ and $\boldsymbol{N}_-=\operatorname{diag}(0,1,2)$. Substituting (2.7) in (2.9) and stripping off the Fourier transform we get the Volterra integral equations

$$K(x,y)\boldsymbol{e}_1=\begin{bmatrix}-\int_x^\infty dz\,\boldsymbol{e}_1^T\mathcal{Q}(z)K(z,z+y-x)\boldsymbol{e}_1\\ -q_{21}(y)-\int_x^y dz\,\boldsymbol{e}_2^T\mathcal{Q}(z)K(z,y)\boldsymbol{e}_1\\ -\frac{1}{2}q_{31}(\frac{1}{2}[x+y])-\int_x^{\frac{1}{2}[x+y]}dz\,\boldsymbol{e}_3^T\mathcal{Q}(z)K(z,x+y-z)\boldsymbol{e}_1\end{bmatrix}, \tag{2.10a}$$

$$K(x,y)\boldsymbol{e}_3=\begin{bmatrix}-\frac{1}{2}q_{13}(\frac{1}{2}[x+y])-\int_x^{\frac{1}{2}[x+y]}dz\,\boldsymbol{e}_1^T\mathcal{Q}(z)K(z,x+y-z)\boldsymbol{e}_3\\ -q_{23}(y)-\int_x^y dz\,\boldsymbol{e}_2^T\mathcal{Q}(z)K(z,y)\boldsymbol{e}_3\\ -\int_x^\infty dz\,\boldsymbol{e}_3^T\mathcal{Q}(z)K(z,z+y-x)\boldsymbol{e}_3\end{bmatrix}, \tag{2.10b}$$

$$J(x,y)\boldsymbol{e}_1=\begin{bmatrix}\int_{-\infty}^x dz\,\boldsymbol{e}_1^T\mathcal{Q}(z)J(z,z+y-x)\boldsymbol{e}_1\\ q_{21}(y)+\int_y^x dz\,\boldsymbol{e}_2^T\mathcal{Q}(z)J(z,y)\boldsymbol{e}_1\\ \frac{1}{2}q_{31}(\frac{1}{2}[x+y])+\int_{\frac{1}{2}[x+y]}^x dz\,\boldsymbol{e}_3^T\mathcal{Q}(z)J(z,x+y-z)\boldsymbol{e}_1\end{bmatrix}, \tag{2.10c}$$

$$J(x,y)\boldsymbol{e}_3=\begin{bmatrix}\frac{1}{2}q_{13}(\frac{1}{2}[x+y])+\int_{\frac{1}{2}[x+y]}^x dz\,\boldsymbol{e}_1^T\mathcal{Q}(z)J(z,x+y-z)\boldsymbol{e}_3\\ q_{23}(y)+\int_y^x dz\,\boldsymbol{e}_2^T\mathcal{Q}(z)J(z,y)\boldsymbol{e}_3\\ \int_{-\infty}^x dz\,\boldsymbol{e}_3^T\mathcal{Q}(z)J(z,z+y-x)\boldsymbol{e}_3\end{bmatrix}, \tag{2.10d}$$

where $\mathcal{Q}(x)=\left(q_{ij}(x)\right)_{i,j=1}^3$ has its entries in $L^1(\mathbb{R})$. Then (2.8) follows from (2.10) by applying Gronwall's inequality [34] and using that $q_{ij}\in L^1(\mathbb{R})$ for (distinct) $i,j=1,2,3$. $\square$

Substituting $y = x$ in (2.10) we obtain

$$K(x,x)\boldsymbol{e}_1 = \begin{bmatrix} \int_x^\infty dz\,\big(q_{12}(z)q_{21}(z) + \tfrac{1}{2}q_{13}(z)q_{31}(z)\big) \\ -q_{21}(x) \\ -\tfrac{1}{2}q_{31}(x) \end{bmatrix}, \tag{2.11a}$$

$$K(x,x)\boldsymbol{e}_3 = \begin{bmatrix} -\tfrac{1}{2}q_{13}(x) \\ -q_{23}(x) \\ \int_x^\infty dz\,\big(\tfrac{1}{2}q_{31}(z)q_{13}(z) + q_{32}(z)q_{23}(z)\big) \end{bmatrix}, \tag{2.11b}$$

$$J(x,x)\boldsymbol{e}_1 = \begin{bmatrix} \int_{-\infty}^x dz\,\big(q_{12}(z)q_{21}(z) + \tfrac{1}{2}q_{13}(z)q_{31}(z)\big) \\ q_{21}(x) \\ \tfrac{1}{2}q_{31}(x) \end{bmatrix}, \tag{2.11c}$$

$$J(x,x)\boldsymbol{e}_3 = \begin{bmatrix} \tfrac{1}{2}q_{13}(x) \\ q_{23}(x) \\ \int_{-\infty}^x dz\,\big(\tfrac{1}{2}q_{31}(z)q_{13}(z) + q_{32}(z)q_{23}(z)\big) \end{bmatrix}. \tag{2.11d}$$

Thus Eqs. (2.11) allow us to compute the **potential entries** $q_{21}(x)$, $q_{31}(x)$, $q_{13}(x)$, and $q_{23}(x)$ directly. Then the remaining potential entries $q_{12}(x)$ and $q_{32}(x)$ can be evaluated with the help of the identities

$$\begin{aligned} q_{12}(x)q_{21}(x) &= -\frac{d}{dx}\boldsymbol{e}_1^T K(x,x)\boldsymbol{e}_1 - \tfrac{1}{2}q_{13}(x)q_{31}(x) \\ &= \frac{d}{dx}\boldsymbol{e}_1^T J(x,x)\boldsymbol{e}_1 - \tfrac{1}{2}q_{13}(x)q_{31}(x), \\ q_{32}(x)q_{23}(x) &= -\frac{d}{dx}\boldsymbol{e}_3^T K(x,x)\boldsymbol{e}_3 - \tfrac{1}{2}q_{31}(x)q_{13}(x) \\ &= \frac{d}{dx}\boldsymbol{e}_3^T J(x,x)\boldsymbol{e}_3 - \tfrac{1}{2}q_{31}(x)q_{13}(x), \end{aligned}$$

provided $q_{21}(x)$ and $q_{23}(x)$ do not vanish. For $\boldsymbol{\sigma}$-focusing potentials commuting or anticommuting with Σ we can alternatively compute $q_{12}(x)$ and $q_{32}(x)$ by taking suitable **signed** complex conjugates of $q_{21}(x)$ and $q_{23}(x)$.

3 Adjoint scattering solutions

In the preceding section we have defined four out of six Jost solutions, derived their Volterra integral equations, and proved their triangular representations. To define the remaining two Jost solutions, we first need to introduce the dual Jost solutions. These dual Jost solutions are the unique solutions of the Volterra integral equations

$$\breve{\Psi}(x,k) = e^{-ikx\Sigma} + \int_x^\infty dy\,\breve{\Psi}(y,k)\mathcal{Q}(y)e^{ik(y-x)\Sigma}, \tag{3.1a}$$

$$\breve{\Phi}(x,k) = e^{-ikx\Sigma} - \int_{-\infty}^x dy\,\breve{\Phi}(y,k)\mathcal{Q}(y)e^{-ik(x-y)\Sigma}, \tag{3.1b}$$

where $(x,k)\in\mathbb{R}^2$. Then $\breve{\Psi}(x,k)$ and $\breve{\Phi}(x,k)$ satisfy the first order systems

$$\breve{v}_x=-\breve{v}(ik\Sigma+\mathcal{Q}),\qquad \breve{v}_x^T=-(ik\Sigma+\mathcal{Q}^T)\breve{v}^T, \tag{3.2}$$

with asymptotic conditions

$$\breve{\Psi}(x,k)=[I_3+o(1)]e^{-ikx\Sigma},\qquad x\to+\infty, \tag{3.3a}$$
$$\breve{\Phi}(x,k)=[I_3+o(1)]e^{-ikx\Sigma},\qquad x\to-\infty. \tag{3.3b}$$

It is then easily verified that

$$\breve{\Psi}(x,k)=\Psi(x,k)^{-1},\qquad \breve{\Phi}(x,k)=\Phi(x,k)^{-1}, \tag{3.4}$$

where $(x,k)\in\mathbb{R}^2$. We now define the dual Faddeev functions

$$\breve{M}(x,k)=e^{ikx\Sigma}\breve{\Psi}(x,k),\qquad \breve{N}(x,k)=e^{ikx\Sigma}\breve{\Phi}(x,k).$$

Then $\breve{M}(x,k)$ and $\breve{N}(x,k)$ satisfy the Volterra integral equations

$$\breve{M}(x,k)=I_3+\int_x^\infty dy\,e^{-ik(y-x)\Sigma}\breve{M}(y,k)\mathcal{Q}(y)e^{ik(y-x)\Sigma}, \tag{3.5a}$$
$$\breve{N}(x,k)=I_3-\int_{-\infty}^x dy\,e^{ik(x-y)\Sigma}\breve{N}(y,k)\mathcal{Q}(y)e^{-ik(x-y)\Sigma}. \tag{3.5b}$$

Thus, proceeding as in Theorem 2.1, the Volterra integral equations

$$\boldsymbol{e}_1^T\breve{M}(x,k)=\boldsymbol{e}_1^T+\int_x^\infty dy\,\boldsymbol{e}_1^T\breve{M}(y,k)\mathcal{Q}(y)e^{-ik(y-x)\boldsymbol{N}_-}, \tag{3.6a}$$
$$\boldsymbol{e}_3^T\breve{M}(x,k)=\boldsymbol{e}_3^T+\int_x^\infty dy\,\boldsymbol{e}_3^T\breve{M}(y,k)\mathcal{Q}(y)e^{ik(y-x)\boldsymbol{N}_+}, \tag{3.6b}$$
$$\boldsymbol{e}_1^T\breve{N}(x,k)=\boldsymbol{e}_1^T-\int_{-\infty}^x dy\,\boldsymbol{e}_1^T\breve{N}(y,k)\mathcal{Q}(y)e^{ik(x-y)\boldsymbol{N}_-}, \tag{3.6c}$$
$$\boldsymbol{e}_3^T\breve{N}(x,k)=\boldsymbol{e}_3^T-\int_{-\infty}^x dy\,\boldsymbol{e}_3^T\breve{N}(y,k)\mathcal{Q}(y)e^{-ik(x-y)\boldsymbol{N}_+}, \tag{3.6d}$$

imply the triangular representations

$$\boldsymbol{e}_1^T\breve{M}(x,k)=\boldsymbol{e}_1^T+\int_x^\infty dy\,e^{-ik(y-x)}\boldsymbol{e}_1^T\breve{K}(y,x), \tag{3.7a}$$
$$\boldsymbol{e}_3^T\breve{M}(x,k)=\boldsymbol{e}_3^T+\int_x^\infty dy\,e^{ik(y-x)}\boldsymbol{e}_3^T\breve{K}(y,x), \tag{3.7b}$$
$$\boldsymbol{e}_1^T\breve{N}(x,k)=\boldsymbol{e}_1^T+\int_{-\infty}^x dy\,e^{ik(x-y)}\boldsymbol{e}_1^T\breve{J}(y,x), \tag{3.7c}$$

$$e_3^T \breve{N}(x,k) = e_3^T + \int_{-\infty}^{x} dy\, e^{-ik(x-y)} e_3^T \breve{J}(y,x), \tag{3.7d}$$

where

$$\sum_{s=1,3} \left(\int_x^{\infty} dy\, \|e_s^T \breve{K}(y,x)\| + \int_{-\infty}^{x} dy\, \|e_s^T \breve{J}(y,x)\| \right) < +\infty.$$

As a result, $e_1^T \breve{N}(x,k)$ and $e_3^T \breve{M}(x,k)$ are analytic in $k \in \mathbb{C}^+$, are continuous in $k \in \mathbb{C}^+ \cup \mathbb{R}$, and tend to e_1 and e_3 as $k \to \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$, whereas $e_1^T \breve{M}(x,k)$ and $e_3^T \breve{N}(x,k)$ are analytic in $k \in \mathbb{C}^-$, are continuous in $k \in \mathbb{C}^- \cup \mathbb{R}$, and tend to e_1 and e_3 as $k \to \infty$ from within $\mathbb{C}^- \cup \mathbb{R}$.

