



Historical review

Early traces of the principle of virtual work in Hellenistic texts: Exegesis of problems 1, 2 and 3 in pseudo-Aristotle's *Mechanica Problemata*Mario Spagnuolo ^a,* , Francesco dell'Isola ^b^a Department of Civil and Environmental Engineering and Architecture (DICAAR), University of Cagliari, Via Marengo 2, Cagliari, 09123, Italy^b Department of Civil Construction-Architectural and Environmental Engineering (DICEAA), University of L'Aquila, Piazzale Ernesto Pontieri, Poggio di Roio, L'Aquila, 67100, Italy

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ABSTRACT

Our aim is to try to trace, in the history of mechanics, the first formulation of the principle of virtual work (PVW). This important question is, of course, connected with the origin of the concept of *kinematics* and its relation with the concept of *dynamics*. Now it is widely accepted that the Principle of Virtual Velocities (later called Virtual Work) was known in a geometrical form by the author of the Greek text *Mechanica Problemata* (*The Mechanical Problems*). Indeed, this text does not appear to be a theoretical treatise but rather a collection of solved exercises, mainly concerning statics, the functioning of machines and some dynamics. In this paper we present our exegesis of the first three problems of the *Mechanica Problemata*, because we believe that deeply understanding its content may allow us to clarify the Greek origin of the Principle of Virtual Work, to locate in space and time the birth of mathematized mechanics and to prove that Renaissance mechanics derives from Greek sources.

Contents

1. Introduction	2
1.1. Applications of principle of virtual work in contemporary scientific progress	2
1.2. Extensions of the principle of virtual work in generalized continua	2
1.3. Motivation for the present study	3
1.4. The <i>Mechanica Problemata</i>	3
1.5. The need of a translation guided by mechanics knowledge	3
1.6. The problem of the authorship	4
1.7. Pristine form of the principle of virtual work	4
1.8. Hellenistic science influence on D'Alembert and Lagrange through Jordanus de Nemore's <i>Opus</i>	5
1.9. The importance of lexicon: from <i>virtus</i> to <i>virtual</i>	5
2. The three first problems	5
2.1. Exegesis of Problem 1	6
2.2. Problem 2 and the study of stable equilibrium	11
2.3. Similarities with Dirichlet-Lyapunov criteria	12
2.4. Suggestive similarities between Hellenistic and modern stability analyses	12
2.5. Problem 3: early form of the Principle of Virtual Work	12
3. Jordanus de Nemore and the reconstruction of the Principle of Virtual Work from Problem 3	13
3.1. Problem 3 as a reference for Jordanus de Nemore	13
3.2. A conjectural reconstruction of the first form of the Principle of Virtual Work	14
3.2.1. The lever system	14
3.2.2. Static equilibrium: Moment balance	14
3.2.3. Virtual work approach	14
4. Conclusion	15
CRediT authorship contribution statement	16

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Declaration of competing interest.....	16
Data availability	16
References.....	16

1. Introduction

The Principle of Virtual Work (PVW) has been considered controversial since its first modern formulation by D'Alembert and Piola [1,2]. In fact, even recently [3] the innovative and heuristic value of the PVW has been denied and alternative postulation schemes have been proposed [4,5]: in these last approaches *ad hoc* balance laws are postulated for each independent kinematical descriptor. This choice leads to a multiplication of basic assumptions, contradicting Occam razor logic principle, and produces serious consistency problems among different balance laws, constitutive equations and boundary conditions. We believe that, instead of introducing artificially in mechanical theories not relevant thermodynamic principles [3,6], it is more logically economic to postulate a form of the PVW and deduce, using the postulated constitutive equations for internal and external work, by integration by parts balance equations and the corresponding boundary conditions [7].

It is therefore to be accepted that, when inventing new models or new theories in mechanics, it is most suitable to start from the postulation of a form of the PVW. To reinforce this argument we believe it is essential to solve a problem in history of mechanics: which was the first postulation scheme used to invent mechanical theories?

We believe that [8] available sources indicate that mathematical mechanics was invented by the school of Archytas of Tarentum [9] and that Archytas based it on the PVW. The present paper aims to give a logical basis to this statement by carefully reading parts of *Mechanica Problemata*.

In fact, even without digging in the pristine sources of mathematical mechanics, we can say that the PVW has long served as a cornerstone in analytical mechanics [10,11], underpinning a wide spectrum of formulations in both classical and modern theories of elasticity and structural analysis. In recent decades, its role has deepened and expanded, becoming not only a conceptual tool for expressing equilibrium in mechanical systems, but also a rigorous foundation for the mathematical analysis of complex boundary value problems (BVPs). In particular, the PVW provides a natural variational framework for formulating weak solutions, characterizing admissible external actions, and deriving natural boundary conditions across a range of higher-order and multiphysics models.

In contemporary mathematical continuum mechanics, the PVW is a tool which cannot be replaced when generalized continuum mechanics models need to be formulated: the following referenced literature will be focused on this specific need and to the question of formulating well-posedness of static and quasi-static boundary value problems. Particular attention will be paid to generalized theories of elasticity that account for microstructure, higher gradients, and surface effects. For instance, in the context of surface elasticity models of arbitrary order, the PVW enables the precise formulation of weak solutions and the identification of admissibility conditions under complex boundary constraints [12,13]. Similar approaches have been employed to analyze existence and uniqueness in micropolar elasticity [14,15], strain gradient elasticity [16], and pantographic lattice models [17], each of which involve differential operators of higher order and thus require a careful mathematical treatment of boundary data.

The PVW is not only crucial for asserting solvability but also instrumental in deriving the compatible natural boundary conditions which emerge from the variational formulation itself and which model the physical principles governing mechanical interaction at surfaces, interfaces, or material discontinuities. In higher-order theories such as second-gradient elasticity or strain gradient fluids, where additional

degrees of freedom are associated with microdeformations or interfacial phenomena, classical boundary conditions are insufficient or inconsistent [18,19]. Here, the PVW allows for a consistent derivation of the correct additional terms that describe surface tractions, moments, and other generalized forces.

Furthermore, recent investigations have highlighted the role of the PVW in characterizing the class of admissible external forces. In advanced mechanical models such as first and second strain gradient elasticity, the set of allowable loadings cannot be specified arbitrarily [20]; instead, their structure must conform to the variational framework defined by the PVW. This admissibility demand guarantees not only mathematical consistency, but also the capacity of modeling physical phenomenology by the resulting boundary value problems [21].

1.1. Applications of principle of virtual work in contemporary scientific progress

The enduring conceptual lineage, which starts from Archytas and arrives up to Paul Germain, is clearly visible in a wide range of contemporary works that use the PVW to formulate complex models in biomechanics, generalized continua, and architected materials. A particularly fertile area of application is bone remodeling, where variational formulations have been used to model the interaction among mechanical stimuli, biological response, and internal structural evolution. These include models accounting for graft resorption in three-dimensional scaffold integration [22], the effects of multiple mechanical sources in remodeling dynamics [23], and the interplay between damage and diffusive bio-mechanical signals [24]. Complementary approaches investigate the adaptive response of bone microstructure under dynamic loads [25], as well as orthotropic models with internal substructure evolution [26]. Further generalizations have led to the formulation of variational models for linear thermo-viscoelastic systems [27], as well as for three-dimensional thermoelasticity with thermal inertia [28].

Parallel efforts in the modeling of architected and metamaterial systems have exploited second and higher gradient theories to describe a rich variety of mechanical behaviors. Variational formulations have been developed for pantographic lattices with complex fiber geometries such as the case treated in [29,30], as well as for the numerical modeling of shear rupture, dissipation, and hysteresis in such structures under bias extension and cyclic loads [31,32]. Other contributions explore lattice shells composed of curved Kirchhoff rods [33], the influence of fiber topologies in reinforced composites [34], discrete formulations of rod dynamics [35], and energy-based control strategies for flexible manipulators [36]. Foundational studies on nonlinear elastica [37] and two-dimensional models of pantographic sheets in 3D motion [38] continue this trajectory. Additional advances include the study of wrinkling in twisted thin films [39] and Cosserat-type micromorphic media [40]. We believe that the previously accounted research works could not have been easily conceived without the systematic use of the PVW.

1.2. Extensions of the principle of virtual work in generalized continua

The theoretical scope of the PVW has found a natural extension in recent studies devoted to generalized continua, where higher-order kinematics and microstructural effects demand a more refined variational treatment as they require novel sets of essential and boundary conditions. Among these contributions, the reconstruction of Piola's method in the context of second- and third-gradient elasticity [41,42] makes explicit the deep structural role of virtual displacements and work conjugate variables in nonclassical theories, where the PVW

provides a irreplaceable framework for parameter identification in complex materials.

The power of variational principles extends beyond classical field theories to novel modeling paradigms such as swarm-inspired continuum dynamics. In this domain, the position-based dynamics developed in [43,44] reinterpret continuum deformation via discrete particle algorithms, with behavior governed by constraints derivable from energy and virtual work considerations. Likewise, the numerical inverse approaches [45,46] underscore how parameter estimation and constitutive modeling hinge on the variational structure of the underlying equations. Such approaches often rely on a minimization of residual virtual work to reconcile models with empirical data, reaffirming the foundational role of PVW in bridging theory and experiment.

Micropolar and gradient elasticity theories, which generalize classical continua by incorporating micro-rotations or higher deformation measures, also benefit from the variational insights rooted in the PVW. New deformation measures for nonlinear and linear micropolar media have been proposed [47,48], building upon earlier work on strain measures in the non-linear micropolar continuum [49]. In parallel, advances in dissipative and frictional models [50–52] have been formulated using energy-based or variational arguments. This is equally true in multiphysics domains, such as electromagneto-elastic materials [53] and architected media exhibiting nonlinear vibrations [54]. Finally, even in the context of nonlocal viscoelasticity in fluids, variational formulations remain a natural and effective language for capturing dispersive and memory effects [55].

These diverse lines of research confirm that the PVW is not merely a relic of classical mechanics, but a generative principle that continues to inspire new modeling strategies across scales, materials, and disciplines. Whether reformulated in geometrically exact language, embedded in numerical algorithms, or extended to nonclassical material behaviors, it persists as the conceptual and formal bedrock of modern continuum theory.

1.3. Motivation for the present study

Taken together, these developments underline the centrality of the Principle of Virtual Work as a unifying tool in mechanics, and in particular in modern continuum mechanics. Its dual role, both as a methodological bridge between classical and generalized theories, and as supplier of a rigorous mathematical apparatus for the analysis of weak formulations, confirms its timeless relevance. This theoretical richness imposes further historical investigation into its origins, evolution, and foundational status. In this light, it becomes compelling to explore whether the conceptual structure of the PVW, so prominent in modern mechanics, might trace its roots back to the early stages of mechanical thought in antiquity.

In the present work, we start undertaking such an inquiry. We return to one of the earliest preserved mechanical texts, the *Mechanica Problemata*, traditionally and wrongly attributed to Aristotle, and conduct a detailed exegesis of its first three problems. Our goal is to demonstrate that a pristine form of the Principle of Virtual Work was not only known in the Hellenistic period but effectively used to reason about equilibrium configurations of levers and balances. By examining the logical structure of the arguments and comparing them with later developments, particularly those of *Jordanus de Nemore* and the modern formalism of virtual displacements, we argue that the PVW has a lineage that predates its Renaissance rediscovery and its Enlightenment formulation. This inquiry, thus, bridges ancient and contemporary approaches to mechanical theory, offering a deeper understanding of how foundational principles emerge, persist, and evolve across the history of science.

1.4. The *Mechanica Problemata*

The *Mechanica Problemata* (MP) is a work that has long been attributed to Aristotle or his school (the author of the MP is sometimes referred to in the literature as Pseudo-Aristotle). Before reading in detail some passages from three of the 35 problems collected in the work, it is necessary to make a few preliminary remarks.

First of all, we would like to point out a matter of the utmost importance: although we strongly believe that the first version of the principle of virtual work is present in the MP, this can be very difficult to verify if (i) one does not read the text carefully, even referring to the Greek original, and (ii) one does not investigate the influence that the MP had on the subsequent literature. We explicitly remark here that the analysis of all secondary sources [56–58] proves without any doubt that MP were considered a scientific textbook to be used in the formation of new scientists without interruption until the publication of the 1758 edition of Lagrange's "Mécanique Analytique".

More specifically, we would like to point out that the translations of ancient Greek texts on scientific subjects are often, if not always, carried out by scholars who are extremely knowledgeable for their humanistic-linguistic skills, but who are almost completely unaware of the technical details of the mathematical mechanics. Among the exceptions, as we shall see, remarkable is Marshall Clagett, author of works of fundamental importance for understanding the development of Science in antiquity and the Middle Ages. The usual result is that the translations of scientific texts are completely unreadable to the contemporary scientist, being logically inconsistent, and are, unavoidably and consequently, considered outdated or too naïve.

The second aspect to be taken into account when analyzing the texts of these mechanical problems is that the MP was written when the main mathematical tool in mechanics was Euclidean geometry. Therefore, many believe that its author was using a primitive version of the Principle of Virtual Work. We believe that the original version of the MP had to undergo several rewrites and reinterpretations before coming down to us in the version known today (that of D'Alembert and Lagrange), which is mainly based on algebraic formalism. In particular, we believe that it is *Jordanus de Nemore* (or *Nemorarius*) who takes the text of problem 3 and rewrites it in a version in which the principle of virtual work is more easily recognizable. But *Jordanus* would not have written his work without having in his hands the believed-primitive version of the principle as used in the MP!

1.5. The need of a translation guided by mechanics knowledge

Within the scholarly landscape, there are a number of relevant contributions that examine the available text constituting the *Mechanica Problemata*. These studies do not confine themselves to a simple translation of the text, but are devoted to contextualizing its contents. However, in spite of the extraordinary efforts made by philologists and classicists, it is important to recognize that the result presented is very often considerably inadequate.

In fact, many of the translations and critical treatises consulted [8, 59–62] suffer from the same fundamental shortcoming: they are the work of scholars who are excellent in linguistic skills but who had little or no knowledge of mathematics, mechanics, physics and other branches of exact sciences. Their lack of in-depth knowledge of the technical aspects of these disciplines contributes to a superficial, incomplete and logically inconsistent interpretation of the original text. Therefore, there is an urgent need of the competences of multidisciplinary experts capable of combining philological expertise with a solid understanding of the scientific disciplines involved in order to produce a more complete and accurate analysis of the MP.

The result of many translations of MP is often a text that seems to lack scientific quality, appearing as just another ancient testimony about levers and balances. They often use a language similar to that of translations of the *Iliad* or *Archilocus' Jambi*. The reality behind it,

however, is quite different. If one really attempts the reading of the text of the MP and to understand its author's intentions, a panorama of scientific knowledge emerges that parallels or overcomes that of figures such as Newton or Galileo. Indeed, a careful comparison between the first book of Newton's *Principia* and the first problem of the MP reveals a surprising affinity. The two texts share so many elements that it becomes undeniable that Newton's *Principia* comes from the scientific tradition of the MP.

