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TITLE OF THE Ph.D. THESIS

Structural Conditions for Global Existence in Some Chemotaxis Systems with Different Logistics

Scientific Disciplinary Sector

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Abstract

This doctoral thesis develops a comprehensive mathematical analysis of a wide class of chemotaxis systems inspired by and extending the classical Keller–Segel model. The models investigated combine attraction–repulsion mechanisms, signal production and consumption, and logistic-type growth and decay effects, of both local and nonlocal type. The central aim is to identify structural conditions ensuring the global existence, boundedness, and qualitative stability of classical solutions, while also exploring the onset of blow-up phenomena.

The study begins with two-signal systems incorporating both attractive and repulsive cues, where the interaction of opposing forces strongly influences system dynamics. By analyzing linear and nonlinear production laws, sensitivity functions, and threshold values of parameters, the thesis demonstrates how repulsive signaling can counteract destabilizing aggregation effects. Special attention is given to nonlinear diffusion and chemotactic sensitivities, as well as to logistic source terms with gradient-dependent damping. A novel contribution in this context is the introduction of a gradient-type nonlinearity in the logistic dampening term—a formulation that has only recently emerged in the literature and is shown here to play a decisive role in regularizing solutions and preventing singularity formation under appropriate conditions.

Further investigations focus on signal-consumption models, motivated by biologically realistic settings where chemoattractants are degraded by cells. These systems are proven to be more stable than production-dominated counterparts, with global boundedness results established under explicit parametric thresholds involving chemotactic sensitivity, initial data, and damping coefficients. The thesis also explores indirect chemotaxis mechanisms, particularly in the context of immune–tumor interactions, where signaling is mediated by an intermediate variable. For these models, conditions for global boundedness and estimates on blow-up times are provided.

Overall, the results highlight a recurring principle: the interplay between nonlinearities, cross-diffusion, and signal regulation dictates whether chemotactic populations evolve toward stable distributions or develop singularities. By constructing and analyzing a broad range of single- and multi-signal, local and nonlocal, parabolic and elliptic systems, this work establishes a framework that enriches both the mathematical theory of chemotaxis and its applications to biologically relevant phenomena such as tissue aggregation, immune responses, and tumor–immune interactions.

While the primary focus lies on global behavior, selected numerical simulations illustrate and complement the analytical results, offering further insight into system dynamics. For solutions exhibiting blow-up behavior, estimates of the blow-up time are obtained through both theoretical analysis and numerical simulations.

The thesis thus contributes both generalizations of classical results and novel perspectives on chemotaxis modeling, advancing the theoretical understanding of nonlinear partial differential equations in biological contexts.

Finally, the thesis is structured into seven chapters, each subdivided into sections and subsections, followed by a section containing the bibliographic references. Fundamentally, the thesis is divided into two principal parts: the first part, comprising Chapters 1, 2, 3, and 4, addresses the preliminaries, research objectives, state of the art, and classical tools instrumental for the subsequent analyses; the second part, consisting of Chapters 5 and 6, presents the original contributions and advancements made in the field during the course of the doctoral research. Specifically, Chapters 5 is dedicated to attraction-repulsion problems, whereas Chapters 6 focuses exclusively on attraction problems. Chapters 5 and 6 collectively encapsulate the peer-reviewed journal articles and/or conference proceedings published throughout the doctoral program. (Figure 1 provides a summary of the foregoing.)

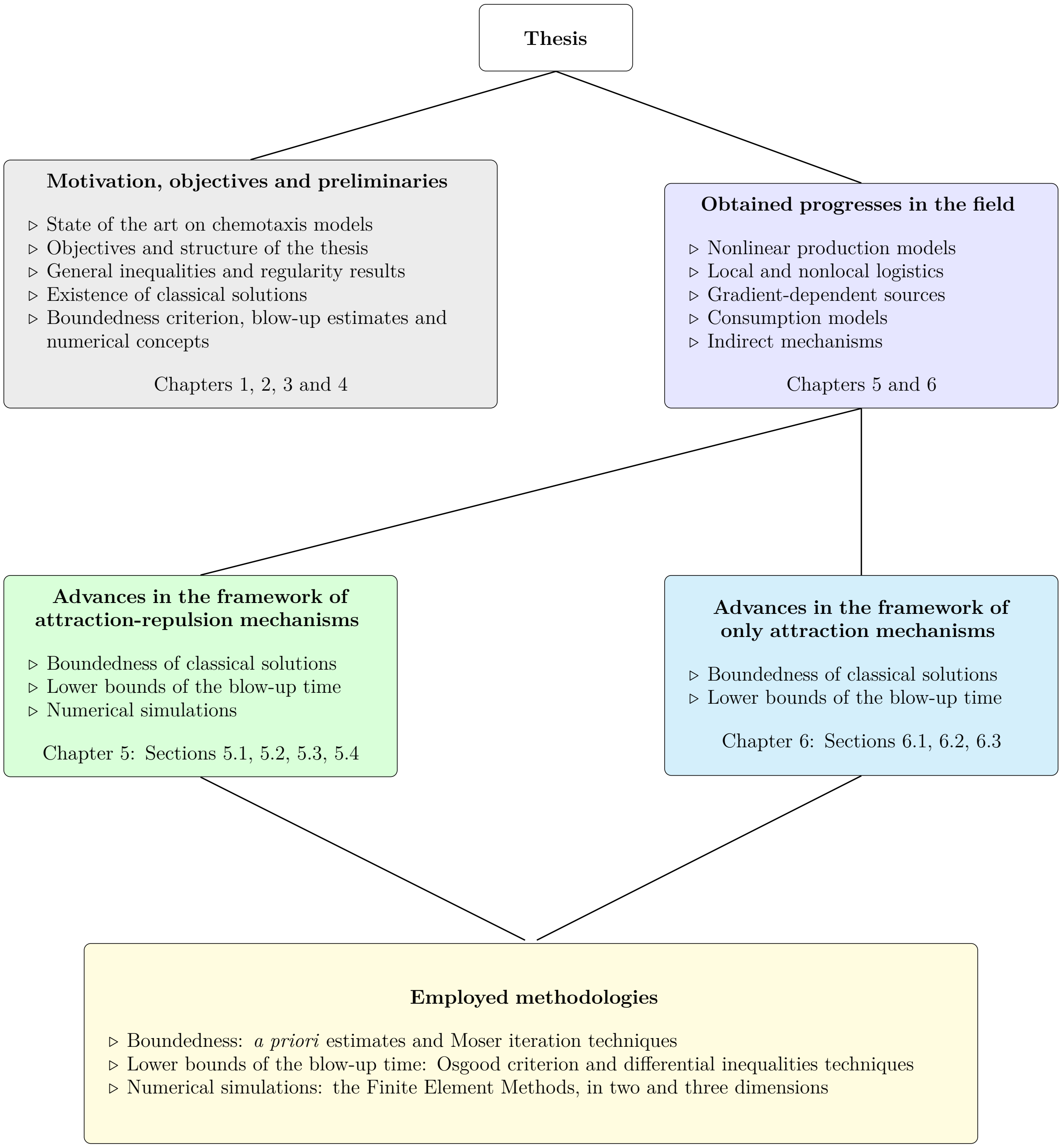

Figure 1: Outline of the thesis and development of key arguments.

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Contents

Chapter 1

Introduction: chemotaxis phenomena and the Keller–Segel model

In this thesis, we will analyze and discuss, from a mathematical point of view, the *chemotaxis* phenomenon. Its formulation dates back to the work proposed by Keller and Segel[1] in [KS70, KS71]. In these articles it is essentially described the mechanism by which bacteria and other unicellular or multicellular organisms direct their movement based on the presence of certain chemical substances in their environment.

Naturally, to describe the model, some elements are essential to represent the environment where the mentioned phenomenon occurs: the two substances involved in the mechanism, their interaction, the initial conditions of these substances, and their behavior near the boundary of the domain. In this sense, the two most famous mathematical models that can potentially translate chemotaxis are the following systems of two coupled partial differential equations with a *drift* term:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) & \text{in } \Omega \times (0, T_{\max}), \\ v_t = \Delta v - v + u, & \text{in } \Omega \times (0, T_{\max}), \\ u_\nu = v_\nu = 0 & \text{on } \partial\Omega \times (0, T_{\max}) \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & x \in \Omega, \end{cases} \tag{1.0.1}$$

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) & \text{in } \Omega \times (0, T_{\max}), \\ v_t = \Delta v - uv, & \text{in } \Omega \times (0, T_{\max}), \\ u_\nu = v_\nu = 0 & \text{on } \partial\Omega \times (0, T_{\max}) \\ u(x,0) = u_0(x), v(x,0) = v_0(x) & x \in \Omega, \end{cases} \tag{1.0.2}$$

In the two systems described above, $u = u(x,t)$ and $v = v(x,t)$ represent the distributions of the cells and the chemical signal at the point in space x (where x belongs to a bounded domain $\Omega \subset \mathbb{R}^n$, with $n \geq 1$, and boundary $\partial\Omega$ is regular) and at time $t > 0$; also, $T_{\max}$ (potentially infinite) indicates the time up to which these distributions are defined. On the other hand, the homogeneous Neumann boundary conditions model the isolation of the environment (i.e., its impenetrability), and the conditions $u_0(x)$ and $v_0(x)$ represent the initial configurations of the cells and the chemical signal, respectively. Specifically, for the first equation (parabolic and evolutive), Δu indicates how cells tend to diffuse and stabilize within the domain and $-\chi\nabla \cdot (u\nabla v)$, with $\chi > 0$, shows how the chemical signal influences this stabilization tendency. The second equation indicates how the chemical signal evolves over time: on the one hand, in (1.0.1) we have a production of the signal; on the other hand, in (1.0.2), the second equation describes its absorption or consumption.

Remark 1.1 (Chemotaxis in bacteria)**.** *As an example, some bacteria, such as E. coli, have numerous flagella per cell (typically 4-10). These can rotate in two ways:*

[1] **Evelyn Fox Keller** (1936-2023) was an American physicist and feminist. She was also a professor of history and philosophy at the Massachusetts Institute of Technology (MIT). **Lee Aaron Segel** (1932–2005) was an applied mathematician, particularly known for his contributions to the spontaneous appearance of order types in convection and chemotaxis problems.

- *Counterclockwise rotation aligns the flagella into a single rotating bundle, so the bacterium swims straight.*
- *Clockwise rotation breaks the flagellar bundle, so each flagellum points in a different direction, causing the bacterium to tumble in place.*

The directions of rotation refer to an observer outside the cell looking at the flagella toward the cell.

1.1 Interpretation of the model: cellular productivity and absorption

The terms *production* and *consumption* are relatively intuitive. Indeed, the equation $v_t = \Delta v - v + u$ indicates exactly an increase in the variable v in terms of another one, u; in other words, an increase in the cell density u corresponds to an increase in the chemical signal v. The opposite situation occurs for $v_t = \Delta v - vu$; in this case, high values of cell density u correspond to low values of the signal v. To provide an adequate interpretation of the models, it is extremely important to establish the sign of χ in the expression $-\chi\nabla \cdot (u\nabla v)$, which, as specified, is *positive*. In the literature, $-\chi u$ is known as chemotactic sensitivity, and its effect on cell movement is of an *attractive* nature, increasing as χ and/or the distribution of v increases.

1.1.1 The production model

Returning to what was discussed for the attractive models, it is quite natural that a production-type mechanism could present instabilities; indeed, as the cell distribution tends to grow, the chemical signal also grows, which in turn increases the attractive effect on the cells. This process, in some cases, can cause the cells to aggregate in one area of the environment. This is referred to as *chemotactic collapse* or *blow-up*, where the movement described above can degenerate into uncontrolled aggregation. (See Figure 1.1.)

From a strictly mathematical perspective, the classic chemotaxis model has been widely studied in recent years (the bibliography on this topic is very extensive; see, for example, [HW05, HP09]). Specifically, chemotactic collapse implies that the solution u is *local in time* (or simply *local*), and at a certain finite time (the so-called *blow-up time*), it becomes unbounded at one or more points of the domain. This instability may depend on the initial conditions, the interaction between the diffusion term and the chemotactic sensitivity term, and also on the chemical signal production term.

Concerning problem (1.0.1), it is the magnitude of the product of sensitivity χ and cell mass $m := \int_\Omega u_0(x)\,dx = \|u_0\|_{L^1(\Omega)}$ that determines the presence of blow-up. If χm is large enough, it may appear for some initial distributions and not for others; conversely, the cell distribution will tend to stabilize over time, regardless of the initial configuration of the population. (See [JL92] for further mathematical insights on this issue.) Mathematically, and in contrast to the previous situation, we will speak of *global solutions in time* (or simply *global*) and, possibly, bounded solutions[2].

If the problem under consideration is analyzed in a one-dimensional domain, blow-up cannot occur; in higher dimensions, however, this is no longer true, and the Keller-Segel model's solution may become unbounded at a finite or infinite time for $n = 2$ or $n \geq 3$ (see [HV97, Nag01] and the references therein).

1.1.2 The consumption model

The version of the consumption model described in problem (1.0.2) is primarily suitable for depicting the banding behavior of bacteria due to chemotactic response, i.e., the distorted movement of bacteria toward the oxygen concentration gradient; this aspect has been experimentally analyzed in [Adl75]. In some cases, it is possible to establish the existence of traveling wave solutions for these models (see [KS71]). While for the production problem we observed that large initial data (specifically u_0) and the chemical signal (especially sensitivity χ) can generate unbounded solutions in finite time, for the consumption problem,

[2]As highlighted in the literature mentioned, there are situations where the solution is global but unbounded as time tends to infinity.

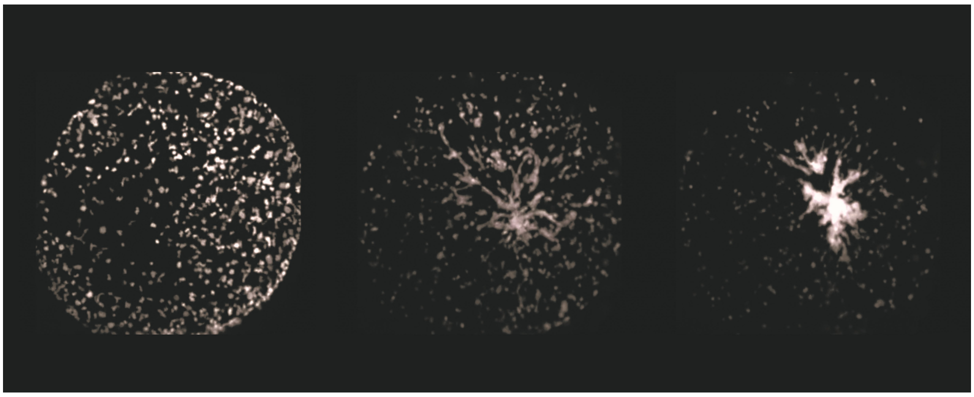


Figure 1.1: The phenomenon of chemotaxis under the microscope; chemotactic collapse situation for a production model.

this possibility is unclear, and obtaining blow-up solutions seems quite complicated. This should not be unexpected, as, as specified, an increase in cells tends to diminish the chemical signal, and in turn, the attractive effect on the cell distribution disappears. As we will see in the next chapter, for $n = 1$ (and also $n = 2$), all solutions are global and bounded, as in the production model; conversely, for higher dimensions, the same holds (see [Tao11]) for sufficiently small values of the product of sensitivity χ and the maximum of $v_0(x)$ (and thus not the mass $m = \int_\Omega u_0(x)\,dx$). Specifically, the possible presence of blow-up phenomena remains unclear at this time.

Chapter 2

Focus of the thesis and its structure

The primary objective of this study is to analyze two distinct variants of the original chemotaxis model.

- The first variant, introduced in Section 2.1, is characterized by the presence of both an attractive signal and a repulsive signal, which naturally exerts an antagonistic effect. This model incorporates both linear and nonlinear formulations for diffusion and signal sensitivity. Furthermore, both signals are assumed to be produced by the cells themselves.
- The second variant, detailed in Section 2.2, considers solely the attraction mechanism; however, in this case, the signal is consumed by the cells. Particular emphasis will be placed on exploring both linear and nonlinear formulations of diffusion and signal sensitivity within this context.

2.1 The attraction-repulsion model with production

This is an example of an attraction–repulsion model with production:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) & \text{in } \Omega \times (0, T_{\max}), \\ \tau v_t = \Delta v - \beta v + f_1(u) & \text{in } \Omega \times (0, T_{\max}), \\ \tau w_t = \Delta w - \delta w + f_2(u) & \text{in } \Omega \times (0, T_{\max}), \end{cases} \tag{2.1.1}$$

defined in a bounded and smooth domain $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, $\chi, \xi, \beta, \delta > 0$, $\tau \in \{0, 1\}$, and some positive functions $f_1 = f_1(s)$ and $f_2 = f_2(s)$, sufficiently regular in their argument $s \geq 0$, together with suitably regular initial data $u_0(x), \tau v_0(x), \tau w_0(x) \geq 0$. Moreover, as in the previous cases, the domain is insulated by Neumann boundary conditions. In addition, $\tau \in \{0, 1\}$ distinguishes between the elliptic ($\tau = 0$) and parabolic ($\tau = 1$) cases.

In line with the celebrated Keller–Segel models, system (2.1.1) idealizes the natural mechanism in which the autonomous diffusion of the cells (Δu), initially distributed according to u_0, is perturbed by the aggregation/repulsion effects from the cross terms $\chi u \nabla v$ and $\xi u \nabla w$ (increasing for larger values of χ and ξ). In addition, the initial distribution τv_0 (respectively, τw_0) of the chemoattractant (respectively, chemorepellent) diffuses, and cells contribute to its production according to the law $f_1(u)$ (respectively, $f_2(u)$).

2.1.1 A view on the state of the art

In the context of the mentioned Keller–Segel model with single productive signal, problem (2.1.1) is a combination of this aggregative signal-production mechanism

$$u_t = \Delta u - \chi \nabla \cdot (u \nabla v) \quad \text{and} \quad \tau v_t = \Delta v - v + u, \quad \text{in } \Omega \times (0, T_{\max}), \tag{2.1.2}$$

and this repulsive signal-production one

$$u_t = \Delta u + \xi \nabla \cdot (u \nabla w) \quad \text{and} \quad \tau w_t = \Delta w - w + u, \quad \text{in } \Omega \times (0, T_{\max}). \tag{2.1.3}$$

Let us discuss some details on these problems and, since we will deal with classical solutions, we confine our attention to selected references; in this sense, we precise that the literature is richer.

From the one hand, if problem (2.1.2) is considered, since the attractive signal v increases with u, the inertial homogenization process of the cells' density could interrupt and very high and spatially concentrated spike formations (the already mentioned chemotactic collapse or blow-up) may appear; this is especially due to the size of χ, the initial mass of the particle distribution, i.e., $m = \int_\Omega u_0(x)\,dx$, and the space dimension. Indeed, if in the one-dimensional setting blow-up phenomena are excluded (see [OY01]), in higher dimensions if $m\chi$ surpasses a certain critical value m_χ, the system might present the aforementioned chemotactic collapse, whereas for $m\chi < m_\chi$ no instability appears in the motion of the cells. There are many contributions dedicated to understanding this scenario. In this regard, in [HV97, JL92, Nag01, Win10a] (and references therein cited), the interested reader can find pointers to the rich literature dealing with the existence and properties of global, uniformly bounded or blow-up (local) solutions to the Cauchy problem associated to (2.1.2). On the other hand, as far as nonlinear segregation chemotaxis models like those we are considering, when in problem (2.1.2) the production term $f_1(u) = u$ is replaced by $f_1(u) \cong u^k$, with $0 < k < \frac{2}{n}$ $(n \geq 1)$, uniform boundedness of all its solutions is proved in [LT16]. Moreover, by resorting to a simplified parabolic-elliptic version in spatially radial contexts, when the second equation is reduced to $0 = \Delta v - \frac{1}{|\Omega|}\int_\Omega f_1(u) + f_1(u)$, with $f_1(u) \cong u^k$, it is known (see [Win18a]) that the same conclusion on the boundedness continues to be valid for any $n \geq 1$ and $0 < k < \frac{2}{n}$, whereas for $k > \frac{2}{n}$ blow-up phenomena may occur.