If the potential is $\boldsymbol{\sigma}$-focusing and $\boldsymbol{\sigma} = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3)$, we get

$$\left[M(x,k^*)e_1\right]^\dagger = \sigma_1 e_1^T \breve{M}(x,k)\boldsymbol{\sigma}, \qquad \left[N(x,k^*)e_1\right]^\dagger = \sigma_1 e_1^T \breve{N}(x,k)\boldsymbol{\sigma}, \tag{3.8a}$$

$$\left[M(x,k^*)e_3\right]^\dagger = \sigma_3 e_3^T \breve{M}(x,k)\boldsymbol{\sigma}, \qquad \left[N(x,k^*)e_3\right]^\dagger = \sigma_3 e_3^T \breve{N}(x,k)\boldsymbol{\sigma}, \tag{3.8b}$$

where k belongs to the appropriate closed upper or lower half-plane. Thus we obtain from (3.7) and (3.8)

$$e_1^T \breve{K}(y,x) = \sigma_1 e_1^T K(x,y)^\dagger \boldsymbol{\sigma}, \qquad e_3^T \breve{K}(y,x) = \sigma_3 e_3^T K(x,y)^\dagger \boldsymbol{\sigma}, \tag{3.9a}$$

$$e_1^T \breve{J}(y,x) = \sigma_1 e_1^T J(x,y)^\dagger \boldsymbol{\sigma}, \qquad e_3^T \breve{J}(y,x) = \sigma_3 e_3^T J(x,y)^\dagger \boldsymbol{\sigma}. \tag{3.9b}$$

On the other hand, for $\boldsymbol{\sigma} = \tilde{\sigma}_s$ $(s = 1, 2)$ we obtain

$$\left[M(x,k^*)e_1\right]^\dagger = e_3^T \breve{M}(x,-k)\tilde{\sigma}_s, \qquad \left[N(x,k^*)e_1\right]^\dagger = e_3^T \breve{N}(x,-k)\tilde{\sigma}_s, \tag{3.10a}$$

$$\left[M(x,k^*)e_3\right]^\dagger = e_1^T \breve{M}(x,-k)\tilde{\sigma}_s, \qquad \left[N(x,k^*)e_3\right]^\dagger = e_1^T \breve{N}(x,-k)\tilde{\sigma}_s, \tag{3.10b}$$

where k belongs to the appropriate closed upper or lower half-plane. Thus we obtain from (3.7) and (3.8)

$$e_1^T \breve{K}(y,x) = e_1^T K(x,y)^\dagger \tilde{\sigma}_s, \qquad e_3^T \breve{K}(y,x) = e_3^T K(x,y)^\dagger \tilde{\sigma}_s, \tag{3.11a}$$

$$e_1^T \breve{J}(y,x) = e_1^T J(x,y)^\dagger \tilde{\sigma}_s, \qquad e_3^T \breve{J}(y,x) = e_3^T J(x,y)^\dagger \tilde{\sigma}_s. \tag{3.11b}$$

Applying Theorem A.1 and using $\operatorname{Tr}(ik\Sigma + \mathcal{Q}) = 0$, we conclude that

$$\Omega^+(x,k) = -\left[e_1^T \breve{N}(x,k)\right]^T \wedge \left[e_3^T \breve{M}(x,k)\right]^T = -\left[e_1^T \breve{\Phi}(x,k)\right]^T \wedge \left[e_3^T \breve{\Psi}(x,k)\right]^T$$

satisfies the AKNS system (1.1), is continuous in $k \in \mathbb{C}^+ \cup \mathbb{R}$, is analytic in $k \in \mathbb{C}^+$, and approaches $e_2 = -(e_1 \wedge e_3)$ as $k \to \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$. Analogously,

$$\Omega^-(x,k) = -\left[e_1^T \breve{M}(x,k)\right]^T \wedge \left[e_3^T \breve{N}(x,k)\right]^T = -\left[e_1^T \breve{\Psi}(x,k)\right]^T \wedge \left[e_3^T \breve{\Phi}(x,k)\right]^T$$

satisfies the AKNS system (1.1), is continuous in $k \in \mathbb{C}^- \cup \mathbb{R}$, is analytic in $k \in \mathbb{C}^-$, and approaches $\boldsymbol{e}_2 = -(\boldsymbol{e}_1 \wedge \boldsymbol{e}_3)$ as $k \to \infty$ from within $\mathbb{C}^- \cup \mathbb{R}$. Consequently,

$$\Xi^+(x,k) = \left[\Psi(x,k)\boldsymbol{e}_1 \ \Omega^+(x,k) \ \Phi(x,k)\boldsymbol{e}_3\right] \tag{3.12a}$$

satisfies the AKNS system (1.1) and $F^+(x,k) = \Xi^+(x,k)e^{-ikx\Sigma}$ is continuous in $k \in \mathbb{C}^+ \cup \mathbb{R}$, is analytic in $k \in \mathbb{C}^+$, and tends to I_3 as $k \to \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$. Analogously,

$$\Xi^-(x,k) = \left[\Phi(x,k)\boldsymbol{e}_1 \ \Omega^-(x,k) \ \Psi(x,k)\boldsymbol{e}_3\right] \tag{3.12b}$$

satisfies the AKNS system (1.1) and $F^-(x,k) = \Xi^-(x,k)e^{-ikx\Sigma}$ is continuous in $k \in \mathbb{C}^- \cup \mathbb{R}$, is analytic in $k \in \mathbb{C}^-$, and tends to I_3 as $k \to \infty$ from within $\mathbb{C}^- \cup \mathbb{R}$.

We have thus found a fundamental system of Jost solutions continuous in $k \in \mathbb{C}^+ \cup \mathbb{R}$ and analytic in $k \in \mathbb{C}^+$ and a fundamental system of Jost solutions continuous in $k \in \mathbb{C}^- \cup \mathbb{R}$ and analytic in $k \in \mathbb{C}^-$.

4 Scattering data

In this section we define the scattering data and derive their main properties.

Since the Jost solutions satisfy the same homogeneous linear first order system (1.1), there exist *transition matrices* $A_l(k)$ and $A_r(k)$, one the inverse of the other, such that

$$\Psi(x,k) = \Phi(x,k)A_l(k), \qquad \Phi(x,k) = \Psi(x,k)A_r(k), \tag{4.1}$$

where $(x,k) \in \mathbb{R}^2$. Similarly, since the dual Jost solutions satisfy the same homogeneous linear first order system (3.2), there exist dual transition matrices $\breve{A}_l(k)$ and $\breve{A}_r(k)$, one the inverse of the other, such that

$$\breve{\Psi}(x,k) = \breve{A}_l(k)\breve{\Phi}(x,k), \qquad \breve{\Phi}(x,k) = \breve{A}_r(k)\breve{\Psi}(x,k), \tag{4.2}$$

where $(x,k) \in \mathbb{R}^2$. Equations (3.4) imply that

$$\breve{A}_l(k) = A_l(k)^{-1} = A_r(k), \qquad \breve{A}_r(k) = A_r(k)^{-1} = A_l(k), \tag{4.3}$$

where $\det A_l(k) = \det A_r(k) = 1$ for every $k \in \mathbb{R}$. For $\boldsymbol{\sigma}$-focusing potentials, where $\boldsymbol{\sigma}$ and Σ commute, we obtain for $k \in \mathbb{R}$ the symmetry relations

$$A_l(k)^\dagger = \boldsymbol{\sigma} A_r(k)\boldsymbol{\sigma}, \qquad A_r(k)^\dagger = \boldsymbol{\sigma} A_l(k)\boldsymbol{\sigma}, \tag{4.4a}$$

thus making $A_l(k)$ and $A_r(k)$ $\boldsymbol{\sigma}$-unitary. For $\boldsymbol{\sigma}$-focusing potentials, where $\boldsymbol{\sigma}$ and Σ anticommute, we obtain instead

$$A_l(k)^\dagger = \boldsymbol{\sigma} A_r(-k)\boldsymbol{\sigma}, \qquad A_r(k)^\dagger = \boldsymbol{\sigma} A_l(-k)\boldsymbol{\sigma}. \tag{4.4b}$$

Premultiplying (2.2) by $e^{-ikx\Sigma}$ and using (4.1), we obtain

$$A_l(k) = I_3 - \int_{-\infty}^{\infty} dy\, e^{-iky\Sigma} \mathcal{Q}(y) M(y,k) e^{iky\Sigma}, \tag{4.5a}$$

$$A_r(k) = I_3 + \int_{-\infty}^{\infty} dy\, e^{-iky\Sigma} \mathcal{Q}(y) N(y,k) e^{iky\Sigma}. \tag{4.5b}$$

As a result, with the help of the notation $A_{l,r}(k)_{ij} = \boldsymbol{e}_i^T A_{l,r}(k) \boldsymbol{e}_j$,

$$A_l(k)_{11} = \boldsymbol{e}_1^T A_l(k) \boldsymbol{e}_1 = 1 - \int_{-\infty}^{\infty} dy\, \boldsymbol{e}_1^T \mathcal{Q}(y) M(y,k) \boldsymbol{e}_1, \tag{4.6a}$$

$$A_r(k)_{33} = \boldsymbol{e}_3^T A_r(k) \boldsymbol{e}_3 = 1 + \int_{-\infty}^{\infty} dy\, \boldsymbol{e}_3^T \mathcal{Q}(y) N(y,k) \boldsymbol{e}_3, \tag{4.6b}$$

are continuous in $k \in \mathbb{C}^+ \cup \mathbb{R}$, are analytic in $k \in \mathbb{C}^+$, and tend to 1 as $k \to \infty$ from within $C^+ \cup \mathbb{R}$, whereas

$$A_r(k)_{11} = \boldsymbol{e}_1^T A_r(k) \boldsymbol{e}_1 = 1 + \int_{-\infty}^{\infty} dy\, \boldsymbol{e}_1^T \mathcal{Q}(y) N(y,k) \boldsymbol{e}_1, \tag{4.6c}$$

$$A_l(k)_{33} = \boldsymbol{e}_3^T A_l(k) \boldsymbol{e}_3 = 1 - \int_{-\infty}^{\infty} dy\, \boldsymbol{e}_3^T \mathcal{Q}(y) M(y,k) \boldsymbol{e}_3, \tag{4.6d}$$

are continuous in $k \in \mathbb{C}^- \cup \mathbb{R}$, are analytic in $k \in \mathbb{C}^-$, and tend to 1 as $k \to \infty$ from within $C^- \cup \mathbb{R}$. The other entries of $A_l(k)$ and $A_r(k)$ are continuous in $k \in \mathbb{R}$ and vanish (for the off-diagonal elements) or tend to 1 (for the 2, 2-elements) as $k \to \pm\infty$. Similarly, postmultiplying (3.1) and using (4.1) and (4.3) we obtain

$$A_r(k) = I_3 + \int_{-\infty}^{\infty} dy\, e^{-iky\Sigma} \breve{M}(y,k) \mathcal{Q}(y) e^{iky\Sigma}, \tag{4.7a}$$

$$A_l(k) = I_3 - \int_{-\infty}^{\infty} dy\, e^{-iky\Sigma} \breve{N}(y,k) \mathcal{Q}(y) e^{iky\Sigma}. \tag{4.7b}$$

Using the asymptotic relations

$$e^{-ikx\Sigma}\Psi(x,k) \to \begin{cases} I_3, & x \to +\infty, \\ A_l(k), & x \to -\infty, \end{cases} \qquad e^{-ikx\Sigma}\Phi(x,k) \to \begin{cases} A_r(k), & x \to +\infty, \\ I_3, & x \to -\infty, \end{cases}$$

$$\breve{\Psi}(x,k)e^{ikx\Sigma} \to \begin{cases} I_3, & x \to +\infty, \\ A_r(k), & x \to -\infty, \end{cases} \qquad \breve{\Phi}(x,k)e^{ikx\Sigma} \to \begin{cases} A_l(k), & x \to +\infty, \\ I_3, & x \to -\infty, \end{cases}$$