This connection is so strong that we hypothesize that Newton may have come into possession of scientific texts from the Greek period and has been "inspired" by them in his research: we believe that a careful search in the documents left to us by Newton will reveal his cultural debt towards Greek sources [63–66]. Whether this is only a suggestive hypothesis or an established truth, the crucial aspect is that the *Principia* contains, in a somehow more comprehensive way, the same kind of reasonings found in the MP: albeit Descartes had already introduced Cartesian geometry, Newton continued to use the Greek geometric methods in all his masterpieces. The correlation which we prove, reinforces the idea that the roots of Newton's scientific vision can be traced directly to the MP tradition, thus highlighting the continuity and evolution of the scientific thought throughout history.

All the translations we have examined are based on the assumption that the corpus of the *Mechanica Problemata* (MP) deals exclusively with levers and balances, an interpretation that turns out to be remarkably limited. Indeed, this view is incomplete, since the text of the problems collected in the MP seems to be only part of a much larger and more varied lost scientific treatise. We are tempted to consider the transmitted manuscript of MP as the notebook of a student who wants to collect the ideas of his teacher, who expresses concepts in a fluent and relentless manner.

Often one notices that the transition from one problem to the next is fluid, dealing with the mathematical modeling of different physical phenomena that can nevertheless be represented by the same mathematical formalism. It is in this context that the genius of the Greek scientific tradition behind MP comes to the fore: discussions are started with a general approach, followed by the presentation of the ideas specific to each application problem. This style of treatment highlights an advanced level of thinking, where the authors not only deals with individual problems, but also offers a broader and more abstract view, recognizing the underlying mathematical connections that permeate the entire work. In this way, a profound understanding of the subject matter is manifested, emphasizing the richness and complexity of the scientific content inherent in the *Mechanica Problemata*. The authors believe to recognize in the intentions of the authors of MP the same intentions manifested by R.P. Feynman: we cannot imagine what would have become the set of his lectures if they were not have been recorded and how would appear the lectures notes if they were not taken by the best physicists of their generation. For more comments about this point we refer to the opinion of Feynman himself [67].

The perception of a possible incongruity between the high complexity of scientific thought and the modest quality of its expression in MP seems to be an opinion shared by W. S. Hett, who acted as skilled translator in the Loeb Classical Library's edition of Aristotle's *Minor Works* [68]. This observation formed the basis of our initial intentions and considerations:

Though the author is astray in some cases, it is most surprising to find how far the science of Applied Mathematics had advanced by this date.

So, if we accept the idea that MP were written by someone taking notes, the pupil was overwhelmed by the enormity and complexity of the topics covered by the teacher and only manages to grasp some aspects of them. However, if this seems too interpretive, we invite you to consider how it is possible that an author capable of certain outbursts of thought could then leave some parts of the treatise as incomplete. Surely MP is not the original work, but a compendium of it!

1.6. The problem of the authorship

A problem of considerable interest, which is not only academic but also of fundamental importance, lies in establishing who the author of the MP may be. This problem is relevant because, depending on who the author is, different kinds of conjectures can be made about what the original work might have been like. It now seems implausible to almost all scholars that the author of the *Mechanica Problemata* is Aristotle. Some still converge on the possibility that he was a member of the Peripatus. We can support the idea that the material author of the text may have been a member of the Peripatus, but we do not believe that the source from which the compendium is drawn comes from that cultural milieu. An important piece of evidence in favor of an author of the Peripatus is the fact that the MP is written in the Attic dialect, the same language spoken by Aristotle and his students. However, there are some very convincing theories that the author came from a completely different cultural milieu. In this context, Thomas Winter, introducing his sometimes questionable translation of the MP [8], presents a very persuasive deductive investigation into the identity of the author of the MP. He begins with a text by Vitruvius, who clearly knew who the author was and never explicitly reveals his identity because, Winter argues, this information must have been common knowledge in Vitruvius' time.

Vitruvius, after having reproduced nearly verbatim many of the results reported in MP, mentions in his bibliography a number of Greek scientists whose works inspired him. Many of them can be associated with the work without much difficulty. This leaves only a few authors from which to choose the one who wrote the MP. By exclusion and by chance, Winter shows that only Archytas of Tarentum could be the author of the *Mechanica Problemata*. There is only one problem (which Winter does not point out): Archytas is from Tarentum, a colony of Sparta where the Doric dialect is spoken. There seems to be, therefore, a linguistic inconsistency in the author's choice. Obviously, this inconsistency can be overcome by accepting the thesis that the direct compiler of the MP is a member of the Peripatus (who therefore writes in Attic dialect) who takes notes from a much larger and more complete text and/or from lectures both derived from the ideas of Archytas of Tarentum. In conclusion, we believe that the original text of Archytas is the common source of our version of the MP and the work of Vitruvius. It is suggestive to imagine that the ideas of Archytas were imported into the Peripatus by Strato of Lampsacus. In fact, both Strato and Archytas belonged to the Pythagorean school. It has to be remarked that Clagett himself thinks it possible that the author of the MP could be Strato [69,70].

1.7. Pristine form of the principle of virtual work

Two premises must be made concerning why we agree with the statement [69] that a pristine form of the Principle of Virtual Work is applied in MP. It will be sufficient for us to read the third problem (for the understanding of which, however, we have also included some notes about the first and second ones) in order to clarify somewhat how the Principle of Virtual Work was used in Hellenistic mechanics. It is now necessary to understand two aspects that particularly concern MP.

Firstly, as we have already noted, this text lacks an organic and/or axiomatic formulation such the one which can be found in Euclid's *Elements*, or in contemporary texts such as those by Aristotle, or in later works such as those by Archimedes. This is due to the fact that the MP are not really a treatise, but rather a series of notes. Clagett [69] is aware of this circumstance. However, he justifies it by conjecturing that in Hellenistic Science there were two different ways to formulate theories: the axiomatic/geometric one, typical of the works by Euclid and Archimedes, and the dynamic one, not based on axiomatics but dealing with specific problems. From Clagett's point of view the MP use the Principle of Virtual Work in an unconscious way, as sometimes the

principles are used in some modern textbooks in Physics. We disagree with Clagett's justification.

Secondly, in the third problem we do not find an explicit formulation of the Principle of Virtual Work. Instead, we see how it can be applied to calculate the equilibrium of levers. In this regard, we would like to point out that the equilibrium of a lever can also be calculated through the study of the balance of moments, since the forces applied to the extremes are given and the moments arms correspond to the arms of the lever. This, in fact, is quite familiar to the contemporary reader, because today we are massively influenced by a practical Engineering modern approach that has abandoned the formulation of equilibrium by means of variational principles and has chosen to embrace that made in terms of the balance of forces and moments. It is peculiar that in the brief discussion presented in Problem 3, there is no trace of such balanceist reasoning. We remark that it is very difficult without Cartesian geometry to introduce the concept of moment of force (however introduced by F. Maurolico in the 16th century [71]), whose mastering became much easier after the introduction of the concept of vector product. It is clear that mechanicians basing their reasonings on geometric techniques will find much easier to introduce and use the Principle of Virtual Work than the balance laws: therefore, it is evidently much more natural for Hellenistic mechanicians to use the Principle of Virtual Work than balance laws. Only scientists having a modernistic prejudice may consider that postulating balance laws is more natural.

1.8. Hellenistic science influence on D'Alembert and Lagrange through Jordanus de Nemore's Opus

It should be pointed out that we find in the MP a pristine version of this principle, which will be reformulated in the following millennia until it arrived to the form accepted by D'Alembert and Lagrange. We do not know whether a more complete and rigorous formulation existed in Hellenistic literature (although the evidence would lead us to believe so), but we can follow the traces over the centuries left by Problem 3. In fact, this problem certainly acted as a source for Jordanus de Nemore, who in his works recalls the same reasoning made by the author of the MP and clarifies it in the context of a problem of equilibrium determination. A comparison with the text of Jordanus de Nemore is also very useful because, as it will turn out, Problem 3 seems lacking and concludes with the simple enunciation of the problem to be solved. From this point of view, it appears to be the text of an exercise without its solution (again this circumstance reinforces our belief that the compiler of MP was a student).

1.9. The importance of lexicon: from *virtus* to *virtual*

A second observation, which closes this brief introductory discussion, concerns the nature of the Principle of Virtual Work as used in MP. Its nature is more similar to the Principle of Virtual Velocities contained in D'Alembert and Lagrange's approach than to the successive version named Principle of Virtual Work. In this sense, one can recognize in Lagrange a strong inspiration that is certainly Greek in origin.

On the other hand, this can also be determined because of Lagrange's choice of the lexicon used: he, in fact, does not speak of "force" but of "power" (while giving to "power" the same meaning as what we give to the word "force"): in the MP the word ισχύς is used, which is translated in the literature with either words. We conjecture that Lagrange expressly chose to refer to a "power" rather than to a "force" wanting to separate the Principle of Virtual Velocities from the standard concept of force. In this principle, in fact, as D'Alembert explained in his *Traité de dynamique* [72] using words that cannot be misunderstood, forces are introduced as mere mathematical objects: "I had never intended to attach to these terms any other idea different from those which result from the Principles that I established".

We add another lexical consideration about the adjective "virtual" appearing in the name "Principle of Virtual Work": as we will see, this principle takes its starting point from the ideas contained (in pristine form) in the MP, but it is then refined over the centuries and a strong boost in the direction of its modern formulation, eventually made by D'Alembert and Lagrange, can be found in the Middle Ages Latin tradition. This tradition receives as a heritage the Aristotelian corpus and, through this latter, the text of the MP. As we have mentioned, Jordanus de Nemore, probably also drawing on Arabic sources [69], takes the same problems of the levers contained in MP and recalculates the equilibrium conditions by introducing the current configuration of the system and estimating, starting from it, the linearized displacements-velocities of the applied weights. This "dynamic" approach makes it possible to calculate the equilibrium based not only on the weights and their positions (which are used in the geometric approach à la Archimedes), but also on the velocities (or, equivalently, the displacements) of the "points of application" of the weights.

Obviously, we cannot distinguish between the eventual original contribution by Jordanus and the lost common source of his works and MP. We believe that it is very likely that Jordanus had access to the same or to an equivalent source that Vitruvius.

The Latin tradition, based above all on texts attributed to Jordanus, will also massively influence later scholars in their choice of names. We have already observed how in Lagrange there is an early nomenclature certainly related to the direct translation from Greek of the word ισχύς (force or power). But it is the Latin tradition that is certainly behind the attribute "virtual". In his notes to the *Liber Karastonis* [69], Clagett observes:

Cf. Aristotle, Physics IV, ch. 4. The 4th, 5th and 6th postulates define "force" (virtus or fortitudo), in the sense of the power of a heavy body to fall downward, or of a light body to rise upward, in a given corporeal medium.

It is clear, then, that the attribute "virtual" derives from the Latin "virtus", which can also be translated as "force". But Clagett points out that this word also expresses the idea of a "movement in power" of the weight in question. We can assume that it is here that the concept of force was first born. Of course, it could be argued that Archimedes, in a period immediately following the formulation of the MP (and probably having its source available), introduces equations of equilibrium which therefore use the concept of force independently of its involvement in the Principle of Virtual Work: but (i) Archimedes is a later scholar than the author of MP source, and (ii) it would not be the first time that consequences of a certain principle are assumed as a starting point for presenting further results after they had been fully demonstrated. Or perhaps, referring to Archimedes, it would be the first time, but certainly not the last. Consider, for example, Newton, who, as the story goes, had introduced the laws of motion using the new techniques of differential calculus, but published his *Principia* only after he had demonstrated everything using Euclidean geometry. We will see later that Newton had almost certainly also read the MP (or its medieval remakes, such as the texts of Jordanus), and that his *Principia* seem to follow the MP text closely. But the fact remains that when a new way of formulating theories is introduced, the first attempt is always to reformulate it in the well-known scientific language already accepted by the scientific community. This may have happened with Archimedes, it certainly happened with Newton, it even happened with Heisenberg when he reformulated Quantum Mechanics in terms of operators rather than states.

2. The three first problems

The translation of the parts of the *Mechanica Problemata* which we analyze is taken from the English translation of this work, appearing in pp 327–411 of the Loeb Classical Library's edition of Aristotle's *Minor Works*, first published in 1936 [68]. In the number of available translations this seems to us the more careful one.

2.1. Exegesis of Problem 1

We read in MP Problem 1:

First of all then a difficulty will arise as to what happens to the balance; why, that is, larger balances are more accurate than smaller ones. The origin of this is the question why that part of the radius of a circle which is farthest from the center moves quicker than the smaller radius which is close to the center, and is moved by the same force.

The question from which the author of the MP starts seems to be a purely practical one: the sensitivity of the balance as a measuring instrument. The author observes that a larger balance, *i.e.* one with longer arms, is able to give more accurate mass measurements than a smaller one. The really interesting thing is that the compiler of MP refers to the ideas of somebody who seems to fully understand the concept of indirect measurement and who is able to identify, in a very concrete way, the right quantities to modify in order to obtain an indirect increase in the accuracy of mass measurement: the clarity of the underlying theoretical framework and the poor quality of the following logical steps are a strong indicator of the fact that MP is a lecture note taken by a compiler who is not fully mastering the subject.

It is astonishing, and meaningful, what happens when reading MP: when the statements, albeit deep and reflecting complex epistemological or methodological concepts, are simple to be formulated then the compiler of MP manages to be understandable. On the contrary, when the reasonings become more technical or more involved, then the compiler of MP is lost, the quality of the text is very bad and the reader loses completely confidence about the reliability of the text.

Coming back to the exegesis of the quoted text: firstly, the measurement of the mass is carried out, as in any arm balance, by means of an angular measurement: one observes the displacement of the balance arms with respect to the horizontal and, by measuring the angle, one is able to obtain the measurement of the weighted mass.

Secondly, the ability of the school of thought that elaborated the sources of the MP is not only of a practical and technological nature: a deep understanding of the theoretical aspects of mathematics, reaching heights of abstraction that could be described as contemporary, manages to reach us via what appears to be a corrupted text. In fact, the observation that a small balance may not give a measurement for masses that are too small, but that such a measurement could be obtained by using a balance with longer arms, shows that the scientific *milieu* underlying MP succeeds in clearly distinguishing between the concept of infinitesimal displacement (or rotation) from that of zero displacement (or rotation). In fact, the fact that in some cases a small balance does not provide a measurement is not due to the possibility of having an object with zero mass, but rather to the practical impossibility of measuring infinitesimal displacements with this instrument. This aspect, which may seem marginal, is of fundamental importance from an epistemological point of view. Those who claim that Greek scientists were not interested in applications of their mathematics should be surprised in discovering that in MP the concept of sensitivity of a physical instrument is clearly addressed. In future investigations we will confute with more arguments the wrongly supposed incapacity of Hellenistic Science to apply abstract concepts to practical problems. We continue to read the text of MP:

The word quicker is used in two senses; if a point covers the same distance as another in a shorter space of time we call it quicker, and also if it covers a greater distance in an equal time. But in our case the greater radius describes a greater circle in equal time; for the circumference outside is greater than the circumference inside.

