Continuum Mech. Thermodyn. (2026) 38:2
https://doi.org/10.1007/s00161-025-01430-8

ORIGINAL ARTICLE

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Positive definiteness of deformation energy as a consistency condition of discrete-to-continuum transition

Received: 23 September 2025 / Accepted: 1 November 2025

Abstract We discuss the correspondence between discrete and continuum models of one-dimensional (1D) elastic structures. We propose positive definiteness as a consistency condition for continuum models. As a result, generalized continuum models may be required, as simple (Cauchy-type) continuum models may be insufficient. This theory is illustrated by a few physical examples.

Keywords Elastic chain · Discrete model · Positivity · Strain gradient elasticity · Generalized models

1 Introduction

Correspondence between discrete and and continuum models plays an important role from the beginning of rational mechanics, see e.g. [1,2]. For example, it is worth to mention the Poisson transition to a continuum medium with Poisson's ratio equal to 1/4 [3]. The relations between discrete models such as lattice dynamics and continuum models was discussed in many research, see e.g. fundamental books [4–8] and more recent reviews [9,10].

The connection between discrete and continuum models remains a relevant topic in multiscale analysis in nanotechnologies, see e.g. Question 9 in [11] and the related discussion. In addition to nano- and micromechanics, significant progress has recently been made in the engineering of new composites, such as beam-lattice metamaterials [12–14]. Beam-lattice materials mimic the properties of a perfect crystalline lattice, such as high strength and a high stiffness-to-weight ratio. Recent advances in additive technologies allow us to produce optimized beam-lattice structures and other architected materials apriori. Unlike classic materials, whose properties can be tuned within a limited range, 3D printing enables the modification of the microstructure of architected materials to achieve the desired response. The mechanics of discrete lattice-type structures has been studied extensively; see, for example, [15–23] and the references therein. In some cases, these metamaterials require generalized continuum models, such as strain gradient elasticity or micropolar/micromorphic media.

In addition to beam-lattice materials, it is worth to mention also origami-type structures [24–27] and other deployable structures [28,29] as examples of discrete-like systems.

Finally, a discrete representation of a continuous body is crucial for analysis of defects and crack nucleation and its propagation, allowing modeling of such effects in lattices [30–32] and chains [33,34].

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Published online: 12 November 2025

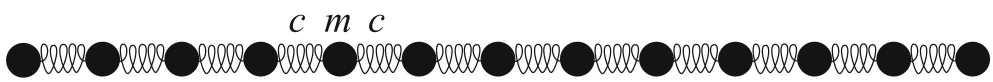

Fig. 1 Mass-spring chain

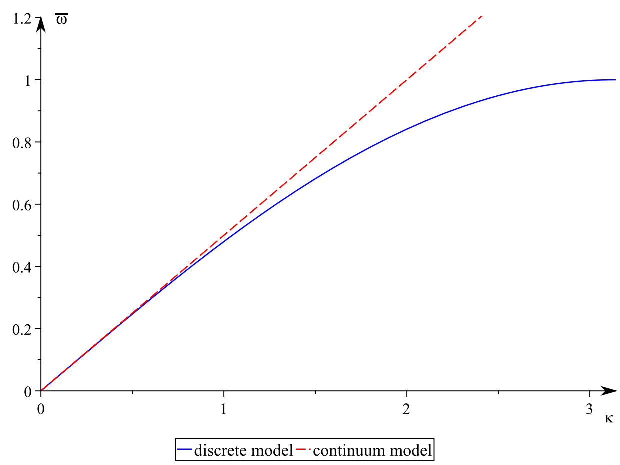

Fig. 2 Dispersion curves

The remainder of the paper is organised as follows. In Sect. 2, we review the equations of motion for an elastic chain with nearest-neighbors and next-nearest-neighbors interactions. Section 3 considers the general 1D statements for local and non-local interactions. Here, we introduce a simple continuum model and formulate conditions for discrete systems that result in the positive definiteness of a simple material's deformation energy. Section 4 illustrates the general theory with three physical examples. Two examples require more general continuum models, as their simple continuum counterparts do not satisfy the positive definiteness condition. The third example demonstrates that the need for a generalised model depends on the stiffness parameters. In the Conclusions section, we summarise the results and discuss their possible generalisations.