**we obtain

$$e^{-ikx\Sigma}\,\Xi^{+}(x,k)\longrightarrow\begin{cases}D_{+}^{\mathrm{up}}(k)=\begin{bmatrix}1 & -A_l(k)_{12} & A_r(k)_{13}\\ 0 & A_l(k)_{11} & A_r(k)_{23}\\ 0 & 0 & A_r(k)_{33}\end{bmatrix}, & x\to+\infty,\\ D_{+}^{\mathrm{dn}}(k)=\begin{bmatrix}A_l(k)_{11} & 0 & 0\\ A_l(k)_{21} & A_r(k)_{33} & 0\\ A_l(k)_{31} & -A_r(k)_{32} & 1\end{bmatrix}, & x\to-\infty,\end{cases}\tag{4.8a}$$

$$e^{-ikx\Sigma}\,\Xi^{-}(x,k)\longrightarrow\begin{cases}D_{-}^{\mathrm{dn}}(k)=\begin{bmatrix}A_r(k)_{11} & 0 & 0\\ A_r(k)_{21} & A_l(k)_{33} & 0\\ A_r(k)_{31} & -A_l(k)_{32} & 1\end{bmatrix}, & x\to+\infty,\\ D_{-}^{\mathrm{up}}(k)=\begin{bmatrix}1 & -A_r(k)_{12} & A_l(k)_{13}\\ 0 & A_r(k)_{11} & A_l(k)_{23}\\ 0 & 0 & A_l(k)_{33}\end{bmatrix}, & x\to-\infty,\end{cases}\tag{4.8b}$$

where

$$\det\Xi^{+}(x,k)=A_l(k)_{11}A_r(k)_{33},\qquad \det\Xi^{-}(x,k)=A_r(k)_{11}A_l(k)_{33}.$$

Computing the inverses of the four 3×3 upper or lower triangular matrices in (4.8) we get

$$\Xi^{+}(x,k)^{-1}e^{ikx\Sigma}\longrightarrow\begin{cases}D_{+}^{\mathrm{up}}(k)^{-1}=\begin{bmatrix}1 & \frac{A_l(k)_{12}}{A_l(k)_{11}} & \frac{A_l(k)_{13}}{A_l(k)_{11}}\\ 0 & \frac{1}{A_l(k)_{11}} & \frac{-A_r(k)_{23}}{A_l(k)_{11}A_r(k)_{33}}\\ 0 & 0 & \frac{1}{A_r(k)_{33}}\end{bmatrix}, & x\to+\infty,\\ D_{+}^{\mathrm{dn}}(k)^{-1}=\begin{bmatrix}\frac{1}{A_l(k)_{11}} & 0 & 0\\ \frac{-A_l(k)_{21}}{A_l(k)_{11}A_r(k)_{33}} & \frac{1}{A_r(k)_{33}} & 0\\ \frac{A_r(k)_{31}}{A_r(k)_{33}} & \frac{A_r(k)_{32}}{A_r(k)_{33}} & 1\end{bmatrix}, & x\to-\infty,\end{cases}\tag{4.9a}$$

$$\Xi^{-}(x,k)^{-1}e^{ikx\Sigma}\longrightarrow\begin{cases}D_{-}^{\mathrm{dn}}(k)^{-1}=\begin{bmatrix}\frac{1}{A_r(k)_{11}} & 0 & 0\\ \frac{-A_r(k)_{21}}{A_r(k)_{11}A_l(k)_{33}} & \frac{1}{A_l(k)_{33}} & 0\\ \frac{A_l(k)_{31}}{A_l(k)_{33}} & \frac{A_l(k)_{32}}{A_l(k)_{33}} & 1\end{bmatrix}, & x\to+\infty,\\ D_{-}^{\mathrm{up}}(k)^{-1}=\begin{bmatrix}1 & \frac{A_r(k)_{12}}{A_r(k)_{11}} & \frac{A_r(k)_{13}}{A_r(k)_{11}}\\ 0 & \frac{1}{A_r(k)_{11}} & \frac{-A_l(k)_{23}}{A_r(k)_{11}A_l(k)_{33}}\\ 0 & 0 & \frac{1}{A_l(k)_{33}}\end{bmatrix}, & x\to-\infty,\end{cases}\tag{4.9b}$$

where the nonzero corner elements have been simplified using the identities

$$e_1^T A_{l,r}(k) A_{r,l}(k) e_3 = e_3^T A_{l,r}(k) A_{r,l}(k) e_1 = 0.$$

Let us now express the second columns of the Jost matrices in the columns of $\Xi^+(x,k)$ and $\Xi^-(x,k)$. Indeed, denoting the second columns of the right-hand sides of (4.9) by the respective columns $\boldsymbol{\alpha}^+(k)$, $\boldsymbol{\beta}^+(k)$, $\boldsymbol{\alpha}^-(k)$, and $\boldsymbol{\beta}^-(k)$, we easily derive that

$$\Psi(x,k) e_2 = \Xi^+(x,k)\boldsymbol{\alpha}^+(k) = \Xi^-(x,k)\boldsymbol{\alpha}^-(k), \tag{4.10a}$$

$$\Phi(x,k) e_2 = \Xi^+(x,k)\boldsymbol{\beta}^+(k) = \Xi^-(x,k)\boldsymbol{\beta}^-(k), \tag{4.10b}$$

where we have employed their $x \to +\infty$ asymptotics to derive (4.10a) and their $x \to -\infty$ asymptotics to derive (4.10b).

Since $\Xi^+(x,k)$ and $\Xi^-(x,k)$ are 3×3 matrix solutions of the same homogeneous linear first order system, there exist scattering matrices $S(k)$ and $\tilde{S}(k)$, one the inverse of the other, such that

$$\Xi^-(x,k) = \Xi^+(x,k) S(k), \qquad \Xi^+(x,k) = \Xi^-(x,k)\tilde{S}(k), \tag{4.11}$$

where $(x,k) \in \mathbb{R}^2$. We call the diagonal entries of $S(k)$ and $\tilde{S}(k)$ *transmission coefficients* and minus their off-diagonal entries *reflection coefficients*. Then

$$S(k) = \begin{bmatrix} \frac{1}{A_l(k)_{11}} & \frac{-A_r(k)_{12}}{A_l(k)_{11}} & \frac{A_l(k)_{13}}{A_l(k)_{11}} \\ \frac{-A_l(k)_{21}}{A_l(k)_{11}A_r(k)_{33}} & \frac{1-A_l(k)_{31}A_r(k)_{13}}{A_l(k)_{11}A_r(k)_{33}} & \frac{-A_r(k)_{23}}{A_l(k)_{11}A_r(k)_{33}} \\ \frac{A_r(k)_{31}}{A_r(k)_{33}} & \frac{-A_l(k)_{32}}{A_r(k)_{33}} & \frac{1}{A_r(k)_{33}} \end{bmatrix}, \tag{4.12a}$$

$$\tilde{S}(k) = \begin{bmatrix} \frac{1}{A_r(k)_{11}} & \frac{-A_l(k)_{12}}{A_r(k)_{11}} & \frac{A_r(k)_{13}}{A_r(k)_{11}} \\ \frac{-A_r(k)_{21}}{A_r(k)_{11}A_l(k)_{33}} & \frac{1-A_r(k)_{31}A_l(k)_{13}}{A_r(k)_{11}A_l(k)_{33}} & \frac{-A_l(k)_{23}}{A_r(k)_{11}A_l(k)_{33}} \\ \frac{A_l(k)_{31}}{A_l(k)_{33}} & \frac{-A_r(k)_{32}}{A_l(k)_{33}} & \frac{1}{A_l(k)_{33}} \end{bmatrix}, \tag{4.12b}$$

Here we have computed $S(k)$ and $\tilde{S}(k)$ by using the $x \to +\infty$ as well as the $x \to -\infty$ asymptotics of $\Xi^\pm(x,k)^{-1}\Xi^\mp(x,k)$ as $x \to \pm\infty$. Moreover, we have relied on the fact that the cofactor matrix of $A_{l,r}(k)$ equals $A_{r,l}(k)$ and that $e_i^T A_{l,r}(k) A_{r,l}(k) e_j = \delta_{ij}$.

By a *spectral singularity* we mean a real zero of any of the functions $A_l(k)_{11}$, $A_r(k)_{11}$, $A_l(k)_{33}$, $A_r(k)_{33}$, $1 - A_l(k)_{31}A_r(k)_{13}$, and $1 - A_r(k)_{31}A_l(k)_{13}$. Assuming the absence of spectral singularities, we denote by w_l and w_r the winding numbers of the curves in the complex plane obtained by computing the values of the functions $1 - A_l(k)_{31}A_r(k)_{13}$ and $1 - A_r(k)_{31}A_l(k)_{13}$ as k runs along the real line from $-\infty$ to $+\infty$. Further, there exist unique functions $\mathbb{A}_l^\pm(k)$ and $\mathbb{A}_r^\pm(k)$ that are continuous in $k \in \mathbb{C}^\pm \cup \mathbb{R}$, are analytic in $k \in \mathbb{C}^\pm$, do not have any zeros $k \in \mathbb{C}^\pm \cup \mathbb{R}$, and tend to 1 as $k \to \infty$ from within $\mathbb{C}^\pm \cup \mathbb{R}$ such that the following Wiener-Hopf factorizations hold true [20, 26]:

$$1 - A_l(k)_{31} A_r(k)_{13} = \left(\frac{k-i}{k+i}\right)^{w_l} \mathbb{A}_l^+(k)\mathbb{A}_l^-(k), \tag{4.13a}$$

$$1 - A_r(k)_{31} A_l(k)_{13} = \left(\frac{k-i}{k+i}\right)^{w_r} \mathbb{A}_r^+(k)\mathbb{A}_r^-(k). \tag{4.13b}$$

Moreover, there exist $\mathbb{K}_l^\pm(y)$ and $\mathbb{K}_r^\pm(y)$ in $L^1(\mathbb{R}^\pm)$ satisfying

$$\mathbb{A}_l^\pm(k) = 1 \pm \int_0^{\pm\infty} dy\, e^{iky}\mathbb{K}_l^\pm(y), \qquad \mathbb{A}_r^\pm(k) = 1 \pm \int_0^{\pm\infty} dy\, e^{iky}\mathbb{K}_r^\pm(y).$$

Using (4.4a) we obtain for $\boldsymbol{\sigma}$-focusing potentials commuting with Σ

$$A_l(k)_{31}^* = \sigma_1\sigma_3 A_r(k)_{13}, \qquad A_r(k)_{31}^* = \sigma_1\sigma_3 A_l(k)_{13},$$

where $k \in \mathbb{R}$. Thus in the absence of spectral singularities the left-hand sides of (4.13) are positive functions of $k \in \mathbb{R}$ and therefore their winding numbers w_l and w_r both vanish. Hence, the Wiener-Hopf factors $\mathbb{A}_l^\pm(k)$ and $\mathbb{A}_r^\pm(k)$ satisfy

$$\mathbb{A}_l^+(k) = \mathbb{A}_l^-(k^*)^*, \qquad \mathbb{A}_r^+(k) = \mathbb{A}_r^-(k^*)^*, \tag{4.14}$$

where $k \in \mathbb{C}^+ \cup \mathbb{R}$. On the other hand, (4.4b) implies that for $\boldsymbol{\sigma} = \tilde{\sigma}_1$ and $\boldsymbol{\sigma} = \tilde{\sigma}_2$ we have

$$A_l(k)_{31}^* = A_r(-k)_{31}, \qquad A_r(k)_{31}^* = A_l(-k)_{31},$$

where $k \in \mathbb{R}$. Thus in the absence of spectral singularities the left-hand sides of (4.13) have identical (but not necessarily zero) winding numbers: $w_l = w_r$.