We start remarking that the greek text reads: which, word by word, means:

it can, either, in fact in a shorter time the same space go through.

We do not understand why the translator wrote “in a shorter space of time” which in this context is extremely confusing. Indeed, a skilled mechanic would have translated “in a shorter interval of time”. Remark that the Greek text with a laudable synthesis simply says “shorter time”.

This is a simple and immediate example of a bug in the available English translations of MP: together with many similar others, it demands that expert mathematicians and mechanics with enough competence of Greek language should accept the challenge of producing good quality translations of Greek mathematical texts.

Coming back to the quoted sentence: the concept of average speed is expressed very clearly, albeit in a language that is completely different from contemporary language (and this should in no way lead one to believe that the content is less correct or less complete than contemporary content). In fact, this concept is introduced by comparing the speeds of two material points: one material point will be said to be quicker than another if (i) it travels the same distance as the other in a shorter time, or if (ii) it travels a greater distance in the same time. This definition seems very natural if one considers the use of geometry and geometric concepts systematically adopted by Hellenistic scientists: in fact, it is suitable for comparing different time intervals by using time representative segments. It has to be remarked that, albeit it has been recognized that Newton and Leibniz have introduced an algebraic definition of velocity based on ratios of increments, Newton himself used systematically the quoted definition taken from MP.

It is important that the concept of average speed is introduced by means of two examples: in fact, they make it possible to state that the sources of the MP clearly had in mind the dual dependence of speed on the distance traveled and the time employed. This aspect may seem trivial, but instead shows that the treatment of the problem under consideration is formulated in both an extremely rational manner and trying to be pedagogically clear.

MP continues as follows:

The reason is that the radius describing the circle is performing two movements. Now whenever a body is moved in two directions in a fixed ratio it necessarily travels in a straight line, which is the diagonal of the figure which the lines arranged in this ratio describe.

Albeit in the Introduction of MP the study of circumferences is sketched, we remark that the compiler of MP seems to be too quick in his kinematical preliminaries: we will dedicate further investigations to clarify this point.

Here one must read the text of the MP problem carefully: “the radius that describes the circle makes two movements”. This seemingly obscure sentence will be, in our opinion, clarified later in MP text, but it is nothing more than a “natural language” expression to introduce the components of displacement. The next sentence, in fact, basically explains the parallelogram method for the sum of two vectors. As we know, plane geometry was developed in the very same period as the composition of the MP and its application in demonstrations of mechanically relevant problems is natural.¹ This way of acting in the demonstration of concepts proper to Physics was preponderant still until Newton, who, despite having, according to the vulgate on the History of Science, introduced, together with Leibniz, the infinitesimal calculus, chose to demonstrate the laws of dynamics by means of geometric constructions. This is perhaps an indirect proof that Newton was in possession of Hellenistic sources and drew heavily on these to formulate [73,74] his Principia.²

In order to follow the next quotation from MP, it is necessary to refer to Fig. 1.

¹ We believe that in reality Euclidean geometry and mathematized mechanics were developed in parallel, one influencing the other: this point deserves further investigations [69].

² We share with Keynes the opinion that Newton has been the last of Middle Age scientists [64].

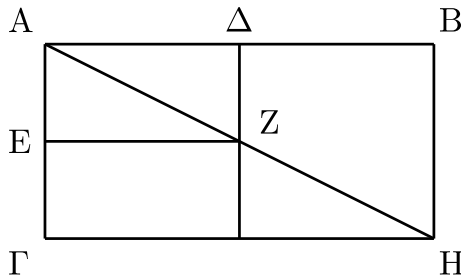


Fig. 1. The figure explaining the composition of displacements in the manuscript of MP transmitted to us.

Let the ratio according to which the body moves be represented by the ratio of AB to $A\Gamma$. Let $A\Gamma$ move towards B while AB be moved towards the position $H\Gamma$; now let A travel to Δ , and let AB travel a distance determined by the point E . Then if the ratio of the movement is that of AB to $A\Gamma$, then the line $A\Delta$ must bear the same ratio to AE . Then the small parallelogram has the same proportions as the larger, so that its diagonal is the same, and the body will move to Z . It can be shown that it will behave in the same way at whatever point its movement be interrupted; it will always be on the diagonal. Conversely it is obvious that an object traveling with two movements along a diagonal will always move in the ratio of the sides of the parallelogram. For with any other proportion it will not travel along the diagonal. But, if a body travels with two movements with no fixed ratio and in no fixed time, it would be impossible for it to travel in a straight line. For suppose it to be a straight line. If this line is drawn as a diagonal and the sides of the parallelogram be filled in, the body must move in the ratio of the sides; this has been demonstrated before. Hence the body that travels in no constant ratio and in no fixed time will not make a straight line. For if it travels in a fixed ratio for a given time, during this time it must move in a straight line, because of what we have already said. So that if it moves in two directions with no fixed ratio and in no fixed time it will be a curve.

Again we find that the text, both for the low quality of the original and for the lack of mechanical competences of the translator, can be understood only if the reader is very familiar with the evoked kinematical concepts. Postponing to future investigations the careful philological analysis of this text, we correctly translate explicitly one sentence, whose correct meaning we could reconstruct both referring to the logical structure of the underlying mechanical concept and to a more careful identification of the meaning of used Greek words. We refer to the sentence: which, instead of being translated as:

but, if a body travels with two movements with no fixed ratio and in no fixed time, it would be impossible for it to travel in a straight line

should be translated, word by word, as:

if in any ratio are conducted two motions in any time, it is impossible that straight may be the motion

Our translation proves that the source of the compiler of MP was stating that, we use a modern nomenclature for being clearer: *if in a motion the components of the displacement are changing their ration, then the trajectory cannot be a straight line*. Of course, the original concept was lost in two translations: from Doric Greek into Attic Greek and from Attic Greek into English. However, together with the intrinsic difficulties of translation the role of the compiler of MP in confusing the original ideas cannot be neglected: we conjecture that the ideas contained in the original work by Archytas were imported in the Peripatos by Strato of Lampsacos, without a big success, as after Strato mechanics seemed to be abandoned as a study discipline in Athens.

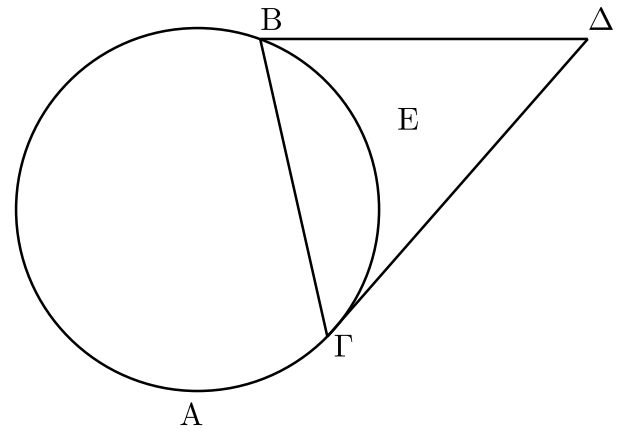


Fig. 2. The figure explaining the circular motion in the manuscript of MP transmitted to us.

To remain close to the referred Greek text, we say that it states that a plane motion is described as being composed of two rectilinear motions along the horizontal, AB , and vertical directions, $A\Gamma$. In modern terms, we would say that a plane motion can be considered as a composition of a horizontal rectilinear motion along abscissa x and another vertical rectilinear motion along ordinate y . The reader will appreciate the enormous economy of thought gained in introducing Cartesian geometry: the author of MP must specify the geometrical setting of his reasoning before talking about kinematics. On the contrary, once Cartesian geometrical/algebraic relationship has been established once forever, the language is quicker albeit the rigor is maintained. In addition, the author of the MP specifies that in order for the composite plane motion to be itself rectilinear, two hypotheses must be verified, one of a geometric nature and the other of a kinematic nature. The first hypothesis, the geometrical one, is that the horizontal and vertical displacement components remain, for each successive step of displacement, in the same ratio.

The second assumption that confusedly emerges from the Greek text is of a kinematic nature and concerns time intervals. In the quoted excerpt, we find evidence that the concept of acceleration can be found in Greek mechanics in a pristine form. In fact, the author of the MP states that in order for the resulting motion to be rectilinear, the time intervals must be fixed. Although this cannot be taken as a definitive proof of the existence of the idea of acceleration, it must be recognized that, in contemporary textbooks, acceleration is introduced precisely as the variation of instantaneous speed, in which there is therefore a variability of the relationship between the spatial interval and the time interval “instant by instant”. Clearly, the echoes of the source of MP have traveled a long way in Western civilization!

MP then continues describing circular motions.

That the line describing a circle moves in two directions simultaneously is obvious from these considerations, and also because that which travels along a straight line is along a perpendicular, so that it again travels along the perpendicular to a point above the center. Let $AB\Gamma$ be a circle, and from the point B above the center let a line be drawn to Δ ; it is joined to the point Γ ; if it traveled with velocities in the ratio of $B\Delta$ to $A\Gamma$ it would move along the diagonal $B\Gamma$.

Also in this paragraph, the idea of sum between displacement vectors is evident. Furthermore, we can clearly see how in this case the author has, in an extremely natural way, applied the graphical method of head-to-tail, in which to sum two vectors, $B\Delta$ and $A\Gamma$, one brings the tail of the latter, Δ , to the point of application of the former, and draws the conjunction from the tail of the former, B , to the head of the latter, Γ . Furthermore, the author uses the discussion

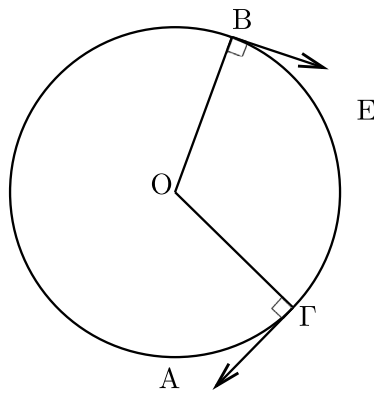


Fig. 3. Description of circular motion.

of the sum of the two vectors (see Fig. 3) to demonstrate (see Fig. 2) that for a motion to be circular, the component displacements must be orthogonal to each other, effectively introducing a centripetal and a tangential component.³

But, as it is, seeing that it is in no such proportion it travels along the arc BEF.

The MP source has demonstrated, with a previous reasoning, that if the displacement components were in a fixed ratio to each other for each displacement step, then the resulting displacement would be uniformly rectilinear (uniformity is ensured by the requirement that the time intervals are also always equal). In the case considered and shown in Fig. 2, on the other hand, the motion studied is characterized by components that do not have a constant relationship with each other, although they remain orthogonal, and therefore do not produce rectilinear motion, but the material point will move along the circumference ABΓ. This concept is discussed in detail in the following paragraph.

Now if of two objects moving under the influence of the same force one suffers more **interference**, and the other less; it is reasonable to suppose that the one suffering the greater **interference** should move more slowly than that suffering less, which seems to take place in the case of the greater and the less of those radii which describe circles from the center. For because the extremity of the less is nearer the fixed point than the extremity of the greater, being attracted towards the center in the opposite direction, the extremity of the lesser radius moves more slowly.

This is a very rich and dense but confused paragraph. Clearly, its interpretation cannot be separated from the interpretation of the two terms ἐκχροούντο (deponent form of the verb ἐκχροῦν, meaning “to push out or to push back”) translated by Hett as “interference” and ἰσχύος (genitive of ἰσχύς) translated as “force”.

Let us first focus on the meaning of the verb ἐκχαρούντο. In fact, it is peculiar that the Loeb Classical Library's edition translates "suffers interference" while Jean De Groot's version refers to ἐκχαρούσις (translatable as "the act of expelling") as a constraint. We would like to emphasize here that both versions cited are the work of a

³ We have remarked by consulting the different versions of MP available in the literature that many versions of the same diagrams are presented. Of course, while the corruption of text is slower, the corruption of diagrams occurs at a higher speed, so that, while MP text remains more or less stable, the appearance of explanatory diagrams changed very much indifferent sources. We, therefore, are not surprised that, in the source we are consulting, the orthogonality condition, which is referred to in the text, is not shown. A beautiful and persuasive discussion of this point can be found in [75].

mostly humanistic/philosophical school that, therefore, fails in accurately recognizing correspondences between ancient and modern scientific language. The authors of the present analysis, on the other hand, have carried out formal scientific studies (theoretical physics) and complement these with a reasonable knowledge of ancient Greek and Latin.

For this reason, we can say, without a shadow of a doubt, that the author of the source of MP does not want to signify either an interference or a constraint, but a “change in trajectory” due to a push back. This is immediately understandable by those who are familiar with the concepts of circular motion, centripetal accelerations and centrifugal forces. In fact, it is known that centripetal acceleration in circular motion depends quadratically on the tangential velocity and inversely on the radius of the circumference along which the motion occurs. Furthermore, if the motion is circular and uniform then the total acceleration will only be the centripetal one that does not produce variation in the modulus of the tangential velocity, but only in its direction.

If we now accept the translation of ἰσχύς as “force” (but other translations are also possible, such as “power, impulse”: it is remarkable that Lagrange in his *Mécanique Analytique* used the word “power” to refer to what we now call force), then we know that, by the second law of dynamics, force is proportional to the acceleration that a body of mass m undergoes. Therefore, we can understand that for the author of MP to specify that “two objects moving under the influence of the same force” corresponds to considering their accelerations equal, mass being equal. This implies, in modern terms, that keeping the centripetal acceleration, expressed by means of the tangential velocity and radius as v^2/R fixed, varying the radius R must also vary the tangential velocity v .

From the above, it follows logically that the material point moving on the circumference of a smaller radius is slower than the one moving on the circumference of a larger radius. The truly remarkable aspect of this discussion is the fact that this observation, which could easily be demonstrated by simple kinematic considerations (greater space traveled in the same time as lesser space implies greater velocity), is instead demonstrated through considerations of dynamics. We conclude that the source of MP had improved Eudoxus purely kinematic description of planetary motions and could give for them a more complete dynamic explanation.

Moreover, this demonstration also indirectly implies an awareness of the dependence of a central force of attraction on the distance from the center. Of course, there is no evidence, in this discussion, of an awareness of the law of the inverse of the square of the distance, but certainly the author of the source of MP had this kind of knowledge in his background.

Why are we so sure of this? As we pointed out earlier, to prove that two points moving on two arcs of a circle corresponding to the same angle but of different radiuses in the same time, it is sufficient to observe that to travel a longer path in the same time as the shorter one, it is necessary for the material points to move at different speeds.

But this is not enough for the source of the MP. This would certainly suffice to describe the motion of an arm balance or a lever, where the ends of the beam are constrained to remain along the circumference. Incidentally, it is probably the thought of a lever that led Jean De Groot [76] to translate ἐκκρούοντο as “is constrained”. Jean De Groot is absolutely far from the idea that the source of the MP could also refer to the motion of the planets. On the other hand, we have no idea what the Loeb Classical Library’s edition means by translating “suffers more interference”: this is a recurring problem when translations of ancient scientific texts are edited by humanistic scholars who have never studied contemporary Science. Indeed, we subsequently find the following text:

This happens with any radius which describes a circle; it moves along a curve naturally in the direction of the tangent, but is attracted to the center contrary to nature.