2 Classic mass-spring chain

First, let us recall the classic model of a structure consisting of masses connecting through linear elastic springs, see Fig. 1. This model was introduced by Lagrange, see discussion in [35]. Nowadays it can be found in may books, see e.g. [4,5,7,8], where the correspondence to the dynamics of an elastic bar was also discussed. The equation of motion has the form

$$m\ddot{u_n} + 2cu_n - cu_{n-1} - cu_{n+1} = 0, \tag{1}$$

where m is the mass and c is the spring stiffness, $u_n = u_n(t)$ are the horizontal displacement of nth mass, t is time, and the overdot stands for the derivative with respect to t. For an infinite chain ($n = 0, \pm 1, \pm 2, \dots$) we can find the solution of (1) in the form

$$u_n = u_0 e^{i(n\kappa - \omega t)} \tag{2}$$

where u_0 is a constant, ω is the frequency, κ is a wave number, and $i = \sqrt{-1}$ is the imaginary unit. Substituting (2) into (1) we came to the dispersion relation

$$\omega = \sqrt{\frac{2c}{m}}\sqrt{1 - \cos\kappa} \equiv 2\sqrt{\frac{c}{m}}\left|\sin\frac{\kappa}{2}\right|. \tag{3}$$

Corresponding dispersion curve is shown in Fig. 2 as a solid curve. Here $\varpi = \sqrt{\frac{m}{c}\frac{\omega}{2}}$.

For derivation of the continuum model of the chain, we assume the existence of a continuum function $u = u(x, t)$ such that $u(x_n, t) = u_n(t)$, $x_n = hn$, h is a distance between masses. In fact, the existence of such

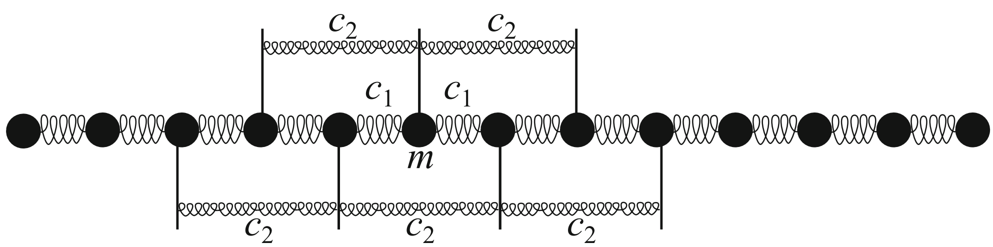

Fig. 3 Mass-spring chain with long-range interactions. Here each particle interacts with four neighbors

a function can be proven using the Fourier transform, see [36,37]. In the following we assume that h is mall with respect to considered wave length, see also Maslov [38]. Then we can use the Taylor series

$$u_{n\pm1} = u_n \pm h u_n' + \frac{h^2}{2} u_n'' \pm \frac{h^3}{6} u_n''' + \frac{h^4}{24} u_n'''' + \cdots + (-1)^m \frac{h^m}{m!} u_n^{(m)} + o(h^m). \tag{4}$$

Restricting ourselves to the order $m = 2$ we transform (1) into the classic wave equation

$$m\ddot{u} = ch^2 u'', \tag{5}$$

which has a harmonic solution $u = u_0 e^{i(\omega t - kx)}$, where k is the wave number. It gives us the classic $\omega - k$ dependence

$$\omega = \sqrt{\frac{c}{m}} \overline{k}, \quad \overline{k} = kh. \tag{6}$$

The graph of (6) is shown in Fig. 2 as the dashed line tangent to the dispersion curve of the discrete chain. Here we assumed that $\overline{k} = \kappa$. Obviously, Eq. (6) cannot describe the dispersion of the discrete chain. In order to reproduce the dispersion of the discrete 1D structure various non-local continuum models were proposed by Askar [7], Kunin [36], Eringen [39], and Maugin [2], see also Andrianov et al. [35], where various models were discussed including Padé approximations.

We should note that the deformation energy is positive definite for both models. Indeed, they are given by

$$\mathcal{E}_{disc} = \sum_n \frac{c}{2} (u_{n+1} - u_n)^2,$$
$$\mathcal{E}_{cont} = \int \frac{ch^2}{2} u'^2 \, dx.$$

Thus, both the discrete and continuum models are consistent in terms of the positive definiteness of the deformation energy. Note that $\mathcal{E}_{cont}$ describes a simple elastic material in the sense of Noll [40]. Indeed, the deformation energy here depends on the first gradient of displacement. We will also refer to such a continuum model as “simple” in the following.

For a one-dimensional chain we can easily take into account long-range interactions [5]. For example, for the chain shown in Fig. 3, the equation of motion takes the form

$$m\ddot{u}_n + 2(c_1 + c_2)u_n - c_1 u_{n-1} - c_1 u_{n+1} - c_2 u_{n-2} - c_2 u_{n+2} = 0, \tag{7}$$

where c_1 and c_2 are the stiffness of the springs. Kunin [36] found that $c_2/c_1 > -0.25$ using the stability analysis. Here we take into account interactions between a particle and its four close neighbors. The corresponding dispersion relation has the form

$$-m\omega^2 + 2(c_1 + c_2) - 2c_1 \cos\kappa - 2c_2 \cos 2\kappa = 0, \tag{8}$$

or, in the dimensionless form as

$$\overline{\omega} = \sqrt{1 - \eta_1 \cos\kappa - \eta_2 \cos 2\kappa}, \tag{9}$$