Under the condition that $w_l = w_r = 0$, let us write the scattering matrices as the sums of a diagonal matrix and an off-diagonal matrix by means of correction factors as follows:

$$S(k) = \left(\mathcal{T}(k) - \tilde{\mathcal{C}}(k)^{-1}\mathcal{R}(k)\right)\mathcal{C}(k), \tag{4.15a}$$

$$\tilde{S}(k) = \left(\tilde{\mathcal{T}}(k) - \mathcal{C}(k)^{-1}\tilde{\mathcal{R}}(k)\right)\tilde{\mathcal{C}}(k), \tag{4.15b}$$

where

$$\mathcal{T}(k) = \mathrm{diag}\left(\frac{1}{A_l(k)_{11}}, \frac{\mathbb{A}_l^+(k)}{A_l(k)_{11}A_r(k)_{33}}, \frac{1}{A_r(k)_{33}}\right),$$

$$\tilde{\mathcal{T}}(k) = \mathrm{diag}\left(\frac{1}{A_r(k)_{11}}, \frac{\mathbb{A}_r^-(k)}{A_r(k)_{11}A_l(k)_{33}}, \frac{1}{A_l(k)_{33}}\right),$$

contain the so-called *corrected transmission coefficients* and

$$\mathcal{C}(k) = \mathrm{diag}\left(1, \mathbb{A}_l^-(k), 1\right),$$
$$\tilde{\mathcal{C}}(k) = \mathrm{diag}\left(1, \mathbb{A}_r^+(k), 1\right),$$

are the so-called *correction factors*. We call the off-diagonal matrices $\mathcal{R}(k)$ and $\tilde{\mathcal{R}}(k)$ the *corrected reflection coefficients*. We then arrive at the following Riemann-Hilbert problems (please note that, as usual, the Riemann-Hilbert problems are formulated along the real axis in the complex k-plane)

$$\Xi^-(x,k)\mathcal{C}(k)^{-1} = \Xi^+(x,k)\mathcal{T}(k) - \Xi^+(x,k)\tilde{\mathcal{C}}(k)^{-1}\mathcal{R}(k), \tag{4.16a}$$
$$\Xi^+(x,k)\tilde{\mathcal{C}}(k)^{-1} = \Xi^-(x,k)\tilde{\mathcal{T}}(k) - \Xi^-(x,k)\mathcal{C}(k)^{-1}\tilde{\mathcal{R}}(k), \tag{4.16b}$$

where

$$\Xi^+(x,k)\tilde{\mathcal{C}}(k)^{-1} = \left[I_3 + \int_0^\infty d\gamma\, e^{ik\gamma}\mathbb{K}^+(x,x+\gamma)\right]e^{ikx\Sigma}, \tag{4.17a}$$
$$\Xi^-(x,k)\mathcal{C}(k)^{-1} = \left[I_3 + \int_0^\infty d\gamma\, e^{-ik\gamma}\mathbb{K}^-(x,x-\gamma)\right]e^{ikx\Sigma}, \tag{4.17b}$$

and

$$\int_0^\infty d\gamma\left(\|\mathbb{K}^+(x,x+\gamma)\| + \|\mathbb{K}^-(x,x-\gamma)\|\right) < +\infty.$$

To take into account the so-called norming constants when formulating the Riemann-Hilbert problems (4.16), we assume that the zeros of $A_l(k)_{11}$ and $A_r(k)_{33}$ in $\mathbb{C}^+$ are simple and nonoverlapping, that the zeros of $A_r(k)_{11}$ and $A_l(k)_{33}$ in $\mathbb{C}^-$ are simple and nonoverlapping, and that the winding numbers w_l and w_r both vanish. Then the zeros k_j of $A_r(k)_{33}A_l(k)_{11}$ in $\mathbb{C}^+$ are simple and the zeros $\tilde{k}_j$ of $A_l(k)_{33}A_r(k)_{11}$ in $\mathbb{C}^-$ are simple. Moreover, the absence of spectral singularities implies that $A_r(k)_{33}$, $A_l(k)_{11}$, $A_l(k)_{33}$, and $A_r(k)_{11}$ do not have any real zeros. As a result, the reduced transmission coefficients $\mathcal{T}(k)$ and $\tilde{\mathcal{T}}(k)$ are continuous in $k \in \mathbb{R}$, while $\tilde{\mathcal{T}}(k)$ is meromorphic in $k \in \mathbb{C}^+$ with simple poles k_j and $\mathcal{T}(k)$ is meromorphic in $k \in \mathbb{C}^-$ with simple poles $\tilde{k}_j$. Letting τ_j stand for the residue of $\tilde{\mathcal{T}}(k)$ at $k = k_j$ and $\tilde{\tau}_j$ for the residue of $\mathcal{T}(k)$ at $k = \tilde{k}_j$, we define the *norming constants* as those off-diagonal 3×3 matrices $\boldsymbol{N}_j$ and $\tilde{\boldsymbol{N}}_j$ such that

$$\Xi^+(x,k_j)\tau_j = +i\,\Xi^+(x,k_j)\tilde{\mathcal{C}}(k_j)^{-1}\boldsymbol{N}_j, \tag{4.18a}$$
$$\Xi^-(x,\tilde{k}_j)\tilde{\tau}_j = -i\,\Xi^-(x,\tilde{k}_j)\mathcal{C}(\tilde{k}_j)^{-1}\tilde{\boldsymbol{N}}_j. \tag{4.18b}$$

We can then write the Riemann-Hilbert problem (4.16) in its final form

$$F^-(x,k)\mathcal{C}(k)^{-1} = F^+(x,k) + \sum_j \frac{F^+(x,k) - F^+(x,k_j)}{k-k_j}\tau_j + i\sum_j \frac{F^+(x,k_j)\tilde{\mathcal{C}}(k_j)^{-1}\boldsymbol{N}_j}{k-k_j} - F^+(x,k)\tilde{\mathcal{C}}(k)^{-1}e^{ikx\Sigma}\mathcal{R}(k)e^{-ikx\Sigma}, \quad (4.19a)$$

$$F^+(x,k)\tilde{\mathcal{C}}(k)^{-1} = F^-(x,k) + \sum_j \frac{F^-(x,k) - F^-(x,\tilde{k}_j)}{k-\tilde{k}_j}\tilde{\tau}_j - i\sum_j \frac{F^-(x,\tilde{k}_j)\mathcal{C}(\tilde{k}_j)^{-1}\tilde{\boldsymbol{N}}_j}{k-\tilde{k}_j} - F^-(x,k)\mathcal{C}(k)^{-1}e^{ikx\Sigma}\tilde{\mathcal{R}}(k)e^{-ikx\Sigma}. \quad (4.19b)$$

5 System of Marchenko integral equations

In this section we convert the Riemann-Hilbert problem (4.19) into two coupled Marchenko systems of integral equations, one satisfied by $\mathbb{K}^+(x,y)$ and the other satisfied by $\mathbb{K}^-(x,y)$. Here we assume that the zeros of $A_l(k)_{11}$ and $A_r(k)_{33}$ in $\mathbb{C}^+$ are simple and nonoverlapping, that the zeros of $A_r(k)_{11}$ and $A_l(k)_{33}$ in $\mathbb{C}^-$ are simple and nonoverlapping, that these four functions as well as $1 - A_{r,l}(k)_{31}A_{l,r}(k)_{13}$ do not have any real zeros, and that the winding numbers w_l and w_r both vanish.

We now observe that the functions $A_l(k)_{11}$ and $A_r(k)_{33}$ belong to $\mathcal{W}^+$, the functions $A_r(k)_{11}$ and $A_l(k)_{33}$ belong to $\mathcal{W}^-$, and the entries of the functions $F^{\pm}(x,k)$ belong to $\mathcal{W}^{\pm}$ for each $x \in \mathbb{R}$. Here $\mathcal{W}^+$ and $\mathcal{W}^-$ are subalgebras of the Wiener algebra $\mathcal{W}$ (see App. B). Letting $\Pi^{\pm}$ be the projections defined by (B.1), we apply Π^- to (4.19a) and Π^+ to (4.19b) and obtain

$$F^-(x,k)\mathcal{C}(k)^{-1} - I_3 = +i\sum_j \frac{F^+(x,k_j)\tilde{\mathcal{C}}(k_j)^{-1}\boldsymbol{N}_j}{k-k_j} - \Pi^-\left(F^+(x,k)\tilde{\mathcal{C}}(k)^{-1}e^{ikx\Sigma}\mathcal{R}(k)e^{-ikx\Sigma}\right), \quad (5.1a)$$

$$F^+(x,k)\tilde{\mathcal{C}}(k)^{-1} - I_3 = -i\sum_j \frac{F^-(x,\tilde{k}_j)\mathcal{C}(\tilde{k}_j)^{-1}\tilde{\boldsymbol{N}}_j}{k-\tilde{k}_j} - \Pi^+\left(F^-(x,k)\mathcal{C}(k)^{-1}e^{ikx\Sigma}\tilde{\mathcal{R}}(k)e^{-ikx\Sigma}\right). \quad (5.1b)$$

We now write (5.1a) as an equation in $\mathcal{W}_0^-$ (definition of $\mathcal{W}_0^-$ is given in App. B) and (5.1b) as an equation in $\mathcal{W}_0^+$, strip off the Fourier transform, and end up with a coupled set of two equations for $\mathbb{K}^{\pm}(x, x \pm \gamma)$. To do so, we apply (4.17), the identities

$$\frac{-i}{k-k_j} = \int_0^\infty ds\, e^{-i(k-k_j)s}, \qquad k \in \mathbb{C}^-,$$

$$\frac{+i}{k-\tilde{k}_j} = \int_0^\infty ds\, e^{i(k-\tilde{k}_j)s}, \qquad k \in \mathbb{C}^+,$$

and the equations

$$\begin{aligned}
e^{ikx\Sigma}\mathcal{R}(k)e^{-ikx\Sigma} &= \begin{bmatrix} 0 & e^{ikx}\mathcal{R}(k)_{12} & e^{2ikx}\mathcal{R}(k)_{13} \\ e^{-ikx}\mathcal{R}(k)_{21} & 0 & e^{ikx}\mathcal{R}(k)_{23} \\ e^{-2ikx}\mathcal{R}(k)_{31} & e^{-ikx}\mathcal{R}(k)_{32} & 0 \end{bmatrix} \\
&= \int_{-\infty}^\infty d\gamma\, e^{-ik\gamma} \begin{bmatrix} 0 & \hat{\mathcal{R}}(\gamma+x)_{12} & \hat{\mathcal{R}}(\gamma+2x)_{13} \\ \hat{\mathcal{R}}(\gamma-x)_{21} & 0 & \hat{\mathcal{R}}(\gamma+x)_{23} \\ \hat{\mathcal{R}}(\gamma-2x)_{31} & \hat{\mathcal{R}}(\gamma-x)_{32} & 0 \end{bmatrix} \\
&= \int_{-\infty}^\infty d\gamma\, e^{-ik\gamma} \left[\hat{\mathcal{R}}(\gamma+[\Sigma_i-\Sigma_j]x)_{ij}\right]_{i,j=1}^3, \qquad (5.2a)
\end{aligned}$$

$$\begin{aligned}
e^{ikx\Sigma}\tilde{\mathcal{R}}(k)e^{-ikx\Sigma} &= \begin{bmatrix} 0 & e^{ikx}\tilde{\mathcal{R}}(k)_{12} & e^{2ikx}\tilde{\mathcal{R}}(k)_{13} \\ e^{-ikx}\tilde{\mathcal{R}}(k)_{21} & 0 & e^{ikx}\tilde{\mathcal{R}}(k)_{23} \\ e^{-2ikx}\tilde{\mathcal{R}}(k)_{31} & e^{-ikx}\tilde{\mathcal{R}}(k)_{32} & 0 \end{bmatrix} \\
&= \int_{-\infty}^\infty d\gamma\, e^{ik\gamma} \begin{bmatrix} 0 & \hat{\tilde{\mathcal{R}}}(\gamma-x)_{12} & \hat{\tilde{\mathcal{R}}}(\gamma-2x)_{13} \\ \hat{\tilde{\mathcal{R}}}(\gamma+x)_{21} & 0 & \hat{\tilde{\mathcal{R}}}(\gamma-x)_{23} \\ \hat{\tilde{\mathcal{R}}}(\gamma+2x)_{31} & \hat{\tilde{\mathcal{R}}}(\gamma+x)_{32} & 0 \end{bmatrix} \\
&= \int_{-\infty}^\infty d\gamma\, e^{ik\gamma} \left[\hat{\tilde{\mathcal{R}}}(\gamma+[\Sigma_j-\Sigma_i]x)_{ij}\right]_{i,j=1}^3, \qquad (5.2b)
\end{aligned}$$