This statement, read consciously, has an enormous importance. It clearly is implied by what we now call the “Principle of Inertia” and which we attribute to Galileo and Newton. It speaks of motion “according to nature” and motion “against nature”. Moreover, it specifies that the motion according to nature has a direction tangent to the circumference, we could say a rectilinear motion, while the motion against nature is in the radial direction.

Now, we must specify that the compiler of MP has the habit of not providing direct definitions, of discussing more than one phenomenon at the same time, of expressing some concepts while denying the opposite. This can lead to confusion, especially if the reader does not have a solid scientific background. Also this time, it would seem as if the compiler of the MP is a student taking notes, transcribing excerpts of a broader and more comprehensive discussion, following a professor who is giving examples while speaking. In this sense, one could interpret Vitruvius’ indirect attribution of the text of the MP to Archytas of Tarentum: in reality, this text comes from the school of Archytas, but does not constitute Archytas’ original text. Obviously, as we have already remarked, this is only a conjecture, but it seems to us a good one: on the other hand, it is quite improbable that someone who is able to speak with knowledge of deep concepts of modern mechanics, but in the 4th century BC, was unable to make a complete and organic draft of it.

Having said this, it is clear that a motion “according to nature” cannot be anything other than uniform rectilinear motion. Such motion is perturbed in the presence of a center of attraction: “but is attracted towards the center contrary to nature”. A center of attraction is nothing other than a center of gravity.

So, rephrasing the words used in the MP in this paragraph, one could write: a body not subject to forces remains in a state of uniform rectilinear motion, from which it deviates if forces intervene.

This would be sufficient to identify Newton’s First Principle of Dynamics, in which, however, inertial systems are mentioned. It may appear that in MP the concept of inertial observer and free motion of a body is not taken into account. So, apparently the modern version of the Principle of Inertia remains more complete. However, if one takes a further step in understanding the content of the source of MP, Archytas, or whoever was the originator of MP tradition, does not write “in the presence of forces”, but specifies that a central field of attraction influences uniform rectilinear motion by deflecting it. And we know that introducing a central force field means not only adding forces, but also the concept of non-inertial system.⁴ And with a single statement, the source of the MP seems to anticipate the conceptual bases which found not only Newtonian mechanics, but also Einstein’s General Relativity, where gravity is a mere non-inertial effect!

The lesser radius always moves in its unnatural direction; for because it is nearer the center which attracts it, it is the more influenced. That the lesser radius moves more than the greater in the unnatural direction in the case of radii describing the circles from a fixed center is obvious from the following considerations.

Here, the author of MP reiterates his knowledge of an analogue of the gravitational field. In fact, he compares the motion of two material points at different distances from the center of attraction, i.e. the center of gravity, and he claims that the closer point is more attracted.

No indication of possible knowledge of the law of the inverse of the square of distance is given, but it is certainly clear that the author knows that the intensity of attraction is inversely proportional to -

⁴ The reader will remind that while studying the motion under the action of central forces it is very useful to introduce centrifugal or centripetal apparent forces: therefore, as also explicitly shown in other parts of MP, the concept of inertial and non inertial forces had to be somehow clear to the most skilled extensors of MP.

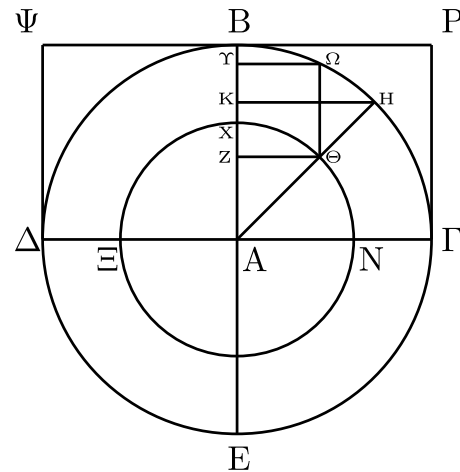


Fig. 4. The figure explaining non-inertial systems in the manuscript of MP transmitted to us.

we would say today - a function of distance. We remind that the MP calls “natural” direction in a circular motion the tangent direction and “unnatural” the radial one: it also states that a stronger force is needed for imposing a circular motion of smaller radius (see Fig. 4). The geometrical demonstrations which follows is a more precise formulation of the previous statement.

Let there be a circle BTEΔ and another smaller one inside it XNME described about the same center A and let the diameters be drawn, the larger ΔΓ and BE and in the smaller circle MX and NE; let the rectangle ΔΨΠΓ be completed. If the radius AB describing the circle returns again to the same position from which it started, namely to AB, it is clearly traveling towards itself. In the same way AX will return to the position AX. But AX travels more slowly than AB, as has been said, because the interference with it is greater, and AX is more interrupted. Let AΘH be drawn, and from the point Θ a perpendicular ΘZ be dropped within the circle to AB; again from Θ let ΘΩ be drawn parallel to AB, and the perpendiculars ΩΥ and HK dropped on AB. Now the lines ΩΥ and ΘZ are equal, but BΥ is less than XZ. For in unequal circles equal straight lines drawn perpendicular to the diameter cut off smaller parts of the diameter in the greater circles, and ΩΥ is equal to ΘZ. Now in the same time in which AΘ travels along the distance XΘ the extremity of the radius BA has described a greater arc than BΩ in the greater circle. For the natural travel is equal, but the unnatural is less; and BΥ is less than XZ: but one would expect them to be in proportion, the two that is whose travel is natural, and the two whose travel is unnatural. The point has actually traveled over HB, which is greater than ΩB. Now in the given time (i.e., that in which AX moves to AΘ) AB must have traveled over the arc HB; for that will be its position, when the proportion between the natural and unnatural movements is true. If, then, the natural movement is greater in the greater circle, the unnatural movement would at that point have the same proportion only in the sense that the point B would travel along the arc BH in the same time as the point X would travel along the arc XΘ. For in that case the natural movement of the point B carries it to H, but its unnatural movement to K. For HK is the perpendicular dropped from H. Then HK is in the same ratio to KB, as ΘZ is to ZX. This will be obvious if B and X are joined respectively to H and Θ. But if the distance traveled by B is either greater or less than HB, the result will not be the same, nor will the proportion between the natural and unnatural movements be the same in the two circles.

There are several observations to be made regarding the geometric demonstration given here. A first remark lies in the apparent change of definition of natural motion and unnatural motion. In the preceding

paragraphs, as we have already underlined, it was clear that a motion of a material point was defined as natural when that point moves in the direction tangent to the circumference; the radial direction, on the other hand, is that according to which the point would move if it were not displaced by tangential velocity.

This is clearly expressed in the preceding paragraphs, where, moreover, as we have seen, the Principle of Inertia is stated and to the concept of unnatural motion is added the connotation of causality that deviates a body initially in (natural) uniform rectilinear motion. It is indisputable that the preceding paragraphs contain this concept, which instead is usually attributed to Middle Age or Renaissance sources.

Here, however, it appears that the example given relates to a different problem. Proving that the point moving on the circle of larger radius is faster than the point moving on the circle of smaller radius is very simple. In fact, this is stated and demonstrated graphically in the first paragraph, where the arclengths $\widehat{X\Theta}$ and $\widehat{B\mathcal{H}}$ are compared. Furthermore, it is pointed out that if the two points were moving at the same speed, then they would have to travel the same arclength in the same time, hence $\widehat{X\Theta}$ for the smaller circle and $\widehat{B\Omega}$ for the larger one. Obviously, if one imagines that the two points are constrained to travel equal angular amplitudes (and this, in fact, is what happens for all points on the arms of a balance or lever) then it will be the case that the one farthest from the center must move faster.

However, we must note that in this paragraph the directions “according to nature” and “against nature” no longer seem to be the tangent and radial directions, but the horizontal and the vertical. This can only correspond to the same definition as in the previous paragraphs if we refer to the radius oriented along the vertical, identified by the segment AB for the great circle and AX for the small one. And, in fact, the discussion that is made starts by considering the radius initially in the vertical position.

But when considering the current configuration, in which the radius has moved from AB to $A\mathcal{H}$ (in the small circumference from AX a $A\Theta$), certainly the global displacement is not reconstructed according to tangential and radial components, but in the global reference system, with vertical and horizontal components. This would still be acceptable if infinitesimal displacements were considered, for which it would be licit to confuse reference and current configuration and, therefore, the tangential component would correspond to a horizontal component and the radial component to a vertical one.

We can, therefore, formulate two hypotheses:

- i. actually, the author of the MP wants to study the problem in small displacements, but the drawing is made by magnifying the quantities in order to make the quantities distinguishable and to show how comparable these quantities are to each other (we stress that also in any contemporary book on Continuum Mechanics, the drawings are made out of scale, so that the infinitesimal quantities seem finite);
- ii. the drawing and its demonstration refer to a collateral problem to that analyzed so far: the decomposition of a uniform circular motion into two simple harmonic motions orthogonal to each other.

If the first hypothesis seems not realistic, we invite the reader to the interesting comparison with the first pages of Isaac Newton's *Principia*. Book I “Of the motion of the bodies” begins with a section that concerns exactly what has been done here with regard to Fig. 1. It considers a repetition of the parallelogram method for the description of a curved trajectory. As we have already discussed in the previous pages, the author of MP argues that if the ratios between the vertical and horizontal components for successive displacements remain constant, then the resulting motion will be uniformly rectilinear (uniformity is ensured by the constancy of the time intervals). The author of MP further argues that when these ratios are no longer constant, then curved motion will result. If we reinterpret, therefore, the paragraph

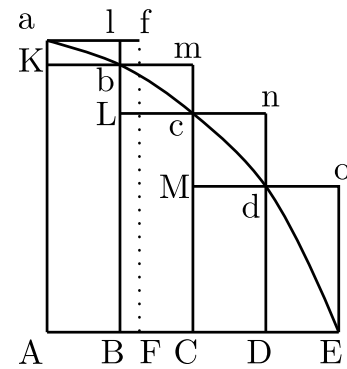


Fig. 5. Figure included in the first Section (Lemma II et seq.) of the first book of Newton's *Principia* to explain the parallelogram method for the description of a curved trajectory.

concerning Fig. 1 by comparing it with that concerning Fig. 5 and with the first Section (Lemma II et seq.) of the first book of Newton's *Principia*, we find the impressive result that the author of the MP introduced in his own fashion a kind of infinitesimal calculus, where the ratio between the components is nothing other than the derivative of the “function” that realizes the curve: if this ratio turns out to be constant (constant derivative), then the function will be a straight line; if this ratio is not constant, then we will have a curve.

We could be accused to have “a modernistic view” of the content of MP, so that we believe to read something that was not in the intention of its authors. Albeit we can base our analysis only on pure logical ground, we instead believe that the corruption introduced by the copyists and subsequent extensors is the cause of the unclarity of the text. Indeed, it is very unlikely that somebody: (i) misunderstanding the nature of a demonstration, (ii) who believes to have demonstrated something which is not even dealt with in the demonstration, could have managed to state in such a precise way some statements of “modern” mechanics.

We should not be too impressed, then, by the hypothesis that the drawing in Fig. 4 realizes an off-scale version of a situation in which displacements are infinitesimal. In that case, then, there would be no difference between the definitions of motion according to nature and against nature given so far and those in this last paragraph.

If, on the other hand, we wish to consider the drawing as being to scale, actually representing two positions of material points obtained from each other by means of a finite displacement, then we must necessarily imagine that the first MP problem represents a very narrow compendium of an extremely broader treatment of central motions in statics and dynamics.

We would like to emphasize, once again, that the style of the MP appears, on a careful reading, to be that of a series of notes rather than of an actual scientific treatise. Of course, this consideration refers not so much to philological and historical parameters (people more qualified in this respect have analyzed this, such as Jean De Groot [76]), but to the logic of the scientific treatment that is presented. Very well-structured works in the scientific field with extreme rigor in terms of method belong to the same period. It therefore seems implausible that the *Mechanica Problemata* were a complete work: rather, they are the notes of a not so skilled disciple.

Specifically, it is known that Archytas of Tarentum wrote in Doric, Tarentum being a colony of Sparta. The *Mechanica Problemata*, on the other hand, are written in Attic and, therefore, its attribution made by T. Winter to Archytas seems weak on linguistic grounds. In any case, Winter's deductive reasoning that by exclusion and expediency attributes the MP to Archytas is very convincing. It is, therefore, possible that the MP are the result of a scientific tradition originating with Archytas and reaching the Peripatetic scholars at some point. It is no coincidence, in fact, that this work was transmitted in the corpus of Aristotle's works and is still sometimes attributed to Aristotle himself.

From what has already been said the reason why the point more distant from the center travels more quickly than the nearer point, though impelled by the same force, and why the greater radius describes the greater arc, is quite obvious.

Here we intend to state an aspect that the author of the MP gives for certain and that in this first problem is taken as a kind of background: we are dealing with a motion in a field of central forces where the only reason why the material point does not collapse into the center of gravity is that it is endowed with an initial velocity in the direction orthogonal to the radial direction. This velocity is the origin of a centrifugal force that opposes the force of attraction towards the center and produces a kind of equilibrium. Whether this is indeed the author's intention is demonstrated by the fact that, depending on the context, he speaks of a repulsion in the radial direction (corresponding, therefore, to the centrifugal force) and a center of attraction, which, on the other hand, is the origin of the force that attracts the material point towards the center.

Why also greater balances are more accurate than smaller ones, is clear from these considerations. The cord which suspends the balance is the center (for it is a fixed point), and the parts on either side of the balance scale are the radii from the center. Now the extremity of the balance scale must move at a greater rate under the influence of the same weight, in as much as it is further from the cord, and consequently in small balances some quantities must make no impression on the senses, but in large balances the movement must be obvious; for there is nothing to prevent a quantity from moving too little for it to be observed by the senses. But in a large balance the same weight makes the movement visible. Some movements are obvious in both cases, but are much more obvious in larger balances, because then the extent of the swing is much greater for the same weight. This is how sellers of purple arrange their weighing machines to deceive, by putting the cord out of the true center, and pouring lead into one arm of the balance, or by employing wood for the side to which they want it to incline taken from the root or from where there is a knot. For the part of the tree in which the root lies is heavier, and a knot is in a sense a root.

Most likely the source of MP was a sophisticated scientific text in which, together with the theory, relevant conceptual problems relative to experimentation were fully discussed. In particular, this source was considering the problem of sensitivity of measurements and was carefully describing so-called sensitivity errors. Moreover, the typical Greek scientific attitude towards applications is exhibited when considering the applications of all theoretical concepts discussed to describe how frauds in balances are conceived.