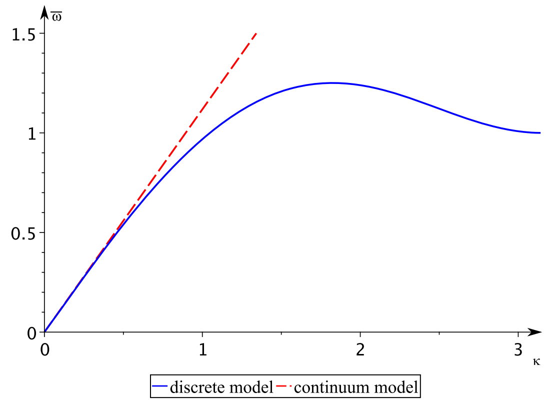

Fig. 4 Dispersion curves for interactions with four neighbors

where

$$\overline{\omega} = \sqrt{\frac{m}{2(c_1 + c_2)}}\omega, \quad \eta_1 = \frac{c_1}{c_1 + c_2}, \quad \eta_2 = \frac{c_2}{c_1 + c_2}.$$

The dispersion curve is shown in Fig. 4 for the case $c_1 = c_2$ ($\eta_1 = \eta_2 = 1/2$).

Even more general chain-type systems were considered in [36,37,41], see also the review [42] and the references therein.

The corresponding continuum model can be introduced in a similar way using (4). The result is given by the equation

$$m\ddot{u} = (c_1 + 4c_2)h^2 u'', \tag{10}$$

which differs from (5) only by a coefficient. The dispersive curve is shown in Fig. 4 as a dashed line. Let us note again that in this case we have positive deformation energies for both discrete and continuum models.

3 General case

3.1 Local interactions

Let us discuss more general equation of motion for an one-dimensional discrete structure considering interactions of nth particle with two closed neighbors. It takes the form

$$m\ddot{u}_n + 2au_n + bu_{n-1} + bu_{n+1} = 0, \tag{11}$$

where a and b are some coefficients. They form a symmetric tri-diagonal matrix $\mathbb{A}$ which is strictly diagonally dominant if

$$|a| > |b|. \tag{12}$$

If $a > 0$, $\mathbb{A}$ is a positive definite matrix. As the quadratic form $\mathcal{E} = \frac{1}{2}(\mathbb{A}\mathbf{u}, \mathbf{u})$ has the meaning of a deformation energy, we can conclude that it is also positive definite. For positive semidefinite deformation energy it is enough to assume the non-strict inequality, i.e. $|a| \geq |b|$. Note that unlike the chain model (1) Eq. (11) has no invariance under translation $u_n \to u_n + C$ with constant C, in general. In general, it depends on the physical meaning of the variables u_n. For example, Parodi [43] presented a system of magnets that results in a similar equation for angles, though it lacks translational invariance.

We are looking for a solution of (11) again in the form (2) that gives us a formula

$$m\omega^2 = 2a + 2b\cos\kappa.$$

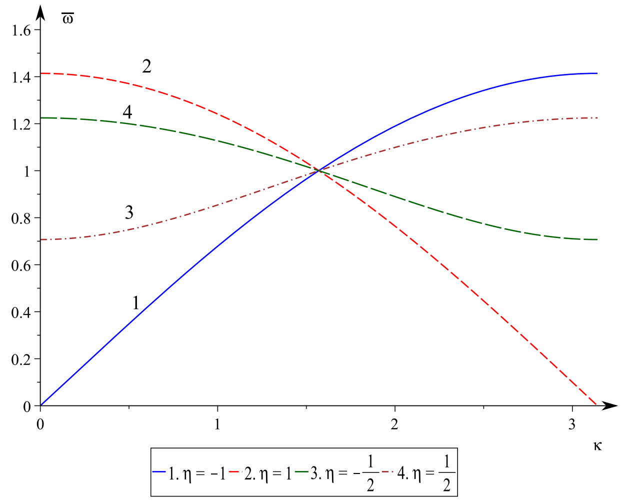

Fig. 5 Dispersion curves in the general case. Curves 1, 2, 3, and 4 correspond to $\eta = -1$, 1, $-1/2$, and $1/2$, respectively

From this we get the dispersion relation

$$\omega = \sqrt{\frac{2a}{m}}\sqrt{1+\eta\cos\kappa}. \tag{13}$$

Depending on the ratio $\eta = b/a \in [-1, 1]$ we have different dispersion curves, see Fig. 5. Here $\overline{\omega} = \omega\sqrt{m/2a}$. Note that the case $\eta = -1$ ($b = -a$) corresponds to the elastic chain, see Eq. (3) and Fig. 2, curve 1. It is interesting to note that if $\eta > 0$ we have an anomalous dispersion, as the phase and group velocities have different signs. Moreover, if $\eta \neq \pm 1$ there exist waves only in a certain range of frequencies: $0 < \min(\omega_\pm) \leq \omega \leq \max(\omega_\pm) < 2\sqrt{\frac{a}{m}}$, where $\omega_\pm = \sqrt{\frac{2a}{m}}\sqrt{1 \pm \eta}$, see curves 3 and 4 in Fig. 5.