where $\Sigma = \operatorname{diag}(\Sigma_1, \Sigma_2, \Sigma_3) = \operatorname{diag}(1, 0, -1)$ and the entries of $\hat{\mathcal{R}}$ and $\hat{\tilde{\mathcal{R}}}$ belong to $L^1(\mathbb{R})$. Consequently, we have arrived at the Marchenko integral equations

$$\begin{aligned}
\mathbb{K}^-(x, x-\gamma) = &-\sum_j \left(e^{ik_j\gamma}\boldsymbol{N}_j + \int_0^\infty d\overline{\gamma}\, e^{ik_j(\gamma+\overline{\gamma})}\mathbb{K}^+(x, x+\overline{\gamma})\boldsymbol{N}_j\right) \\
&- \left[\hat{\mathcal{R}}(\gamma+[\Sigma_i-\Sigma_j]x)_{ij}\right]_{i,j=1}^3 \\
&- \int_0^\infty d\overline{\gamma}\, \mathbb{K}^+(x, x+\overline{\gamma}) \left[\hat{\mathcal{R}}(\overline{\gamma}+\gamma+[\Sigma_i-\Sigma_j]x)_{ij}\right]_{i,j=1}^3,
\end{aligned} \qquad (5.3a)$$

$$\begin{aligned}
\mathbb{K}^+(x, x+\gamma) = &-\sum_j \left(e^{-i\tilde{k}_j\gamma}\tilde{\boldsymbol{N}}_j + \int_0^\infty d\overline{\gamma}\, e^{-i\tilde{k}_j(\gamma+\overline{\gamma})}\mathbb{K}^-(x, x-\overline{\gamma})\tilde{\boldsymbol{N}}_j\right) \\
&- \left[\hat{\tilde{\mathcal{R}}}(\gamma+[\Sigma_j-\Sigma_i]x)_{ij}\right]_{i,j=1}^3 \\
&- \int_0^\infty d\overline{\gamma}\, \mathbb{K}^-(x, x-\overline{\gamma}) \left[\hat{\tilde{\mathcal{R}}}(\overline{\gamma}+\gamma+[\Sigma_j-\Sigma_i]x)_{ij}\right]_{i,j=1}^3.
\end{aligned} \qquad (5.3b)$$

By using (5.2b) it is immediate to observe that (5.3a) and (5.3b) can be written as the respective equations

$$\mathbb{K}^-(x,x-\gamma)+\omega^+(\gamma;x)+\int_0^\infty d\overline{\gamma}\,\mathbb{K}^+(x,x+\overline{\gamma})\omega^+(\overline{\gamma}+\gamma;x)=0, \tag{5.4a}$$

$$\mathbb{K}^+(x,x+\gamma)+\omega^-(\gamma;x)+\int_0^\infty d\overline{\gamma}\,\mathbb{K}^-(x,x-\overline{\gamma})\omega^-(\overline{\gamma}+\gamma;x)=0, \tag{5.4b}$$

where

$$\omega^+(\gamma;x)=\sum_j e^{ik_j\gamma}\boldsymbol{N}_j+\left[\hat{\mathcal{R}}(\gamma+[\Sigma_i-\Sigma_j]x)_{ij}\right]_{i,j=1}^3, \tag{5.5a}$$

$$\omega^-(\gamma;x)=\sum_j e^{-i\tilde{k}_j\gamma}\tilde{\boldsymbol{N}}_j+\left[\hat{\tilde{\mathcal{R}}}(\gamma+[\Sigma_j-\Sigma_i]x)_{ij}\right]_{i,j=1}^3, \tag{5.5b}$$

are off-diagonal matrices whose entries belong to $L^1(\mathbb{R}^+)$.

Using (2.7) and the definitions of $F^\pm(x,k)$, we obtain [cf. (3.12)]

$$\begin{cases} F^+(x,k)\tilde{\mathcal{C}}(k)^{-1}\boldsymbol{e}_1=M(x,k)\boldsymbol{e}_1,\\ F^+(x,k)\tilde{\mathcal{C}}(k)^{-1}\boldsymbol{e}_3=N(x,k)\boldsymbol{e}_3,\\ F^-(x,k)\mathcal{C}(k)^{-1}\boldsymbol{e}_1=N(x,k)\boldsymbol{e}_1,\\ F^-(x,k)\mathcal{C}(k)^{-1}\boldsymbol{e}_3=M(x,k)\boldsymbol{e}_3. \end{cases}$$

Consequently,

$$\begin{cases} \mathbb{K}^+(x,x)\boldsymbol{e}_1=K(x,x)\boldsymbol{e}_1,\ \mathbb{K}^+(x,x)\boldsymbol{e}_3=J(x,x)\boldsymbol{e}_3,\\ \mathbb{K}^-(x,x)\boldsymbol{e}_1=J(x,x)\boldsymbol{e}_1,\ \mathbb{K}^-(x,x)\boldsymbol{e}_3=K(x,x)\boldsymbol{e}_3, \end{cases}$$

can be expressed in the potentials $q_{ij}(x)$ by means of (2.11).

Let us now discuss the norming constants $\boldsymbol{N}_j$ and $\tilde{\boldsymbol{N}}_j$. Starting from the residues $\tau_j=\mathrm{diag}(\tau_{1j},\tau_{2j},\tau_{3j})$ and $\tilde{\tau}_j=\mathrm{diag}(\tilde{\tau}_{1j},\tilde{\tau}_{2j},\tilde{\tau}_{3j})$, we see that τ_{2j} and $\tilde{\tau}_{2j}$ are nonzero and that exactly one of τ_{1j} and τ_{3j} and exactly one of $\tilde{\tau}_{1j}$ and $\tilde{\tau}_{3j}$ is nonzero. Now put $\Xi^+=\mathrm{col}(\phi,\chi,\psi)$ and $\Xi^-=\mathrm{col}(\overline{\psi},\overline{\chi},\overline{\phi})$. If $\tau_{1j}\neq 0$ and $\tau_{3j}=0$, the eigenfunctions at the eigenvalue k_j are linear combinations of ϕ and χ, meaning that $\psi\notin L^2(\mathbb{R})^{3\times 1}$. Hence,

$$\begin{cases} -i\tau_{1j}\phi=\boldsymbol{N}_{j;21}\chi+\boldsymbol{N}_{j;31}\psi,\\ -i\tau_{2j}\chi=\boldsymbol{N}_{j;12}\phi+\boldsymbol{N}_{j;32}\psi,\\ 0=\boldsymbol{N}_{j;13}\phi+\boldsymbol{N}_{j;23}\chi, \end{cases}$$

which implies that $\boldsymbol{N}_{j;31}=\boldsymbol{N}_{j;32}=0$ and the linear system

$$\begin{bmatrix}\phi & \chi\end{bmatrix}\begin{bmatrix} i\tau_{1j} & \boldsymbol{N}_{j;12} & \boldsymbol{N}_{j;13}\\ \boldsymbol{N}_{j;21} & i\tau_{2j} & \boldsymbol{N}_{j;23}\end{bmatrix}=0_{2\times 1}$$

is governed by a 2×3 matrix of rank 2. On the other hand, if $\tau_{1j} = 0$ and $\tau_{3j} \neq 0$, the eigenfunctions at the eigenvalue k_j are linear combinations of χ and ψ, meaning that $\phi \notin L^2(\mathbb{R})^{3\times 1}$. Hence,

$$\begin{cases} 0 = \boldsymbol{N}_{j;21}\chi + \boldsymbol{N}_{j;31}\psi, \\ -i\tau_{2j}\chi = \boldsymbol{N}_{j;12}\phi + \boldsymbol{N}_{j;32}\psi, \\ -i\tau_{3j}\psi = \boldsymbol{N}_{j;13}\phi + \boldsymbol{N}_{j;23}\chi, \end{cases}$$

which implies that $\boldsymbol{N}_{j;12} = \boldsymbol{N}_{j;13} = 0$ and the linear system

$$\begin{bmatrix} \chi & \psi \end{bmatrix} \begin{bmatrix} \boldsymbol{N}_{j;21} & i\tau_{2j} & \boldsymbol{N}_{j;23} \\ \boldsymbol{N}_{j;31} & \boldsymbol{N}_{j;32} & i\tau_{3j} \end{bmatrix} = 0_{1\times 2}$$

is governed by a 2×3 matrix of rank 2.

Let us summarize how to evaluate the potential entries from the solutions of the Marchenko equations (5.4). Indeed, let us first construct the Marchenko equations (5.4) from the norming constants and reflection coefficients by using (5.5). Next, we obtain $K(x, x)\boldsymbol{e}_i$ and $J(x, x)\boldsymbol{e}_i$ ($i = 1, 3$) as the first and third columns of the Marchenko solutions $\mathbb{K}^{\pm}(x, y)$ for $y = x$. Finally, we compute the potential entries from $K(x, x)\boldsymbol{e}_i$ and $J(x, x)\boldsymbol{e}_i$ ($i = 1, 3$) by using (2.11).

6 Time evolution of scattering data

In this section we study the time evolution of the scattering data of the AKNS system (1.1) with $\Sigma = \mathrm{diag}(1, 0, -1)$ associated with the solution of the generalized long-wave-short-wave equations by means of the inverse scattering transform.

Let $\boldsymbol{V}(x, t)$ be a nonsingular 3×3 matrix solution of the first order systems

$$\boldsymbol{V}_x = \boldsymbol{X}\boldsymbol{V}, \qquad \boldsymbol{V}_t = \boldsymbol{T}\boldsymbol{V}, \tag{6.1}$$

where $(\boldsymbol{X}, \boldsymbol{T})$ is the Lax pair given by $\boldsymbol{X} = ik\Sigma + \mathcal{Q}$ and $\boldsymbol{T} = (ik)^2 A + ikB + C$ and A, B, and C are given by (1.3). Then $\boldsymbol{V}(x, t)$ can be postmultiplied in (6.1) by any nonsigular matrix independent of $(x, t) \in \mathbb{R}^2$. Under the assumption that $L(x, t)$, $K(x, t)$, $S(x, t)$, and $T(x, t)$ vanish as $x \to \pm\infty$, we see that $\boldsymbol{T}(k; x, t) \to (ik)^2 A$ as $x \to \pm\infty$. For any invertible matrix solution $W(x, t)$ of (1.1) there exists an invertible matrix C_W, not depending on x, such that

$$W = \boldsymbol{V}C_W^{-1}.$$

Then

$$\begin{aligned} W_t &= \boldsymbol{V}_t C_W^{-1} - \boldsymbol{V}C_W^{-1}[C_W]_t C_W^{-1} = \boldsymbol{T}\boldsymbol{V}C_W^{-1} - \boldsymbol{V}C_W^{-1}[C_W]_t C_W^{-1} \\ &= \boldsymbol{T}W - W[C_W]_t C_W^{-1} \end{aligned}$$

implies the important identity

$$[C_W]_t C_W^{-1} = W^{-1}\boldsymbol{T}W - W^{-1}W_t, \tag{6.2}$$

where the left-hand side (and hence also the right-hand side) does not depend on x. We may therefore exploit the asymptotic behavior of W and $\boldsymbol{T}$ as $x \to \pm\infty$ to write the right-hand side of (6.2) in a simplified way. Using that Σ and $A = \frac{i}{3}\operatorname{diag}(1, -2, 1)$ commute, we thus get as $x \to \pm\infty$

$$[C_\Psi]_t C_\Psi^{-1} = e^{-ikx\Sigma}[I_3 + o(1)][(ik)^2 A + o(1)]e^{ikx\Sigma}[I_3 + o(1)] = -k^2 A, \tag{6.3a}$$

$$[C_\Phi]_t C_\Phi^{-1} = e^{-ikx\Sigma}[I_3 + o(1)][(ik)^2 A + o(1)]e^{ikx\Sigma}[I_3 + o(1)] = -k^2 A. \tag{6.3b}$$

Hence, $C_\Psi(k;t) = e^{-k^2 tA}C_\Psi(k;0)$ and $C_\Phi(k;t) = e^{-k^2 tA}C_\Phi(k;0)$.