2.2. Problem 2 and the study of stable equilibrium

Problem 2 proposes to study another aspect of lever equilibrium (or of a beam supported at a point in general). It is possible to find equilibrium conditions for a beam with a fixed point (for simplicity, the geometric center, assuming that the section is homogeneous and the material is isotropic). It should be noted, however, that depending on how the equilibrium is obtained, it may be, in modern terms, stable or unstable. The text of the problem offers a very simple case: given the same equilibrium conditions, a beam resting on a fulcrum will be in unstable equilibrium (see Fig. 7), while a suspended beam will be in stable equilibrium (see Fig. 6). We have in modern times the Ljapunov criteria to quantitatively characterize these phenomenological observations.

The explanation given in this discussion of the difference between stable and unstable equilibrium clearly involves the concept of the extremal value of the potential energy, and in this sense is surprisingly similar to Ljapunov's modern treatment. In fact, one considers the part of the beam that is above the perpendicular in a configuration in which the beam has been tilted by a certain angle: this consists in studying the monotony of the potential energy, the minimum of which is found in the equilibrium position.

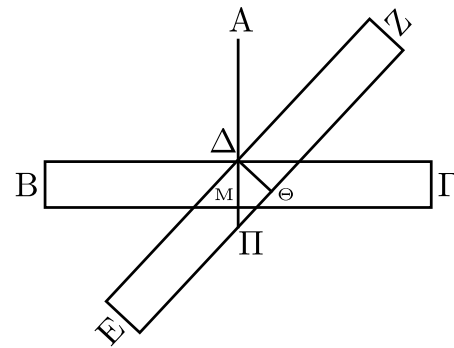


Fig. 6. The figure explaining equilibrium of a suspended beam in the manuscript of MP transmitted to us.

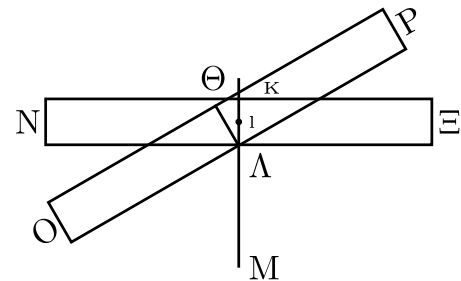


Fig. 7. The figure explaining equilibrium of a beam resting on a fulcrum in the manuscript of MP transmitted to us.

If the cord supporting a balance is fixed from above, when after the beam has inclined the weight is removed, the balance returns to its original position. If, however, it is supported from below, then it does not return to its original position. Why is this? It is because, when the support is from above (when the weight is applied) the larger portion of the beam is above the perpendicular. For the cord is the perpendicular. So that the greater weight must swing downwards until the line dividing the beam coincides with the perpendicular, because the greater weight now lies in the raised part of the beam. Let the beam be a straight one represented by $B\Gamma$, and the cord be $A\Delta$.

When this is driven downwards the perpendicular will be represented by $A\Delta M$, if the weight is attached in the direction of B . The face B will then adopt the position E , and the face Γ that of Z , so that the line bisecting the beam at first was in the position of the perpendicular ΔM , but when the weight was attached took up the position $\Delta\Theta$. Consequently that part of the beam in its position EZ which is outside the perpendicular AM will exceed half the beam by ΘP . If, then, the weight is removed from the arm E , the arm Z must be depressed, for the arm E is the smaller. If, then, the cord is attached from above, the balance returns again to its original position. If, however, the support is from below, the opposite results; for now the portion of the beam which is lower than the perpendicular dividing it is more than half; consequently it does not return to its place; for the part rising above is lighter. Let the straight beam be represented by $N\Xi$, the perpendicular being $K\Lambda M$, and this bisects $N\Xi$.

When the weight is attached to arm N , N will take up the position O and Ξ will take up the position P , while $K\Lambda$ will go to $\Theta\Lambda$, so that KO is greater than ΛP by $\Theta K\Lambda$. Now when the weight is removed the beam must keep its new position; for the excess over half the beam beyond K acts as a weight and depresses the beam.

2.3. Similarities with Dirichlet-Lyapunov criteria

Problem 2 involves reasoning that, in some way, we recognize as extremely modern. Indeed, it fundamentally concerns the stability of a physical system. Two possibilities are considered: the same physical system is constrained in different ways, resulting in a stable equilibrium state in one case and an unstable one in the other. What fundamentally changes is well explicated in the problem text: it is the amount of matter that, in the current configuration, is above the horizontal line passing through the point where the beam is constrained.

It is clear that the extensors of the MP had a good understanding of two fundamental concepts of what we now define as modern mechanics, even though they did not have the suitable terminology for a systematic treatment: (i) the concept of gravitational potential energy and the fact that it is related to the mass of the body, as well as its elevation; (ii) the idea that a stable equilibrium corresponds to a minimum point of potential energy. However, the clarity of the argument shows that the source of the MP was much more precise and rigorous.

The extensors of MP do not explicitly mention potential energy or mass, but unequivocally express the fact that the difference between the two equilibria, one unstable (the beam supported on the fulcrum), the other stable (the beam hanging from the wire), is discriminated by the amount of matter above the horizontal line in the current configuration.

In terms more familiar to us, in the case of stable equilibrium, the current configuration's potential energy is greater than that of the initial configuration, which is thus minimal; in the other case, the current configuration has less potential energy than the initial configuration, and the equilibrium is therefore unstable.

This, in much more mathematical terms, was rationalized by Dirichlet-Lyapunov in the 19th century. We believe that the essence of the discussion remains the same.

Lyapunov's treatise [77], *The General Problem of the Stability of Motion*, published in 1892, addresses the stability of solutions to differential equations, providing rigorous methods to determine the behavior of dynamic systems in the vicinity of equilibrium points. His primary contributions include the formulation of Lyapunov's direct method and the second (indirect) method, both of which offer ways to ascertain stability without solving the differential equations explicitly.

It is emphasized that even in the case of Problem 2, the determination of equilibrium is achieved without solving equations, but by making energetic considerations. The languages used are extremely (and obviously) different, but the result is the same.

2.4. Suggestive similarities between Hellenistic and modern stability analyses

Our interpretation of the MP Second Problem cannot be considered too daring. In fact, in the works of Archimedes [78] it is well attested that these stability analyses were common in Hellenistic mechanics. Therefore, the strong similarities between the analysis of the stability of the lever and the modern direct Lyapunov method are not a modernistic interpretation of the presented text. For underlining these similarities, we report here the main ideas of the direct Lyapunov method.

The direct Lyapunov method involves constructing a Lyapunov function $V(x)$, a scalar function analogous to the concept of energy in physical systems. This function must satisfy certain conditions to demonstrate stability:

1. **Positive Definite:** $V(x) > 0$ for all $x \neq 0$ and $V(0) = 0$. This implies that the function is positive for all states except at the equilibrium point, where it is zero.
2. **Negative Definite Derivative:** The time derivative of $V(x)$ along the trajectories of the system, denoted as $\dot{V}(x)$, must be negative definite, i.e., $\dot{V}(x) < 0$ for all $x \neq 0$. This condition ensures that $V(x)$ decreases over time, indicating that the system's trajectories are moving towards the equilibrium point.

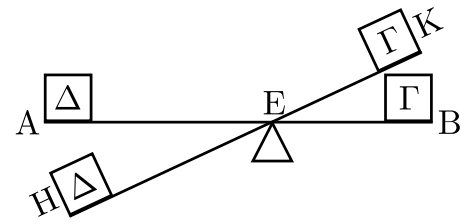


Fig. 8. The figure explaining the Principle of Virtual Work in the manuscript of MP transmitted to us.

If these conditions are met, the equilibrium point is said to be asymptotically stable, meaning that the system will return to equilibrium after small perturbations.

It has however to be remarked that in Greek texts no explicit reference to stability of motion is found. Instead, the stable configuration is perturbed with what will be later called *virtual displacements*. This point has been dealt in greater detail in Section 1.9.

2.5. Problem 3: early form of the Principle of Virtual Work

Problem 3 applies the concepts mentioned in Problem 1 to a practical case: consider two weights placed on the two arms of a lever, the fulcrum of which is not placed in a median position between the two extremes.

Why is it that small forces can move great weights by means of a lever, as was said at the beginning of the treatise, seeing that one naturally adds the weight of the lever? For surely the smaller weight is easier to move, and it is smaller without the lever. Is the lever the reason, being equivalent to a beam with its cord attached below, and divided into two equal parts?

The problem the author poses is fundamentally the following: it is clear that a small force is needed to move a small weight compared to the force needed to move a large weight; why, on the other hand, can a large weight also be moved by means of a small force if a lever is introduced? This “baroque” manner of expression does not detract from the importance of the question asked.

For the fulcrum acts as the attached cord: for both these remain stationary, and act as a center. But since under the impulse of the same weight the greater radius from the center moves the more rapidly, and there are three elements in the lever, the fulcrum, that is the cord or center, and the two weights, the one which causes the movement, and the one that is moved; now the ratio of the weight moved to the weight moving it is the inverse ratio of the distances from the center. Now the greater the distance from the fulcrum, the more easily it will move.

The concepts already discussed in Problem 1 are briefly recalled here: clearly the advantage of using the lever is due to the distance of the weight from the fulcrum. This concept is expressed by considering that there is no equilibrium and that therefore one of the two weights, the one placed at a greater distance from the fulcrum, is responsible for the movement (see Fig. 8). But it is evident that this consideration can easily be used in the circumstance where one wants instead to study the equilibrium of the system: this is what Jordanus will do more than a millennium later, clearly drawing on the text of this problem.

The reason has been given before that the point further from the center describes the greater circle, so that by the use of the same force, when the motive force is farther from the lever, it will cause a greater movement. Let AB be the bar, Γ be the weight, and Δ the moving force, E the fulcrum; and let H be the point to which the moving force travels and K the point to which Γ the weight moved travels.

The reason why the case of non-equilibrium is considered in this problem is immediately understandable when reading the text of the problem carefully. Although Fig. 8 uses Δ and Γ to denote two identical objects (and thus ascribable to two weights placed on the ends of the lever), the text calls Γ the weight moved by the lever and Δ the force used to set the system in motion. This can be interpreted as the MP attempt to provide theoretical notions and practical concepts at the same time.

3. Jordanus de Nemore and the reconstruction of the Principle of Virtual Work from Problem 3

Archimedes' influence during the Middle Ages is well known and widely acknowledged. However, while his approach to mechanical problems, in his works which were transmitted to us, was framed in terms of axiomatic principles concerning forces, similar to those found in Euclid's *Elements*, the *Mechanica Problemata* offers a contrasting methodology. MP is rooted in the concept of virtual velocity and relies fundamentally on the Principle of Virtual Velocities.⁵ Some modern scholars argue that basing mechanics on forces is more abstract than using virtual work. Yet, it is not clear why changing the foundational postulates would produce a superior theory, especially if it is believed that both systems are meant to be logically equivalent.⁶

To better understand this distinction, it is worth contrasting the axiomatic approach⁷ with the method grounded in the Principle of Virtual Work. Both Archimedes and the MP dedicate significant attention to the mechanics of levers, a focus that makes sense given their engineering relevance. However, the reasoning behind the two approaches differs substantially. The MP implicitly applies the Principle of Virtual Work, whereas Archimedes' framework builds on explicit equilibrium conditions.⁸

In Archimedean mechanics, one essentially takes a “snapshot” of the system, such as the lever in Problem 3, estimating the weights, their distances from the pivot point, and the associated angles at a specific moment. This leads to an *ad hoc* formulation of what we would now call a moment balance. In contrast, the Virtual Work approach requires a mental jump: it demands imagining small, hypothetical displacements or velocities starting from the current configuration. Essentially it consists in introducing virtual motion: Clagett, in his seminal studies, highlighted this conceptual difference.

In modern continuum mechanics, it is well established that one can derive balance laws from the Principle of Virtual Work [79–81]. However, these same balance laws cannot be directly postulated without first identifying the correct physical quantities to be balanced.

⁵ Albeit differently believed by some authors, the Principle of Virtual Velocities is exactly the same as the Principle of Virtual Work: there was simply a name change which occurred after Lagrange. The term “virtual velocities” (in latin: *velocitates virtuales*) was used by Johann Bernoulli (1717) in correspondence with Pierre Varignon. Lagrange (1788) used *vitesse virtuelle* in his *Mécanique Analytique*, while Jean-Baptiste Fourier (1798) in his *Mémoire sur la Statique*, unpublished but cited by later works, introduced the expression “travail virtuel”, followed by Louis Poincaré (1806) in his *Éléments de Statique*.

⁶ Trusdellians always claim that the PVW can be proven as a theorem in the framework of their postulation: therefore, they believe that the two postulations are equivalent and claim (sic!) that the one based on forces is more physically understandable. However, as we discuss in the Introduction and in the Conclusion, the PVW seems to have a much wider scope, as it allows for formulating models where more balance laws than those of force and moment of force are needed.

⁷ This approach is called “geometric” by Clagett [69]. Here Clagett refers to the meaning of the word “geometric”, as used in Hellenistic Science: its appropriate translation would be “axiomatic” or “mathematical”.

⁸ We explicitly remark here that the equilibrium conditions used by Archimedes [10] reduce to some geometrical conditions: the reader is warned about the ambiguity which may arise with the nomenclature of Clagett, who unfortunately calls “geometrical” alternative approach based on the PVW.

Thus, the Principle of Virtual Work is arguably more foundational or “primitive” from a logical standpoint. The presence of this principle in Hellenistic texts like the MP reinforces its significance and legitimacy: when inventing new models, it is probably most suitable to base them on the PVW also if one considers that Mechanics was invented at first in this way [82].

It is suggestive, albeit completely conjectural, to imagine that Jordanus de Nemore did work to the rediscovery of the PVW while being at the court of Frederick II Hohenstaufen. As Marshall Clagett notes [69, pp. 64–65], the mathematical sophistication of Jordanus' *De ratione ponderis* aligns with the intellectual atmosphere fostered by Frederick II, particularly in the court's exposure to Arabic and Greek scientific traditions through active translation efforts. Clagett cautiously posits the possibility of Jordanus' association with this *milieu*, emphasizing the emperor's documented interest in mechanics and Arabic science [69, pp. 71–72]. Paul Lawrence Rose [83, pp. 32–33] adds that while no direct biographical link exists, the mathematical culture of the period reflects a broader pattern of cross-cultural fertilization that may have shaped Jordanus' algebraic work, such as *De numeris datis*. While the evidence remains circumstantial, the convergence of Arabic, Greek, and Latin traditions at Frederick's court renders the hypothesis intellectually compelling, even if historically unconfirmed.

We conclude by remarking [69] that very few biographical information about Jordanus are available in the literature: to the point where some authors even believe that Jordanus de Nemore, Jordanus Nemorarius and Jornanus of Nemi could be three different persons. Our malicious consideration is that, as Tartaglia was “elaborating” his works, it could be that Tartaglia voluntarily erased the available information about him.

3.1. Problem 3 as a reference for Jordanus de Nemore

As previously discussed, the MP laid the groundwork for much of the mechanical thinking that followed in the centuries often labeled the *Dark Ages*. Far from being intellectually barren, this period helped to prepare the way for the Renaissance. In particular, the work of *Jordanus de Nemore* (also called *Nemorarius* or *di Nemi*) reveals a strong link to ideas found in the MP. According to Clagett, Jordanus inherited and elaborated upon the Hellenistic mechanical tradition, merging pseudo-Aristotelian (i.e. the theory contained in MP) dynamics with the mathematical statics of Archimedes.