The case $\tau = 0$

For linear growth of the chemoattractant and the chemorepellent, $f_1(u) = \alpha u$, $\alpha > 0$, and $f_2(u) = \gamma u$, $\gamma > 0$, we have that $\Theta := \chi\alpha - \xi\gamma$, value of the difference between the attraction and repulsion impacts, is such that whenever $\Theta < 0$ (repulsion-dominated regime), in any dimension all solutions to the model are globally bounded, whereas for $\Theta > 0$ (attraction-dominated regime) and $n = 2$ unbounded solutions can be detected (see [GJZ18, LL16b, TW13, Vig19, YGZ17] for some details on the issue). Indeed, for more general expressions of the production laws, respectively f and g generalizing the prototypes $f_1(u) = \alpha u^k$, $k > 0$, and $f_2(u) = \gamma u^l$, $l > 0$, as far as we know, the following are the most recent results, valid for $n \geq 2$ and derived in [Vig21]: For every $\alpha, \beta, \gamma, \delta, \chi > 0$, and $l > k \geq 1$ (resp. $k > l \geq 1$), there exists $\xi^* > 0$ (resp. $\xi_* > 0$) such that if $\xi > \xi^*$ (resp. $\xi \geq \xi_*$), any sufficiently regular initial datum $u_0(x) \geq 0$ (resp. $u_0(x) \geq 0$ small in some Lebesgue space) emanates a unique classical solution, uniformly bounded in time. Moreover, the same conclusion remains valid for any $\alpha, \beta, \gamma, \delta, \chi, \xi > 0$, any sufficiently regular $u_0(x) \geq 0$ if the hypotheses $0 < k < 1$ and $l = 1$ are accomplished.

The case $\tau = 1$

First, we point out that in the context of real applications, model (2.1.1) was introduced in [LCREKM03] for one-dimensional settings and linear proliferation, to describe the aggregation of microglia observed in Alzheimer's disease.

Moreover, from the stricter mathematical point of view, in [TW13] it is proved that in two-dimensional domains, for $f_1(u) = \alpha u$ and $f_2(u) = \gamma u$, sufficiently smooth initial data generate global-in-time bounded solutions whenever (recall $\Theta = \chi\alpha - \xi\gamma$)

$$\Theta < 0 \quad \text{and} \quad \beta = \delta \quad \text{or} \quad \Theta < 0 \quad \text{and} \quad -\frac{\chi^2\alpha^2(\beta-\delta)^2}{2\Theta\beta^2 C}\int_\Omega u_0(x) \leq 1, \quad \text{for some } C > 0.$$

The nonlocal case

If we take into account this nonlocal version of model (2.1.1)

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w) & \text{in } \Omega\times(0,T_{\max}),\\ 0 = \Delta v - \frac{1}{|\Omega|}\int_\Omega f_1(u) + f_1(u) & \text{in } \Omega\times(0,T_{\max}),\\ 0 = \Delta w - \frac{1}{|\Omega|}\int_\Omega f_2(u) + f_2(u) & \text{in } \Omega\times(0,T_{\max}),\\ \int_\Omega v(x,t)\,dx = \int_\Omega w(x,t)\,dx = 0 & t\in(0,T_{\max}), \end{cases} \tag{2.1.4}$$

in [LL21] it is established that if $k > l$ and $k > \frac{2}{n}$, there are initial data producing solutions blowing up at some finite time while, for any $l > 0$, if $0 < k < \frac{2}{n}$ such a phenomenon is prevented and all solutions are globally bounded in time.

Remark 2.1. *For the sake of clarity, in this model v represents the deviation of the chemical signal (rather than the chemical substance itself, as previously indicated). In particular, since the deviation measures the difference between the signal concentration and its mean, rigorously in the model we would substitute v with $\bar{v}$, being $\bar{v} = v - \frac{1}{|\Omega|}\int_\Omega v$. We avoid this relabeling, but we observe that conversely to what happens to the cell and chemical densities (which are nonnegative) the deviation changes sign. In particular, from its definition, it is immediately seen that its mean is zero (as fixed in the last assumptions of model* (2.1.4)*), which in turn ensures the uniqueness of the solution for the Poisson equation under homogeneous Neumann boundary conditions. Exactly the same explanation is valid for w.*

2.1.2 The role of the logistic source

When the first equation of the Keller–Segel model is replaced with

$$u_t = \nabla \cdot (D(u)\nabla u - S(u)\nabla v + T(u)\nabla w) + h(u), \tag{2.1.5}$$

for some $D(u), S(u), T(u)$ and $h(u)$, the overall mathematical analysis is in some sense more manageable. Indeed, for standard situations where $D(u) \simeq u^{m_1}, S(u) \simeq u^{m_2}, T(u) \simeq u^{m_3}$ and $h(u) \simeq \lambda u - \mu u^{\vartheta}$ (where $m_1, m_2, m_3, \lambda, \mu, \vartheta$ assume some real values), the technical machinery involving functional and algebraic inequalities is more inclined to work exactly due to the leeway given by those parameters.

In particular, the term $h(u)$ is called *logistic source*: it models the situation where the quantity of the cells increases or decreases in the domain over time. Intuitively, adding this term can result in the situation where cells are produced more than they are eliminated: this leads to an increase in cell mass and, consequently, to a chemotactic collapse over time. On the contrary, we are interested in the case where cells are eliminated more than they are produced: in this way, the decrease of biological material may prevent the occurrence of the blow-up phenomenon.

2.2 The only attraction model with consumption and production

2.2.1 The attraction model with consumption

The mathematical formulation of this model is given as the following problem:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) & \text{in } \Omega \times (0, T_{\max}), \\ v_t = \Delta v - uv & \text{in } \Omega \times (0, T_{\max}). \end{cases} \tag{2.2.1}$$

This system describes the situation where, due to the negative term $-uv$ in the second equation, the attractive signal is progressively consumed by the cells. Naturally, this suggests that v remains bounded over time.

The described phenomenon is quite different from the production case: for system (2.2.1), it has been shown that the existence of global or blow-up solutions is independent of the size if initial data u_0. Specifically, Tao ([Tao11]) proves that for sufficiently regular initial data u_0 and v_0, if

$$0 < \chi \|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{6(n+1)}, \tag{2.2.2}$$

then problem (2.2.1) possesses a unique global classical solution that remains uniformly bounded. (See [BK17] for an improvement of this condition.) The question of whether blow-up solutions exist for large initial data v_0 or large chemotactic parameter χ, which do not satisfy (2.2.2), remains open.

Further progress on the chemotaxis-consumption model (2.2.1) has been made. In [TW12b], weak solutions that become smooth after some time are constructed for sufficiently regular initial data in the three-dimensional setting. For $n = 2$, globally bounded classical solutions with convergence properties are also shown to exist.

Moreover, by interpreting the system as a special case of the general coupled chemotaxis-fluid model proposed by Goldstein in [TCD+05], where the fluid does not contribute directly, [Win12] discusses the existence of global classical and weak solutions for $n = 2$ and $n = 3$, respectively. In addition, [Win14] investigates the stabilization properties of the two-dimensional solutions. Further results on existence and asymptotic behavior of solutions for the three-dimensional version of the coupled system are provided in [Win16a] and [Win17].

The logistic degradation in the consumption model

In order to contrast the undesired blow-up singularities, more complete formulations of these systems with nontrivial sources have been considered. For instance, let us consider the system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + h(u) & \text{in } \Omega \times (0, T_{\max}), \\ \tau v_t = \Delta v - v + u & \text{in } \Omega \times (0, T_{\max}), \end{cases}$$

endowed with homogeneous Neumann boundary conditions and sufficiently regular initial data, and perturbed by a logistic-type source term, $h(u) \simeq \lambda u - \mu u^\gamma$, with $\lambda \in \mathbb{R}$ and $\mu > 0$ as constants.

In [Lan15], the existence of global weak solutions is established for the case $\chi = \tau = 1$ and $\gamma = 2$, for any arbitrarily small value of $\mu > 0$. Moreover, if $n = 3$, these solutions become classical after some time, provided that λ is not too large. On the other hand, for $\mu, \tau > 0$, $\gamma = 2$, and $\chi > 0$, the author in [Win10b] proves that if μ is sufficiently large, the same system possesses a unique bounded and global-in-time classical solution.

Finally, for $\gamma > 1$, $\tau = 1$, and $\chi > 0$, the global existence of very weak solutions, as well as their boundedness properties and long-time behavior, are discussed in [Vig16], [Vig17], and [VW17].

As explicitly derived in [BBNS10], generalized kinetic theory methods can be intrinsically employed to obtain the continuous models presented here from the underlying description at the scale of cells. In this sense, the evolution of u, which has so far been considered (and idealizes a diffusion with infinite speed of propagation, which is definitely not lifelike), is a special case of the equation

$$u_t = \nabla \cdot (D(u) \nabla u - S(u, v) \nabla v) \quad \text{in } \Omega \times (0, T_{\max}).$$

This latter formulation includes a very general framework where the diffusion D and sensitivity S may be described by different expressions. For instance, some special cases with singular sensitivity are noteworthy. Specifically, in the one-dimensional setting, if $D(u)$ is a positive constant and $S(u, v) = \frac{\chi u}{v}$, results on global existence for initial-boundary value problems tied to (2.2.1) with arbitrary initial data have been derived in [TWW13]. In higher dimensions, the analysis is more complex, and only partial results are known (see [Win16b] and [Win18c] for interesting achievements involving generalized and renormalized solutions in bounded domains of $\mathbb{R}^n$, $n \geq 2$).

Additionally, in [Lan17] and [Liu18], with the respective choices $D(u, v) \geq \delta u^{m_1 - 1}$, $\delta > 0$, and $S(u, v) = \frac{u}{v}$, and $D(u, v) \equiv 1$ and $S(u, v) \simeq \frac{\lambda u (u+1)^{m_2 - 1}}{v}$, $\lambda > 0$, for appropriate values of m_1 and m_2 (depending on the dimension n), global classical solutions have been studied.

To the best of our knowledge, significant results in the direction of this investigation, tied to (2.2.1) under a perturbation for the evolution of u through a logistic source, are as follows. In a bounded and smooth domain of $\mathbb{R}^n$, $n \geq 1$, and under homogeneous Neumann boundary conditions, for the system

$$\begin{cases} u_t = \nabla \cdot (D(u) \nabla u - S(u) \nabla v) + h(u) & \text{in } \Omega \times (0, T_{\max}), \\ v_t = \Delta v - uv & \text{in } \Omega \times (0, T_{\max}), \end{cases} \tag{2.2.3}$$

these results have been achieved (the list is not exhaustive):

i) For $D(u) = \delta > 0$, $S(u) = \chi u$ ($\chi > 0$), and $h(u) \leq \lambda u - \mu u^\gamma$ (with $h(0) \geq 0$, $\lambda \geq 0$, $\mu > 0$, and $\gamma > 1$), and sufficiently regular initial data (u_0, v_0), the system admits a unique global and bounded classical solution for suitable small $\chi \|v_0\|_{L^\infty(\Omega)}$ (see [BK16]).

ii) For $D(u) \equiv 1$, $S(u) = \chi u$ ($\chi > 0$), $\lambda \in \mathbb{R}$, $h(u) = \lambda u - \mu u^2$, and sufficiently regular initial data (u_0, v_0), the system admits a unique global and bounded classical solution for μ large enough compared to $\chi \|v_0\|_{L^\infty(\Omega)}$, and a weak solution for arbitrary $\mu > 0$ (see [LW17]).

iii) For $D(u) \equiv (u+1)^{m_1-1}$, $S(u) = \chi u(u+1)^{m_2-1}$ ($\chi > 0$), $m_1, m_2, \lambda \in \mathbb{R}$, $h(u) = \lambda u - \mu u^2$, it is proved that for nonnegative and sufficiently regular initial data (u_0, v_0), the corresponding initial-boundary value problem admits a unique globally bounded classical solution, provided $m_2 < \frac{m_1+1}{2}$ and μ is larger than a quantity depending on $\chi\|v_0\|_{L^\infty(\Omega)}$ (see [MV18]).

2.2.2 Indirect chemotaxis consumption model

The tumor–immune cell interactions chemotaxis system can be interpreted within the context of predator-prey models featuring indirect prey-taxis. In this model, lymphocytes (the predators) move toward gradients of chemical released by cancer cells (the prey), rather than directly towards the higher densities of cancer cells themselves. Tello and Wrzosek [TW16] were the first to introduce what are known as indirect prey-taxis models, where the movement of prey is not considered. They examined the global bounded solution for $n \geq 1$ and investigated the asymptotic behavior for $n \leq 2$. Later, Wang et al. [WW20] examined the global existence and asymptotic behavior of the following predator-prey system with indirect prey taxis

$$\begin{cases} u_t = d_1\Delta u - \nabla\cdot(\chi(w)u\nabla w) + bug(v) - uh(u), \\ w_t = d_2\Delta w - \mu w + rv, \\ v_t = d_3\Delta v + f(v) - ug(v). \end{cases} \tag{2.2.4}$$

and they provided the convergence rates of the solution. In the presence of nonlinear diffusion and chemosensitivity in (2.2.4), the existence of a classical solution was established by Xing et al. [XZP21] under certain additional parameter assumptions. The authors also demonstrated the asymptotic behavior of the solution under various predation conditions.

In contrast to the previous prey–taxis system, Ahn and Yoon [AY21] analyzed a similar system with indirect predator–taxis. They considered the following system

$$\begin{cases} u_t = d_1\Delta u - \nabla\cdot(\chi u\nabla c) + f_1(u) - vg(u,v), \\ c_t = d_2\Delta c + \alpha v - \beta c, \\ v_t = d_3\Delta v + rvg(u,v) - k(v)v, \end{cases} \tag{2.2.5}$$

and provided results on the existence of solutions under the assumption of quadratic decay of predator for $n \leq 2$. Additionally, by constructing an appropriate Lyapunov functional, they demonstrated stability results and showed that a large chemosensitivity can lead to the occurrence of pattern formations via linear stability analysis.

Immune chemotaxis models offer valuable insights into immune cell migration mechanisms, enhancing our understanding of immune responses, inflammation, and related diseases. These models have diverse applications in immunology research, drug development, and immunotherapy. Lymphocytes are one of the major immune cells, which play a vital role in the immune system. They are produced from bone marrow stem cells and are present in blood and lymph tissues. Lymphocytes work together with other immune cells to protect the body against various microorganisms. They include B cells, which produce antibodies to fight foreign microorganisms, and T cells, which regulate immune responses. Most of the T cells require assistance from another immune cell to be activated. Activated T cells, such as Natural Killer cells (NK cells), that specialize in destroying cancer cells and virus-infected cells. The dynamics of the above phenomena are mathematically described by Hu and Tao [HT20] with the help of nonlinear parabolic partial differential equations inspired by [MRF19]

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v), \\ v_t = \Delta v + w - v - uv, \\ w_t = \Delta w - uw + w(1-w). \end{cases} \tag{2.2.6}$$

The authors investigated well-posedness, solvability and properties of classical solutions to system (2.2.6) in the case where Ω is a two-dimensional domain. Specifically, they showed that for any $\chi > 0$ and sufficiently regular initial data, the solutions are globally bounded. Moreover, if χ is small enough and $\frac{1}{|\Omega|}\int_\Omega u_0 < 1$, then the solution converges to nontrivial equilibrium points as $t \to \infty$. On the other hand, if $\frac{1}{|\Omega|}\int_\Omega u_0 \geq 1$, the solution converges to the semi-trivial equilibrium point.

2.3 Gradient-dependent sources $h = h(u, \nabla u)$

Recently, there has been an increasing interest in the study of chemotactic systems incorporating gradient-dependent source terms. These investigations aim to elucidate how external sources, modulated by the gradient of chemical concentrations, influence the movement and interactions of cellular populations. As noted in [Sou96], the evolution of a species within a habitat is governed not only by birth and natural death processes but also by accidental deaths, which can be modeled by the term $-|\nabla u|^\gamma$. Within the existing literature, including works such as [CW89, Fil91, KP89], there are significant results addressing gradient nonlinearities in semilinear heat equations subject to Dirichlet boundary conditions, particularly with respect to conditions that guarantee or prevent blow-up phenomena. Specifically, the source term

$$h = h(u, \nabla u) = \lambda u^\rho - \mu u^k - c|\nabla u|^\gamma \tag{2.3.1}$$

represents a reaction exhibiting a dual dissipative effect: one arising from the zero-order term u^k, and the other from the first-order gradient term $|\nabla u|^\gamma$. Furthermore, in [ILV24], where the authors relate the model to real biological and ecological mechanisms, a production chemotaxis problem with a gradient-dependent logistic source is analyzed, yielding a boundedness criterion for the solution dependent on the parameter γ.

2.4 Aims of the investigation and structure of the thesis

By considering all that has been written previously and taking a look at the most general model for the chemotaxis phenomenon

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u - S(u)\nabla v + T(u)\nabla w) + h(u, \nabla u) & \text{in } \Omega \times (0, T_{\max}), \\ \tau v_t = \Delta v + E(u, v), & \text{in } \Omega \times (0, T_{\max}), \\ \tau w_t = \Delta w + G(u, w), & \text{in } \Omega \times (0, T_{\max}), \\ u_\nu = v_\nu = w_\nu = 0, & \text{on } \partial\Omega \times (0, T_{\max}), \\ u(x, 0) = u_0(x),\ \tau v(x, 0) = \tau v_0(x), \tau w(x, 0) = \tau w_0(x) & x \in \Omega, \end{cases}$$

we now present the overall structure of this thesis along with its primary objectives.
Chapter 3 compiles a selection of classical and well-established results that will be instrumental in supporting the arguments and proofs throughout the thesis.
Chapter 4 is devoted to fundamental aspects common to all the studies presented herein, namely:

- the existence and uniqueness of solutions;
- extension and dichotomy criteria;
- the boundedness of the total biological mass;
- the boundedness criterion;

We also address analytical techniques for deriving lower bounds on the blow-up time and briefly introduce the concept of numerical simulations.
In Chapter 5 we investigate the attraction-repulsion chemotaxis model with signal production. More specifically:

- Section 5.1 focuses on establishing conditions for the boundedness of solutions under the assumptions $D(u) = 1$, $S(u) = \chi u$, $T(u) = \xi u$, and $h(u, \nabla u) = 0$. For the second and third equations, nonlinear signal production terms of the form $E(u, v) = -v + u^k$ and $G(u, w) = -v + u^l$ are considered.
- Section 5.2 addresses the derivation of conditions guaranteeing the unboundedness of solutions in certain $L^p(\Omega)$-spaces, assuming unboundedness in the $L^\infty(\Omega)$-norm. The model incorporates laws given by $D(u) = (u+1)^{m_1-1}$, $S(u) = \chi u(u+1)^{m_2-1}$, $T(u) = \xi u(u+1)^{m_3-1}$, and $h(u, \nabla u) = \lambda u - \mu u^k$, with nonlinear signal production as in the previous section.

- Section 5.3 is dedicated to identifying conditions ensuring boundedness of solutions where the diffusion, sensitivity, and signal production laws are nonlinear, specifically $D(u) = (u+1)^{m_1-1}$, $S(u) = \chi u(u+1)^{m_2-1}$, $T(u) = \xi u(u+1)^{m_3-1}$, and $h(u, \nabla u) = \lambda u - \mu u^r$.
- Section 5.4 generalizes the results presented in [ILV24] by incorporating nonlinear laws for $D(u), S(u)$, and $T(u)$, as well as introducing a gradient-dependent nonlinearity in the logistic term described in (2.3.1). The signal production laws considered are also nonlinear.