Let us discuss the continuum analogous of (11). Using again the Taylor series (4) we transform (11) into a

$$m\ddot{u} + 2(a+b)u + bh^2u'' + b\frac{h^4}{12}u'''' \ldots + o(h^m) = 0 \tag{14}$$

with even m. Depending on m we have the series equations of motion, for example,

$$m = 2: \quad m\ddot{u} + 2(a+b)u + bh^2u'' = 0, \tag{15}$$

$$m = 4: \quad m\ddot{u} + 2(a+b)u + bh^2u'' + b\frac{h^4}{12}u'''' = 0. \tag{16}$$

Let us note that Eq. (15) is hyperbolic only if $b < 0$, for $b > 0$ it is elliptic, whereas Eq. (16) is parabolic as all the others for $m > 2$. Eqs. (15) and (16) have the following dispersion relations

$$m = 2: \quad \omega = \sqrt{\frac{2a}{m}}\sqrt{1+\eta-\frac{1}{2}\eta h^2k^2}, \tag{17}$$

$$m = 4: \quad \omega = \sqrt{\frac{2a}{m}}\sqrt{1+\eta-\frac{1}{2}\eta h^2k^2+\frac{1}{24}\eta h^4k^4}. \tag{18}$$

The corresponding dispersion curves are shown in Fig. 6 for the same values of η as in Fig. 5. As in the previous case, we use the correspondence $\kappa = hk$. One can see from (17) that for $m = 2$ and $b > 0$ there is a value of $k^* = \sqrt{2(1+\eta)/\eta}$, such that $\omega = 0$ and ω became imaginary if $k > k^*$. Instead, for $m = 4$ and $b > 0$ the frequency is positive for any k.

We can extend this analysis for any value of m. Real values of ω exist for all range of k if $m = 4, 8, 12$, i.e. for $m = 4n$, whereas for $m = 2, 6, 10$, etc., ω became imaginary if k exceeds a certain value, see [44]. Similar observations was also done in [45,46].

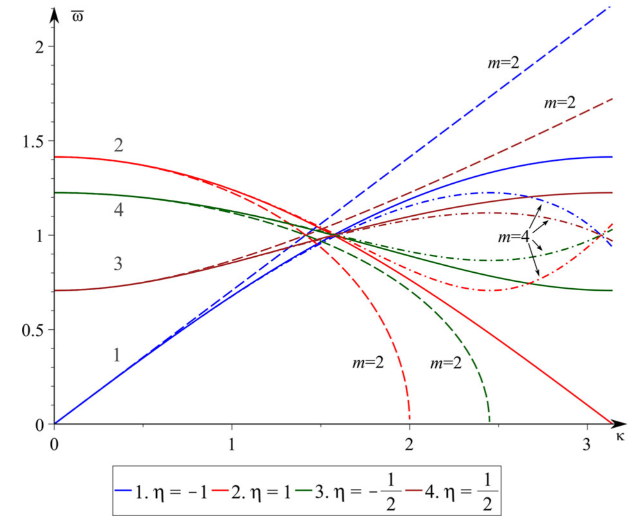

Fig. 6 Dispersion curves for $m = 2$ (dashed curves) and for $m = 4$ (dash-dot curves). Solid curves relate to (13). Here $\eta = -1$, 1, $-1/2$, and 1/2, respectively

Moreover, one can also see the difference in the positive definiteness of the deformation energies

$$\mathcal{E}_{m=2} = \int \left[(a+b)u^2 - \frac{1}{2}bh^2(u')^2 \right] dx,$$
$$\mathcal{E}_{m=4} = \int \left[(a+b)u^2 - \frac{1}{2}bh^2(u')^2 + b\frac{h^4}{24}(u'')^2 \right] dx.$$

Indeed, $\mathcal{E}_{m=2}$ is not positive definite if $b > 0$, whereas $\mathcal{E}_{m=4}$ has this property. In the case of negative b we have the opposite situation: $\mathcal{E}_{m=2}$ is positive definite while $\mathcal{E}_{m=4}$ is not. So one can see that consistency of the continuum model with the discrete ones is not guaranteed, in general. Hereafter, we will refer to models with energy dependent on spatial derivatives of order greater than one as 1D strain gradient elastic materials.

In summary, we can answer the following question: when does a 'naive' continualisation, i.e. a transition to a simple material, keep the deformation energy positive definite? This can be formulated as follows:

Proposition 1 *The discrete model given by Eq.* (11) *with a positive definite matrix* $\mathbb{A}$ *corresponds to a simple 1D continuum model with positive deformation energy if and only if* b *is negative and* $|b| \le a$.