Numerous works are attributed to Jordanus. Clagett highlights a few, but it is worth noting that a popular version of his work, the *Opusculum de ponderositate*, was edited by Niccolò Tartaglia and printed in Venice in 1565 by Curzio Troiano. Unfortunately, this edition is riddled with mistakes. Tartaglia even claimed in the preface to have “corrected and enhanced” the text with new figures. Ironically, these illustrations often contradict the actual content, revealing Tartaglia's overconfidence and limited understanding.⁹

Clagett and Moody [69] identify three main works attributed to Jordanus: *Elementa Jordani super demonstrationem ponderum*, *Liber Jordani de ponderibus*, and *Liber Jordani de ratione ponderis*. Although all three address similar topics, they do so in slightly varied ways. Of these, the *De ratione ponderis* seems most closely aligned with the MP, especially in its treatment of levers as described in Problem 3. Clagett provides an English translation of Jordanus' text of *Quaestio Sexta* in the *De ratione ponderis*, which reads:

If the arms of a balance are proportional to the weights suspended, in such manner that the heavier weight is suspended from the shorter arm, the weights will have equal positional gravity. Let the balance beam be

⁹ It may be possible that the mistakes were introduced by Tartaglia, who, as underlined also by Heiberg, managed to mess also the works of Archimedes and Euclid [84].

ACB, as before, and the suspended weights a and b ; and let the ratio of b to a be as the ratio of AC to BC . I say that the balance will not move in either direction. For let it be supposed that it descends on the side of B ; and let the line DCE be drawn obliquely to the position of ACB . If then the weight d , equal to a , and the weight e equal to b , are suspended, and if the line DG is drawn vertically downward and the line EH vertically upward, it is evident that the triangles DCG and ECH are similar, so that the proportion of DC to CE is the same as that of DG to EH . But DC is to CE as b is to a ; therefore DG is to EH as b is to a . Then suppose CL to be equal to CB and to CE , and let l be equal in weight to b ; and draw the perpendicular LM . Since then LM and EH are shown to be equal, DG will be to LM as b is to a , and as l is to a . But, as has been shown, a and l are inversely proportional to their contrary (upward) motions. Therefore, what suffices to lift a to D , will suffice to lift l through the distance LM . Since therefore l and b are equal, and LC is equal to CB , l is not lifted by b ; and consequently a will not be lifted by b , which is what is to be proved.

This passage clearly demonstrates that Jordanus sought to revive the insights found in the MP source materials. His more refined arguments,¹⁰ compared with the MP, imply that he had access to sources beyond the MP itself. Clagett questions whether Jordanus actually read the MP, and it seems more likely that Jordanus drew from other, related texts. These may well have belonged to the same tradition referenced by Vitruvius, possibly originating from the now-lost writings of Archytas.

3.2. A conjectural reconstruction of the first form of the Principle of Virtual Work

As we have frequently remarked along this work, the *Mechanica Problemata*, particularly in its Problem 3, offers a valuable yet implicit foundation upon which one might reconstruct an early conceptual form of the Principle of Virtual Work. Although this ancient text lacks an axiomatic formulation and does not explicitly articulate the PVW in Archimedean terms, its structure and logical intent suggest an underlying variational reasoning. When placed in dialogue with the later and more mathematically developed works of Jordanus de Nemore, a path emerges towards deducing a plausible conjecture for the primitive, lost, Hellenistic form of the Principle.

Problem 3 addresses the equilibrium of a lever, describing configurations in which weights suspended at unequal distances from the fulcrum produce balance. Rather than employing an explicit law of moments, the argumentation is geometric and relies on proportions derived from *virtual* displacements. The key idea is that if a system is in equilibrium, then an infinitesimal displacement (conceived either physically or conceptually) would result in no net work done by the system's forces. This reasoning remains embedded in geometric language and devoid of higher abstraction: however, it mirrors the logic that underpins the PVW.

Jordanus de Nemore, writing in the 13th century, offers a more explicit and structured analysis of this topic. In his *De ratione ponderis*, he advances arguments that bear a clear resemblance to those found in MP, albeit presented with greater detail. Crucially, Jordanus introduces reasoning based on linearized displacements, explicitly considering the relative movement of suspended weights. He does so by framing hypothetical shifts from the reference configuration, and examining the implications for equilibrium. These are recognizably virtual displacements, and the corresponding evaluations of mechanical action strongly resemble what would, centuries later, be named as virtual work.

Taken together, these two bodies of work form the basis for a conjectural reconstruction of the PVW in its earliest form. While neither

source articulates the principle in the precise variational language of d'Alembert or Lagrange, both evince a form of reasoning in which balance arises from the cancellation of opposing effects through imagined, infinitesimal displacements.

Hence, one may reasonably conjecture that the ancient compilers of MP, though lacking the analytical tools of mechanics, possessed an intuitive and operational grasp of what we now term virtual work. Their formulations, which arrived to us as fragmentary and indirect, underpins the principle's essential logic.

It is now possible to conjecture two alternative flows of scientific ideas transmission. The first flow would have been originated from a lost axiomatic text, due to Archytas of Tarentum and/or his school, with epigones deteriorating the original message by focusing on the application of theory: this alternative will be explored in future investigations. The second flow could be originated by not-so-formal original text, which, after some elaboration, produced more mathematically rigorous text, which are lost and where a precise formulation of the PVW was presented. The PVW, in this second view, did not originate as a formal axiom but rather as a heuristic insight grounded in observation and geometric reasoning, gradually acquiring its abstract and universal form through centuries of mathematical refinement.

Both these conjectures, which will be explored by means of both textual exegesis and structural comparison, suggest that the PVW is deeply rooted in classical mechanical thought: its origins are not a product of modern invention, but a rediscovery and formalization of ancient insight.

3.2.1. The lever system

Consider a rigid, massless beam resting on a fulcrum, with two masses suspended at different points (see Fig. 9). Let us denote these points as A and B , with the fulcrum located at E . The distances from A and B to the fulcrum are $d_A = |\vec{r}_A| = AE$ and $d_B = |\vec{r}_B| = EB$ respectively. Let weights $\vec{W}_A$ and $\vec{W}_B$ be suspended at A and B .

3.2.2. Static equilibrium: Moment balance

In modern statics, equilibrium is often addressed by requiring the net moment about the fulcrum to vanish. Using counter-clockwise moments as positive, the balance condition reads:

$$\vec{W}_A \times \vec{r}_A + \vec{W}_B \times \vec{r}_B = \vec{0} \quad (1)$$

This expresses the condition that the sum of moments due to each force with respect to the fulcrum is zero. It provides a direct, algebraic tool to solve for unknown weights or distances when some are given.

3.2.3. Virtual work approach

The Principle of Virtual Work provides an alternative and more general framework. Instead of balancing moments, we consider infinitesimal virtual displacements of the system and require that the total virtual work done by the forces be zero if the system is in equilibrium. Let us apply a small virtual rotation ϑ to the beam about point E . Under this rotation:

$$\text{Vertical displacement of } A : \Delta y_A = -\overline{AE} \cdot \vartheta$$

$$\text{Vertical displacement of } B : \Delta y_B = \overline{EB} \cdot \vartheta$$

Here, we use the approximation $\sin(\vartheta) \approx \vartheta$ valid for infinitesimally small angles. The virtual work done by each force is the product of the force and its displacement. Summing the contributions of both weights, we write:

$$\mathcal{L}_{\text{TOT}} = W_A \cdot (-\overline{AE} \cdot \vartheta) + W_B \cdot (\overline{EB} \cdot \vartheta) \quad (2)$$

$$= -W_A d_A \cdot \vartheta + W_B d_B \cdot \vartheta \quad (3)$$

For equilibrium, we require that $\mathcal{L}_{\text{TOT}} = 0$. Factoring out the arbitrary but non-zero ϑ , we are left with:

$$-W_A d_A + W_B d_B = 0 \quad (4)$$

¹⁰ A careful reading of the just quoted passage and its comparison with the text of Problem 3 in MP clearly proves that Jordanus delineates in greater detail the formulation of the PVW.

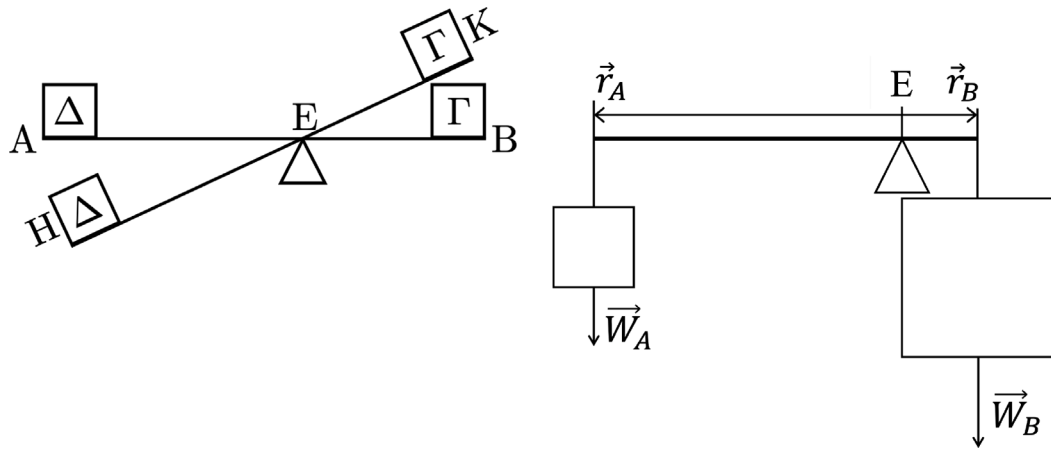


Fig. 9. Equilibrium of lever system as addressed in *Mechanica Problemata* (left) and in modern texts (right).

which is equivalent to the condition of moment balance.

The above derivation illustrates a key insight: the Principle of Virtual Work implies the required balance conditions. Gabrio Piola in his fundamental works [85,86] has proven the following general result

Piola's Theorem: *when considering as virtual displacements the rigid body motions then the PVW implies the balance of forces and the balance of moments of forces.*

For this reason in Italian and French literature [11] sometimes the balance of forces is called “Equilibrium under translation” and the balance of moments of forces “Equilibrium under rotation”.

However, the reverse of Piola's Theorem is not generally true: in [87,88] it is proven that balance of forces and balance of moment forces are not sufficient to characterize equilibrium configurations for generalized continua.

Moreover, the PVW highlights the importance of configurations and virtual displacements, offering a conceptual richness not present in the purely algebraic moment method. In historical context, this shows that the advanced reasoning embedded in ancient treatises like the *Mechanica Problemata* implicitly include variational principles: as indicated by the content of Problem 2.

4. Conclusion

The close examination of the first three problems of the *Mechanica Problemata*, conducted with attention to both the philological nuances and the underlying mechanical logic, has brought to light compelling evidence that the conceptual foundation of the Principle of Virtual Work was not a modern invention *ex nihilo*, but rather a rediscovery and refinement of insights already present in Hellenistic science and transmitted to us in a fragmented form.

Our analysis has shown that the MP, though traditionally viewed as a compilation of mechanical curiosities or an ancillary Aristotelian text, in fact reflects the existence of a surprisingly advanced body of mechanical reasoning. In *Problem 1*, the distinction between infinitesimal and zero displacement, the understanding of sensitivity in measuring instruments, and the clear articulation of average velocity reveal a sophisticated grasp of concepts that are epistemologically central to modern kinematics. *Problem 2* reflects not only an early knowledge of stability theory, expressed in terms remarkably akin to the Dirichlet-Lyapunov criteria, but also introduces energetic reasoning about equilibrium without relying on any formal calculus. Moreover, the same existence of *Problem 2*, and its use of energy minimization principle, in the same text where the PVW is formulated indicates that the sources of MP had established the relationship between these two principles.

The *Problem 3* is the focal point of our study, and allows for the reconstruction of the pristine form of the PVW: in future investigations we believe to be able to prove that Lagrange formulated the Principle of Virtual Velocities by completing such a reconstruction.

From these elements emerges a picture of Hellenistic science that is far removed from the caricature of Greek thought as purely contemplative and disengaged from empirical inquiry. On the contrary, what we find is a mechanical theory that is geometrical rather than algebraic, variational rather than balance-based, and rooted in axiomatic formalism. This theory exhibits a capacity to reason from dynamic intuition (e.g. virtual velocities, conceptual displacements, and force-distance relationships) based on geometrical concepts differently from what done in modern literature, where the algebraic tools and calculus formulations developed only in the early modern period are used. We explicitly remark here that both Jordanus de Nemore and Lagrange presented their version of the PVW in a form which is intermediate between the geometrical one found in MP and the algebraic one found for instance in Piola.

An important consequence regarding history of science of our work is the possibility of re-evaluating the genealogical continuity between ancient and modern science. It is no longer plausible to claim that the PVW, as formulated by Lagrange and d'Alembert, emerged *ex abrupto* from Enlightenment rationalism. Instead, we have shown that it is the culmination of a process of recovery, rearticulation, and algebraization of ideas already present in antiquity. The transmission of this knowledge, fragmentary and indirect, occurred via medieval authors such as Jordanus de Nemore, who reformulated Hellenistic insights into Latin *scholastic* terminology, likely drawing not only from Archimedean sources but also from the intellectual heritage embodied in the MP. Jordanus' use of proportional reasoning between displacements and forces mirrors almost exactly the argument in *Problem 3*, albeit in a more systematic and numerically structured way. His works represent a transitional phase in which the variational reasoning of ancient science is reinterpreted through the framework of pre-Renaissance mathematics.

This reinterpretation is not merely a philological curiosity but reveals an important methodological divide: between a science based on balance laws (as in Archimedean statics), and a science based on principles of virtual displacement and hypothetical motion (as in the MP and later in Lagrangian mechanics). The former takes equilibrium as a static postulate, calculable through finite geometry; the latter presupposes the potential for motion, evaluates the effects of infinitesimal virtual displacements, and locates equilibrium in the vanishing of total work. Simple linear algebra reasoning can easily prove the equivalence of the mathematical equations for equilibrium found in both approaches: however, we believe that the heuristic capacity of the PVW is superior.

It is no accident that Newton, despite his access to calculus and analytic geometry, chose in the *Principia* to return to the synthetic, geometric method: a method resonant with the Hellenistic tradition that likely “inspired” him. The MP, with its recursive geometrical reasoning, appears in several passages, which seem to anticipate Newtonian constructions. We suggest, provocatively but with grounding, that Newton may have drawn, either directly or indirectly, on such texts in forming his mechanical worldview.