Chapter 6 focuses on the purely attractive version of the system, in which the signal is consumed by the cells (i.e., $E(u, v) = -uv$). This chapter is organized as follows:

- Section 6.1 considers the case $D(u) = 1$, $S(u) = \chi u$, and $h(u, \nabla u)$ as in (2.3.1). The objective is to establish boundedness criteria depending on the parameter γ and, where applicable, on additional parameters such as $\|u_0\|_{L^1(\Omega)}$, $\|v_0\|_{L^\infty(\Omega)}$, μ, χ, and c.
- Section 6.2 pursues similar goals as the previous section, but with nonlinear laws

  $$D(u) = (u+1)^{m_1-1}, \quad S(u) = \chi u(u+1)^{m_2-1},$$

  and the same form of $h(u, \nabla u)$.
- Section 6.3 examines an indirect attraction model as described by (2.2.6). The main goals are to derive boundedness conditions for solutions and to establish lower bounds for blow-up times.

In the following pages, for each work examined in this thesis, a table is provided containing: a concise abstract, a reference to the corresponding publication, and an indication of the section in which the work is discussed.

Section 5.1	Related paper: [CFV23b]
Some refined criteria toward boundedness in an attraction-repulsion chemotaxis system with nonlinear productions	
Appl. Anal., (2023)	

We study some zero-flux attraction-repulsion chemotaxis models, with nonlinear production rates for the chemorepellent and the chemoattractant, whose formulation can be schematized as:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) & \text{in } \Omega \times (0, T_{\max}), \\ \tau v_t = \Delta v - \varphi(t, v) + f_1(u) & \text{in } \Omega \times (0, T_{\max}), \\ \tau w = \Delta w - \psi(t, w) + f_2(u) & \text{in } \Omega \times (0, T_{\max}). \end{cases}$$

In this problem, $f_1(u)$, $f_2(u)$ are reasonably regular functions generalizing, respectively, the prototypes $f_1(u) = \alpha u^k$ and $f_2(u) = \gamma u^l$, for some $k, l, \alpha, \gamma > 0$ and all $u \geq 0$. Moreover, $\varphi(t, v)$ and $\psi(t, w)$ have specific expressions, $\tau \in \{0, 1\}$ and $\Theta := \chi\alpha - \xi\gamma$. We establish that u, v and w are uniformly bounded in $\Omega \times (0, \infty)$ in the following cases:

I) For $\varphi(t, v) = \beta v$, $\beta > 0$, $\psi(t, w) = \delta w$, $\delta > 0$ and $\tau = 0$, provided

- I.1) $k < l$;
- I.2) $k, l \in \left(0, \frac{2}{n}\right)$;
- I.3) $k = l$ and $\Theta < 0$, or $l = k \in \left(0, \frac{2}{n}\right)$ and $\Theta \geq 0$.

II) For $\varphi(t, v) = \beta v$, $\beta > 0$, $\psi(t, w) = \delta w$, $\delta > 0$ and $\tau = 1$, whenever

- II.1) $l, k \in \left(0, \frac{1}{n}\right]$;
- II.2) $l, k \in \left(\frac{1}{n}, \frac{1}{n} + \frac{2}{n^2+4}\right)$.
- II.3) $l \in \left(\frac{1}{n}, \frac{1}{n} + \frac{2}{n^2+4}\right)$ and $k \in \left(0, \frac{1}{n}\right]$, or $k \in \left(\frac{1}{n}, \frac{1}{n} + \frac{2}{n^2+4}\right)$ and $l \in \left(0, \frac{1}{n}\right]$;

III) For $\varphi(t, v) = \frac{1}{|\Omega|} \int_\Omega f_1(u)$ and $\psi(t, w) = \frac{1}{|\Omega|} \int_\Omega f_2(u)$ and $\tau = 0$, under the assumptions $k < l$ or I.3).

In particular, in this paper we partially improve what derived in [Vig21] and solve an open question given in [LL21]. Finally, the research is complemented with numerical simulations in bi-dimensional domains.

Section 5.2	Related paper: [CFV23a]
Properties of given unbounded solutions to a class of chemotaxis models	
Stud. Appl. Math., (2023)	

We analyze a zero-flux chemotaxis system essentially described as

$$
\begin{cases}
u_t = \nabla \cdot ((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v \\
\qquad +\xi u(u+1)^{m_3-1}\nabla w) + \lambda u - \mu u^k & \text{in } \Omega \times (0, T_{\max}), \\
0 = \Delta v - m_1(t) + u^\alpha & \text{in } \Omega \times (0, T_{\max}), \\
0 = \Delta w - m_2(t) + u^\beta & \text{in } \Omega \times (0, T_{\max}).
\end{cases}
$$

Under specific relations involving the above parameters, one of these always requiring some largeness conditions on $m_2 + \alpha$,

i) we prove that any given solution, blowing-up at some finite time $T_{\max}$, becomes unbounded also in $L^{\mathfrak{p}}(\Omega)$-norm, for all $\mathfrak{p} > \frac{n}{2}(m_2 - m_1 + \alpha)$;

ii) we give lower bounds T (depending on $\int_\Omega u_0^{\bar{p}}$) of $T_{\max}$ for the aforementioned solutions in some $L^{\bar{p}}(\Omega)$-norm, being $\bar{p} = \bar{p}(n, m_1, m_2, m_3, \alpha, \beta) \geq \mathfrak{p}$.

Section 5.3	Related paper: [CDFF24]
Uniform-in-time boundedness in a class of local and nonlocal nonlinear attraction-repulsion chemotaxis models with logistics	
Nonlinear Anal. Real World Appl., (2024)	

The following fully nonlinear attraction-repulsion and zero-flux chemotaxis model is studied:

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v \\ \qquad +\xi u(u+1)^{m_3-1}\nabla w\big) + \lambda u - \mu u^r & \text{in } \Omega \times (0, T_{\max}), \\ \tau v_t = \Delta v - \phi(t,v) + f_1(u) & \text{in } \Omega \times (0, T_{\max}), \\ \tau w_t = \Delta w - \psi(t,w) + f_2(u) & \text{in } \Omega \times (0, T_{\max}). \end{cases}$$

herein λ, μ, r proper positive numbers, $m_1, m_2, m_3 \in \mathbb{R}$, and $f_1(u)$ and $f_2(u)$ regular functions that generalize the prototypes $f_1(u) \simeq u^k$ and $f_2(u) \simeq u^l$, for some $k, l > 0$ and all $u \geq 0$. We are interested in deriving sufficient conditions implying globality (i.e., $T_{\max} = \infty$) and boundedness (i.e., $\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \leq C$ for all $t \in (0,\infty)$) of solutions. This is achieved in the following scenarios:

- For $\phi(t,v)$ proportional to v and $\psi(t,w)$ to w, whenever $\tau = 0$ and provided one of the following conditions

  (I) $m_2 + k < m_3 + l$,

  (II) $m_2 + k < r$,

  (III) $m_2 + k < m_1 + \frac{2}{n}$

  is accomplished or $\tau = 1$ in conjunction with one of these restrictions

  (i) $\max\{m_2 + k, m_3 + l\} < r$,

  (ii) $\max\{m_2 + k, m_3 + l\} < m_1 + \frac{2}{n}$,

  (iii) $m_2 + k < r$ and $m_3 + l < m_1 + \frac{2}{n}$,

  (iv) $m_2 + k < m_1 + \frac{2}{n}$ and $m_3 + l < r$;

- For $\phi(t,v) = \frac{1}{|\Omega|}\int_\Omega f_1(u)$ and $\psi(t,w) = \frac{1}{|\Omega|}\int_\Omega f_2(u)$, whenever $\tau = 0$ if moreover one among (I), (II), (III) is fulfilled.

This research partially improves and extends some results derived in Section (5.1) and in [JJL23, RL20, CY22].

Section 5.4	Related paper: [LASCV25]
Dissipative gradient nonlinearities prevent δ-formations in local and nonlocal attraction-repulsion chemotaxis models	
Stud. Appl. Math., (2025)	

We study a class of zero-flux attraction-repulsion chemotaxis models, characterized by nonlinearities laws for the diffusion of the cell density u, the chemosensitivities and the production rates of the chemoattractant v and the chemorepellent w. Additionally, a source involving also the gradient of u is incorporated. The related mathematical formulation reads as:

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v \\ \qquad +\xi u(u+1)^{m_3-1}\nabla w\big) + \lambda u^\rho - \mu u^k - c|\nabla u|^\gamma & \text{in } \Omega \times (0, T_{\max}), \\ \mathscr{P}(v,u) = 0 & \text{in } \Omega \times (0, T_{\max}), \\ \mathscr{Q}(w,u) = 0 & \text{in } \Omega \times (0, T_{\max}). \end{cases}$$

Herein $\rho, k, \gamma, \lambda, \mu, c$ are proper positive numbers and $m_1, m_2, m_3 \in \mathbb{R}$. Depending on the expressions of $\mathscr{P}(v,u)$ and $\mathscr{Q}(w,u)$, we will deal with issues tied to *local* or *nonlocal* models. Our overall study touches two different aspects: we derive sufficient conditions so to ensure boundedness of solutions and finally we develop numerical simulations giving insights on the evolution of the system. In the specific, for $\tau \in \{0,1\}$, and $f_1(u)$ and $f_2(u)$ reasonably regular functions generalizing, respectively, the prototypes $f_1(u) = u^\alpha$ and $f_2(u) = u^\beta$, for some $\alpha, \beta > 0$ and all $u \geq 0$, by defining $\Theta_\tau = \max\left\{1, \frac{n}{n+1}(m_2+\alpha), \tau\frac{n}{n+1}(m_3+\beta)\right\}$ this will be established:

(I) For $\mathscr{P}(v,u) = P^\tau_\alpha(v,u) := \tau v_t - \Delta v + v - f_1(u)$, $\mathscr{Q}(w,u) = Q^\tau_\beta(w,u) := \tau w_t - \Delta w + w - f_2(u)$, for $1 \leq \rho < k$ and $\Theta_\tau < \gamma \leq 2$, the system is globally solvable (i.e., $T_{\max} = \infty$) and the solution is bounded.

(II) For $\mathscr{P}(v,u) = P_\alpha(v,u) := \Delta v - \frac{1}{|\Omega|}\int_\Omega f_1(u) + f_1(u)$ and $\mathscr{Q}(w,u) = Q_\beta(w,u) := \Delta w - \frac{1}{|\Omega|}\int_\Omega f_2(u) + f_2(u)$, for $\tau = 0$, $1 \leq \rho < k$ and $\Theta_\tau < \gamma \leq 2$ the same conclusion as in (I) applies.

(III) Numerical simulations show that the suppression of the condition $\Theta_\tau < \gamma \leq 2$ may lead to only local solutions (i.e., $T_{\max}$ finite) which precisely blow-up at $T_{\max}$.

This work, inter alia, generalizes the result in [ILV24], where the linear and only attraction version of the model is addressed.

Section 6.1	Related paper: [Col25]
Boundedness criteria for a chemotaxis consumption model with gradient nonlinearities	
J. Math. Anal. Appl., (2025)	

This work deals with the consumption chemotaxis problem

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot u\nabla v + \lambda u - \mu u^2 - c|\nabla u|^\gamma, & \text{in } \Omega \times (0, T_{\max}),\\ v_t = \Delta v - uv, & \text{in } \Omega \times (0, T_{\max}),\end{cases}$$

for $\chi, \lambda, \mu, c > 0$, $T_{\max} \in (0, \infty]$ and for u_0, v_0 positive initial data with a certain regularity. We will show that the problem has a unique and uniformly bounded classical solution for $\gamma \in \left(\frac{2n}{n+1}, 2\right]$. Moreover, we have the same result for $\gamma = \frac{2n}{n+1}$ and a condition that involves the parameters c, μ, n, χ and the initial data.

Section 6.2	Related paper: [CV25]
Boundedness in a nonlinear chemotaxis-consumption model with gradient terms	
To apper in SEMA-SIMAI SPRINGER SERIES volume "Analysis, approximation and control of chemotaxis models"	

We study a chemotaxis-consumption mechanism described by

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v\big) & \\ \qquad +\lambda u - \mu u^2 - c|\nabla u|^\gamma, & \text{in } \Omega \times (0, T_{\max}),\\ v_t = \Delta v - uv, & \text{in } \Omega \times (0, T_{\max}).\end{cases}$$

In order to control the concentration of such a population, sources involving gradient nonlinearities, which introduce a dampening effect on the model, are considered. We derive conditions on some data of the problem so to ensure the boundedness of related solutions. This work extends the research presented in [MV18] and in Section 6.1, where the same nonlinear model without gradient terms and its linear version with gradient sources has been, respectively, addressed.

Section 6.3	Related paper: [GCDFN24]
Global existence and lower bounds in a class of tumor-immune cell interactions chemotaxis systems	
Discrete Contin. Dyn. Syst. Ser. S., (2024)	

This work investigates the properties of classical solutions to this class of chemotaxis systems

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (u\nabla v), & \text{in } \Omega \times (0, T_{\max}),\\ \tau v_t = \Delta v + \alpha w - \beta v - \gamma uv, & \text{in } \Omega \times (0, T_{\max}),\\ \tau w_t = \Delta w - \delta uw + \mu w(1-w), & \text{in } \Omega \times (0, T_{\max}),\end{cases}$$

with $\tau \in \{0, 1\}$, which models interactions between tumor and immune cells. Our focus is on examining the global existence and explosion of such solutions in bounded domains of $\mathbb{R}^n$, $n \geq 3$, and under Neumann boundary conditions. We distinguish between two scenarios: one where all equations are parabolic and another where only one equation is parabolic while the rest are elliptic. Boundedness is demonstrated under smallness assumptions on the initial data in the former scenario, while no such constraints are necessary in the latter.

Chapter 3

Preliminaries and necessary tools

In this chapter, we present a comprehensive collection of fundamental algebraic and functional inequalities that play a crucial role in the analytical framework of this thesis. We also review key theoretical results on the properties of solutions to elliptic and parabolic partial differential equations. Both these aspects provide the mathematical foundation for the chemotaxis models studied in the subsequent chapters.

In case it will be not directly mentioned, unless otherwise specified, from this point onward all constants appearing below are assumed to be positive. Moreover, we shall introduce a sequence of further positive constants denoted by c_i, $i = 1, 2, \dots$, whose indexing will be reset at the beginning of each section.

3.1 Algebraic inequalities

We begin with the first topic, concerning fundamental algebraic inequalities, which will be repeatedly used in the analysis that follows.

Proposition 3.1 (Young's inequality)**.** *Let $a, b \geq 0$ and $\alpha \in (0,1)$. Then, for every $\epsilon > 0$ there exists a positive constant $C_Y = C_Y(\epsilon, \alpha)$ such that:*

$$a^\alpha b^{1-\alpha} \leq \epsilon a + C_Y b. \tag{3.1.1}$$

Proof. Let us consider the function $f(x) = e^x$. It is straightforward to see that $f''(x) > 0$ for all x. This implies, for every $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab = e^{\ln(a)}e^{\ln(b)} = e^{\frac{1}{p}\ln(a^p)+\frac{1}{q}\ln(b^q)} \leq \frac{1}{p}e^{\ln(a^p)} + \frac{1}{q}e^{\ln(b^q)} = \frac{a^p}{p} + \frac{b^q}{q}.$$

We now set $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$, so to have

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b. \tag{3.1.2}$$

Consequently

$$a^\alpha b^{1-\alpha} = \frac{\epsilon^\alpha a^\alpha}{\alpha^\alpha}\frac{\alpha^\alpha b^{1-\alpha}}{\epsilon^\alpha} = \left(\frac{\epsilon a}{\alpha}\right)^\alpha \left(\frac{\alpha^{\frac{\alpha}{1-\alpha}} b}{\epsilon^{\frac{\alpha}{1-\alpha}}}\right)^{1-\alpha} \leq \epsilon a + \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{\epsilon^{\frac{\alpha}{1-\alpha}}} b,$$

where, in the last step, we used (3.1.2), finding

$$C_Y(\epsilon, \alpha) = \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{\epsilon^{\frac{\alpha}{1-\alpha}}}. \tag{3.1.3}$$

□

Lemma 3.2. *Let $A, B \geq 0$, $d_1, d_2 > 0$ and $s > 0$. Then for some $d, d_3 > 0$ we have*

$$A^{d_1} + B^{d_2} \geq 2^{-d}(A+B)^d - d_3, \tag{3.1.4}$$

and

$$(A+B)^s \leq \max\{1, 2^{s-1}\}(A^s + B^s). \tag{3.1.5}$$

Moreover, let $d_4, d_5 > 0$ be such that $d_4 + d_5 < 1$. Then for all $\epsilon > 0$ there exists a positive constant d_4 such that

$$A^{d_4} B^{d_5} \leq \epsilon(A+B) + d_4. \tag{3.1.6}$$

Proof. The proofs are available, respectively, in [MV18, Lemma 3.3], [Jam14, Theorem 1] and [FV21, Lemma 4.3]. □

Lemma 3.3. *Let y be a positive real number verifying $y \leq k(y^l + 1)$ for some $k > 0$ and $0 < l < 1$. Then $y \leq \max\left\{1, (2k)^{\frac{1}{1-l}}\right\}$.*

Proof. For $y < 1$ is trivial. If $y \geq 1$ then $y^l \geq 1$ and, subsequently, $y \leq k(y^l + 1) \leq 2ky^l$. So we have the thesis. □

3.2 Functional inequalities

In this section, we continue by presenting a collection of functional inequalities that will play a crucial role in the calculations of the subsequent chapters.

Proposition 3.4 (Hölder's inequality)**.** *Let $1 \leq p \leq q \leq +\infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let also $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}. \tag{3.2.1}$$

Proof. See [Bre11, 92]. □

Proposition 3.5 (Gagliardo–Nirenberg inequality)**.** *Let $n \in \mathbb{N}$, $0 \leq p, q, r \leq +\infty$ and $\theta \in [0,1]$ such that*

$$\frac{1}{p} = \left(\frac{1}{r} - \frac{1}{N}\right)\theta + \frac{1-\theta}{q} \quad \textit{and} \quad 0 \leq \theta \leq 1.$$

Let $\Omega \subset \mathbb{R}^n$, with Ω smooth, $n \geq 1$, and $f\colon \Omega \to \mathbb{R}$ such that $f \in L^q(\Omega)$ and its gradient is contained in $L^r(\Omega)$. Then, for every $s > 0$ there exists a positive constant C_{GN} such that

$$\|f\|_{L^p(\Omega)} \leq C_{GN}\Big(\|\nabla f\|^{\theta}_{L^r(\Omega)} \|f\|^{1-\theta}_{L^q(\Omega)} + \|f\|_{L^s(\Omega)}\Big). \tag{3.2.2}$$

Proof. See [Nir59]. □

Lemma 3.6. *Let $q \in [1, \infty)$. Then for any $f \in C^2(\bar{\Omega})$ satisfying $f\frac{\partial f}{\partial \nu} = 0$ on $\partial\Omega$, the following inequality holds*

$$\|\nabla f\|^{2q+2}_{L^{2q+2}(\Omega)} \leq 2(4q^2 + n)\|f\|^2_{L^\infty(\Omega)} \big\||\nabla f|^{q-1} D^2 f\big\|^2_{L^2(\Omega)}. \tag{3.2.3}$$

Proof. See [LW17]. □

Lemma 3.7. *Let Ω be a bounded convex domain in $\mathbb{R}^3$. Then for any nonnegative $f \in C^1(\bar{\Omega})$ and for every $\varepsilon > 0$ it holds*

$$\int_\Omega f^3 \leq A_1 \left(\int_\Omega f^2\right)^{\frac{3}{2}} + \frac{A_2}{\varepsilon^3}\left(\int_\Omega f^2\right)^3 + A_3 \varepsilon \int_\Omega |\nabla f|^2, \tag{3.2.4}$$

where

$$A_1 = \frac{3^{\frac{3}{2}}}{2\rho^{\frac{3}{2}}}, \quad A_2 = \frac{3^3}{4^{\frac{15}{4}}}\left(1 + \frac{d}{\rho}\right)^{\frac{3}{2}}, \quad A_3 = \sqrt{2}\left(1 + \frac{d}{\rho}\right)^{\frac{3}{2}},$$

and

$$\rho := \min_{\partial\Omega} x \cdot \nu > 0, \qquad d := \max_{\partial\Omega} |x|.$$

Proof. The proof is a combination of the result in [PPVP10, Lemma A.2] and known inequalities. (See details in [Vig19]). □

3.3 Properties of solutions to elliptic and parabolic equations

We now turn to the study of the properties of solutions to elliptic and parabolic partial differential equations, presenting the relevant results in a form tailored to our specific objectives, rather than in their most classical or general formulations.