Proof Indeed, from the positive definiteness of $\mathbb{A}$, we know that $a + b$ is nonnegative. Therefore, the first term in $\mathcal{E}_{m=2}$ is also nonnegative. If b is negative, then the second term in the expression $\mathcal{E}_{m=2}$ is positive and $\mathcal{E}_{m=2}$ is positive definite. However, if b is positive, then, even $a + b > 0$, $\mathcal{E}_{m=2}$ is not positive definite. This follows from the properties of Sobolev's spaces [47,48]. □

3.2 Nonlocal interactions: four closed neighbors

Let us now discuss the more general case of four neighbors. The equation of motion now takes the form

$$m\ddot{u}_n + 2au_n + bu_{n-1} + bu_{n+1} + cu_{n-2} + cu_{n+2} = 0, \tag{19}$$

with coefficients a, b, and c. If

$$a > 0, \quad \text{and} \quad a > |b| + |c|,$$

then the corresponding five-diagonal matrix is symmetric and positive definite, as it has strictly diagonal dominance. For the weak diagonal dominance it is enough to have non-strong inequality. So the corresponding deformation energy is positive definite. In general, Eq. (19) has no translation invariance. This is similar to Eq. (11).

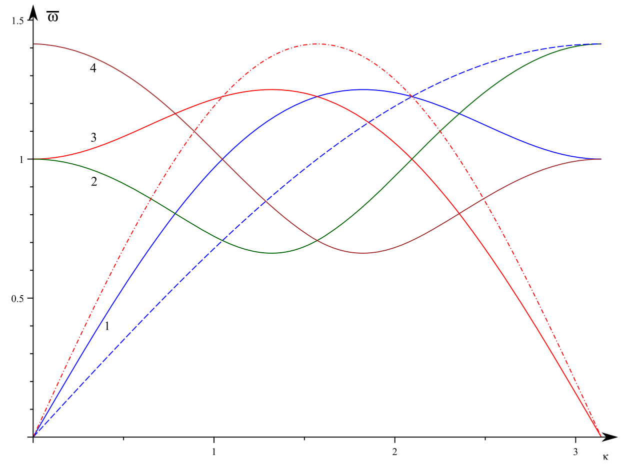

Fig. 7 Dispersion curves for four neighbors. Curves 1, 2, 3, and 4 correspond to $\eta_1 = -1/2$, $\eta_2 = -1/2$; $\eta_1 = -1/2$, $\eta_2 = 1/2$; $\eta_1 = 1/2$, $\eta_2 = -1/2$; and $\eta_1 = 1/2$, $\eta_2 = 1/2$, respectively. The dashed curve relates to the local case ($\eta_1 = 1$, $\eta_2 = 0$) whereas the dash-dot curve to the case only long interactions ($\eta_1 = 0$, $\eta_2 = -1$)

Looking for a solution of (19) in the form (2) we get the dispersion relation

$$\omega = \sqrt{\frac{2a}{m}}\sqrt{1 + \eta_1 \cos\kappa + \eta_2 \cos 2\kappa}, \tag{20}$$

where

$$\eta_1 = \frac{b}{a}, \quad \eta_2 = \frac{c}{a},$$

and $1 - |\eta_1| - |\eta_2| > 0$. The corresponding dispersion curves are shown in Fig. 7 for the limit values of η_1 and η_2. The dashed curve relates to the local case ($\eta_2 = 0$) whereas the dash-dot curve to the case only long interactions ($\eta_1 = 0$).

The transition to a continuum model can be performed in a similar way to that described in the previous section. Using Taylor's series (4) we obtain the formula

$$m\ddot{u} + 2(a+b+c)u + (b+4c)h^2u'' + \frac{h^4}{12}(b+16c)u'''' + \cdots + o(h^m) = 0. \tag{21}$$

From this, we can derive a series of equations of motion for different values of m. For example, we have

$$m = 2: \quad m\ddot{u} + 2(a+b+c)u + (b+4c)h^2u'' = 0, \tag{22}$$

$$m = 4: \quad m\ddot{u} + 2(a+b+c)u + (b+4c)h^2u'' + (b+16c)\frac{h^4}{12}u'''' = 0. \tag{23}$$

These equations differ from (15) and (16) only by coefficients, so the corresponding dispersion relations almost coincide with (17) and (18). This similarity enables us to formulate the following:

Proposition 2 *The non-local discrete model given by Eq. (19) with condition*

$$a + b + c \geq 0$$

corresponds to a simple 1D continuum model with positive deformation energy if and only if $b+4c$ *is negative.*

Note that the non-local interactions do not always require enhancement models such as the strain gradient elasticity.

4 Examples

To illustrate general considerations, let us look at some physical examples.