Furthermore, our work also highlights the severe limitations of prior translations and interpretations of the *Mechanica Problemata*, many of which have been carried out by scholars of impeccable philological training but without adequate mechanical knowledge. These translations often obscure the conceptual clarity of the original by introducing vague or misleading terms, thus masking the physical depth of the text. Only a multidisciplinary approach, combining historical linguistics, mathematical physics, and the epistemology of science, can hope to recover the full significance of these texts: we have read and sometimes proposed modifications of the available translations. Such an approach reveals the MP not as a marginal work of speculative mechanics, but as a fragment of a lost theoretical system, perhaps originating from figures like *Archytas of Tarentum* or *Strato of Lampsacus*, whose insights prefigure key elements of the scientific revolution.

Finally, from an epistemological perspective, this study suggests a fundamental revision of how we conceptualize the history of scientific principles. The PVW is often presented in modern textbooks as a refined product of analytical mechanics, grounded in abstract calculus and differential formalism. Our analysis shows that, in fact, its conceptual core (the use of hypothetical, infinitesimal displacements to evaluate equilibrium) predates the calculus and survives through centuries in geometric, linguistic, and diagrammatic forms. It is not an abstract product of Enlightenment rationalism, but a deeply rooted scientific formalism that evolved in the workshops of Greek mechanicians, was preserved and transmitted in medieval treatises, and was finally formalized in modern notation.

Thus, what appears in Lagrange's *Mécanique Analytique* as the PVW is not a historical beginning, but a conclusion: the codification of a tradition whose origins stretch back more than two millennia. Our hope is that this work contributes not only to the reevaluation of a single mechanical principle but also to a broader reassessment of the continuity, depth, and complexity of ancient science.

This historical reassessment of the Principle of Virtual Work finds further confirmation in the expanding domain of contemporary mechanics, where variational methods must serve as the cornerstone of increasingly complex and microstructured models. Recent studies have explored the role of virtual power principles in problems involving imperfect interfaces and interfacial sliding. These themes are echoed in the analysis of dynamic properties and size effects in lattice materials [89]. In each case, variational formulations grounded in the PVW offer a powerful framework for capturing discontinuities, micro-inertia, and generalized boundary effects.

Metamaterials represent another fertile arena where generalized continuum theories flourish. These include studies on wave dispersion in nonlinear pantographic beams [90], one-dimensional second gradient continua [91], and hemivariational formulations for granular materials with evolving anisotropy [92]. The computational identification of material parameters in additively manufactured metamaterials [93] and micromechanics-based damage models for strain gradient solids [94] further highlight the centrality of variational reasoning. Comprehensive reviews on mechanical and acoustic metamaterials [95,96], as well as on micromorphic continua derived from granular microstructures [97], provide a theoretical backdrop for modeling phenomena such as defect-induced material behavior [98], brittle fracture [99], equilibrium in articulated beam networks [100], and paradoxes in classical elasticity [101].

This methodological unification continues across a wide spectrum of applications: from volumetric growth processes [102,103], to fluid

diffusion-induced structural aging [104], from symmetry classifications in piezoelectricity [105], to the probabilistic mechanics of multiscale materials [106]. The use of PVW-inspired variational techniques is evident in the experimental identification of strain gradient length scales [107], the design and control of origami metamaterials [108, 109], frequency band-gap detection in microstructured media [110], and benchmark solutions in topology optimization [111]. In each case, the PVW does not merely offer an abstract theoretical formulation, but acts as a generative principle, enabling the synthesis, analysis, and refinement of material models, from generalized elasticity [112, 113] to dissipative systems [52], orthodontic biomechanics [114], dynamic plate theories [115], tensorial stress analysis [116], and complex contact problems with wear [117]. In modern metamaterial design, this variational legacy is embodied in the study of axial-transverse loaded beams [118–121], Galerkin approximations for tetraskelion systems [122], all of which extend and enrich the historical trajectory of the PVW, reaffirming its central role in the epistemology of mechanics.

CRediT authorship contribution statement

Mario Spagnuolo: Conceptualization, Formal analysis, Investigation, Methodology, Visualization, Writing – original draft, Writing – review & editing. **Francesco dell'Isola:** Conceptualization, Formal analysis, Investigation, Methodology, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

- [1] F. dell'Isola, U. Andreaus, L. Placidi, At the origins and in the vanguard of peridynamics, non-local and higher-gradient continuum mechanics: an underestimated and still topical contribution of Gabrio Piola, *Math. Mech. Solids* 20 (8) (2015) 887–928.
- [2] F. dell'Isola, A. Della Corte, R. Esposito, L. Russo, Some cases of unrecognized transmission of scientific knowledge: from antiquity to Gabrio Piola's peridynamics and generalized continuum theories, in: *Generalized Continua As Models for Classical and Advanced Materials*, Springer, 2016, pp. 77–128.
- [3] F. dell'Isola, A.D. Corte, I. Giorgio, Higher-gradient continua: The legacy of Piola, Mindlin, Sedov and Toupin and some future research perspectives, *Math. Mech. Solids* 22 (4) (2017) 852–872.
- [4] C. Truesdell, W. Noll, The non-linear field theories of mechanics, in: *The Non-Linear Field Theories of Mechanics*, Springer, 2004, pp. 1–579.
- [5] G. Capriz, Continua with latent microstructure, in: *Analysis and Thermomechanics: A Collection of Papers Dedicated To W. Noll on His Sixtieth Birthday*, Springer, 1987, pp. 161–174.
- [6] B.D. Coleman, W. Noll, The thermodynamics of elastic materials with heat conduction and viscosity, in: *The Foundations of Mechanics and Thermodynamics: Selected Papers*, Springer, 1974, pp. 145–156.
- [7] N. Auffray, F. dell'Isola, V.A. Eremeyev, A. Madeo, G. Rosi, Analytical continuum mechanics à la Hamilton–Piola least action principle for second gradient continua and capillary fluids, *Math. Mech. Solids* 20 (4) (2015) 375–417.
- [8] T.N. Winter, The mechanical problems in the corpus of Aristotle, *Fac. Publ. Class. Relig. Stud. Dep.* 68 (2007).
- [9] M. Spagnuolo, M. Stolz, S.R. Eugster, F. dell'Isola, Exegesis of some excerpts of *mechanica problemata*, in: *The Principle of Virtual Work in the Sources of Mechanical Sciences: A Reappraisal of Pseudo-Aristotelian Mechanica Problemata and Germain's Article on Second Gradient Continuum Mechanics*, Springer, 2024, pp. 1–31.
- [10] G. Vailati, Il principio dei lavori virtuali da aristotele a erone d'alessandria, *Atti Della R. Accad. Delle Sci. Torino XXXII* (1898) 113–128.
- [11] G. Colonnetti, *I Fondamenti Della Statica*, UTET, Torino, 1927.

- [12] V.A. Eremeyev, L.P. Lebedev, Mathematical study of boundary-value problems within the framework of Steigmann–Ogden model of surface elasticity, *Contin. Mech. Thermodyn.* 28 (1) (2016) 407–422.
- [13] V.A. Eremeyev, L.P. Lebedev, M.J. Cloud, On weak solutions of boundary value problems within the surface elasticity of n th order, *ZAMM-J. Appl. Math. Mech./Z. Angew. Math. Mech.* 101 (3) (2021) e202000378.
- [14] V.A. Eremeyev, L.P. Lebedev, V. Konopińska-Zmysłowska, On solvability of initial boundary-value problems of micropolar elastic shells with rigid inclusions, *Math. Mech. Solids* 27 (9) (2022) 1800–1812.
- [15] V.A. Eremeyev, L.P. Lebedev, Existence of weak solutions in elasticity, *Math. Mech. Solids* 18 (2) (2013) 204–217.
- [16] V.A. Eremeyev, S.A. Lurie, Y.O. Solyaev, F. dell'Isola, On the well posedness of static boundary value problem within the linear dilatational strain gradient elasticity, *Z. Angew. Math. Und Phys.* 71 (2020) 1–16.
- [17] V.A. Eremeyev, F. dell'Isola, C. Boutin, D. Steigmann, Linear pantographic sheets: existence and uniqueness of weak solutions, *J. Elasticity* 132 (2018) 175–196.
- [18] F. dell'Isola, P. Seppecher, The relationship between edge contact forces, double forces and interstitial working allowed by the principle of virtual power, *C. R. Acad. Sci. Sér. IIb Méc. Phys. Astron.* (1995) 7.
- [19] F. dell'Isola, P. Seppecher, Edge contact forces and quasi-balanced power, *Meccanica* 32 (1) (1997) 33–52.
- [20] P. Germain, The method of virtual power in the mechanics of continuous media, I: Second-gradient theory, *Math. Mech. Complex Syst.* 8 (2) (2020) 153–190.
- [21] V.A. Eremeyev, L.P. Lebedev, M.J. Cloud, On admissible external loadings within the first and second strain gradient elasticity, *Math. Mech. Solids* (2025) 10812865251324562–1–15.
- [22] R. Allena, D. Scerrato, A. Bersani, I. Giorgio, Simulating bone healing with bio-resorbable scaffolds in a three-dimensional system: insights into graft resorption and integration, *C. R. Méc.* 353 (2025) 479–497.
- [23] N. Branecka, M. Shamshezzadeh, M.E. Yildizdag, I. Giorgio, A bone remodeling model involving two mechanical stimuli originated from shear and normal load conditions within the 3D continuum mechanics framework, *Contin. Mech. Thermodyn.* 37 (1) (2025) 7.
- [24] D. Addessi, F. D'Annibale, L. Placidi, I. Giorgio, A bone remodeling approach encoding the effect of damage and a diffusive bio-mechanical stimulus, *Contin. Mech. Thermodyn.* 36 (4) (2024) 993–1012.
- [25] R. Allena, D. Scerrato, A.M. Bersani, I. Giorgio, Functional adaptation of bone mechanical properties using a diffusive stimulus originated by dynamic loads in bone remodelling, *Z. Angew. Math. Phys.* 75 (3) (2024) 85.
- [26] I. Giorgio, F. dell'Isola, U. Andreaus, A. Misra, An orthotropic continuum model with substructure evolution for describing bone remodeling: an interpretation of the primary mechanism behind Wolff's law, *Biomech. Model. Mechanobiol.* 22 (6) (2023) 2135–2152.
- [27] I. Giorgio, A variational formulation for one-dimensional linear thermoviscoelasticity, *Math. Mech. Complex Syst.* 9 (4) (2022) 397–412.
- [28] I. Giorgio, L. Placidi, A variational formulation for three-dimensional linear thermoelasticity with 'thermal inertia', *Meccanica* 59 (2024) 1745–1756.
- [29] A. Ciallella, F. D'Annibale, D. Del Vescovo, I. Giorgio, Deformation patterns in a second-gradient lattice annular plate composed of "spira mirabilis" fibers, *Contin. Mech. Thermodyn.* 35 (4) (2023) 1561–1580.
- [30] E. Turco, M. Golaszewski, A. Cazzani, N.L. Rizzi, Large deformations induced in planar pantographic sheets by loads applied on fibers: experimental validation of a discrete Lagrangian model, *Mech. Res. Commun.* 76 (2016) 51–56.
- [31] A. Ciallella, D. Pasquali, F. D'Annibale, I. Giorgio, Shear rupture mechanism and dissipation phenomena in bias-extension test of pantographic sheets: Numerical modeling and experiments, *Math. Mech. Solids* 27 (10) (2022) 2170–2188.
- [32] A. Ciallella, D. Pasquali, M. Golaszewski, F. D'Annibale, I. Giorgio, A rate-independent internal friction to describe the hysteretic behavior of pantographic structures under cyclic loads, *Mech. Res. Commun.* 116 (2021) 103761.
- [33] I. Giorgio, Lattice shells composed of two families of curved Kirchhoff rods: an archetypal example, topology optimization of a cycloidal metamaterial, *Contin. Mech. Thermodyn.* 33 (4) (2021) 1063–1082.
- [34] I. Giorgio, A. Ciallella, D. Scerrato, A study about the impact of the topological arrangement of fibers on fiber-reinforced composites: some guidelines aiming at the development of new ultra-stiff and ultra-soft metamaterials, *Int. J. Solids Struct.* 203 (2020) 73–83.
- [35] I. Giorgio, A discrete formulation of Kirchhoff rods in large-motion dynamics, *Math. Mech. Solids* 25 (5) (2020) 1081–1100.
- [36] I. Giorgio, D. Del Vescovo, Energy-based trajectory tracking and vibration control for multilink highly flexible manipulators, *Math. Mech. Complex Syst.* 7 (2) (2019) 159–174.
- [37] D. Baroudi, I. Giorgio, A. Battista, E. Turco, L.A. Igumnov, Nonlinear dynamics of uniformly loaded elastica: Experimental and numerical evidence of motion around curled stable equilibrium configurations, *ZAMM-J. Appl. Math. Mech./Z. Angew. Math. Mech.* 99 (7) (2019) e201800121.
- [38] I. Giorgio, N.L. Rizzi, U. Andreaus, D.J. Steigmann, A two-dimensional continuum model of pantographic sheets moving in a 3D space and accounting for the offset and relative rotations of the fibers, *Math. Mech. Complex Syst.* 7 (4) (2019) 311–325.
- [39] F.-F. Wang, T. Wang, X. Zhang, Y. Huang, I. Giorgio, F. Xu, Wrinkling of twisted thin films, *Int. J. Solids Struct.* 262 (2023) 112075.
- [40] I. Giorgio, M. De Angelo, E. Turco, A. Misra, A Biot–Cosserat two-dimensional elastic nonlinear model for a micromorphic medium, *Contin. Mech. Thermodyn.* 32 (5) (2020) 1357–1369.
- [41] R. Fedele, Approach à la Piola for the equilibrium problem of bodies with second gradient energies. Part II: Variational derivation of second gradient equations and their transport, *Contin. Mech. Thermodyn.* 34 (5) (2022) 1087–1111.
- [42] R. Fedele, Third-gradient continua: nonstandard equilibrium equations and selection of work conjugate variables, *Math. Mech. Solids* 27 (10) (2022) 2046–2072.
- [43] R. dell'Erba, Position-based dynamic of a particle system: a configurable algorithm to describe complex behaviour of continuum material starting from swarm robotics, *Contin. Mech. Thermodyn.* 30 (5) (2018) 1069–1090.
- [44] R. dell'Erba, On how swarm robotics can be used to describe particle system's deformation, *Contin. Mech. Thermodyn.* 34 (4) (2022) 955–975.
- [45] B.E. Abali, W.H. Müller, Determination of material coefficients for a non-linear viscous fluid by a numerical inverse analysis and its verification with a finite element simulation, *Exp. Methods Numer. Simul. Eng. Sci.* 1 (2012) 55–58.
- [46] M. De Angelo, E. Barchiesi, I. Giorgio, B.E. Abali, Numerical identification of constitutive parameters in reduced-order bi-dimensional models for pantographic structures: application to out-of-plane buckling, *Arch. Appl. Mech.* 89 (7) (2019) 1333–1358.
- [47] G. La Valle, S. Massoumi, A new deformation measure for micropolar plates subjected to in-plane loads, *Contin. Mech. Thermodyn.* 34 (2022) 243–257.
- [48] G. La Valle, A new deformation measure for the nonlinear micropolar continuum, *Z. Angew. Math. Phys.* 73 (2) (2022) 78.
- [49] W. Pietraszkiewicz, V. Eremeyev, On natural strain measures of the non-linear micropolar continuum, *Int. J. Solids Struct.* 46 (3–4) (2009) 774–787.
- [50] L. Murcia Terranova, C. Cardillo, G. Aretusi, An enhanced beam model incorporating a hysteresis-based solid friction damping mechanism for cementitious materials, *Contin. Mech. Thermodyn.* 37 (1) (2025) 2.
- [51] M. Cuomo, L. Contrafatto, L. Greco, A cohesive interface model with degrading friction coefficient, *Math. Mech. Complex Syst.* 12 (2) (2024) 113–133.
- [52] A.M. Bersani, P. Caresa, A. Ciallella, Numerical evidence for the approximation of dissipative systems by gyroscopically coupled oscillator chains, *Math. Mech. Complex Syst.* 10 (3) (2022) 265–278.
- [53] A. Amir-Hezaveh, P. Karimi, M. Ostoj-Starzewski, IBVP for electromagneto-elastic materials: variational approach, *Math. Mech. Complex Syst.* 8 (1) (2020) 47–67.
- [54] E. Turco, E. Barchiesi, Kinematically triggered nonlinear vibrations of Hencky-type pantographic sheets, *Math. Mech. Complex Syst.* 9 (3) (2022) 311–335.
- [55] J. Goddard, On linear non-local thermo-viscoelastic waves in fluids, *Math. Mech. Complex Syst.* 6 (4) (2018) 321–338.
- [56] B. Baldi, In *Mechanica Aristotelis Problemata Exercitationes, Typis & sumptibus viduae Ioannini Albini, Moguntiae*, 1621.
- [57] M. Abattout, Nutaf min Al-Hiyal: a partial arabic version of pseudo-aristotle's problemata mechanica, *Early Sci. Med.* 6 (2) (2001) 96–122.
- [58] M.R.R. de León, La influencia de la Meccanica Problemata en la arquitectura de Joseph de Porres, *Avance* 3 (3) (2013) 26–31.
- [59] P.L. Rose, S. Drake, The pseudo-aristotelian questions of mechanics in renaissance culture, *Stud. Renaiss.* 18 (1971) 65–104.
- [60] J. De Groot, Modes of explanation in the Aristotelian mechanical problems, *Early Sci. Med.* 14 (1–3) (2009) 22–42.
- [61] J. De Groot, Aristotle's Empiricism: Experience and Mechanics in the Fourth Century BC, *Parthenon Publishing*, Las Vegas, 2014.
- [62] M.A. Coxhead, A close examination of the pseudo-Aristotelian mechanical problems: The homology between mechanics and poetry as technē, *Stud. Hist. Philos. Sci. Part A* 43 (2) (2012) 300–306.
- [63] I. Newton, Isaac Newton letter to Robert Hooke, 1675, Simon Gratz autograph collection.
- [64] M. White, Isaac Newton: The Last Sorcerer, Fourth Estate, London, 1998.
- [65] S. Kean, Newton, the last magician, *Humanities* 32 (1) (2011).
- [66] T. Morrison, The enigma of Isaac Newton: Scientist, theologian, alchemist and prophet, *Int. J. Humanit.* 5 (7) (2007).
- [67] R. Feynman, R. Leighton, M. Sands, M. Gottlieb, The Feynman Lectures on Physics, Pearson/Addison-Wesley, Boston, 2006.
- [68] Aristotle, Minor Works: On Colours. On Things Heard. Physiognomics. On Plants. On Marvellous Things Heard. Mechanical Problems. On Indivisible Lines. The Situations and Names of Winds. On Melissus, Xenophanes, Gorgias. Translated by W. S. Hett. Loeb Classical Library 307, Harvard University Press, Cambridge, MA, 1936.
- [69] E.A. Moody, M. Clagett, The Medieval Science of Weights (Scientia De Ponderibus): Treatises Ascribed to Euclid, Archimedes, Thabit ibn Qurra, Jordanus de Nemore and Blasius of Parma, University of Wisconsin Press, Madison, 1952.
- [70] M. Clagett, The Science of Mechanics in the Middle Ages, The University of Wisconsin Press, Madison, 1961.