3.3.1 Comparison principles

Given $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ a bounded and smooth domain and $T > 0$, for $i, j = 1, \dots, n$ let $a_{ij} = a_{ij}(x,t) = a_{ji}(x,t) = a_{ji}, b_i = b_i(x,t), c = c(x,t)$ belonging to $C^1(\bar{\Omega} \times [0,T])$ and such that there exists a constant $\theta > 0$ complying with

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2 \text{ for all } (x,t) \in \Omega \times [0,T] \text{ and for all } \xi \in \mathbb{R}^n. \tag{3.3.1}$$

Subsequently, for $z = z(x,t)$, with $(x,t) \in \bar{\Omega} \times (0,T)$, let us define the operators

$$Lz = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial z}{\partial x_j}\right) + \sum_{i=1}^{n} b_i \frac{\partial z}{\partial x_i} + cz = -\nabla \cdot (A\nabla z) + b \cdot \nabla z + cz, \tag{3.3.2}$$

and

$$\left(\frac{\partial}{\partial t} + L\right) z = z_t + Lz,$$

being $A = A(x,t) = [a_{ij}(x,t)]$ a matrix-valued function obeying the uniformly ellipticity condition (3.3.1), and $b = b(x,t) = [b_i(x,t)]$ a vector-valued function.

Under the above preparations, we have these results.

Proposition 3.8 (Elliptic comparison principle)**.** *For $c \geq 0$, let*

$$z, w \in C^0\left(\overline{\Omega} \times [0,T]\right) \cap C^{2,1}\left(\bar{\Omega} \times (0,T)\right)$$

such that

$$\begin{cases} Lz \leq Lw & \text{in } \Omega \times (0,T), \\ A\nabla z \cdot \nu = A\nabla w \cdot \nu = 0 & \text{on } \partial\Omega \times (0,T), \\ z(x,0) \leq w(x,0). \end{cases}$$

Then $z(x,t) \leq w(x,t)$ for all $(x,t) \in \bar{\Omega} \times [0,T]$.

Proof. See [Eva10, §6.4.1] or [CIL92, Theorem 7.9], supported by Hopf's Lemma [Fri58, Theorem 1]. ☐

Similarly we have this

Proposition 3.9 (Parabolic comparison principle)**.** *Let*

$$z, w \in C^0\left(\overline{\Omega} \times [0,T]\right) \cap C^{2,1}\left(\bar{\Omega} \times (0,T)\right)$$

such that

$$\begin{cases} z_t + Lz \leq w_t + Lw & \text{in } \Omega \times (0,T), \\ A\nabla z \cdot \nu = A\nabla w \cdot \nu = 0 & \text{on } \partial\Omega \times (0,T), \\ z(x,0) \leq w(x,0). \end{cases}$$

Then $z(x,t) \leq w(x,t)$ for all $(x,t) \in \bar{\Omega} \times [0,T]$.

Proof. See [Eva10, §7.1.4] or [CIL92, Theorem 8.2], supported by Hopf's Lemma [Fri58, Theorem 1]. ☐

Remark 3.10. *We shall adopt the following terminology, which will be used in connection with Propositions 3.8 and 3.9. Let f be a given function.*

We say that a function $\underline{z}$ is a subsolution *of the equation $Lz = f$ if and only if $L\underline{z} \leq f$. Conversely a function $\overline{z}$ is a* supersolution *if and only if $L\overline{z} \geq f$.*

Analogous definitions apply to the parabolic equation $z_t + Lz = f$.

3.3.2 Elliptic regularity

Lemma 3.11. *For some $\delta \in (0,1)$ and $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ is a bounded domain of class $C^{2+\delta}$, with boundary $\partial\Omega$. If $\psi \in C^{\delta}(\bar{\Omega})$, then the solution $z \in C^{2+\delta}(\bar{\Omega})$ of*

$$\begin{cases} -\Delta z + z = \psi & \text{in } \Omega, \\ z_\nu = 0 & \text{on } \partial\Omega, \end{cases}$$

is such that

$$\nabla z \in L^\infty(\Omega). \tag{3.3.3}$$

Moreover for any $\hat{c}, \rho > 0$, there is $C_\rho > 0$ such that

$$\hat{c} \int_\Omega z^q \le \rho \int_\Omega \psi^q + C_\rho \Big(\int_\Omega \psi \Big)^q. \tag{3.3.4}$$

Proof. A detailed proof of (3.3.4) is available in [Vig21, Lemma 3.1]. ☐

3.3.3 Parabolic regularity

Lemma 3.12. *For some $\delta \in (0,1)$ and $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ is a bounded domain of class $C^{2+\delta}$, with boundary $\partial\Omega$. For every $q > n$, $T \in (0,\infty]$, $g \in L^q([0,T); L^q(\Omega))$ and $z_0 \in W^{2,q}(\Omega)$, with $\partial_\nu z_0 = 0$ on $\partial\Omega$, every solution $z \in W^{1,q}_{loc}([0,T); L^q(\Omega)) \cap L^q_{loc}([0,T); W^{2,q}(\Omega))$ of*

$$\begin{cases} z_t = \Delta z - z + g & \text{in } \Omega \times (0,T), \\ \partial_\nu z = 0 & \text{on } \partial\Omega \times (0,T), \\ z(\cdot,0) = z_0 & \text{on } \Omega, \end{cases}$$

satisfies for some $C_P > 0$

$$\int_0^t e^s \int_\Omega |\Delta z(\cdot,s)|^q \, ds \le C_P \Big[1 + \int_0^t e^s \int_\Omega |g(\cdot,s)|^q \, ds \Big] \quad \text{for all } t \in (0,T). \tag{3.3.5}$$

Additionally, if $g \in L^\infty([0,T); L^q(\Omega))$ then

$$\nabla z \in L^\infty((0,T); L^\infty(\Omega)). \tag{3.3.6}$$

Proof. For (3.3.5) we refer to [ILV24], whereas for (3.3.6) we can invoke [HW05, Lemma 4.1] and the embedding $W^{1,\frac{nq}{(n-q)_+}}(\Omega) \subset L^\infty(\Omega)$, valid for $q > n$. ☐

3.4 An ODI comparison principle

We will also make use of the following general comparison principle for Ordinary Differential Equations.

Lemma 3.13. *Let $T > 0$ and $\phi : (0,T) \times \mathbb{R}^+_0 \to \mathbb{R}$. If $0 \le y \in C^0([0,T)) \cap C^1((0,T))$ is such that*

$$y' \le \phi(t,y) \quad \text{for all } t \in (0,T),$$

and there is $y_1 > 0$ with the property that whenever $y > y_1$ for some $t \in (0,T)$ one has that $\phi(t,y) \le 0$, then

$$y \le \max\{y_1, y(0)\} \quad \text{on } (0,T).$$

Proof. For $y_0 = y(0)$, we distinguish the cases $y_0 < y_1$ and $y_0 \ge y_1$, proving that, respectively, the sets

$$S_{y_1} := \{t \in (0,T) \mid y(t) > y_1\} \quad \text{and} \quad S_{y_0} := \{t \in (0,T) \mid y(t) > y_0\}$$

are empty. In particular, we will establish only that $S_{y_1} = \emptyset$, being the reasons for S_{y_0} similar to prove.

If by contradiction there exists some $t_0 \in S_{y_1}$, then the continuity of y and $y_0 < y_1$ would provide an interval $I = (\underline{t}, \bar{t})$ (with possibly $t_0 = \bar{t}$) such that $y_1 < y(\underline{t}) < y(\bar{t})$, $y_1 < y(t)$ on I; in turn, by hypothesis, $\phi(t,y) \le 0$ for all $t \in I$. Now, by invoking Lagrange theorem we might find a proper $\xi \in I$ such

$$0 < \frac{y(\bar{t}) - y(\underline{t})}{\bar{t} - \underline{t}} = y'(\xi) \le \phi(\xi, y) \le 0.$$

This inconsistency gives the claim. ☐

3.5 Elements of analytic semigroup theory

The mathematical analysis of the Keller-Segel system relies heavily on the theory of analytic semigroups. Since the system is fundamentally parabolic, the linear diffusion part generates an analytic semigroup $(e^{t\Delta})_{t\geq 0}$ on suitable Banach spaces, such as $L^p(\Omega)$. The defining characteristic of analytic semigroups is their instantaneous regularizing effect: for any $t > 0$, the operator $e^{t\Delta}$ maps the initial data into the domain of the generator, effectively smoothing out the solution. This property allows us to reformulate the differential system as an integral equation via the variation of constants formula and to study the well-posedness of the problem in terms of mild solutions.
Thanks to [Win10a, Lemma 1.3] we know that there exist positive constants C_f and $\hat{C}_f$ such that, for all $t > 0$ and $f \in L^1(\Omega)$ such that $\int_\Omega f = 0$

$$\left\|e^{t\Delta} f\right\|_{L^\infty(\Omega)} \leq C_f\left(1 + t^{-\frac{n}{2}}\right)\|f\|_{L^1(\Omega)} \tag{3.5.1}$$

and for all $t > 0$, $r > 1$ and $f \in (L^p(\Omega))^n$

$$\left\|e^{t\Delta}\nabla \cdot f\right\|_{L^\infty(\Omega)} \leq \hat{C}_f\left(1 + t^{-\frac{1}{2}-\frac{n}{2r}}\right)\|f\|_{L^r(\Omega)}. \tag{3.5.2}$$

In Addition, as in [HW05, Section 2], we define

$$A_q w \coloneqq -\Delta w \quad \text{where} \quad w \in D(A_q) = \left\{\psi \in W^{2,q}(\Omega) : \frac{\partial \psi}{\partial \nu} = 0 \text{ on } \partial\Omega\right\}.$$

It is known that:

$$D\big((A_q + 1)^\beta\big) \hookrightarrow W^{1,q}(\Omega) \quad \text{if } \beta > \frac{1}{2}, \tag{3.5.3}$$

and, for every $1 \leq p < q \leq \infty$ and $w \in L^p(\Omega)$, there exists a positive constant $\hat{c}$ such that, for some $\hat{\mu} > 0$,

$$\left\|(A_q + 1)^\beta e^{-tA_q} w\right\|_{L^q(\Omega)} \leq \hat{c}t^{-\beta-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} e^{(1-\hat{\mu})t}\|w\|_{L^p(\Omega)}. \tag{3.5.4}$$

We will make use of these particular inequalities.

Chapter 4

Existence of classical solutions: extension and boundedness criteria, a piori estimates, estimates of blow-up time and numerical concepts

A fundamental initial step in the analysis of a chemotaxis model is to rigorously establish the existence and uniqueness of classical solutions to the system. Such results ensure that the problem is mathematically well-posed and that the model's predictions are both reliable and reproducible.

In this chapter, our objective is to prove local-in-time existence and uniqueness of sufficiently regular solutions to a very general chemotactic system, under general assumptions on the parameters and on the initial data. The proof framework we employ is sufficiently robust to be adapted for all chemotaxis problems addressed in this thesis.

Following the well-posedness result, we discuss an extension criterion characterizing the maximal existence interval for solutions, highlighting the blow-up alternatives.

In addition, we introduce a boundedness criterion, which provides conditions under which boundedness of solutions in certain Lebesgue spaces implies boundedness in the supremum norm. This criterion is an essential tool to study global existence and long-term behavior. We then establish the uniform boundedness of the total mass of the cell population under suitable parameter conditions, a key property for controlling solution growth and further qualitative analysis.

Finally, we provide a brief discussion on the derivation of a lower bound for the maximal existence time in cases where the solution exhibits finite-time blow-up, and we introduce the method employed for the numerical simulations.

With these foundational results, we set the stage for subsequent sections that delve into more detailed qualitative properties of chemotaxis phenomena.

4.1 Existence and uniqueness

In this section, we concentrate on a general chemotaxis model that encapsulates the fundamental characteristics encountered throughout this thesis. In particular, we examine system ($\mathcal{L}_\tau$) (the same as in Section 5.4), which incorporates nonlinear diffusion, nonlinear attractive and repulsive sensitivities, and logistic growth with gradient-dependent damping. Our primary objective is to rigorously establish the existence and uniqueness of solutions for this broadly formulated chemotaxis problem. The methodology developed herein is sufficiently robust to be adapted to all related problems discussed in this work.

Let us consider the following

$$\begin{cases} u_t = \nabla\cdot\big((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v + \xi u(u+1)^{m_3-1}\nabla w\big) & \\ \qquad +\lambda u^\rho - \mu u^k - c|\nabla u|^\gamma & \text{in } \Omega\times(0,T_{\max}),\\ \tau v_t = \Delta v - v + f_1(u) & \text{in } \Omega\times(0,T_{\max}),\\ \tau w_t = \Delta w - w + f_2(u) & \text{in } \Omega\times(0,T_{\max}),\\ u_\nu = v_\nu = w_\nu = 0 & \text{on } \partial\Omega\times(0,T_{\max}),\\ u(x,0)=u_0(x), \tau v(x,0)=\tau v_0(x), \tau w(x,0)=\tau w_0(x) & x\in\bar\Omega, \end{cases} \tag{$\mathcal{L}_\tau$}$$

where we assume

$$\begin{cases} f_1, f_2 : [0,\infty)\to\mathbb{R}^+, \text{ with } f_1, f_2\in C^1([0,\infty)),\\ \text{for some } \delta\in(0,1) \text{ and } n\in\mathbb{N}, \Omega\subset\mathbb{R}^n \text{ is a bounded domain of class } C^{2+\delta},\\ u_0, v_0, w_0 : \bar\Omega\to\mathbb{R}^+, \text{ with } u_0, v_0, w_0 \in C^{2+\delta}_\nu(\bar\Omega) = \{\phi\in C^{2+\delta}(\bar\Omega) : \phi_\nu = 0 \text{ on } \partial\Omega\}. \end{cases} \tag{4.1.1}$$

We will prove the following result.

Theorem 4.1. *For $\tau\in\{0,1\}$, let Ω, $f_1, f_2, u_0, \tau v_0, \tau w_0$ comply with hypotheses in* (4.1.1). *Additionally, let $\chi,\xi,\lambda,\mu,c>0$, and $m_1,m_2,m_3\in\mathbb{R}$, $k,\rho,\gamma\geq 1$. Then there exist $T>0$ and a unique triple of nonnegative functions (u,v,w), with*

$$(u,v,w)\in C^{2+\delta,1+\frac{\delta}{2}}(\bar\Omega\times[0,T])\times C^{2+\delta,\tau+\frac{\delta}{2}}(\bar\Omega\times[0,T])\times C^{2+\delta,\tau+\frac{\delta}{2}}(\bar\Omega\times[0,T]),$$

solving problem $(\mathcal{L}_\tau)$.

Proof. Let us start with the *existence* issue. For any $R>0$, let us consider for some $0<T\leq 1$, to be precised later on, the closed bounded convex subset of $C^{\delta,\frac{\delta}{2}}(\bar\Omega\times[0,T])$

$$S_T = \{0\leq u\in C^{\delta,\frac{\delta}{2}}(\bar\Omega\times[0,T]) : \|u(\cdot,t)-u_0\|_{C^\delta(\bar\Omega)}\leq R, \text{ for all } t\in[0,T]\}.$$

Once an element $\tilde u$ of S_T is picked, from the properties of f_1 and f_2, one has that $f_1(\tilde u), f_1(\tilde u)\in C^{\delta,\frac{\delta}{2}}(\bar\Omega\times[0,T])$; in the specific, the associated solution v and w to problems

$$\begin{cases} \tau v_t - \Delta v + v = f_1(\tilde u) & \text{in } \Omega\times(0,T),\\ v_\nu = 0 & \text{on } \partial\Omega\times(0,T),\\ \tau v(x,0) = \tau v_0(x), \end{cases} \tag{4.1.2}$$

and

$$\begin{cases} \tau w_t - \Delta w + w = f_2(\tilde u) & \text{in } \Omega\times(0,T),\\ w_\nu = 0 & \text{on } \partial\Omega\times(0,T),\\ \tau w(x,0) = \tau w_0(x), \end{cases} \tag{4.1.3}$$

are, thanks to $\tau v_0, \tau w_0\in C^{2+\delta}_\nu(\bar\Omega)$ and classical elliptic and parabolic regularity results ([Bre11, Theorem 9.33], [GT83, Theorem 6.31] for $\tau=0$, and [Lun95, Corollary 5.1.22], [LSU88, Theorem IV.5.3] for $\tau=1$), such that

$$v,w\in C^{2+\delta,\tau+\frac{\delta}{2}}(\bar\Omega\times[0,T]) \quad\text{and}\quad \sup_{t\in[0,T]}\|v(\cdot,t)\|_{C^{2+\delta}(\bar\Omega)} + \sup_{t\in[0,T]}\|w(\cdot,t)\|_{C^{2+\delta}(\bar\Omega)}\leq H, \tag{4.1.4}$$

where $H = H(f_1, f_2, \tilde u, \tau v_0, \tau w_0)>0$ is estimated by $\tilde u$, so essentially, $H=H(R)$. Now, given these gained properties of ∇v and ∇w, considered the smoothness of $\zeta\longmapsto(1+\zeta)^\vartheta$ for all $\vartheta\in\mathbb{R}$ and $\zeta\geq 0$, let us define

$$\begin{cases} \varphi_1 : \Omega\times(0,T)\to\mathbb{R}^+,\\ \varphi_2 : \Omega\times(0,T)\to\mathbb{R}^n,\\ \varphi_3 : \Omega\times(0,T)\times\mathbb{R}^{1+n}\to\mathbb{R}, \end{cases}$$

such that
$$\begin{cases}\varphi_1(x,t) := (\tilde{u}(x,t)+1)^{m_1-1},\\ \varphi_2(x,t) := -\chi(\tilde{u}(x,t)+1)^{m_2-1}\nabla v(x,t) + \xi(\tilde{u}(x,t)+1)^{m_3-1}\nabla w(x,t),\\ \varphi_3(x,t,u,\nabla u) := \lambda u^\rho(x,t) - \mu u^k(x,t) - c|\nabla u(x,t)|^\gamma,\end{cases} \tag{4.1.5}$$

and also this system:

$$\begin{cases} u_t = \nabla\cdot\big((\tilde{u}+1)^{m_1-1}\nabla u - \chi u(\tilde{u}+1)^{m_2-1}\nabla v + \xi u(\tilde{u}+1)^{m_3-1}\nabla w\big) & \\ \qquad +\lambda u^\rho - \mu u^k - c|\nabla u|^\gamma & \text{in } \Omega\times(0,T),\\ u_\nu = 0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x) & x\in\overline{\Omega}.\end{cases} \tag{4.1.6}$$

In the specific, with the positions in (4.1.5), problem (4.1.6) reads as

$$\begin{cases} u_t = \mathcal{A}u + \varphi(x,t,u,\nabla u) = \varphi_1\Delta u + u\nabla\cdot\varphi_2 + (\nabla\varphi_1+\varphi_2)\cdot\nabla u + \varphi_3 & \text{in } \Omega\times(0,T),\\ \mathcal{B}_1 u = u_\nu = 0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x) & x\in\overline{\Omega},\end{cases}$$

and it is seen that φ complies with [Lun95, (7.3.5)], exactly using the notation therein. As a consequence, by recalling that $u_0 \in C^{2+\delta}_\nu(\bar{\Omega})$, [Lun95, Theorem 7.3.3] ensures the existence of some $0<T<1$, as above, such that problem (4.1.6) has a unique solution

$$u \in C^{2+\delta,1+\frac{\delta}{2}}(\bar{\Omega}\times[0,T]).$$

In particular, this produces some positive constant $K_1 = K_1(\tilde{u},\nabla v,\nabla w,u_0)$, and henceforth $K_1 = K_1(R)$, with the property that

$$\|u\|_{C^{\delta,\frac{\delta}{2}}(\bar{\Omega}\times[0,T])} \le K_1,$$

so that (for suitable K_2) from

$$\|u(\cdot,t)-u_0\|_{C^\delta(\bar{\Omega})} \le K_2 T^{\frac{\delta}{2}}\|u\|_{C^{\delta,\frac{\delta}{2}}(\bar{\Omega}\times[0,T])} \qquad \text{for all } t\in[0,T],$$

we obtain for $K = K_1K_2$

$$\|u(\cdot,t)-u_0\|_{C^\delta(\bar{\Omega})} \le KT^{\frac{\delta}{2}} \quad \text{for all} \quad t\in[0,T].$$

In this way, we deduce that

$$\text{for} \quad T \le \left(\frac{R}{K}\right)^{\frac{2}{\delta}}, \quad \|u(\cdot,t)-u_0\|_{C^\delta(\bar{\Omega})} \le R \quad \text{for all} \quad t\in[0,T].$$

With $u=u(x,t)$ in our hands, let L be the linear operator constructed as in (3.3.2) with

$$A = A(x,t) = -(u+1)^{m_1-1}I, \qquad b = b(x,t) = \chi(u+1)^{m_2-1}\nabla v - \xi(u+1)^{m_3-1}\nabla w,$$
$$c = c(x,t) = -\nabla\cdot b, \qquad f = f(x,t) = \lambda u^\rho - \mu u^k - c|\nabla u|^\gamma,$$

where I is the identity matrix, and consider the linear problem in the unknown $U=U(x,t)$

$$U_t + LU = f \qquad \text{in } \Omega\times(0,T).$$

Since $\underline{U}\equiv 0$ is a subsolution of the previous equation and $\overline{U}=u$ a supersolution, by means of Proposition 3.9 and Remark 3.10 we have that the solution u remains nonnegative in $\Omega\times(0,T)$. So, for values of T as above, the map $\Phi(\tilde{u})=u$ where u solves problem (4.1.6), is such that $\Phi(S_T)\subset S_T$. Additionally, Φ is bounded in $C^{2+\delta,1+\frac{\delta}{2}}(\bar{\Omega}\times[0,T])$, and compact in $C^{\delta,\frac{\delta}{2}}(\bar{\Omega}\times[0,T])$, because the [Bre11, Ascoli–Arzelà Theorem 4.25] implies that the natural embedding of $C^{2+\delta,\tau+\frac{\delta}{2}}(\bar{\Omega}\times[0,T])$ into $C^{\delta,\frac{\delta}{2}}(\bar{\Omega}\times[0,T])$. Let u be the fixed point of Φ asserted by the Schauder's fixed point theorem: first, Propositions 3.8 and 3.9,

$f_1(u), f_2(u) \geq 0$ in problems (4.1.2) and (4.1.3) also imply $v, w \geq 0$ in $\Omega \times (0,T)$; secondly, as seen, u, v, w have the claimed regularity.