Recalling the definitions (4.1) of the transition matrices, $A_l = \Phi^{-1}\Psi$ and $A_r = \Psi^{-1}\Phi$, we compute

$$\begin{aligned}
[A_l]_t &= \Phi^{-1}\Psi_t - \Phi^{-1}\Phi_t\Phi^{-1}\Psi \\
&= \Phi^{-1}\Psi_t - \Phi^{-1}\Phi_t A_l \\
&= \Phi^{-1}\left(\boldsymbol{T}\Psi - \Psi[C_\Psi]_t C_\Psi^{-1}\right) - \Phi^{-1}\left(\boldsymbol{T}\Phi - \Phi[C_\Phi]_t C_\Phi^{-1}\right)A_l \\
&= \Phi^{-1}\boldsymbol{T}\Psi - A_l[C_\Psi]_t C_\Psi^{-1} - \Phi^{-1}\boldsymbol{T}\Phi A_l + [C_\Phi]_t C_\Phi^{-1} A_l \\
&= [C_\Phi]_t C_\Phi^{-1} A_l - A_l[C_\Psi]_t C_\Psi^{-1} = k^2(A_l(k)A - AA_l(k)).
\end{aligned}$$

Consequently, using that $A_r(k;t) = A_l(k;t)^{-1}$ we obtain

$$A_l(k;t) = e^{-k^2 tA}A_l(k;0)e^{k^2 tA}, \quad A_r(k;t) = e^{-k^2 tA}A_r(k;0)e^{k^2 tA}. \tag{6.4}$$

Since $A = \frac{i}{3}\operatorname{diag}(1, -2, 1)$, we see that the diagonal and corner elements of $A_l(k;t)$ and $A_r(k;t)$ are time independent whereas the remaining four elements are not. More precisely,

$$A_{l,r}(k;t) = \begin{bmatrix} A_{l,r}(k;0)_{11} & e^{-ik^2 t}A_{l,r}(k;0)_{12} & A_{l,r}(k;0)_{13} \\ e^{ik^2 t}A_{l,r}(k;0)_{21} & A_{l,r}(k;0)_{22} & e^{ik^2 t}A_{l,r}(k;0)_{23} \\ A_{l,r}(k;0)_{31} & e^{-ik^2 t}A_{l,r}(k;0)_{32} & A_{l,r}(k;0)_{33} \end{bmatrix}. \tag{6.5}$$

With the help of (4.12) we obtain for the scattering matrix

$$S(k;t) = \begin{bmatrix} S(k;0)_{11} & e^{-ik^2 t}S(k;0)_{12} & S(k;0)_{13} \\ e^{ik^2 t}S(k;0)_{21} & S(k;0)_{22} & e^{ik^2 t}S(k;0)_{23} \\ S(k;0)_{31} & e^{-ik^2 t}S(k;0)_{32} & S(k;0)_{33} \end{bmatrix}, \tag{6.6a}$$

$$\tilde{S}(k;t) = \begin{bmatrix} \tilde{S}(k;0)_{11} & e^{-ik^2 t}\tilde{S}(k;0)_{12} & \tilde{S}(k;0)_{13} \\ e^{ik^2 t}\tilde{S}(k;0)_{21} & \tilde{S}(k;0)_{22} & e^{ik^2 t}\tilde{S}(k;0)_{23} \\ \tilde{S}(k;0)_{31} & e^{-ik^2 t}\tilde{S}(k;0)_{32} & \tilde{S}(k;0)_{33} \end{bmatrix}. \tag{6.6b}$$

As a result, in the absence of spectral singularities the Wiener-Hopf factors $\mathbb{A}_{l,r}^{\pm}(k)$ in (4.13) are time independent as are the corrected transmission coefficients $\mathcal{T}(k)$ and $\tilde{\mathcal{T}}(k)$ and the correction factors $\mathcal{C}(k)$ and $\tilde{\mathcal{C}}(k)$. The corrected reflection coefficients $\mathcal{R}(k;t)$ and $\tilde{\mathcal{R}}(k;t)$ are time varying and satisfy

$$\mathcal{R}(k;t) = e^{-k^2tA}\mathcal{R}(k;0)e^{k^2tA}, \qquad \tilde{\mathcal{R}}(k;t) = e^{-k^2tA}\tilde{\mathcal{R}}(k;0)e^{k^2tA}.$$

Let us now derive the time evolution of the norming constants $\boldsymbol{N}_j$ and $\tilde{\boldsymbol{N}}_j$ under the assumption that the zeros k_j of $A_l(k)_{11}$ and $A_r(k)_{33}$ in $\mathbb{C}^+$ are simple and nonoverlapping, the zeros $\tilde{k}_j$ of $A_r(k)_{11}$ and $A_l(k)_{33}$ in $\mathbb{C}^-$ are simple and nonoverlapping, and the winding numbers w_l and w_r both vanish. We now recall that $\Xi^+(x,k)$ and $\Xi^-(x,k)$ are 3×3 matrix solutions of (1.1) satisfying the asymptotic conditions [cf. (4.8)]

$$e^{-ikx\Sigma}\,\Xi^+(x,k;t) \simeq \begin{cases} D_+^{\mathrm{up}}(k;t), & x\to+\infty,\\ D_+^{\mathrm{dn}}(k;t), & x\to-\infty,\end{cases} \tag{6.7a}$$

$$e^{-ikx\Sigma}\,\Xi^-(x,k;t) \simeq \begin{cases} D_-^{\mathrm{dn}}(k;t), & x\to+\infty,\\ D_-^{\mathrm{up}}(k;t), & x\to-\infty,\end{cases} \tag{6.7b}$$

where the four matrices $D_\pm^{\mathrm{up}}(l;t)$ and $D_\pm^{\mathrm{dn}}(k;t)$ satisfy the time evolutions

$$D_\pm^{\mathrm{up}}(k;t) = e^{-k^2tA}D_\pm^{\mathrm{up}}(k;0)e^{k^2tA}, \quad D_\pm^{\mathrm{dn}}(k;t) = e^{-k^2tA}D_\pm^{\mathrm{dn}}(k;0)e^{k^2tA}. \tag{6.8}$$

Hence,

$$\Xi^+(x,k;t) = \Psi(x,k;t)D_+^{\mathrm{up}}(k;t) = \Phi(x,k;t)D_+^{\mathrm{dn}}(k;t), \tag{6.9a}$$

$$\Xi^-(x,k;t) = \Psi(x,k;t)D_-^{\mathrm{dn}}(k;t) = \Phi(x,k;t)D_-^{\mathrm{up}}(k;t). \tag{6.9b}$$

Moreover [cf. (4.11)],

$$S(k;t) = D_+^{\mathrm{up}}(k;t)^{-1}D_-^{\mathrm{dn}}(k;t) = D_+^{\mathrm{dn}}(k;t)^{-1}D_-^{\mathrm{up}}(k;x), \tag{6.10a}$$

$$\tilde{S}(k;t) = D_-^{\mathrm{up}}(k;t)^{-1}D_+^{\mathrm{dn}}(k;t) = D_-^{\mathrm{dn}}(k;t)^{-1}D_+^{\mathrm{up}}(k;x). \tag{6.10b}$$

The time evolutions (6.6) are immediate from (6.8) and (6.9). Finally, using (6.8) we obtain

$$\begin{aligned}
&[C_{\Xi^+}(k;t)]_t C_{\Xi^+}(k;t)^{-1} = D_+^{\mathrm{up}}(k;t)^{-1}\left(-k^2AD_+^{\mathrm{up}}(k;t) - [D_+^{\mathrm{up}}(k;t)]_t\right)\\
&= e^{-k^2tA}D_+^{\mathrm{up}}(k;0)^{-1}e^{k^2tA}\Big(-k^2Ae^{-k^2tA}D_+^{\mathrm{up}}(k;0)e^{k^2tA}\\
&- e^{-k^2tA}\left[-k^2AD_+^{\mathrm{up}}(k;0) + D_+^{\mathrm{up}}(k;0)k^2A\right]e^{k^2tA}\Big)\\
&= e^{-k^2tA}D_+^{\mathrm{up}}(k;0)^{-1}\left(-k^2AD_+^{\mathrm{up}}(k;0) + k^2AD_+^{\mathrm{up}}(k;0) - D_+^{\mathrm{up}}(k;0)k^2A\right)e^{k^2tA}
\end{aligned}$$

$$= -e^{-k^2tA}D_+^{\mathrm{up}}(k;0)^{-1}D_+^{\mathrm{up}}(k;0)k^2Ae^{k^2tA}$$
$$= -e^{-k^2tA}(k^2A)e^{k^2tA} = -k^2A. \tag{6.11a}$$

Similarly,

$$[C_{\Xi^-}(k;t)]_tC_{\Xi^-}(k;t)^{-1} = -k^2A. \tag{6.11b}$$

Letting τ_j stand for the residue of $\tilde{\mathcal{T}}(k)$ at $k = k_j$ and $\tilde{\tau}_j$ for the residue of $\mathcal{T}(k)$ at $k = \tilde{k}_j$, we define the *norming constants* as those off-diagonal 3×3 matrices $\boldsymbol{N}_j(t)$ and $\tilde{\boldsymbol{N}}_j(t)$ such that

$$\Xi^+(x,k_j;t)\tau_j = +i\ \Xi^+(x,k_j;t)\tilde{\mathcal{C}}(k_j)^{-1}\boldsymbol{N}_j(t), \tag{6.12a}$$
$$\Xi^-(x,\tilde{k}_j;t)\tilde{\tau}_j = -i\ \Xi^-(x,\tilde{k}_j;t)\mathcal{C}(\tilde{k}_j)^{-1}\tilde{\boldsymbol{N}}_j(t), \tag{6.12b}$$

where τ_j and $\tilde{\tau}_j$ are time independent but the norming constants are not.

Using (6.11a), $[\Xi^+]_t = \boldsymbol{T}\Xi^+ - \Xi^+[C_\Xi]_tC_{\Xi^+}^{-1}$, and the time independence of τ_j we get

$$\Big(\boldsymbol{T}(k_j)\Xi^+(x,k_j;t) - \Xi^+(x,k_j;t)(-k_j^2A)\Big)\,\tau_j = i\,\Big(\boldsymbol{T}(k_j)\Xi^+(x,k_j;t)$$
$$-\ \Xi^+(x,k_j;t)(-k_j^2A)\Big)\,\tilde{\mathcal{C}}(k_j)^{-1}\boldsymbol{N}_j(t) + i\ \Xi^+(x,k_j;t)\tilde{\mathcal{C}}(k_j)^{-1}[\boldsymbol{N}_j]_t,$$

which on deleting the (identical) first terms on either side leads to the identity

$$\Xi^+(x,k_j;t)k_j^2A\tau_j = i\ \Xi^+(x,k_j;t)k_j^2A\tilde{\mathcal{C}}(k_j)^{-1}\boldsymbol{N}_j(t) + i\ \Xi^+(x,k_j;t)\tilde{\mathcal{C}}(k_j)^{-1}[\boldsymbol{N}_j]_t,$$

thus implying

$$\Xi^+(x,k_j;t)\tau_jk_j^2A = i\ \Xi^+(x,k_j;t)\tilde{\mathcal{C}}(k_j)^{-1}k_j^2A\boldsymbol{N}_j(t) + i\ \Xi^+(x,k_j;t)\tilde{\mathcal{C}}(k_j)^{-1}[\boldsymbol{N}_j]_t.$$

With the help of (6.12a) we obtain in turn

$$i\ \Xi^+(x,k_j;t)\tilde{\mathcal{C}}(k_j)^{-1}\boldsymbol{N}_j(t)k_j^2A = i\ \Xi^+(x,k_j;t)\tilde{\mathcal{C}}(k_j)^{-1}k_j^2A\boldsymbol{N}_j(t)$$
$$+\,i\ \Xi^+(x,k_j;t)\tilde{\mathcal{C}}(k_j)^{-1}[\boldsymbol{N}_j]_t.$$

Therefore,

$$\boldsymbol{N}_j(t)k_j^2A = k_j^2A\boldsymbol{N}_j(t) + [\boldsymbol{N}_j]_t,$$

and hence

$$[\boldsymbol{N}_j]_t = \boldsymbol{N}_j(t)k_j^2A - k_j^2A\boldsymbol{N}_j(t).$$

Consequently, we have derived the time evolution of the norming constants

$$\boldsymbol{N}_j(t) = e^{-k_j^2 tA}\boldsymbol{N}_j(0)e^{k^2 tA}. \tag{6.13a}$$

In the same way we derive from (6.12b)

$$\tilde{\boldsymbol{N}}_j(t) = e^{-\tilde{k}_j^2 tA}\tilde{\boldsymbol{N}}_j(0)e^{\tilde{k}_j^2 tA} \tag{6.13b}$$

Let us consider the two special cases of the norming constants considered at the end of Sect. 5. Put $\tau_j = \mathrm{diag}(\tau_{1j}, \tau_{2j}, \tau_{3j})$. On one hand, if $\tau_{1j} \neq 0$ and $\tau_{3j} = 0$, we obtain the rank two matrix

$$i\tau_j + \boldsymbol{N}_j(t) = \begin{bmatrix} i\tau_{1j} & e^{-ik_j^2 t}\boldsymbol{N}_{j;12}(0) & \boldsymbol{N}_{j;13}(0) \\ e^{ik_j^2 t}\boldsymbol{N}_{j;21}(0) & i\tau_{2j} & e^{ik_j^2 t}\boldsymbol{N}_{j;23}(0) \\ 0 & 0 & 0 \end{bmatrix}.$$

On the other hand, if $\tau_{1j} = 0$ and $\tau_{3j} \neq 0$, we obtain the rank two matrix

$$i\tau_j + \boldsymbol{N}_j(t) = \begin{bmatrix} 0 & 0 & 0 \\ e^{ik_j^2 t}\boldsymbol{N}_{j;21}(0) & i\tau_{2j} & e^{ik_j^2 t}\boldsymbol{N}_{j;23}(0) \\ \boldsymbol{N}_{j;31}(0) & e^{-ik_j^2 t}\boldsymbol{N}_{j;32}(0) & i\tau_{3j} \end{bmatrix}.$$

A Linear equations for wedge products

The following result can presumably be derived using the methods discussed in [7]. Here we include this result with full proof. This result has been applied in [33], where an unspecific reference to [7] was given.