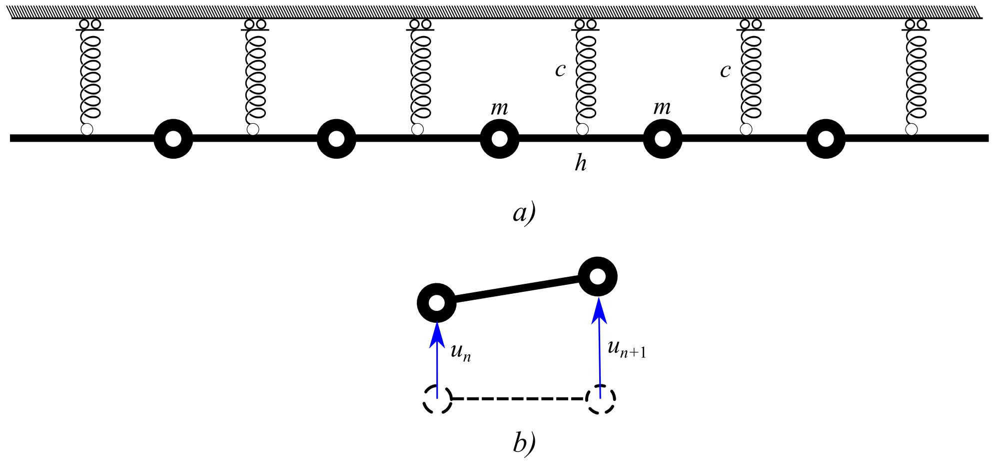

Fig. 8 Discrete chain suspended on springs **(a)** and motion of two masses **(b)**

4.1 Chain of suspended masses

Let us consider the 1D structure shown in Fig. 8. It consists of masses m concentrated in pivots connected by massless bars suspended on springs of stiffness c. Denoting the vertical displacement of nth mass as u_n we arrive at the following expressions of the kinetic and elastic energies

$$\mathcal{K} = \frac{m}{2}\sum_n \dot{u}_n^2, \quad \mathcal{E} = \frac{c}{8}\sum_n (u_n + u_{n+1})^2 . \tag{24}$$

The equations of motion follow from the Lagrange equation of second kind

$$\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{u}_n} + \frac{\partial \mathcal{E}}{\partial u_n} = 0, \tag{25}$$

which can a be transformed into

$$m\ddot{u}_n + \frac{c}{4}(2u_n + u_{n-1} + u_{n+1}) = 0. \tag{26}$$

Eq. (26) has the form of Eq. (11) with $a = c/4$ and $b = c/4$. As a result, we have $\eta \equiv b/a = 1$ and the corresponding dispersion curve is already given in Fig. 5 as curve 2.

As $b > 0$ the conditions of Proposition 1 are violated. So the discrete system shown in Fig. 8 requires an enhanced continuum model. The minimum admissible model described by Eq. (16) which transforms into

$$m\ddot{u} + cu + \frac{c}{4}h^2 u'' + c\frac{h^4}{48}u'''' = 0.$$

Note that this structure was analysed differently in [49, p. 72].

4.2 Discrete beam on an elastic foundation

As an example of a nonlocal discrete system let us consider the elastic chain shown in Fig. 9. This is Hencky's beam [50] with an elastic foundation added. We introduce the vertical displacements of nth mass by u_n and angles of rotations α_n as shown in Fig. 9 b). For small motions we have $u_n = u_{n-1} + h\alpha_n$, so we get

$$\alpha_n = \frac{u_{n-1} - u_n}{h}.$$

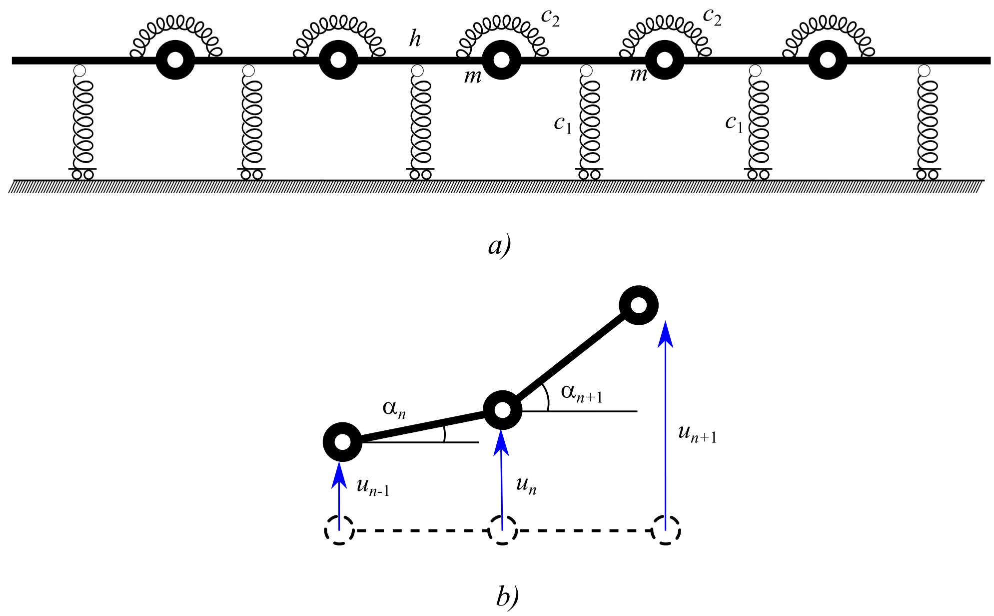

Fig. 9 Discrete beam (Hencky's model) on an elastic foundation **(a)**, motion of three masses and corresponding angles **(b)**