As to *uniqueness*, let (u_1, v_1, w_1) and (u_2, v_2, w_2) be two different nonnegative classical solutions of problem $(\mathcal{L}_\tau)$ in $\Omega \times (0,T)$ with the same initial data $u_1(x,0) = u_2(x,0) = u_0(x)$ and $\tau v_1(x,0) = \tau v_2(x,0) = \tau v_0(x)$ and $\tau w_1(x,0) = \tau w_2(x,0) = \tau w_0(x)$. By confining our attention to the case $\tau = 0$ (being the case $\tau = 1$ similar, as we will indicate later), we thus have from (4.1.3)–(4.1.6) and $i = 1,2$ these six problems:

$$\begin{cases} -\Delta v_i + v_i = f_1(u_i) & \text{in } \Omega \times (0,T), \\ (v_i)_\nu = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{4.1.7}$$

$$\begin{cases} -\Delta w_i + w_i = f_2(u_i) & \text{in } \Omega \times (0,T), \\ (w_i)_\nu = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{4.1.8}$$

and

$$\begin{cases} (u_i)_t = \nabla \cdot ((u_i+1)^{m_1-1}\nabla u_i - \chi u_i (u_i+1)^{m_2-1}\nabla v_i) & \text{in } \Omega \times (0,T), \\ \qquad + (\xi u_i (u_i+1)^{m_3-1}\nabla w_i) & \\ \qquad + \lambda u_i^\rho - \mu u_i^k - c|\nabla u_i|^\gamma & \\ (u_i)_\nu = 0 & \text{on } \partial\Omega \times (0,T), \\ u_i(x,0) = u_0(x) & x \in \overline{\Omega}. \end{cases} \tag{4.1.9}$$

Now we set

$$s_1 = \min\left\{\|u_1\|_{L^\infty(\Omega\times(0,T))}, \|u_2\|_{L^\infty(\Omega\times(0,T))}\right\}, \quad s_2 = \max\left\{\|u_1\|_{L^\infty(\Omega\times(0,T))}, \|u_2\|_{L^\infty(\Omega\times(0,T))}\right\},$$

and introduce for $s \geq 0$ the function

$$\theta(s) = \begin{cases} \log(s+1) & \text{if } m_1 = 0, \\ \dfrac{(s+1)^{m_1}}{m_1} & \text{if } m_1 \neq 0, \end{cases} \quad \text{with } \ m_1 \in \mathbb{R}.$$

Noting that $\theta'(s) > 0$ for all $s \geq 0$, we define, in view of forthcoming applications of the Mean Value Theorem, the positive T-dependent constants

$$\begin{cases} C_1 = \max\limits_{[s_1,s_2]} |f_1'|, \\ C_2 = \max\limits_{[s_1,s_2]} |f_2'|, \end{cases} \text{and } \ C_3 = \begin{cases} \max\limits_{[s_1,s_2]} \theta' & \text{if } (u_1-u_2)\Delta(u_1-u_2) \geq 0, \\ \min\limits_{[s_1,s_2]} \theta' & \text{if } (u_1-u_2)\Delta(u_1-u_2) \leq 0. \end{cases} \tag{4.1.10}$$

Under such circumstances, by considering problems (4.1.7), we have that $(v_1 - v_2)$ solves, for some $\bar{u} \in (u_1, u_2)$, the equation $-\Delta(v_1 - v_2) + (v_1 - v_2) = f_1(u_1) - f_1(u_2) = f_1'(\bar{u})(u_1 - u_2)$, with Neumann boundary conditions. In light of this, through Young's inequality and taking into account (4.1.10), testing against $(v_1 - v_2)$, the mentioned equation provides the following estimate

$$\begin{aligned} \int_\Omega |\nabla(v_1 - v_2)|^2 + \int_\Omega (v_1 - v_2)^2 &= \int_\Omega (f_1(u_1) - f_1(u_2))(v_1 - v_2) \leq C_1 \int_\Omega |(u_1 - u_2)(v_1 - v_2)| \\ &\leq \int_\Omega (v_1 - v_2)^2 + c_1 \int_\Omega (u_1 - u_2)^2 \quad \text{on } t \in (0,T), \end{aligned}$$

inferring

$$\int_\Omega |\nabla(v_1 - v_2)|^2 \leq c_1 \int_\Omega (u_1 - u_2)^2 \quad \text{for all } \ t \in (0,T). \tag{4.1.11}$$

Similarly, by considering problems (4.1.8) we arrive at

$$\int_\Omega |\nabla(w_1 - w_2)|^2 \leq c_2 \int_\Omega (u_1 - u_2)^2 \quad \text{for all } \ t \in (0,T). \tag{4.1.12}$$

We also get, on $(0,T)$, from problems (4.1.9)

$$
\begin{aligned}
\frac{d\mathcal{F}}{dt} = \frac{1}{2}\frac{d}{dt}\int_\Omega (u_1-u_2)^2 = &\int_\Omega (u_1-u_2)(u_1-u_2)_t\\
= &-\int_\Omega \nabla(u_1-u_2)\cdot\big((u_1+1)^{m_1-1}\nabla u_1-(u_2+1)^{m_1-1}\nabla u_2\big)\\
&+\chi\int_\Omega \nabla(u_1-u_2)\cdot\big(u_1(u_1+1)^{m_2-1}\nabla v_1-u_2(u_2+1)^{m_2-1}\nabla v_2\big)\\
&-\xi\int_\Omega \nabla(u_1-u_2)\cdot\big(u_1(u_1+1)^{m_3-1}\nabla w_1-u_2(u_2+1)^{m_3-1}\nabla w_2\big)\\
&+\int_\Omega (u_1-u_2)(\lambda u_1^\rho-\lambda u_2^\rho-\mu u_1^k+\mu u_2^k-c|\nabla u_1|^\gamma+c|\nabla u_2|^\gamma)
\end{aligned}
\tag{4.1.13}
$$

where the first, second, third, and fourth integrals at the right-hand side are denoted, for simplicity, by I_1, I_2, I_3 and I_4, respectively. We now estimate these separate terms as follows (recall (4.1.10), again, and the definition of θ):

$$
\begin{aligned}
I_1 &= -\int_\Omega \nabla(u_1-u_2)\cdot\nabla\left(\theta(u_1)-\theta(u_2)\right) = \int_\Omega \Delta(u_1-u_2)(\theta(u_1)-\theta(u_2))\\
&= \int_\Omega \Delta(u_1-u_2)\theta'(\overline{u})(u_1-u_2) \leq C_3\int_\Omega (u_1-u_2)\Delta(u_1-u_2)\\
&= -C_3\int_\Omega |\nabla(u_1-u_2)|^2 \quad \text{for all } t\in(0,T).
\end{aligned}
\tag{4.1.14}
$$

On the other hand, once more the smoothness of $\zeta\longmapsto(1+\zeta)^\vartheta$ for all $\vartheta\in\mathbb{R}$ and $\zeta\geq 0$, makes that for some C_4 and C_5, and for $j\in\{m_2,m_3\}$

$$
\begin{cases}
|u_1(u_1+1)^{j-1}-u_2(u_2+1)^{j-1}|\leq C_4|u_1-u_2| & \text{in } \Omega\times(0,T),\\
u_2(u_2+1)^{j-1}\leq C_4 & \text{in } \Omega\times(0,T);
\end{cases}
$$

furthermore

$$
\begin{cases}
|\nabla v_1|\leq C_5 \text{ and } |\nabla v_2|\leq C_5 & \text{in } \Omega\times(0,T),\\
|\nabla w_1|\leq C_5 \text{ and } |\nabla w_2|\leq C_5 & \text{in } \Omega\times(0,T).
\end{cases}
$$

For similar reasons, there exists C_6 such that for all $\rho\geq 1$

$$
|u_1^\rho-u_2^\rho|\leq C_6|u_1-u_2| \quad \text{and} \quad ||\nabla u_1|^\rho-|\nabla u_2|^\rho|\leq C_6||\nabla u_1|-|\nabla u_2|| \quad \text{in } \Omega\times(0,T). \tag{4.1.15}
$$

Henceforth, starting from the identity on $(0,T)$

$$
I_2 = \chi\int_\Omega \nabla(u_1-u_2)\cdot\left((u_1(u_1+1)^{m_2-1}-u_2(u_2+1)^{m_2-1})\nabla v_1+u_2(u_2+1)^{m_2-1}\nabla(v_1-v_2)\right),
$$

and using Hölder's inequality and relation (3.1.5) we obtain

$$
\begin{aligned}
I_2^2 \leq &\chi^2\int_\Omega |(u_1(u_1+1)^{m_2-1}-u_2(u_2+1)^{m_2-1})\nabla v_1+u_2(u_2+1)^{m_2-1}\nabla(v_1-v_2)|^2\int_\Omega|\nabla(u_1-u_2)|^2\\
\leq &2\chi^2\left(\int_\Omega |(u_1(u_1+1)^{m_2-1}-u_2(u_2+1)^{m_2-1})\nabla v_1|^2+\int_\Omega|u_2(u_2+1)^{m_2-1}\nabla(v_1-v_2)|^2\right)\int_\Omega|\nabla(u_1-u_2)|^2\\
\leq &2\chi^2C_4^2\left(C_5^2\int_\Omega (u_1-u_2)^2+\int_\Omega|\nabla(v_1-v_2)|^2\right)\int_\Omega|\nabla(u_1-u_2)|^2 \quad \text{for all } t\in(0,T).
\end{aligned}
$$

Taking the square root, we get by Young's inequality

$$
\begin{aligned}
I_2 &\leq \chi C_4\sqrt{2}\left(C_5^2\int_\Omega (u_1-u_2)^2+\int_\Omega|\nabla(v_1-v_2)|^2\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla(u_1-u_2)|^2\right)^{\frac{1}{2}}\\
&\leq \frac{C_3}{3}\int_\Omega|\nabla(u_1-u_2)|^2+c_3\int_\Omega(u_1-u_2)^2+c_3\int_\Omega|\nabla(v_1-v_2)|^2 \quad \text{on } (0,T).
\end{aligned}
\tag{4.1.16}
$$

Similarly, from

$$
\begin{aligned}
I_3 =&\xi \int_\Omega \nabla(u_1-u_2)\cdot\Big(\big(u_2(u_2+1)^{m_3-1}-u_1(u_1+1)^{m_3-1}\big)\nabla w_2\\
&+u_1(u_1+1)^{m_3-1}\nabla(w_2-w_1)\Big) \quad \text{for every } t\in(0,T),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
I_3 &\le \xi C_5\sqrt{2}\Big(C_5^2\int_\Omega(u_1-u_2)+\int_\Omega|\nabla(w_1-w_2)|^2\Big)^{\frac12}\Big(\int_\Omega|\nabla(u_1-u_2)|^2\Big)^{\frac12}\\
&\le \frac{C_3}{3}\int_\Omega|\nabla(u_1-u_2)|^2+c_4\int_\Omega(u_1-u_2)^2+c_4\int_\Omega|\nabla(w_1-w_2)|^2 \quad \text{on } (0,T).
\end{aligned}
\tag{4.1.17}
$$

As to the addendum

$$
I_4 := \int_\Omega (u_1-u_2)(\lambda u_1^\rho-\lambda u_2^\rho-\mu u_1^k+\mu u_2^k-c|\nabla u_1|^\gamma+c|\nabla u_2|^\gamma) \quad \text{on } (0,T), \tag{4.1.18}
$$

analogous computations based on the Mean Value Theorem lead to

$$
I_4 \le C_6\int_\Omega(u_1-u_2)^2+C_6\int_\Omega|u_1-u_2|\,\big||\nabla u_1|-|\nabla u_2|\big| \quad \text{on } (0,T), \tag{4.1.19}
$$

and in turn Young's inequality, using that $(|\nabla u_1|-|\nabla u_2|)^2\le|\nabla(u_1-u_2)|^2$, provides

$$
I_4 \le c_5\int_\Omega(u_1-u_2)^2+\frac{C_3}{3}\int_\Omega|\nabla(u_1-u_2)|^2 \quad \text{for all } t\in(0,T). \tag{4.1.20}
$$

Now a rearrangement of the derived bounds (4.1.11)–(4.1.20) yields a suitable positive constant $\tilde{C}$ such that

$$
\mathcal{F}'(t) := \frac12\frac{d}{dt}\int_\Omega(u_1-u_2)^2 \le \tilde{C}\int_\Omega(u_1-u_2)^2=\tilde{C}\mathcal{F}(t), \qquad t\in(0,T).
$$

We then exploit the nonnegativity of $\mathcal{F}(t)$, the initial condition $\mathcal{F}(0)=0$ (based on $u_1(\cdot,0)=u_2(\cdot,0)$), and Lemma 3.13 to prove that $\mathcal{F}(t)\equiv 0$. As a result, $u_1-u_2=0$ on $\Omega\times(0,T)$ and consequently, recalling again problems (4.1.7) and (4.1.8), we manifestly get $v_1-v_2=0$ and $w_1-w_2=0$, as well on $\Omega\times(0,T)$; the proof is concluded. For more details we refer to [CW08, Nag95, Wan16, WD10]. □

Remark 4.2. *The validity of Theorem 4.1 extends beyond the setting explicitly stated therein and remains applicable under several notable modifications of the model. In particular:*

- *Signal consumption instead of production: If the chemical signal is consumed by the population rather than produced, the conclusion still holds, as the argument relies on elliptic and parabolic regularity estimates, which remain unaffected by this change.*
- *Absence of repulsive effects: The case of a purely attractive system, corresponding to the choice $\xi=0$, is likewise covered, since the proof does not require the presence of a repulsive term.*
- *Lack of logistic growth: When the logistic term is omitted, i.e., by setting the parameters $\lambda=\mu=c=0$, the result continues to apply without modification.*

Therefore, by virtue of Theorem 4.1, a unique classical solution exists for every problem addressed herein.

4.2 The extension criterion

From Theorem 4.1, we have this consequence:

Corollary 4.3. *Under the hypotheses of Theorem 4.1, there exists $T_{\max} \in (0, \infty]$ such that the solution (u, v, w) obeys this dichotomy criterion:*

$$
\begin{aligned}
&\textit{either} \quad T_{\max} = \infty, \quad \textit{or} \\
&T_{\max} < \infty \quad \textit{and} \quad \limsup_{t \to T_{\max}} \Big(\|u(\cdot,t)\|_{C^{2+\delta}(\bar{\Omega})} + \|v(\cdot,t)\|_{C^{2+\delta}(\bar{\Omega})} + \|w(\cdot,t)\|_{C^{2+\delta}(\bar{\Omega})} \Big) = \infty.
\end{aligned} \tag{4.2.1}
$$

Proof. Taking T as initial time and $u(\cdot, T)$ as the initial condition, Theorem 4.1 would provide a solution $(\hat{u}, \hat{v}, \hat{w})$ defined on $\bar{\Omega} \times [T, \hat{T}]$, for some $\hat{T} > 0$, which by uniqueness would be the prolongation of (u, v, w) (exactly, from $\bar{\Omega} \times [0, T]$ to $\bar{\Omega} \times [0, \hat{T}]$). This procedure can be repeated up to construct a maximal interval time $[0, T_{\max})$ of existence, in the sense that either $T_{\max} = \infty$, or if $T_{\max} < \infty$ no solution belonging to $C^{2+\delta, \tau + \frac{\delta}{2}}(\bar{\Omega} \times [0, T_{\max}])$ may exist and henceforth relation (4.2.1) has to be fulfilled. □

With the above criterion in our hands, in the following result, we will make use of the so-called Hadamard growth condition, which imposes constraints on terms of the equation in order to control the behavior of its solutions. In this contexts, such conditions are crucial for guaranteeing that solutions do not grow too fast. (We also cite [PV93] for analyses related to general evolution equations involving Hadamard-type growth conditions.)

Proposition 4.4. *Let $\gamma \in [1, 2]$. If $u \in L^\infty((0, T_{\max}), L^\infty(\Omega))$, then $T_{\max} = \infty$.*

Proof. By contradiction let $T_{\max}$ be finite. Naturally, it is sufficient to show that boundedness of u in $L^\infty(\Omega)$ for $t \in (0, T_{\max})$, entails finiteness of $\|u(\cdot,t)\|_{C^{2+\delta}(\bar{\Omega})} + \|v(\cdot,t)\|_{C^{2+\delta}(\bar{\Omega})} + \|w(\cdot,t)\|_{C^{2+\delta}(\bar{\Omega})}$ on $[0, T_{\max}]$. In fact, from (4.2.1), we would have an inconsistency.