Theorem A.1 *Let $\boldsymbol{u}$ and $\boldsymbol{w}$ be two vector solutions of the differential systems $\boldsymbol{u}_x = B\boldsymbol{u}$ and $\boldsymbol{w}_x = B\boldsymbol{w}$, where B is a constant 3×3 matrix. Then $\tilde{\boldsymbol{v}} = \boldsymbol{u} \wedge \boldsymbol{w}$ is a vector solution of the differential system*

$$\tilde{\boldsymbol{v}}_x = \left([Tr\,B]I_3 - B^T\right)\tilde{\boldsymbol{v}}.$$

Proof Consider the real or complex 3×3 matrices

$$B = \begin{bmatrix} a_1 & b_1 & b_2 \\ c_1 & a_2 & d_1 \\ c_2 & d_2 & a_3 \end{bmatrix}, \quad B^T = \begin{bmatrix} a_1 & c_1 & c_2 \\ b_1 & a_2 & d_2 \\ b_2 & d_1 & a_3 \end{bmatrix}.$$

Consider the two first order column vector equations

$$\boldsymbol{u}_x = B\boldsymbol{u}, \qquad \boldsymbol{w}_x = B\boldsymbol{w}.$$

Put $\tilde{\boldsymbol{v}} = \boldsymbol{u} \wedge \boldsymbol{w}$. Then

$$
\begin{aligned}
\tilde{\boldsymbol{v}}_x &= \boldsymbol{u}_x \wedge \boldsymbol{w} + \boldsymbol{u} \wedge \boldsymbol{w}_x = (B\boldsymbol{u}) \wedge \boldsymbol{w} + \boldsymbol{u} \wedge (B\boldsymbol{w}) \\
&= \begin{bmatrix} a_1u_1 + b_1u_2 + b_2u_3 \\ c_1u_1 + a_2u_2 + d_1u_3 \\ c_2u_1 + d_2u_2 + a_3u_3 \end{bmatrix} \wedge \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \wedge \begin{bmatrix} a_1w_1 + b_1w_2 + b_2w_3 \\ c_1w_1 + a_2w_2 + d_1w_3 \\ c_2w_1 + d_2w_2 + a_3w_3 \end{bmatrix} \\
&= \begin{bmatrix} \boxed{\begin{matrix} (c_1u_1 + a_2u_2 + d_1u_3)w_3 - (c_2u_1 + d_2u_2 + a_3u_3)w_2 \\ +u_2(c_2w_1 + d_2w_2 + a_3w_3) - u_3(c_1w_1 + a_2w_2 + d_1w_3) \end{matrix}} \\ \boxed{\begin{matrix} (c_2u_1 + d_2u_2 + a_3u_3)w_1 - (a_1u_1 + b_1u_2 + b_2u_3)w_3 \\ +u_3(a_1w_1 + b_1w_2 + b_2w_3) - u_1(c_2w_1 + d_2w_2 + a_3w_3) \end{matrix}} \\ \boxed{\begin{matrix} (a_1u_1 + b_1u_2 + b_2u_3)w_2 - (c_1u_1 + a_2u_2 + d_1u_3)w_1 \\ +u_1(c_1w_1 + a_2w_2 + d_1w_3) - u_2(a_1w_1 + b_1w_2 + b_2w_3) \end{matrix}} \end{bmatrix} \\
&= \begin{bmatrix} a_2(\boldsymbol{u} \wedge \boldsymbol{w})_1 + a_3(\boldsymbol{u} \wedge \boldsymbol{w})_1 - c_1(\boldsymbol{u} \wedge \boldsymbol{w})_2 - c_2(\boldsymbol{u} \wedge \boldsymbol{w})_3 \\ a_1(\boldsymbol{u} \wedge \boldsymbol{w})_2 + a_3(\boldsymbol{u} \wedge \boldsymbol{w})_2 - b_1(\boldsymbol{u} \wedge \boldsymbol{w})_1 - d_2(\boldsymbol{u} \wedge \boldsymbol{w})_3 \\ a_1(\boldsymbol{u} \wedge \boldsymbol{w})_3 + a_2(\boldsymbol{u} \wedge \boldsymbol{w})_3 - b_2(\boldsymbol{u} \wedge \boldsymbol{w})_1 - d_1(\boldsymbol{u} \wedge \boldsymbol{w})_2 \end{bmatrix} \\
&= ([\mathrm{Tr}\, B]I_3 - B^T)(\boldsymbol{u} \wedge \boldsymbol{w}) = ([\mathrm{Tr}\, B]I_3 - B^T)\tilde{\boldsymbol{v}},
\end{aligned}
$$

where $\mathrm{Tr}\, B = a_1 + a_2 + a_3$. □

B Wiener algebras

By the (continuous) Wiener algebra $\mathcal{W}$ we mean the complex vector space of constants plus Fourier transforms of L^1-functions

$$\mathcal{W} = \{c + \hat{h} : c \in \mathbb{C},\ h \in L^1(\mathbb{R})\}$$

endowed with the norm $|c| + \|h\|_1$. Here we define the Fourier transform as follows: $(\mathcal{F}h)(k) = \hat{h}(k) = \int_{-\infty}^{\infty} dy\, e^{iky} h(y)$. The invertible elements of the commutative Banach algebra $\mathcal{W}$ with unit element are exactly those $c + \hat{h} \in \mathcal{W}$ for which $c \neq 0$ and $c + \hat{h}(k) \neq 0$ for each $k \in \mathbb{R}$ [16, 17, 22].

The algebra $\mathcal{W}$ has the two closed subalgebras $\mathcal{W}^+$ and $\mathcal{W}^-$ consisting of those $c + \hat{h}$ such that h is supported on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively. The invertible elements of $\mathcal{W}^\pm$ are exactly those $c + \hat{h} \in \mathcal{W}^\pm$ for which $c \neq 0$ and $c + \hat{h}(k) \neq 0$ for each $k \in \mathbb{C}^\pm \cup \mathbb{R}$ [16, 17, 22]. Letting $\mathcal{W}_0^\pm$ and $\mathcal{W}_0$ stand for the (nonunital) closed subalgebras of $\mathcal{W}^\pm$ and $\mathcal{W}$ consisting of those $c + \hat{h}$ for which $c = 0$, we obtain the direct sum decompositions

$$\mathcal{W} = \mathbb{C} \oplus \mathcal{W}_0^+ \oplus \mathcal{W}_0^-, \qquad \mathcal{W}_0 = \mathcal{W}_0^+ \oplus \mathcal{W}_0^-.$$

By $\Pi_\pm$ we now denote the (bounded) projections of $\mathcal{W}$ onto $\mathcal{W}_0^\pm$ along $\mathbb{C}\oplus\mathcal{W}_0^\mp$. Then Π_+ and Π_- are complementary projections. In fact,

$$(\Pi_\pm f)(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\zeta\, \frac{f(\zeta)}{\zeta-(k\pm i0^+)}, \tag{B.1}$$

where $f\in\mathcal{W}_0\cap L^p(\mathbb{R})$ for some $p\in(1,+\infty)$. These direct sum decompositions coupled with the Fourier transform can be schematically represented as follows:

$$\begin{array}{ccccc} L^1(\mathbb{R}) & = & L^1(\mathbb{R}^-) & \oplus & L^1(\mathbb{R}^+) \\ \big\downarrow \mathcal{F} & & \big\downarrow \mathcal{F} & & \big\downarrow \mathcal{F} \\ \mathcal{W}_0 & = & \mathcal{W}_0^- & \oplus & \mathcal{W}_0^+ \end{array}$$

Now observe that $\mathcal{F}$ acts as an isometric linear one-to-one correspondence from $L^1(\mathbb{R})$ onto $\mathcal{W}_0$. If we define the norm of $\mathbb{C}\oplus L^1(\mathbb{R})$ as $\|c+h\| = |c|+\|h\|_1$, we obtain the direct sum decomposition

$$L^1(\mathbb{R}) = L^1(\mathbb{R}^+)\oplus L^1(\mathbb{R}^-),$$

where the projection $\mathcal{F}^{-1}\Pi_\pm\mathcal{F}$ is the restriction of an arbitrary $h\in L^1(\mathbb{R})$ to the half-line $\mathbb{R}^\pm$.

The following result is most easily proved using the Gelfand theory of commutative Banach algebras [16, 17, 22].

Theorem B.1 *If for some complex number h_∞ and some $h\in L^1(\mathbb{R})$ the Fourier transform $h_\infty+\int_{-\infty}^{\infty} dz\, e^{ikz}h(z)\neq 0$ for every $k\in\mathbb{R}$ and if $h_\infty\neq 0$, then there exists $\kappa\in L^1(\mathbb{R})$ such that*

$$\frac{1}{h_\infty+\int_{-\infty}^{\infty} dz\, e^{ikz}h(z)} = \frac{1}{h_\infty}+\int_{-\infty}^{\infty} dz\, e^{ikz}\kappa(z)$$

for every $k\in\mathbb{R}$.

C Nonreal spectrum

In this appendix we prove that the nonreal spectrum of the AKNS system (1.1) consists of eigenvalues of finite algebraic multiplicity which can only accumulate on the real k-line. The fact that the diagonal matrix Σ has a zero diagonal entry forces us to consider the three operator polynomials

$$L(\lambda) = -\partial_x I_3 + i\lambda\Sigma + \mathcal{Q}, \tag{C.1a}$$

$$\tilde{L}(\lambda) = -\partial_x I_3 + i\lambda\Sigma - \mathcal{Q}, \tag{C.1b}$$

$$L_0(\lambda) = -\partial_x I_3 + i\lambda\Sigma, \tag{C.1c}$$

where $\mathcal{Q} \in L^1(\mathbb{R})^{3\times 3}$. In fact, the noninvertibility of Σ prevents us from studying the spectra of the linear operators $i\Sigma^{-1}\partial_x I_3$ and $i\Sigma^{-1}(\partial_x I_3 \mp \mathcal{Q})$ instead, as done before in [14, 25].