- [71] E. Giusti, Maurolico et archimède: sources et datation du 1er livre du de momentis aequalibus, in: *Medieval and Classical Traditions and the Renaissance of Physico-Mathematical Sciences in the 16th Century: Proceedings of the XXth International Congress of History of Science (Liège, 20-26 July 1997)* Vol. VIII, 2001, pp. 33–40.
- [72] J.L.R. d'Alembert, *Traité de dynamique*, David l'aîné, Libraire, Rue Saint Jacques à la Plume d'or, Paris, 1743.
- [73] I. Newton, *Philosophiae Naturalis Principia Mathematica*, iussu Societatis Regiae ac typis Josephi Streater, London, 1687.
- [74] L. Russo, *The Forgotten Revolution. How Science Was Born in 300 BC and Why it Had to Be Reborn*, Springer, Berlin, Heidelberg, 2004.
- [75] R. Netz, Greek mathematical diagrams: Their use and their meaning, *For Learn. Math.* 18 (3) (1998) 33–39.
- [76] J. De Groot, S. Berryman, Aristotle's empiricism: Experience and mechanics in the 4th century BC, *Aestimation: Sources Stud. Hist. Sci.* 11 (2014) 248–252.
- [77] A.M. Lyapunov, The general problem of the stability of motion, *Internat. J. Control* 55 (3) (1992) 531–534.
- [78] S.T.L. Heath, *The Works of Archimedes*, Cambridge University Press, Cambridge, 1897.
- [79] V.A. Eremeyev, H. Altenbach, Equilibrium of a second-gradient fluid and an elastic solid with surface stresses, *Meccanica* 49 (2014) 2635–2643.
- [80] V.A. Eremeyev, G. Rosi, S. Naili, Surface/interfacial anti-plane waves in solids with surface energy, *Mech. Res. Commun.* 74 (2016) 8–13.
- [81] V.A. Eremeyev, G. Rosi, S. Naili, Transverse surface waves on a cylindrical surface with coating, *Internat. J. Engrg. Sci.* 147 (2020) 103188.
- [82] R. Fedele, L. Placidi, F. Fabbrocino, A review of inverse problems for generalized elastic media: formulations, experiments, synthesis, *Contin. Mech. Thermodyn.* 36 (6) (2024) 1413–1453.
- [83] P.L. Rose, *The Italian Renaissance of Mathematics: Studies on Humanists and Mathematicians from Petrarch to Galileo*, Librairie Droz, Genève, 2013.
- [84] M. Spagnuolo, F. dell'Isola, B. Gerber, A.M. Cazzani, Translation of heiberg's prolegomena, in: *Evaluation of Scientific Sources in Mechanics: Heiberg's Prolegomena To the Works of Archimedes and Hellinger's Encyclopedia Article on Continuum Mechanics*, Springer, 2021, pp. 75–97.
- [85] F. dell'Isola, G. Maier, U. Perego, U. Andreaus, R. Esposito, S. Forest, *The Complete Works of Gabrio Piola: Volume I*, Springer, Cham, 2014.
- [86] F. dell'Isola, U. Andreaus, A. Cazzani, R. Esposito, L. Placidi, U. Perego, G. Maier, P. Seppecher, *The Complete Works of Gabrio Piola: Volume II*, Springer, Cham, 2019.
- [87] F. dell'Isola, A. Madeo, P. Seppecher, Cauchy tetrahedron argument applied to higher contact interactions, *Arch. Ration. Mech. Anal.* 219 (3) (2016) 1305–1341.
- [88] R. Fedele, Piola's approach to the equilibrium problem for bodies with second gradient energies. Part I: First gradient theory and differential geometry, *Contin. Mech. Thermodyn.* 34 (2) (2022) 445–474.
- [89] G. Mancusi, F. Fabbrocino, L. Feo, F. Fraternali, Size effect and dynamic properties of 2D lattice materials, *Compos. Part B: Eng.* 112 (2017) 235–242.
- [90] E. Barchiesi, M. Laudato, F. Di Cosmo, Wave dispersion in non-linear pantographic beams, *Mech. Res. Commun.* 94 (2018) 128–132.
- [91] E. Barchiesi, S.R. Eugster, L. Placidi, Pantographic beam: A complete second gradient 1d-continuum in plane, *Z. Angew. Math. Phys.* 70 (5) (2019) 1–24.
- [92] D. Timofeev, E. Barchiesi, A. Misra, L. Placidi, Hemivariational continuum approach for granular solids with damage-induced anisotropy evolution, *Math. Mech. Solids* 26 (5) (2021) 738–770.
- [93] B.E. Abali, E. Barchiesi, Additive manufacturing introduced substructure and computational determination of metamaterials parameters by means of the asymptotic homogenization, *Contin. Mech. Thermodyn.* 33 (4) (2021) 993–1009.
- [94] L. Placidi, E. Barchiesi, A. Misra, D. Timofeev, Micromechanics-based elastoplastic-damage energy formulation for strain gradient solids with granular microstructure, *Contin. Mech. Thermodyn.* 33 (5) (2021) 2213–2241.
- [95] E. Barchiesi, M. Spagnuolo, L. Placidi, Mechanical metamaterials: A state of the art, *Math. Mech. Solids* 24 (1) (2019) 212–234.
- [96] E. Barchiesi, F. Di Cosmo, M. Laudato, A review of some selected examples of mechanical and acoustic metamaterials, *Discret. Contin. Model. Complex Metamaterials* (2020) 52–102.
- [97] A. Misra, L. Placidi, F. dell'Isola, E. Barchiesi, Identification of a geometrically nonlinear micromorphic continuum via granular micromechanics, *Z. Angew. Math. Phys.* 72 (2021) 1–21.
- [98] Y. Solyaev, S. Lurie, E. Barchiesi, L. Placidi, On the dependence of standard and gradient elastic material constants on a field of defects, *Math. Mech. Solids* 25 (1) (2020) 35–45.
- [99] L. Placidi, E. Barchiesi, Energy approach to brittle fracture in strain-gradient modelling, *Proc. R. Soc. A: Math. Phys. Eng. Sci.* 474 (2210) (2018) 20170878.
- [100] E. Barchiesi, F. dell'Isola, A.M. Bersani, E. Turco, Equilibria determination of elastic articulated duoskelion beams in 2D via a riks-type algorithm, *Int. J. Non-Linear Mech.* 128 (2021) 103628.
- [101] N. Rezaei, E. Barchiesi, D. Timofeev, C.A. Tran, A. Misra, L. Placidi, About the solution of a paradox related to axial pull out of a bar from a concrete cylindrical elastic domain in standard first gradient 3D isotropic elasticity, *Mech. Res. Commun.* 126 (2022) 103870.
- [102] A. Grillo, S. Di Stefano, An a posteriori approach to the mechanics of volumetric growth, *Math. Mech. Complex Syst.* 11 (1) (2023) 57–86.
- [103] A. Grillo, S. Di Stefano, Comparison between different viewpoints on bulk growth mechanics, *Math. Mech. Complex Syst.* 11 (2) (2023) 287–311.
- [104] A. Scrofani, E. Barchiesi, B. Chiaia, A. Misra, L. Placidi, Fluid diffusion related aging effect in a concrete dam modeled as a timoshenko beam, *Math. Mech. Complex Syst.* 11 (2) (2023) 313–334.
- [105] M. Olive, N. Auffray, Symmetry classes in piezoelectricity from second-order symmetries, *Math. Mech. Complex Syst.* 9 (1) (2021) 77–105.
- [106] C. Soize, An overview on uncertainty quantification and probabilistic learning on manifolds in multiscale mechanics of materials, *Math. Mech. Complex Syst.* 11 (1) (2023) 87–174.
- [107] N. Rezaei, J. Riesselmann, A. Misra, D. Balzani, L. Placidi, A procedure for the experimental identification of the strain gradient characteristic length, *Z. Angew. Math. Phys.* 75 (2024) 35.
- [108] E. Turco, E. Barchiesi, F. dell'Isola, Nonlinear dynamics of origami metamaterials: energetic discrete approach accounting for bending and in-plane deformation of facets, *Z. Angew. Math. Phys.* 74 (1) (2023) 26.
- [109] E. Barchiesi, F. dell'Isola, P. Seppecher, E. Turco, A beam model for duoskelion structures derived by asymptotic homogenization and its application to axial loading problems, *Eur. J. Mech. A Solids* 98 (2023) 104848.
- [110] L. Placidi, M.G. El Sherbiny, P. Baragatti, Experimental investigation for the existence of frequency band gap in a microstructure model, *Math. Mech. Complex Syst.* 9 (4) (2022) 413–421.
- [111] K.Y. Ko, Y. Solyaev, Explicit benchmark solution for topology optimization of variable-thickness plates, *Math. Mech. Complex Syst.* 11 (3) (2023) 381–392.
- [112] L. Placidi, L. Greco, S. Bucci, E. Turco, N.L. Rizzi, A second gradient formulation for a 2D fabric sheet with inextensible fibres, *Z. Angew. Math. Phys.* 67 (2016) 114.
- [113] U. Andreaus, F. dell'Isola, I. Giorgio, L. Placidi, T. Lekszycki, N.L. Rizzi, Numerical simulations of classical problems in two-dimensional (non) linear second gradient elasticity, *Internat. J. Engrg. Sci.* 108 (2016) 34–50.
- [114] M. Tepedino, The mechanical role of the periodontal ligament for developing mathematical models in orthodontics, *Math. Mech. Complex Syst.* 11 (4) (2023) 525–539.
- [115] H. Askes, A.R. Wallace, Dynamic Bergan–Wang theory for thick plates, *Math. Mech. Complex Syst.* 10 (2) (2022) 191–204.
- [116] K. Heiduschke, On tensor projections, stress or stretch vectors and their relations to Mohr's three circles, *Math. Mech. Complex Syst.* 12 (2) (2024) 173–216.
- [117] D. Ponomarev, A generalised time-evolution model for contact problems with wear and its analysis, *Math. Mech. Complex Syst.* 10 (3) (2022) 279–319.
- [118] A. Cazzani, F. Stochino, E. Turco, On the whole spectrum of timoshenko beams. Part I: a theoretical revision, *Z. Angew. Math. Phys.* 67 (2) (2016) 24.
- [119] A. Cazzani, F. Stochino, E. Turco, On the whole spectrum of timoshenko beams. Part II: further applications, *Z. Angew. Math. Phys.* 67 (2) (2016) 25.
- [120] A. Cazzani, F. Stochino, E. Turco, An analytical assessment of finite element and isogeometric analyses of the whole spectrum of timoshenko beams, *ZAMM-J. Appl. Math. Mech./Z. Angew. Math. Mech.* 96 (10) (2016) 1220–1244.
- [121] E. Barchiesi, Equilibria of axial-transversely loaded homogenized duoskelion beams, *Math. Mech. Complex Syst.* 12 (3) (2024) 283–309.
- [122] R. McAvoy, E. Barchiesi, Towards the Galerkin approximation of tetraskelion metamaterials, *Contin. Mech. Thermodyn.* 37 (1) (2025) 6.