First, we have boundedness of $\|v(\cdot,t)\|_{C^{1+\delta}(\bar{\Omega})}$, and of $\|w(\cdot,t)\|_{C^{1+\delta}(\bar{\Omega})}$ on $[0, T_{\max}]$, as consequence of our hypothesis $u \in L^\infty((0, T_{\max}); L^\infty(\Omega))$, once [Lie87, Theorem 1.2] (if $\tau = 1$) or, e.g., from [Fri69, Theorem I.19.1] (if $\tau = 0$) and the Sobolev embedding immersion $W^{2,p}(\Omega) \subset C^{1+\delta}(\bar{\Omega})$, with arbitrarily large p, are invoked. On the other hand, according to the nomenclature in [Lie87], letting $X = (x, t)$, $A(X, u, \nabla u) = -(u+1)^{m_1 - 1}\nabla u + \chi u(u+1)^{m_2 - 1}\nabla v(x,t) - \xi u(u+1)^{m_3 - 1}\nabla w(x,t)$ and $B(X, u, \nabla u) = -\lambda u^\rho + \mu u^k + c|\nabla u|^\gamma$, we conclude from [Lie87, Theorem 1.2] that $\|u(\cdot,t)\|_{C^{1+\delta}(\bar{\Omega})}$ is finite as well on $[0, T_{\max}]$, by using that u is a bounded solution to

$$
\begin{cases}
u_t + \nabla \cdot A(X, u, \nabla u) + B(X, u, \nabla u) = 0 & \text{in } \Omega \times (0, T_{\max}), \\
A(X, u, \nabla u) \cdot \nu = 0 & \text{on } \partial\Omega \times (0, T_{\max}), \\
u(x, 0) = u_0(x) & x \in \bar{\Omega},
\end{cases}
$$

exactly for $\gamma \in [1, 2]$ (see in the specific [Lie87, (1.2*c*), Theorem 1.2]). From this gained regularity of u and the relative bound, bootstrap arguments already developed in Corollary 4.3, in conjunction with the regularity of the initial data $(u_0, \tau v_0, \tau w_0)$ for model $(\mathcal{L}_\tau)$, give

$$
\sup_{t \in [0, T_{\max}]} \Big(\|u(\cdot,t)\|_{C^{2+\delta}(\bar{\Omega})} + \|v(\cdot,t)\|_{C^{2+\delta}(\bar{\Omega})} + \|w(\cdot,t)\|_{C^{2+\delta}(\bar{\Omega})} \Big) < \infty.
$$

□

Remark 4.5. *Consider the case $c = 0$, where the logistic term does not include the gradient-dependent component. In this scenario, the condition on the parameter γ can be omitted, as the gradient-dependent term is absent. This simplification enables the attainment of the Hadamard condition to complete the proof without recourse to [Lie87, Theorem 1.2]. So if $c = 0$ or $c \neq 0$ and $\gamma \in [1, 2]$, the dichotomy criterion can be expressed as*

$$
\begin{aligned}
&\textit{either} \quad T_{\max} = \infty, \quad \textit{or} \\
&T_{\max} < \infty \quad \textit{and} \quad \limsup_{t \to T_{\max}} \big(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \big) = \infty.
\end{aligned} \tag{4.2.2}
$$

4.3 The boundedness criterion

In Proposition 4.4, we established that the boundedness of u guarantees that $T_{\max} = \infty$. A natural question arises: how can one ensure the boundedness of u? To this end, we introduce a fundamental analytical instrument, namely the *boundedness criterion*. By means of some Moser iteration techniques, this result allows one to upgrade the boundedness of u in the $L^p(\Omega)$ norm, obtained through *a priori* estimates for some $p > 1$, to boundedness in the stronger $L^\infty(\Omega)$ norm.

Lemma 4.6. *Consider the following system*

$$\begin{cases} u_t \leq \nabla \cdot (D(x,t,u)\nabla u) + \nabla \cdot f(x,t) + g(x,t), & x \in \Omega,\ t \in (0,T_{\max}), \\ \partial_\nu u(x,t) \leq 0 & x \in \partial\Omega,\ t \in (0,T_{\max}), \end{cases} \tag{A.1}$$

where we require that

- $\Omega \subset \mathbb{R}^n$*,* $n \geq 1$*, is a bounded domain;*
- *the diffusion* D *is such that*
$$D \in C^1\big(\bar{\Omega} \times [0,T_{\max}) \times [0,\infty)\big) \quad \text{and} \quad D \geq 0; \tag{A.2}$$
- *there exist* $m \in \mathbb{R}$*,* $s_0 \geq 1$ *and* $\delta > 0$ *such that*
$$D(x,t,s) \geq \delta s^{m-1} \quad \text{for all } x \in \Omega,\ t \in (0,T_{\max}) \text{ and } s \geq s_0; \tag{A.3}$$
- *the functions* f *and* g *are such that*
$$f \in C^0\big((0,T_{\max}); C^0(\bar{\Omega}) \cap C^1(\Omega)\big) \quad \text{and} \quad g \in C^0(\Omega \times (0,T_{\max})); \tag{A.4}$$
with
$$f \cdot \nu \leq 0 \quad \text{on } \partial\Omega \times (0,T_{\max}), \tag{A.5}$$
and
$$f \in L^\infty((0,T_{\max}); L^{q_1}(\Omega)) \quad \text{and} \quad g \in L^\infty((0,T_{\max}); L^{q_2}(\Omega)), \tag{A.6}$$
for some $q_1 > n+2$ *and* $q_2 > \frac{n+2}{2}$*;*
- *the function* u *is a nonnegative classical solution of* (A.1) *such that*
$$u \in C^0\big(\bar{\Omega} \times [0,T_{\max})\big) \cap C^{2,1}\big(\bar{\Omega} \times (0,T_{\max})\big) \cap L^\infty((0,T_{\max}); L^p(\Omega)), \tag{A.7}$$
where
$$p > \max\left\{1 - m\frac{(n+1)q_1 - (n+2)}{q_1 - (n+2)}, 1 - \frac{m}{1 - \frac{n}{n+2}\frac{q_2}{q_2-1}}, \frac{n(1-m)}{2}\right\}. \tag{A.8}$$

Then there exists $C > 0$*, only depending on* m, δ, Ω*,* $\|f\|_{L^\infty((0,T_{\max});L^{q_1}(\Omega))}$*,* $\|g\|_{L^\infty((0,T_{\max});L^{q_2}(\Omega))}$*,* $\|u\|_{L^\infty((0,T_{\max});L^p(\Omega))}$ *and* $\|u(0)\|_{L^\infty(\Omega)}$ *such that*

$$\|u(t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0,T_{\max}).$$

Proof. See [TW12a]. □

Remark 4.7. *The value of* p *in condition* (A.8) *is not optimal due to the nonlinear nature of the diffusion term in* (A.1)*. In the case of linear diffusion, an optimal value for* p *exists, namely* $p = \frac{n}{2}$*. In Section 6.1, where a continuity argument is employed, we establish the boundedness criterion for this particular case by utilizing semigroup theory.*

4.3.1 The boundedness of the mass: a crucial a priori estimate

In accordance with Lemma 4.6, the boundedness of solutions to a certain class of evolutionary equations can be established by demonstrating that $u \in L^\infty((0, T_{\max}), L^p(\Omega))$ for some sufficiently large exponent p. As a preliminary step toward this goal, it is essential to verify at least the weaker integrability condition $u \in L^\infty\big((0, T_{\max}), L^1(\Omega)\big)$. Indeed, if

$$\lim_{t\to T_{\max}} \|u(\cdot,t)\|_{L^1(\Omega)} = \infty, \quad \text{then also} \quad \lim_{t\to T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty,$$

and uniform boundedness cannot be achieved.

Within the framework of our analysis, and in particular when considering system $(\mathcal{L}_\tau)$, the following key result guarantees that the required integrability condition is fulfilled.

Proposition 4.8. *If $k > \rho$, the u-component of the solution (u, v, w) to system $(\mathcal{L}_\tau)$ is such that the mass is uniformly bounded in time; formally $u \in L^\infty((0, T_{\max}); L^1(\Omega))$, and precisely*

$$\int_\Omega u \le M := \max\left\{\int_\Omega u_0(x)\,dx, \left(\frac{\lambda}{\mu}|\Omega|^{k-\rho}\right)^{\frac{1}{k-\rho}}\right\} \quad \text{on } [0, T_{\max}). \tag{4.3.1}$$

Proof. An integration of the first equation in problem $(\mathcal{L}_\tau)$ and an application of the divergence theorem give, thanks to the Neumann boundary conditions,

$$\frac{d}{dt}\int_\Omega u =: y' \le \lambda \int_\Omega u^\rho - \mu \int_\Omega u^k \quad \text{in } (0, T_{\max}). \tag{4.3.2}$$

Now, thanks to the Hölder inequality, we have for $k > \rho \ge 1$

$$-\int_\Omega u^k \le -|\Omega|^{\frac{\rho-k}{\rho}}\left(\int_\Omega u^\rho\right)^{\frac{k}{\rho}} \quad \text{and} \quad -\int_\Omega u^\rho \le -|\Omega|^{1-\rho}\left(\int_\Omega u\right)^\rho \quad \text{in } (0, T_{\max}). \tag{4.3.3}$$

By combining expressions (4.3.2) and (4.3.3) we arrive for $\gamma(t) = \lambda \int_\Omega u^\rho(x,t)dx \ge 0$ on $(0, T_{\max})$ at this initial problem

$$\begin{cases} y' \le \gamma(t)\left(1 - \frac{\mu}{\lambda}|\Omega|^{\rho-k}y^{k-\rho}\right) & \text{in } (0, T_{\max}),\\ y(0) = \int_\Omega u_0(x)dx, \end{cases}$$

so concluding by invoking Lemma 3.13 with

$$T = T_{\max},\ \phi(t,y) = \gamma(t)\left(1 - \frac{\mu}{\lambda}|\Omega|^{\rho-k}y^{k-\rho}\right),\ y(0) = \int_\Omega u_0(x)\,dx \ \text{ and } \ y_1 = \left(\frac{\lambda}{\mu}|\Omega|^{k-\rho}\right)^{\frac{1}{k-\rho}}.$$

□

4.4 Lower bounds for the blow-up time

If the solution u is unbounded at finite time, as in (4.2.2), we can achieve a lower bound for $T_{\max}$. The aim is deriving, for a functional $\varphi(t)$, a first-order differential inequality (ODI) of the form

$$\varphi'(t) \le \Psi(\varphi(t)) \quad \text{for } t \in (0, T_{\max}).$$

Crucially, the function $\Psi(\tau)$, for any fixed $\tau_0 > 0$, satisfies the classical Osgood criterion (cf. [Osg98]), expressed as

$$\int_{\tau_0}^\infty \frac{d\tau}{\Psi(\tau)} < \infty, \tag{4.4.1}$$

which ensures the integrability condition necessary for controlling the growth of φ. By integrating the ODI over the interval $(0, T_{\max})$, and assuming that

$$\limsup_{t\to T_{\max}} \varphi(t) = \infty,$$

one deduces the following explicit lower bound for the maximal time of existence $T_{\max}$:

$$T := \int_{\varphi(0)}^{\infty} \frac{d\varphi}{\Psi(\varphi)} \leq T_{\max}.$$

This integral bound consequently guarantees the existence of a safe time interval $[0, T)$ on which classical solutions to the model persist. This result plays a fundamental role in the qualitative analysis of the underlying system by providing a rigorous criterion for the continuation of solutions prior to possible blow-up phenomena.

4.5 The Finite Element Method

In this thesis, which is primarily analytical in nature, we do not delve into the details of the Finite Element Method (FEM). However, we rely on it (specifically in its Galerkin formulation) as a computational tool to validate theoretical results and to explore the qualitative behavior of solutions in cases where analytical techniques are insufficient to draw conclusions.

The Finite Element Method is a widely used numerical approach for approximating solutions to partial differential equations (PDEs). The core idea is to reformulate the PDE as a variational problem and to approximate the solution by projecting it onto a finite-dimensional subspace, typically spanned by piecewise polynomial basis functions defined on a mesh of the domain. This leads to a discrete system of algebraic equations that can be solved numerically. Here we provide only this brief outline; for a rigorous and comprehensive treatment, we refer the reader to [BS08].

Chapter 5

Advances in the framework of attraction-repulsion mechanisms

In this chapter, we will discuss four different problems related to the attraction-repulsion mechanism. We also remind that the solution (u, v, w) to each problem, emanating from the initial data $(u_0, \tau v_0, \tau w_0)$ such that $u_0, v_0, w_0 : \bar{\Omega} \to \mathbb{R}^+$, with

$$u_0, v_0, w_0 \in C_\nu^{2+\delta}(\bar{\Omega}) = \{\phi \in C^{2+\delta}(\bar{\Omega}) : \phi_\nu = 0 \text{ on } \partial\Omega\},$$

exists and it is unique.

5.1 [CFV23b] Some refined criteria toward boundedness in an attraction-repulsion chemotaxis system with nonlinear productions

5.1.1 Presentation of the model

This section is devoted to the analysis of the following Cauchy problems, of both local and nonlocal type:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) & \text{in } \Omega \times (0, T_{\max}), \\ \tau v_t = \Delta v - \beta v + f_1(u) & \text{in } \Omega \times (0, T_{\max}), \\ \tau w_t = \Delta w - \delta w + f_2(u) & \text{in } \Omega \times (0, T_{\max}), \\ u_\nu = v_\nu = w_\nu = 0 & \text{on } \partial\Omega \times (0, T_{\max}), \\ u(x,0) = u_0(x),\ \tau v(x,0) = \tau v_0(x),\ \tau w(x,0) = \tau w_0(x) & x \in \bar{\Omega}, \end{cases} \tag{5.1.1}$$

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) & \text{in } \Omega \times (0, T_{\max}), \\ 0 = \Delta v - \frac{1}{|\Omega|} \int_\Omega f_1(u) + f_1(u) & \text{in } \Omega \times (0, T_{\max}), \\ 0 = \Delta w - \frac{1}{|\Omega|} \int_\Omega f_2(u) + f_2(u) & \text{in } \Omega \times (0, T_{\max}), \\ \int_\Omega v(x,t)\, dx = \int_\Omega w(x,t)\, dx = 0 & t \in (0, T_{\max}), \end{cases} \tag{5.1.2}$$

defined in a bounded and smooth domain Ω of $\mathbb{R}^n$, with $n \geq 2$, $\chi, \xi, \beta, \delta > 0$ and $f_1 = f_1(s)$ and $f_2 = f_2(s)$ as in (4.1.1) essentially behaving as s^k and s^l, for some $k, l > 0$.

Remark 5.1. *We recall that, in problem* (5.1.2)*, the functions v and w do not describe the actual distributions of the biological signals but rather their deviations (see Remark 2.1).*

5.1.2 Claims of the Theorems

Once these hypotheses are fixed

$$0 \leq f_1(s) \leq \alpha s^k \qquad \text{and} \qquad \gamma_0 (1+s)^l \leq f_2(s) \leq \gamma_1 (1+s)^l, \tag{5.1.3}$$

for some $\alpha, k, l > 0$, and $\gamma_1 \geq \gamma_0 > 0$, what follows is shown.

Theorem 5.2. *Let Ω be a smooth and bounded domain of $\mathbb{R}^n$, with $n \geq 2$, $\tau = 0$ and χ, ξ, β, δ positive. Moreover, for some $\alpha, \gamma_0, \gamma_1 > 0$, let f_1 and f_2 comply with assumptions* (5.1.3) *in at least one of these situations:*

1. *$k < l$;*
2. *$k, l \in \left(0, \frac{2}{n}\right)$;*
3. *$k = l$ and $\Theta_0 := \chi\alpha - \xi\gamma_0 < 0$, or $l = k \in \left(0, \frac{2}{n}\right)$ and $\Theta_0 \geq 0$.*

Then the solution to problem (5.1.1) *is global and uniformly bounded.*

Theorem 5.3. *Let Ω be a smooth and bounded domain of $\mathbb{R}^n$, with $n \geq 2$, $\tau = 1$ and χ, ξ, β, δ positive. Moreover, for some $\alpha, \gamma_0, \gamma_1 > 0$, let f_1 and f_2 comply with assumptions* (5.1.3) *in at least one of these situations:*

1. *$l, k \in \left(0, \frac{1}{n}\right]$;*
2. *$l \in \left(\frac{1}{n}, \frac{1}{n} + \frac{2}{n^2+4}\right)$ and $k \in \left(0, \frac{1}{n}\right]$, or $k \in \left(\frac{1}{n}, \frac{1}{n} + \frac{2}{n^2+4}\right)$ and $l \in \left(0, \frac{1}{n}\right]$;*
3. *$l, k \in \left(\frac{1}{n}, \frac{1}{n} + \frac{2}{n^2+4}\right)$.*

Then the solution to problem (5.1.1) *is global and uniformly bounded.*

Theorem 5.4. *Let Ω be a smooth and bounded domain of $\mathbb{R}^n$, with $n \geq 2$, and χ, ξ positive. Moreover, for some $\alpha, \gamma_0, \gamma_1 > 0$, let f_1 and f_2 comply with hypotheses* (5.1.3)*, for $k < l$ or for $k = l$, provided that $\Theta_0 := \chi\alpha - \xi\gamma_0$ complies with either $\Theta_0 < 0$, or $\Theta_0 \geq 0$ and $l = k \in \left(0, \frac{2}{n}\right)$. Then the solution to problem* (5.1.2) *is global and uniformly bounded.*

Remark 5.5 (Solving an open question and improvements of known results)**.** *Let us give these comments:*

- ▷ *After [LL21, Theorem 1.2], precisely in [LL21, Remark 1.2], the authors leave an open question by textually writing: "we still have no ideas for the behavior of the solution in the case $k > \frac{2}{n}$ and $k < l$.". The statement of Theorem 5.4 provides the answer to this question and these two theorems give a more complete picture on the behavior of solutions to model* (5.1.2)*.*
- ▷ *The condition $l > k > 0$ derived in Theorem 5.2, and yielding boundedness to model* (5.1.1)*, improves that in [Vig21, Theorem 4.4].*

5.1.3 Main properties of the solution

Lemma 5.6. *The function u is such that*

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx = m \text{ on } (0, T_{\max}). \tag{5.1.4}$$

Moreover

$$f_1(u) \in L^\infty((0,T_{\max}); L^{\frac{1}{k}}(\Omega)) \quad \text{and} \quad f_2(u) \in L^\infty((0,T_{\max}); L^{\frac{1}{l}}(\Omega)), \text{ whenever } k, l \in (0,1]. \tag{5.1.5}$$

Proof. An application of Proposition 4.8, when the first equation has no logistic, leads to

$$\frac{d}{dt}\int_\Omega u(x,t)\,dx = 0.$$

In addition, from bound (5.1.4), we have (5.1.5) by noticing that

$$\int_\Omega f_1(u)^{\frac{1}{k}} \leq \alpha^{\frac{1}{k}} \int_\Omega u = c_1 \quad \text{and similarly } \int_\Omega f_2(u)^{\frac{1}{l}} \leq c_2 \text{ for all } t \in (0, T_{\max}). \tag{5.1.6}$$

□

Inclusions (5.1.5), as well as relations (5.1.6), will be of crucial importance when we analyze properties of solutions to the fully parabolic model (5.1.1). In particular, parabolic regularity theory allows to derive this result, which provides some uniform bounds for $\|v(\cdot,t)\|_{W^{1,r}(\Omega)}$ and $\|w(\cdot,t)\|_{W^{1,s}(\Omega)}$, with $r, s \geq 1$.