Let $\mathcal{H}$ stand for the complex Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}) \oplus H^1(\mathbb{R}) \oplus L^2(\mathbb{R}),$$

where the direct sums are orthogonal and $H^1(\mathbb{R})$ is the first Sobolev space. Thus for nonreal λ there exists a one-to-one correspondence between $f \in \mathcal{H}$ and $w \in L^2(\mathbb{R})^{3\times 1}$ which is given by the equalities $L_0(\lambda)w = f$ and

$$w_1(x) = \begin{cases} -\int_{-\infty}^{x} dy\, e^{i\lambda(x-y)} f_1(y), & \lambda \in \mathbb{C}^+, \\ +\int_{x}^{\infty} dy\, e^{-i\lambda(y-x)} f_1(y), & \lambda \in \mathbb{C}^-, \end{cases} \tag{C.2a}$$

$$w_2(x) = -\int_{-\infty}^{x} dy\, f_2(y) = +\int_{x}^{\infty} dy\, f_2(y), \tag{C.2b}$$

$$w_3(x) = \begin{cases} +\int_{x}^{\infty} dy\, e^{i\lambda(y-x)} f_3(y), & \lambda \in \mathbb{C}^+, \\ -\int_{-\infty}^{x} dy\, e^{-i\lambda(x-y)} f_3(y), & \lambda \in \mathbb{C}^-. \end{cases} \tag{C.2c}$$

C.1 Bounded and integrable potentials

For $\lambda \in \mathbb{C}^\pm$ and bounded potential $\mathcal{Q}$, the two operators $L(\lambda)L_0(\lambda)^{-1}$ and $L_0(\lambda)^{-1}L(\lambda)$ are bounded on the respective Hilbert spaces $L^2(\mathbb{R})^{3\times 1}$ and $\mathcal{H}$. Indeed, using that $\mathcal{H}$ is continuously imbedded in $L^2(\mathbb{R})^{3\times 1}$, we get the two diagrams of bounded linear operators

$$L^2(\mathbb{R})^{3\times 1} \xrightarrow{L_0(\lambda)^{-1}} \mathcal{H} \xrightarrow{\text{imbedding}} L^2(\mathbb{R})^{3\times 1} \xrightarrow{\mathcal{Q}} L^2(\mathbb{R})^{3\times 1}$$
$$\mathcal{H} \xrightarrow{\text{imbedding}} L^2(\mathbb{R})^{3\times 1} \xrightarrow{\mathcal{Q}} L^2(\mathbb{R})^{3\times 1} \xrightarrow{L_0(\lambda)^{-1}} \mathcal{H}$$

The identities $L(\lambda)L_0(\lambda)^{-1} = \mathbf{1} + \mathcal{Q}L_0(\lambda)^{-1}$ and $L_0(\lambda)^{-1}L(\lambda) = \mathbf{1} + L_0(\lambda)^{-1}\mathcal{Q}$, where $\mathbf{1}$ stands for the identity operator on a suitable function space, then imply the boundedness of $L(\lambda)L_0(\lambda)^{-1}$ and $L_0(\lambda)^{-1}L(\lambda)$.

If $\mathcal{Q}$ is bounded and belongs to $L^1(\mathbb{R})^{3\times 3}$, we rely on the polar decompositions

$$\mathcal{Q}(x) = U_l(x)(\mathcal{Q}(x)^\dagger \mathcal{Q}(x))^{1/2} = (\mathcal{Q}(x)\mathcal{Q}(x)^\dagger)^{1/2} U_r(x), \tag{C.3}$$

where $U_{l,r}(x)$ are unitary matrices and the square roots are nonnegative hermitian matrices such that $(\mathcal{Q}(x)^\dagger \mathcal{Q}(x))^{1/4}$ and $(\mathcal{Q}(x)\mathcal{Q}(x)^\dagger)^{1/4}$ belong to $L^2(\mathbb{R})^{3\times 3}$. Then it is easily verified that $(\mathcal{Q}^\dagger \mathcal{Q})^{1/4} L_0(\lambda)^{-1} U_l (\mathcal{Q}^\dagger \mathcal{Q})^{1/4}$ is a Hilbert-Schmidt operator and hence a compact operator on $L^2(\mathbb{R})^{3\times 3}$. Indeed, letting $K(x, y; \lambda)$ stand for the integral kernel of this integral operator and using that for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ we have the estimate

$$\|K(x,y;\lambda)\|_{\text{HS}}^2 = \left[\frac{1}{4} + e^{-2|x-y||\text{Im}\,\lambda|}\right] \|\mathcal{Q}(x)\|\,\|\mathcal{Q}(y)\|,$$

we easily see that $(\mathcal{Q}^\dagger\mathcal{Q})^{1/4}L_0(\lambda)^{-1}U_l(\mathcal{Q}^\dagger\mathcal{Q})^{1/4}$ is a Hilbert-Schmidt operator with Hilbert-Schmidt norm bounded above by $\frac{1}{2}\sqrt{5}\,\|\mathcal{Q}\|_1$, irrespective of the choice of $0 \neq \lambda \in \mathbb{C}$. In the same way we prove that

$$(\mathcal{Q}\mathcal{Q}^\dagger)^{1/4}U_r L_0(\lambda)^{-1}(\mathcal{Q}\mathcal{Q}^\dagger)^{1/4}$$

is a Hilbert-Schmidt operator and hence a compact operator on $L^2(\mathbb{R})^{3\times 1}$. In combination with the assumed boundedness of $\mathcal{Q}$, it follows that

$$\left[\mathcal{Q}L_0(\lambda)^{-1}\right]^2 = \underbrace{U_l(\mathcal{Q}^\dagger\mathcal{Q})^{1/4}}_{\text{bounded}}\underbrace{(\mathcal{Q}^\dagger\mathcal{Q})^{1/4}L_0(\lambda)^{-1}U_l(\mathcal{Q}^\dagger\mathcal{Q})^{1/4}}_{\text{Hilbert-Schmidt}}\underbrace{(\mathcal{Q}^\dagger\mathcal{Q})^{1/4}L_0(\lambda)^{-1}}_{\text{bounded}} \tag{C.4}$$

is a Hilbert-Schmidt and hence a compact operator on $L^2(\mathbb{R})^{3\times 1}$, especially since $\mathcal{H}$ is continuously imbedded in $L^2(\mathbb{R})^{3\times 1}$ and hence $L_0(\lambda)^{-1}$ is bounded on $L^2(\mathbb{R})^{3\times 1}$ for $\lambda \in \mathbb{C}^\pm$. Consequently, for $\lambda \in \mathbb{C}^\pm$ the operator $L(\lambda)L_0(\lambda)^{-1}$ on $L^2(\mathbb{R})^{3\times 1}$ is an additive perturbation of the identity by an operator whose square is a compact operator. As a result, all $\lambda \in \mathbb{C}^\pm$ for which

$$\mathbf{1} - [\mathcal{Q}L_0(\lambda)^{-1}]^2 = L(\lambda)L_0(\lambda)^{-1}\tilde{L}(\lambda)L_0(\lambda)^{-1} = \tilde{L}(\lambda)L_0(\lambda)^{-1}L(\lambda)L_0(\lambda)^{-1}$$

is not boundedly invertible form a set of isolated eigenvalues of finite algebraic multiplicity contained in a strip of the type $\{\lambda \in \mathbb{C} : 0 < |\text{Im}\,\lambda| < \varepsilon\}$ that can only accumulate on the real line or at infinity [21, Thm. I5.1] (also [19]).

We easily verify that, for all $\lambda \in \mathbb{C}^\pm$, $\mathbf{1} - [\mathcal{Q}L_0(\lambda)^{-1}]^2$ is boundedly invertible iff both $L(\lambda)L_0(\lambda)^{-1}$ and $\tilde{L}(\lambda)L_0(\lambda)^{-1}$ are boundedly invertible. Thus the points $\lambda \in \mathbb{C}^\pm$ for which the operator $L(\lambda)L_0(\lambda)^{-1}$ is not boundedly invertible form a set of isolated eigenvalues of finite algebraic multiplicity contained in a strip of the type $\{\lambda \in \mathbb{C} : 0 < |\text{Im}\,\lambda| < \varepsilon\}$ that can only accumulate on the real line or at infinity.

Let us now define the concept of eigenfunction. Let $\mathcal{Q}$ be bounded and belong to $L^1(\mathbb{R})^{3\times 3}$. For $\lambda_0 \in \mathbb{C}^\pm$ the vector function $0 \not\equiv v \in \mathcal{H}$ is called an *eigenfunction* of the pencil $L(\lambda)$ at the eigenvalue λ_0 if

$$L(\lambda_0)v = \mathbf{0},$$

where $\mathbf{0}$ stands for the zero element of $\mathcal{H}$. This is equivalent to stating the existence of a nontrivial $w \in L^2(\mathbb{R})^{3\times 1}$ such that

$$L(\lambda_0)L_0(\lambda_0)^{-1}w = \mathbf{0}.$$

The connection between $v \in \mathcal{H}$ and $w \in L^2(\mathbb{R})^{3\times 1}$ is given by $v = L_0(\lambda_0)^{-1}w$. We call w an eigenfunction of the rational pencil

$$W(\lambda) = L(\lambda)L_0(\lambda)^{-1}$$

at the eigenvalue λ_0.

C.2 Unbounded and integrable potentials

The problem is what to do if $\mathcal{Q} \in L^1(\mathbb{R})^{3\times 3}$ but is not bounded. In this case we define for $\lambda \in \mathbb{C}^\pm$

$$W_l(\lambda) = \mathbf{1} + (\mathcal{Q}^\dagger \mathcal{Q})^{1/4} L_0(\lambda)^{-1} U_l (\mathcal{Q}^\dagger \mathcal{Q})^{1/4}, \tag{C.5a}$$

$$W_r(\lambda) = \mathbf{1} + (\mathcal{Q}\mathcal{Q}^\dagger)^{1/4} U_r L_0(\lambda)^{-1} (\mathcal{Q}\mathcal{Q}^\dagger)^{1/4}, \tag{C.5b}$$

which are Hilbert-Schmidt [cf. (C.4)] and hence compact perturbations of the identity on $L^2(\mathbb{R})^{3\times 1}$. Since $\mathcal{Q} = U_l(\mathcal{Q}^\dagger \mathcal{Q})^{1/2} = (\mathcal{Q}\mathcal{Q}^\dagger)^{1/2}U_r$ implies $U_l(\mathcal{Q}^\dagger \mathcal{Q})^{1/4} = (\mathcal{Q}\mathcal{Q}^\dagger)^{1/4}U_r$, we can write (C.5) in the form

$$W_l(\lambda) = \mathbf{1} + (\mathcal{Q}^\dagger \mathcal{Q})^{1/4} L_0(\lambda)^{-1} (\mathcal{Q}\mathcal{Q}^\dagger)^{1/4} U_r, \tag{C.6a}$$

$$W_r(\lambda) = \mathbf{1} + U_l (\mathcal{Q}^\dagger \mathcal{Q})^{1/4} L_0(\lambda)^{-1} (\mathcal{Q}\mathcal{Q}^\dagger)^{1/4}. \tag{C.6b}$$

Since the unitary matrices U_l and U_r coincide if $\det \mathcal{Q} \neq 0$ and can be made to coincide if $\det \mathcal{Q} = 0$, we see that $W_l(\lambda)$ and $W_r(\lambda)$ are unitarily equivalent for each nonreal λ. We observe that the points $\lambda \in \mathbb{C}^\pm$ where $W_{l,r}(\lambda)$ is not boundedly invertible, are eigenvalues of finite algebraic multiplicity which are confined to a strip of the type $\{\lambda \in \mathbb{C} : 0 < |\operatorname{Im}\lambda| < \varepsilon\}$ and can only accumulate on the real line or at infinity [21, Thm. I5.1] (also [19]).

Using that, for two bounded linear operators T and S, $\mathbf{1} - \lambda TS$ and $\mathbf{1} - \lambda ST$ are both boundedly invertible or neither boundedly invertible, we see that, for $\lambda \in \mathbb{C}^\pm$, the operators $L(\lambda)L_0(\lambda)^{-1}$, $L_0(\lambda)^{-1}L(\lambda)$, $W_l(\lambda)$, and $W_r(\lambda)$ are all boundedly invertible or none of them boundedly invertible whenever $\mathcal{Q}$ is bounded. Thus we have generalized the concept of eigenvalue from the bounded $\mathcal{Q} \in L^1(\mathbb{R})^{3\times 3}$ case to the general $\mathcal{Q} \in L^1(\mathbb{R})^{3\times 3}$ case.

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Declarations

Conflict of interest On behalf of both authors, the corresponding author states that there is no Conflict of interest.

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