Introducing kinetic and potential energies as follows

$$\begin{aligned}\mathcal{E} &= \frac{c_1}{2}\sum_n (u_n + u_{n+1})^2 + \frac{c_2 h^2}{2}\sum_n (\alpha_n - \alpha_{n+1})^2 \\ &= \frac{c_1}{2}\sum_n (u_n + u_{n+1})^2 + \frac{c_2}{2}\sum_n (2u_n - u_{n-1} - u_{n+1})^2, \end{aligned} \tag{27}$$

$$\mathcal{K} = \frac{m}{2}\sum_n \dot{u}_n^2, \tag{28}$$

where c_1 and c_2 are stiffness of springs. Using (25) we get the equation of motion in the form

$$m\ddot{u}_n + \frac{c_1}{4}(2u_n + u_{n-1} + u_{n+1}) + c_2(6u_n - 4u_{n-1} - 4u_{n+1} + u_{n+2} + u_{n-2}) = 0, \tag{29}$$

which coincides with (19) if $a = c_1/4 + 3c_2$, $b = c_1/4 - 4c_2$, and $c = c_2$. As $b + 4c = c_1/4 > 0$ the conditions of Proposition 2 are also violated.

If $c_2 = 0$ we have the model discussed in the previous section 4.1. If $c_1 = 0$ we get exactly the Hencky model [50]

$$m\ddot{u}_n + c_2(6u_n - 4u_{n-1} - 4u_{n+1} + u_{n+2} + u_{n-2}) = 0. \tag{30}$$

Now $a = 3c_2$, $b = -4c_2$, and $c = c_2$. It is interesting that, in that case, we have $b + 4c = 0$. This means that Eq. (22) becomes an ordinary differential equation, whereas Eq. (23) takes the standard form of an equation of motion for Euler-Bernoulli beam on an elastic foundation [51]. The latter is a classic 1D strain gradient model.

One can say that it is obvious that the discrete model of Hencky's type cannot be described using a simple continuum model. In the next section, however, we demonstrate that this is not always the case.

4.3 Discrete beam on an elastic foundation with rotational stiffness

let us consider the Hencky-type beam resting on an elastic foundation which resists only to rotations, see Fig. 10.

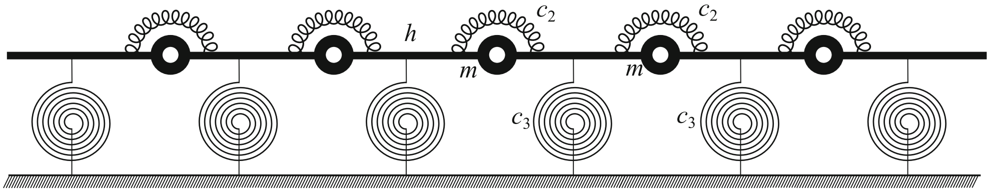

Fig. 10 Discrete beam (Hencky's model) on an elastic foundation with rotational resistance

Now the potential energy is given by

$$\begin{aligned}\mathcal{E} &= \frac{c_2 h^2}{2}\sum_n (\alpha_n - \alpha_{n+1})^2 + \frac{c_3 h^2}{2}\sum_n \alpha_n^2 \\ &= \frac{c_2}{2}\sum_n (2u_n - u_{n-1} - u_{n+1})^2 + \frac{c_3}{2}\sum_n (u_{n+1} - u_n)^2, \end{aligned} \tag{31}$$

where c_3 and c_2 are again stiffness of springs and the kinetic energy is given by Eq. (28). The corresponding equation of motion takes the form

$$m\ddot{u}_n + c_2\left(6u_n - 4u_{n-1} - 4u_{n+1} + u_{n+2} + u_{n-2}\right) + c_3\left(2u_n - u_{n-1} - u_{n+1}\right) = 0. \tag{32}$$

It coincides with (19) with $a = c_3 + 3c_2$, $b = -c_3 - 4c_2$, and $c = c_2$. Now $b + 4c = -c_3 < 0$, so the conditions of Proposition 2 are satisfied. As a result, the corresponding model of simple material has a positive definite energy. Another structure with the same equations for small motions was discussed in [52].