Lemma 5.7. *We have that v and w are such that*

$$\int_\Omega |\nabla v(\cdot,t)|^r \leq c_3 \quad \text{on } (0,T_{\max}) \begin{cases} \text{for all } r \in [1,\infty) & \text{if } k \in \left(0,\frac{1}{n}\right], \\ \text{for all } r \in \left[1,\frac{n}{nk-1}\right) & \text{if } k \in \left(\frac{1}{n},1\right], \end{cases}$$

and

$$\int_\Omega |\nabla w(\cdot,t)|^s \leq c_4 \quad \text{on } (0,T_{\max}) \begin{cases} \text{for all } s \in [1,\infty) & \text{if } l \in \left(0,\frac{1}{n}\right], \\ \text{for all } s \in \left[1,\frac{n}{nl-1}\right) & \text{if } l \in \left(\frac{1}{n},1\right]. \end{cases}$$

Proof. The proof takes into account inclusions (5.1.5) and, depending on τ, it is consequence of (3.3.3) or (3.3.6). □

5.1.4 A priori estimates and proof of the theorems

First, our objective is to prove the boundedness criterion for our problem:

Lemma 5.8. *If $u \in L^\infty((0,T_{\max}),L^p(\Omega))$ for all $p>1$, then $u \in L^\infty((0,\infty),L^\infty(\Omega))$.*

Proof. The function $u = u(x,t)$, with $(x,t) \in \Omega \times (0,T_{\max})$, classically solves problem (A.1) in Lemma 4.6 for

$$D(x,t,u) = 1, \quad f(x,t) = -\chi u \nabla v + \xi u \nabla w, \quad g(x,t) = 0.$$

Specifically, by making use of the Neumann boundary conditions, we can see that (A.2)–(A.5) are complied with. It is trivial to see that the second requirement in (A.6) is verified for any $q_2 > 1$. Since, by hypotheses, $u \in L^\infty((0,T_{\max});L^p(\Omega))$ for every $p>1$, we have that $\int_\Omega u^p$ is uniformly bounded on $(0,T_{\max})$ for p arbitrary large (without relabeling it) and henceforth conditions (A.7) and (A.8) are fulfilled. As to the first assumption of (A.6), if $\tau = 0$ in problem (5.1.1), from the obtained inclusion $u \in L^\infty((0,T_{\max});L^p(\Omega))$ we have $f_1(u) \in L^\infty((0,T_{\max});L^p(\Omega))$, and in turn relation (3.3.3) provides $\nabla v \in L^\infty((0,T_{\max});L^\infty(\Omega))$ (where $\psi = f_1$), and similarly $\nabla w \in L^\infty((0,T_{\max});L^\infty(\Omega))$ (where $\psi = f_2$). For $\tau = 1$ we directly invoke (3.3.6) (where $g = f_1$ or $g = f_2$); in both cases we have that, for any $q_1 > 1$, $f \in L^\infty((0,T_{\max});L^{q_1}(\Omega))$. As a consequence of what explained, we have that $u \in L^\infty((0,T_{\max}),L^\infty(\Omega))$ by virtue of Lemma 4.6. Applying Proposition 4.4, together with the considerations presented in Remark 4.5, we arrive at the desired conclusion. □

So our goal in this section is showing the inclusion $u \in L^\infty((0,T_{\max});L^p(\Omega))$, for all $p>1$, since we can take advantage from Lemma 5.8 and directly obtain that, indeed, $u \in L^\infty((0,\infty);L^\infty(\Omega))$; at this point, regularity theories applied to the equations of v and w entail that also v, w belong to $L^\infty((0,\infty);L^\infty(\Omega))$.

To do this, we are going to establish an absorptive differential inequality for the functional $\varphi(t) := \displaystyle\int_\Omega u^p$. To be precise, we aim at deriving this inequality for φ

$$\varphi'(t) + c_5 \int_\Omega |\nabla u^{\frac{p}{2}}|^2 \leq c_6 \quad \text{on } (0,T_{\max}). \tag{5.1.7}$$

In fact, we can successively exploit the Gagliardo–Nirenberg inequality (see Proposition 3.5) combined with (3.1.5) (used in the sequel without mentioning), so to write

$$\int_\Omega u^p = ||u^{\frac{p}{2}}||^2_{L^2(\Omega)} \leq c_7 ||\nabla u^{\frac{p}{2}}||^{2\theta}_{L^2(\Omega)} ||u^{\frac{p}{2}}||^{2(1-\theta)}_{L^{\frac{2}{p}}(\Omega)} + c_7 ||u^{\frac{p}{2}}||^2_{L^{\frac{2}{p}}(\Omega)} \qquad \text{for all } t \in (0,T_{\max}),$$

with

$$0 < \theta = \frac{\frac{np}{2}(1-\frac{1}{p})}{1-\frac{n}{2}+\frac{np}{2}} < 1.$$

In turn, by taking into consideration bound (5.1.4), the above inequality leads to

$$\int_\Omega u^p \le c_8\Big(\int_\Omega |\nabla u^{\frac{p}{2}}|^2\Big)^\theta + c_8 \quad \text{on } (0,T_{\max}), \tag{5.1.8}$$

so that by manipulating it and inserting the result into (5.1.7), we can observe also by virtue of (3.1.4) that φ satisfies this initial problem

$$\begin{cases} \varphi'(t) \le c_9 - c_{10}\varphi^{\frac{1}{\theta}}(t) & \text{for all } t\in(0,T_{\max}),\\ \varphi(0)=\int_\Omega u_0^p. \end{cases}$$

Consequently, Lemma 3.13 implies that $\displaystyle\int_\Omega u^p \le \max\left\{\varphi(0), \left(\frac{c_9}{c_{10}}\right)^\theta\right\} =: L$ for all $t\in(0,T_{\max})$, providing the desired inclusion.

Analysis of solutions to model (5.1.1) with $\tau = 0$; proof of Theorem 5.2

Lemma 5.9. *Let either* $0<k<l$ *or* $k,l\in\left(0,\frac{2}{n}\right)$. *Then for any* $p>\max\left\{l, \frac{l(nl-2)}{n}, \frac{n}{2}\right\}$, u *fulfills for some* $L>0$

$$\int_\Omega u^p \le L \quad \textit{for all} \quad t\in(0,T_{\max}). \tag{5.1.9}$$

Moreover, we have the same conclusion if $k=l$ *and* $\chi\alpha-\xi\gamma_0<0$, *or* $l=k\in\left(0,\frac{2}{n}\right)$ *and* $\chi\alpha-\xi\gamma_0\ge 0$.

Proof. By testing the first equation of problem (5.1.1) with u^{p-1}, using its boundary conditions and taking into account the second and the third equation with $\tau=0$, we have on $(0,T_{\max})$

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^p =& p\int_\Omega u^{p-1}u_t\\
=&-p(p-1)\int_\Omega u^{p-2}|\nabla u|^2-(p-1)\chi\int_\Omega u^p(v-f_1(u))+(p-1)\xi\int_\Omega u^p(w-f_2(u))\\
\le&-p(p-1)\int_\Omega u^{p-2}|\nabla u|^2+(p-1)\chi\int_\Omega u^pf_1(u)+(p-1)\xi\int_\Omega u^pw-(p-1)\xi\int_\Omega u^pf_2(u).
\end{aligned}$$

Now, we recall the definitions of f_1 and f_2 given in (5.1.3), exploit $u<(u+1)$ and inequality (3.1.5) so to obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^p \le& -p(p-1)\int_\Omega u^{p-2}|\nabla u|^2+(p-1)\chi\alpha\int_\Omega u^{p+k}\\
&+(p-1)\xi\int_\Omega u^pw-(p-1)\xi\gamma_0\int_\Omega u^p(u+1)^l\\
\le& -p(p-1)\int_\Omega u^{p-2}|\nabla u|^2+(p-1)\chi\alpha\int_\Omega u^{p+k}\\
&+(p-1)\xi\int_\Omega u^pw-(p-1)\xi\gamma_0\int_\Omega u^{p+l} \quad \text{for all } t\in(0,T_{\max}).
\end{aligned} \tag{5.1.10}$$

As to the third integral on the right-hand side of (5.1.10), an application of Young's inequality, relation (3.3.4) with $z=w$ and $q=\frac{p}{l}+1$ and bound (3.1.5) provide for $\epsilon_1,\sigma,\tilde\sigma>0$

$$\begin{aligned}
(p-1)\xi\int_\Omega u^pw \le& \epsilon_1\int_\Omega u^{p+l}+c_{11}\int_\Omega w^{\frac{p}{l}+1}\\
\le& \epsilon_1\int_\Omega u^{p+l}+\sigma\int_\Omega (f_2(u))^{\frac{p}{l}+1}+c_{12}\left(\int_\Omega f_2(u)\right)^{\frac{p}{l}+1}\\
\le& \epsilon_1\int_\Omega u^{p+l}+\sigma\int_\Omega (\gamma_1(u+1)^l)^{\frac{p}{l}+1}+c_{12}\left(\int_\Omega \gamma_1(u+1)^l\right)^{\frac{p}{l}+1}\\
\le& \epsilon_1\int_\Omega u^{p+l}+\tilde\sigma\int_\Omega u^{l(\frac{p}{l}+1)}+c_{13}\left(\int_\Omega u^l\right)^{\frac{p}{l}+1}+c_{14} \quad \text{on } (0,T_{\max}).
\end{aligned} \tag{5.1.11}$$

If $l \in (0,1]$ the last integral in (5.1.11) is controlled by a constant thanks Hölder's inequality and the mass conservation property (5.1.4). On the other hand, if $l > 1$ we can estimate the term $\left(\int_\Omega u^l\right)^{\bar{p}+1}$ by applying the Gagliardo–Nirenberg inequality, in this way:

$$c_{13}\left(\int_\Omega u^l\right)^{\frac{p+l}{l}} = c_{13}\|u^{\frac{p}{2}}\|^{\frac{2(p+l)}{p}}_{L^{\frac{2l}{p}}(\Omega)} \leq c_{15}\|\nabla u^{\frac{p}{2}}\|^{\frac{2(p+l)}{p}\theta_1}_{L^2(\Omega)}\|u^{\frac{p}{2}}\|^{\frac{2(p+l)}{p}(1-\theta_1)}_{L^{\frac{2}{p}}(\Omega)} + c_{15}\|u^{\frac{p}{2}}\|^{\frac{2(p+l)}{p}}_{L^{\frac{2}{p}}(\Omega)},$$

for all $t \in (0, T_{\max})$, where for p as in our assumptions we have

$$0 < \theta_1 = \frac{1-\frac{1}{l}}{1+\frac{2}{np}-\frac{1}{p}} < 1 \quad \text{and} \quad 0 < \frac{(p+l)}{p}\theta_1 < 1.$$

Hence, by recalling the mass conservation property (5.1.4), the Young and above inequalities entail for any $\epsilon_2 > 0$

$$c_{13}\left(\int_\Omega u^l\right)^{\frac{p+l}{l}} \leq c_{16}\Big(\int_\Omega |\nabla u^{\frac{p}{2}}|^2\Big)^{\frac{(p+l)}{p}\theta_1} + c_{16} \leq \epsilon_2\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_{17} \quad \text{on } (0, T_{\max}). \tag{5.1.12}$$

As a consequence, in both case, by putting estimates (5.1.11) and (5.1.12) into (5.1.10) we get for all $t \in (0, T_{\max})$

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p \leq &-p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + (p-1)\chi\alpha\int_\Omega u^{p+k}\\ &+(\epsilon_1+\tilde{\sigma}-(p-1)\xi\gamma_0)\int_\Omega u^{p+l} + \epsilon_2\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_{18}.\end{aligned} \tag{5.1.13}$$

Now we analyze the case $0 < k < l$ and by applying the Young inequality to the second integral at the right-hand side of the above estimate yields for $\epsilon_3 > 0$

$$(p-1)\chi\alpha\int_\Omega u^{p+k} \leq \epsilon_3\int_\Omega u^{p+l} + c_{19} \quad \text{for all } t \in (0, T_{\max}),$$

so that, by plugging the latter into (5.1.13), and also in view of the identity

$$\int_\Omega u^{p-2}|\nabla u|^2 = \frac{4}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \quad \text{on } (0, T_{\max}),$$

we have

$$\frac{d}{dt}\int_\Omega u^p \leq \left(\epsilon_2 - \frac{4(p-1)}{p}\right)\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + (\epsilon_1+\tilde{\sigma}+\epsilon_3-(p-1)\xi\gamma_0)\int_\Omega u^{p+l} + c_{20} \quad \text{on } (0, T_{\max}).$$

Finally, by choosing $\epsilon_1, \epsilon_2, \epsilon_3, \tilde{\sigma}$ such that $\epsilon_2 < \frac{4(p-1)}{p}$ and $\epsilon_1 + \tilde{\sigma} + \epsilon_3 \leq (p-1)\xi\gamma_0$, we arrive at similar inequality as in (5.1.7). (Note that in the case $l \in (0,1]$ one can directly take $\epsilon_2 = 0$.)

Let turn our attention to the case $l, k \in \left(0, \frac{2}{n}\right)$. Thanks to the Gagliardo–Nirenberg inequality and to the mass conservation property (5.1.4), the estimate of the second integral of the right-hand side of (5.1.13) becomes

$$\begin{aligned}(p-1)\chi\alpha\int_\Omega u^{p+k} =&(p-1)\chi\alpha\|u^{\frac{p}{2}}\|^{\frac{2(p+k)}{p}}_{L^{\frac{2(p+k)}{p}}(\Omega)}\\ \leq& c_{21}\|\nabla u^{\frac{p}{2}}\|^{\frac{2(p+k)}{p}\theta_2}_{L^2(\Omega)}\|u^{\frac{p}{2}}\|^{\frac{2(p+k)}{p}(1-\theta_2)}_{L^{\frac{2}{p}}(\Omega)} + c_{21}\|u^{\frac{p}{2}}\|^{\frac{2(p+k)}{p}}_{L^{\frac{2}{p}}(\Omega)}\\ \leq& c_{22}\left(\int_\Omega |\nabla u^{\frac{p}{2}}|^2\right)^{\frac{(p+k)}{p}\theta_2} + c_{22} \quad \text{on } (0, T_{\max}),\end{aligned}$$

where

$$\theta_2 = \frac{\frac{p}{2} - \frac{p}{2(p+k)}}{\frac{p}{2} + \frac{1}{n} - \frac{1}{2}} \in (0,1).$$

Since $\frac{(p+k)}{p}\theta_2 < 1$ for $0 < k < \frac{2}{n}$, an application of the Young inequality gives for $\epsilon_4 > 0$

$$(p-1)\chi\alpha \int_\Omega u^{p+k} \le \epsilon_4 \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_{23} \quad \text{for all } t \in (0, T_{\max}). \tag{5.1.14}$$

By plugging this gained estimate into bound (5.1.13), we reason as above and the arbitrariness of the constants associated to the terms $\int_\Omega |\nabla u^{\frac{p}{2}}|^2$ and $\int_\Omega u^{p+l}$ allows us to establish for $\varphi(t)$ an estimate as in (5.1.7).

Finally, if we take $k = l$ in (5.1.13), we arrive at the following estimate for all $t \in (0, T_{\max})$:

$$\frac{d}{dt}\int_\Omega u^p \le -p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + ((p-1)(\chi\alpha - \xi\gamma_0) + \epsilon_1 + \tilde{\sigma})\int_\Omega u^{p+l} + \epsilon_2 \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_{18}.$$

If $\chi\alpha - \xi\gamma_0 < 0$, for proper constant $\epsilon_1, \tilde{\sigma}$, we can make $(p-1)(\chi\alpha - \xi\gamma_0) + \epsilon_1 + \tilde{\sigma} \le 0$ and consequently neglect the term associated to $\displaystyle\int_\Omega u^{p+l}$; on the other hand if $\chi\alpha - \xi\gamma_0 \ge 0$ and $l \in \left(0, \frac{2}{n}\right)$ we may treat such integral similarly to (5.1.14). In both cases we conclude. $\square$

Analysis of solutions to model (5.1.2); proof of Theorem 5.4

Lemma 5.10. *Let $l > 1$ and $k < l$. Then there exists $q > l$ such that whenever $p > \max\{\frac{q}{n}(qn-2), \frac{n}{2}\}$ we have that u obeys the same relation in (5.1.9), possibly with a different constant L.*

Moreover, the same conclusion holds true whenever either $l \in (0,1]$ and $k < l$, or $k = l$ and $\chi\alpha - \xi\gamma_0 < 0$, or $l = k \in \left(0, \frac{2}{n}\right)$ and $\chi\alpha - \xi\gamma_0 \ge 0$.

Proof. By reasoning as in the proof of Lemma 5.9, we deduce the following estimate on $(0, T_{\max})$:

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^p =& -p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 - (p-1)\chi\int_\Omega u^p\left(\frac{1}{|\Omega|}\int_\Omega f_1(u) - f_1(u)\right)\\
&+ (p-1)\xi\int_\Omega u^p\left(\frac{1}{|\Omega|}\int_\Omega f_2(u) - f_2(u)\right)\\
\le& -p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + (p-1)\chi\alpha\int_\Omega u^{p+k} + \frac{(p-1)\xi}{|\Omega|}\int_\Omega f_2(u)\int_\Omega u^p\\
&-(p-1)\xi\gamma_0\int_\Omega u^{p+l}\\
\le& -p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + (p-1)\chi\alpha\int_\Omega u^{p+k} + c_{24}\int_\Omega u^p + c_{25}\int_\Omega u^l\int_\Omega u^p\\
&-(p-1)\xi\gamma_0\int_\Omega u^{p+l}.
\end{aligned} \tag{5.1.15}$$

Since $k < l$, by exploiting Young's inequality we obtain for $\epsilon_5 > 0$

$$(p-1)\chi\alpha\int_\Omega u^{p+k} \le \epsilon_5\int_\Omega u^{p+l} + c_{26} \quad \text{on } (0, T_{\max}).$$

If $0 < l \le 1$, the product of the two integrals $\int_\Omega u^l \int_\Omega u^p$ in (5.1.15) can be treated by exploiting the mass conservation property (5.1.4) and bound (5.1.8), so to get for $\epsilon_6 > 0$

$$\int_\Omega u^l \int_\Omega u^p \le c_{27}\int_\Omega u^p \le \epsilon_6 \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_{28} \quad \text{for all } t \in (0, T_{\max}).$$

On the other hand, if $l > 1$ a twice application of the Hölder inequality gives

$$\int_\Omega u^l \int_\Omega u^p \le c_{29}\left[\left(\int_\Omega u^q\right)^{\frac{p}{q}+1}\right]^{\frac{l}{p+q}} \left(\int_\Omega u^{p+l}\right)^{\frac{p}{p+l}} \quad \text{on } (0, T_{\max}).$$

The restriction $q > l$ implies $\frac{l}{p+q} + \frac{p}{p+l} < 1$, so that by invoking inequality (3.1.6) we can get for $\epsilon_7 > 0$

$$\int_\Omega u^l \int_\Omega u^p \leq \epsilon_7 \int_\Omega u^{p+l} + \epsilon_7 \left(\int_\Omega u^q\right)^{\frac{p+q}{q}} + c_{30} \quad \text{on } (0, T_{\max}).$$

At this point, we can manipulate the term $\left(\int_\Omega u^q\right)^{\frac{p+q}{q}}$ as in (5.1.12). Rephrasing estimate (5.1.15) in terms of what derived, the same motivations used in Lemma 5.9 give the claim. □

Analysis of solutions to model (5.1.1) with $\tau = 1$; proof of Theorem 5.3

Lemma 5.11. *For any $p > 1$ and $r, s > 2$, we have that u is such that for $\epsilon_8, \epsilon_9 > 0$*

$$\frac{d}{dt}\int_\Omega u^p \leq (\epsilon_8 + \epsilon_9 - p(p-1)) \int_\Omega u^{p-2}|\nabla u|^2 + \int_\Omega |\nabla v|^r + \int_\Omega |\nabla w|^s + c_{34} \int_\Omega u^{\frac{pr}{r-2}} + c_{36} \int_\Omega u^{\frac{ps}{s-2}}, \tag{5.1.16}$$

for all $t \in (0, T_{\max})$.