Finally, taking into account both the rotational and tensile springs related to the foundation, we arrive at the following equation:

$$\begin{aligned} m\ddot{u}_n &+ \frac{c_1}{4}\left(2u_n + u_{n-1} + u_{n+1}\right) + c_2\left(6u_n - 4u_{n-1} - 4u_{n+1} + u_{n+2} + u_{n-2}\right) \\ &+ c_3\left(2u_n - u_{n-1} - u_{n+1}\right) = 0. \end{aligned} \tag{33}$$

We now have the following: $a = c_1/4 + 3c_2 + c_3$, $b = c_1/4 - 4c_2 - c_3$, $c = c_2$. and $b + 4c = c_1/4 - c_3$. Therefore, the sign of $b + 4c$ depends on c_1 and c_3. If $c_1 > 4c_3$, it is positive; otherwise, it is negative. Consequently, this combined structure may or may not have the positive definite energy of the corresponding 1D simple material.

5 Conclusions and discussion

In linear elasticity, positive definiteness of the strain energy is a natural requirement [53–56]. In this paper, we consider it to be a consistency condition for the continuous modelling of a discrete structure. A continuum model can be considered consistent, or even admissible, if the deformation energy is positive definite. It is now well known that some discrete or semi-discrete structures require a generalized continuum model, such as strain-gradient continuum or other generalized media, because they cannot be described within a simple material model (see, for example, [15,19,57–61] for strain gradient elasticity and [17,18,62–65] for other generalized media). By analyzing the dispersion relations of certain 1D discrete cases, we formulate conditions for the consistency of the corresponding continuum models. Note that we have restricted ourselves to a minimal level of consistency without attempting to describe the dispersion relation within the continuum model in full. For the latter problem, we refer the reader to [7,35,36,39] and the references therein. Another point of view on a consistent continuum model could be causality [66] or stability [67–70]. We emphasize here that stability is directly related to positive definiteness. Positive definiteness, as a consistency (stability) requirement, can also be applied to strongly non-local models [2]. In particular, a general method for generating consistent nonlocal continuum models was proposed in [71,72].

The main point of this paper is to emphasize that the positivity of the elastic energy of a discrete system is not automatically transferred to the related continuum model. This property must be verified, and the continuum model must be adjusted accordingly. For example, possible inconsistent models of beam-lattice pantographic structures were indicated in [73]. Moreover, the positive definiteness of the deformation energy of the corresponding continuum can be defined in a proper functional space, which may be nontrivial, as in [74]. In this paper, we restrict ourselves to 1D structures. Obviously, the same situation, or even a more complex one, could happen in the 2D and 3D cases. Therefore, one should be aware of the positivity check as a consistency condition.

In this paper, linear elastic interactions between the masses were considered. The long-wavelength approximation was used to obtain the unique continuum limit. However, even in the case of 1D chains [75] and 2D lattices [76,77] with equivalent particles, one continuum limit may be insufficient for the correct description of dynamics, especially in the case of nonlinear coupling between the particles. Various continuum limits allow for the obtaining of different nonlinear equations accounting either for a long-wave propagation and interaction of long and short waves or interaction between acoustic and optical modes. This can be a way for further studies.

The consistency between discrete and continuum models can be examined from a computational perspective. When using the finite element method (FEM) or any other discretisation-based methods, we expect the solution to converge from the 'approximated' discrete model to the 'exact' continuum model solution. For such convergence to occur, the deformation energy must be positive-definite, among other conditions (see, for example, [48,53,78,79]). Therefore, one cannot expect convergence if this condition is violated, even in statics. In other words, a well-posed discrete model may not have a continuum limit within an inconsistent continuum model. This is also well known in the FEM literature as the consistency condition [79]. In contrast to FEM or other methods, we have an inverse situation. In FEM, for a given continuum model, one can propose a series of consistent discrete schemes. Here, for a given discrete structure, we can propose a series of consistent continuum models. As we have shown, in some cases, the latter should include only generalized continuum models, such as strain gradient elasticity. In a sense, we are dealing with continuous approximations of difference operators, as in [44,80]. The deep connections between the finite difference and finite element methods and the enriched continuum models were discussed in [81]. In general, the continualisation of finite difference operators is not unique. This can result in a simple (Cauchy-type) model, a weak non-local model (such as Toupin–Mindlin strain gradient elasticity) or a strong non-local model (such as Eringen's non-local elasticity). For more information on the difference between weak and strong non-local models, we refer to [2]. Here, we restrict ourselves to consistent weak non-local models. In general, consistent non-local models can be introduced using the techniques described in [35,36,38,42,71,72].

Acknowledgements V.A.E. acknowledges the support within the project "Metamaterials design and synthesis with applications to infrastructure engineering" funded by the MUR Progetti di Ricerca di Rilevante Interesse Nazionale (PRIN) Bando 2022 – grant 20228CPHN5. He also thanks the University of Tel Aviv for the support and hospitality during the visiting professorship program in 2025. I.B. acknowledges the support of the Israeli Ministry of Science and Technology, grant No. 0007313 – Project Title "Elasticity and Dynamics of Origami-based Mechanical Metamaterials".

Data availability No datasets were generated or analysed during the current study.

Funding Open access funding provided by Università degli Studi di Cagliari within the CRUI-CARE Agreement.

Declarations

Conflict of interest The authors declare no competing interests.

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