Proof. Testing procedures imply the following estimate on $(0, T_{\max})$

$$\frac{d}{dt}\int_\Omega u^p = -p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + p(p-1)\chi \int_\Omega u^{p-1}\nabla u \cdot \nabla v - p(p-1)\xi \int_\Omega u^{p-1}\nabla u \cdot \nabla w \tag{5.1.17}$$

whereas a twice application of Young's inequality infers for $\epsilon_8 > 0$ and for all $t \in (0, T_{\max})$

$$\begin{aligned} p(p-1)\chi \int_\Omega u^{p-1}\nabla u \cdot \nabla v \leq& \epsilon_8 \int_\Omega u^{p-2}|\nabla u|^2 + c_{31}\int_\Omega u^p |\nabla v|^2 \\ \leq& \epsilon_8 \int_\Omega u^{p-2}|\nabla u|^2 + \int_\Omega |\nabla v|^r + c_{32}\int_\Omega u^{\frac{pr}{r-2}}, \end{aligned} \tag{5.1.18}$$

and for $\epsilon_9 > 0$ on $(0, T_{\max})$

$$\begin{aligned} -p(p-1)\xi \int_\Omega u^{p-1}\nabla u \cdot \nabla w \leq& \epsilon_9 \int_\Omega u^{p-2}|\nabla u|^2 + c_{33}\int_\Omega u^p |\nabla w|^2 \\ \leq& \epsilon_9 \int_\Omega u^{p-2}|\nabla u|^2 + \int_\Omega |\nabla w|^s + c_{34}\int_\Omega u^{\frac{ps}{s-2}}. \end{aligned} \tag{5.1.19}$$

The claim is deduced by simply plugging relations (5.1.18) and (5.1.19) into bound (5.1.17). □

Lemma 5.12. *Let either $l, k \in \left(0, \frac{1}{n}\right] = I$, or $l, k \in \left(\frac{1}{n}, \frac{1}{n} + \frac{2}{n^2+4}\right) = J$, or $l \in I$ and $k \in J$ (or viceversa). Then for some $p > \frac{n}{2}$ and some $L_1 > 0$, we have that u complies with*

$$\int_\Omega u^p \leq L_1 \quad \textit{for all} \quad t \in (0, T_{\max}).$$

Proof. Let us set $p_0 = \frac{n}{2}$, $\rho_0 = \frac{2}{n}$ and $\phi(p, \rho, \sigma) := \frac{p\sigma}{\sigma-2} - (p + \rho)$, which is defined for all $p > 1$, $\rho > 0$ and $\sigma > 2$. Since $\phi(p_0, \rho_0, \sigma) = \frac{n^2-2\sigma+4}{n(\sigma-2)} < 0$ for all $\sigma > \frac{n^2+4}{2}$, by continuity there are $p > p_0$ and $\rho < \rho_0$ such that $\phi(p, \rho, \sigma) < 0$, i.e.

$$\frac{p\sigma}{\sigma - 2} < p + \rho. \tag{5.1.20}$$

Being our scope controlling the terms $\int_\Omega |\nabla v|^r$, $\int_\Omega u^{\frac{pr}{r-2}}$ and $\int_\Omega |\nabla w|^s$, $\int_\Omega u^{\frac{ps}{s-2}}$ appearing in (5.1.16), let us take into consideration the regularity of v and w, and in particular the functional spaces where ∇v and ∇w belong, and let us study three different cases.

- Case $k, l \in I$. From Lemma 5.7, it is known that $\nabla v \in L^\infty((0,T_{\max}); L^r(\Omega))$ and $\nabla w \in L^\infty((0,T_{\max}); L^s(\Omega))$ for all $r, s > 1$, so that we may pick $r = s = \sigma > \frac{n^2+4}{2} > 2$, and by applying (5.1.20) we have

$$
\begin{aligned}
p(p-1)\chi \int_\Omega u^{p-1}\nabla u \cdot \nabla v \leq & \epsilon_8 \int_\Omega u^{p-2}|\nabla u|^2 + \int_\Omega |\nabla v|^r + c_{32} \int_\Omega u^{\frac{pr}{r-2}} \\
\leq & \epsilon_8 \int_\Omega u^{p-2}|\nabla u|^2 + \int_\Omega u^{p+\rho} + c_{35} \quad \text{for all } t \in (0, T_{\max}),
\end{aligned}
\tag{5.1.21}
$$

and

$$
\begin{aligned}
-p(p-1)\xi \int_\Omega u^{p-1}\nabla u \cdot \nabla w \leq & \epsilon_9 \int_\Omega u^{p-2}|\nabla u|^2 + \int_\Omega |\nabla w|^s + c_{34} \int_\Omega u^{\frac{ps}{s-2}} \\
\leq & \epsilon_9 \int_\Omega u^{p-2}|\nabla u|^2 + \int_\Omega u^{p+\rho} + c_{36} \quad \text{on } (0, T_{\max}).
\end{aligned}
\tag{5.1.22}
$$

- Case $k \in I$ and $l \in J$ (or, viceversa, $k \in J$ and $l \in I$). We discuss the first situation, for which accordingly to Lemma 5.7, we have $\nabla v \in L^\infty((0,T_{\max}); L^r(\Omega))$, for all $r \geq 1$ and $\nabla w \in L^\infty((0,T_{\max}); L^s(\Omega))$ for all $1 \leq s < \frac{n}{nl-1}$. (The second case is totally analogous.) Clearly estimate (5.1.21) still holds, while for the validity of bound (5.1.22) we have to check that $\frac{n}{nl-1} > \frac{n^2+4}{2}$ for all $l \in J$. Since $\Lambda(l) := \frac{n}{nl-1}$ is decreasing in J and $\Lambda\left(\frac{1}{n} + \frac{2}{n^2+4}\right) = \frac{n^2+4}{2}$, we conclude.

- Case $l, k \in J$. By following for k the same arguments used previously for l, we can ensure that (5.1.21) and (5.1.22) are satisfied.

As a consequence of this, by putting bounds (5.1.21) and (5.1.22) into estimate (5.1.17) and by manipulating the term $\int_\Omega u^{p+\rho}$, with $\rho < \frac{2}{n}$ as the integral in (5.1.14), we are in the position to deduce the claim. □

5.1.5 Numerical simulations

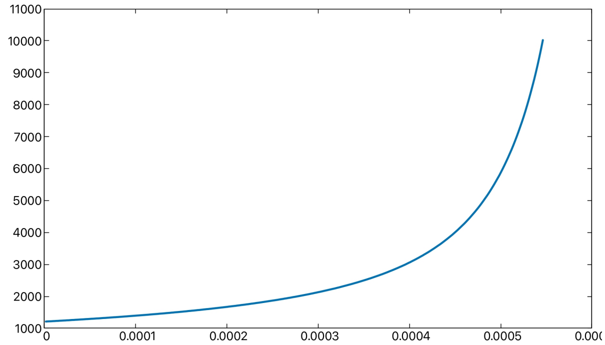
Numerical simulations of taxis-driven models are by no means an easy goal and addressing technical and computational details connected with this analysis is a very challenging issue for the researchers in this field. This is essentially due to the fact that solutions to such systems may exhibit interesting mathematical properties (as for instance blow-up) which cannot be easily reproduced in a discrete framework; indeed, spurious oscillations may appear during the simulations and this might lead to inconsistent results.

In this section we numerically test the presented theoretical results by simulating system (5.1.1) in two dimensions, for both the elliptic and parabolic version. The simulations are used to detect specific data and values of the parameters which delineate regions where different behaviors of solutions may manifest. We are, in particular, interested in two specific scenarios: solutions are bounded and their asymptotic behavior converges to a constant stationary solution, or they suffer from chemotactic blow-up in finite time.

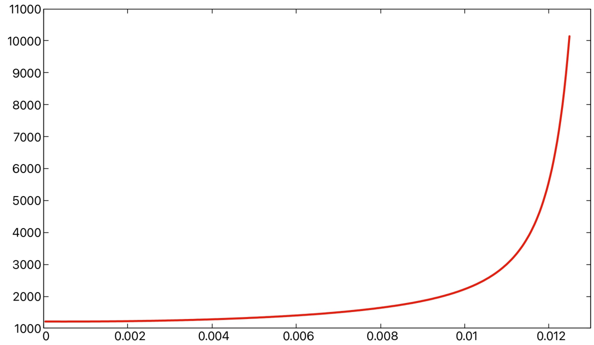
To this aim we introduce a Finite Element Method (FEM) space discretization, defined on a triangular mesh of the domain Ω, with first order piecewise polynomials. On the other hand, we consider an implicit Euler scheme with time discretization Δt, and the algorithm at each time step consists of two operations: First we compute v and w as solutions of the (uncoupled) nonlinear equations in (5.1.1) modeling the productions of these signals, being u explicitly given from the previous time step. Successively, the values of u at any node of the mesh are updated by solving the chemotaxis equation, exactly using the gained quantities of the repulsive and attractive substances at those nodes. Explicitly we have this

Algorithm 5.13. *Let j indicate the time step. For $j = 0, 1, \dots$, we distinguish the cases $\tau = 0$ and $\tau = 1$:*

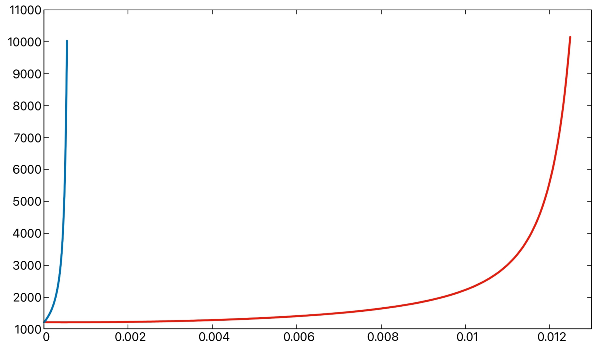
- ▷ $\tau = 0$. *Given u_j, we compute v_j and w_j. Then we obtain u_{j+1}, by using (u_j, v_j, w_j).*
- ▷ $\tau = 1$. *Given (u_j, v_j, w_j), we compute v_{j+1} and w_{j+1}. Then we obtain u_{j+1}, by using (u_j, v_{j+1}, w_{j+1}).*



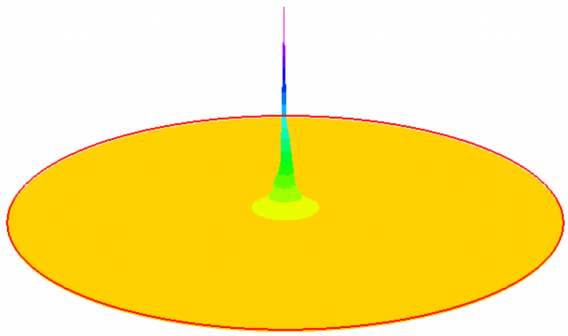
(a) Evolution in time of $\max u$.



(b) Evolution in time of $\max u$.



(c) Comparison of $\max u$.



(d) Representation of δ-formations.

Figure 5.1: **Blow-up case.** Simulations of model (5.1.1) for $\tau = 0$ (blue color) and $\tau = 1$ (red color). Evolution of the cells' density: max ≈ 0.00054 (top left), $T_{\max} \approx 0.0012$ (top right), comparison of behaviors (bottom left), qualitative representation of finite time blow-up at a spatial point (bottom right). The quantitative values are indicated on the axes.

As to the blow-up scenario, we fix this threshold criterion:

- ▷ *If for some j it holds that $\max u_j \geq 10^4$, then $T_{max} \approx j\Delta t$.*

The computations are developed by means of the the open source library FreeFem++ (see [Hec12]).

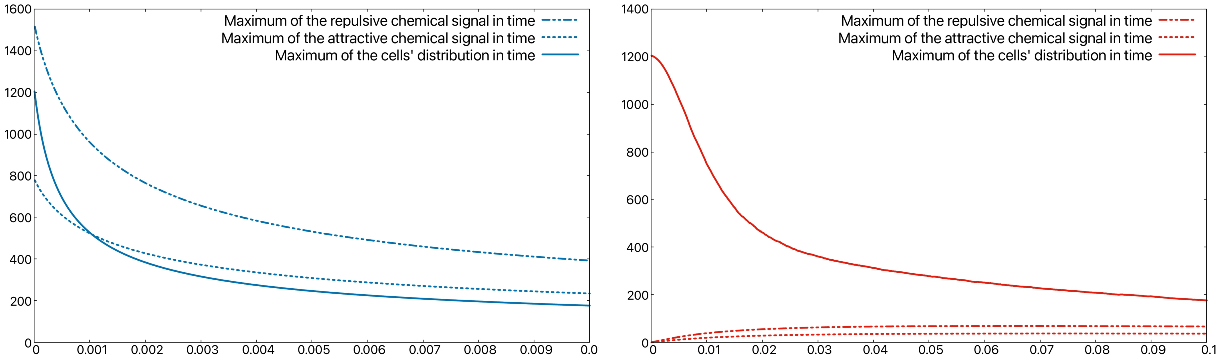
Comparing blow-up and boundedness scenarios for the cases $\tau = 0$ and $\tau = 1$

We consider a mesh of the unit circle $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 81\}$ composed by 43148 triangular elements, and a time discretization $\Delta t = 10^{-5}$. We take $\chi = \xi = 1$, $f_1(u) = \alpha u^k$, $\gamma_0 = \gamma_1$ and, thence, $f_2(u) = \gamma_0(1+u)^l$, $\Theta_0 = \chi\alpha - \xi\gamma_0 = 0$ and the Gaussian bell-shaped initial values

$$\begin{cases} u_0(x_1, x_2) = 15e^{-(x_1^2+x_2^2)}(81 - (x_1^2 + x_2^2)) & \text{if } \tau = 0, \\ u_0(x_1, x_2) = 15e^{-(x_1^2+x_2^2)}(81 - (x_1^2 + x_2^2)), v_0(x_1, x_2) = w_0(x_1, x_2) = e^{-(x_1^2+x_2^2)} & \text{if } \tau = 1. \end{cases}$$

Starting by highly spiky initial cells' density (whose value is 1215), our objective is finding production rates for the chemorepellent and chemoattractant so that either the value of such a spike becomes higher and the appearance of δ-formations is observed, or decreases and stabilizes. As to the first goal (for which we adopted an adaptive mesh refinement), we choose k, l outside the boundedness ranges in Theorems 5.2 and 5.3, and precisely $k = 1.1$ and $l = 0.9$. For both models ($\tau = 0$ and $\tau = 1$) it is seen (Figure 5.1) that the cell distribution uncontrollably increases, up to an explosion at finite time. Nevertheless, a significant difference between the corresponding values of the blow-up time is seen, and this difference can be well appreciated in Subfigure 5.1c.

Oppositely, without changing u_0, we can prevent the previous singularity by suitably changing the values of k and l. Specifically, for $k = 1.1$ and $l = 1.2$ the model does not present any gathering effect



(a) Evolution in time of $\max u$, $\max v$ and $\max w$.

(b) Evolution in time of $\max u$, $\max v$ and $\max w$.

Figure 5.2: **Global boundedness case.** Simulations of model (5.1.1) for $\tau = 0$ (blue color) and $\tau = 1$ (red color). Evolution of the cells' density and the chemical signals. Whilst for $\tau = 1$, we have that $\max v_0$ and $\max w_0$ are given, for $\tau = 0$ they are computed accordingly to what said in Algorithm 5.13. The quantitative values are indicated on the axes.

and the cells and chemical densities remain bounded in time. We point out that the result obtained with these values of k and l is consistent with assumption (1) in Theorem 5.2, but it cannot be discussed in the frame of Theorem 5.3. Finally, Figure 5.2 quantifies the maximum values of u, v and w; in particular, it is possible to observe how the trend of the curves representing $\max u$, $\max v$ and $\max w$ approach the plateau and they become constant throughout the time. Actually (data not shown) they eventually achieve the constant equilibrium $(u, v, w) = \left(\frac{m}{|\Omega|}, \frac{\alpha}{\beta}\left(\frac{m}{|\Omega|}\right)^k, \frac{\gamma}{\delta}\left(1 + \frac{m}{|\Omega|}\right)^l\right)$, which manifestly solves the stationary problem

$$\begin{cases} 0 = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) & \text{in } \Omega \times (0, \infty), \\ 0 = \Delta v - \beta v + \alpha u^k & \text{in } \Omega \times (0, \infty), \\ 0 = \Delta w - \delta w + \gamma_0 (1 + u)^l & \text{in } \Omega \times (0, \infty), \\ u_\nu = v_\nu = w_\nu = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases}$$

Analyzing the influence of the parameters k and l on $T_{\max}$

With Ω, Δt, $f_1(u)$, $f_2(u)$ and Θ_0 introduced above, in Table 5.1 we schematize heuristic analyses of some simulations by particularly indicating the value of the blow-up time for different initial configurations and some parameter sweep for k, l, Θ_0. The first row of the table was inserted to show the consistency of the numerical simulations with the theoretical conclusion of Theorem 5.2, when item (3) is considered. The other situations are of natural interpretation. The second and third rows give hints about the increasing of the blow-up time when k decreases; in particular, for $k > l$, but $k < 1 \left(= \frac{2}{2}\right)$, gathering effects are appreciable. (Recall that for model (5.1.2), blow-up in planar domain is ensured for $k > l$ and $k > 1$.) If, indeed, we observe the part of the table dealing with the parabolic case, from the one hand boundedness is achieved for $k = l = 0.5$, accordingly to the assumption (1) of Theorem 5.3, but from the other hand the same might occur for higher values. A further conclusion can be extrapolated by the fourth and sixth rows: indeed, one can observe that whenever the initial distribution of the chemoattractant v_0 presents a higher maximum, the related blow-up time decreases.

	u_0	v_0	w_0	k	l	Θ_0	$T_{\max}$
	$h(x_1,x_2)$			0.5	0.5	0.36	$+\infty$
$\tau=0$	$h(x_1,x_2)$			1.2	1	-0.2	0.00035
	$h(x_1,x_2)$			0.8	0.6	-0.64	0.0188
	$h(x_1,x_2)$	$e^{-(x_1^2+x_2^2)}$	$e^{-(x_1^2+x_2^2)}$	1	0.8	-0.2	0.025
$\tau=1$	$h(x_1,x_2)$	$e^{-(x_1^2+x_2^2)}$	$e^{-(x_1^2+x_2^2)}$	0.5	0.5	0.52	$+\infty$
	$h(x_1,x_2)$	$5e^{-(x_1^2+x_2^2)}$	$e^{-(x_1^2+x_2^2)}$	1	0.8	-0.2	0.022
	$h(x_1,x_2)$	$5e^{-(x_1^2+x_2^2)}$	$e^{-(x^2+x_2^2)}$	0.8	0.8	-0.58	$+\infty$

Table 5.1: Approximated values of $T_{\max}$ in terms of the sizes of k, l, Θ_0 and the initial data, where $h(x_1,x_2) = 15e^{-(x_1^2+x_2^2)}(81 - x_1^2 - x_2^2)$. With $T_{\max} = +\infty$ we mean that the solution is globally bounded.

5.2 [CFV23a] Properties of given unbounded solutions to a class of chemotaxis models

5.2.1 The formulation of the mathematical problem

In this section we are interested in properties of unbounded classical solutions to this problem

$$
\begin{cases}
u_t = \nabla \cdot ((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v & \\
\qquad +\xi u(u+1)^{m_3-1}\nabla w) + \lambda u - \mu u^k & \text{in } \Omega \times (0, T_{\max}), \\
0 = \Delta v - m_1(t) + f_1(u) & \text{in } \Omega \times (0, T_{\max}), \\
0 = \Delta w - m_2(t) + f_2(u) & \text{in } \Omega \times (0, T_{\max}), \\
u_\nu = v_\nu = w_\nu = 0 & \text{on } \partial\Omega \times (0, T_{\max}), \\
u(x,0) = u_0(x) & x \in \bar{\Omega}, \\
\int_\Omega v(x,t)\, dx = \int_\Omega w(x,t)\, dx = 0 & \text{for all } t \in (0, T_{\max}).
\end{cases}
\tag{5.2.1}
$$

Additionally, the functions $m_i(t)$ are defined for compatibility in the second and third equation (by integrating these over Ω) in terms of $f_i(u)$ themselves, exactly as

$$
m_1(t) = \frac{1}{|\Omega|}\int_\Omega f_1(u) \quad \text{and} \quad m_2(t) = \frac{1}{|\Omega|}\int_\Omega f_2(u). \tag{5.2.2}
$$

with f_1 and f_2 as in (4.1.1). Herein we assume that for all $s \geq 0$, $\alpha, \beta > 0$ and some $k_1, k_2, k_3 > 0$,

$$
f_1(s) \leq k_1(s+1)^\alpha \quad \text{and} \quad f_2(s) \leq k_2(s+1)^\beta. \tag{5.2.3}
$$

We remind that this is a nonlocal model: v and w are the deviation of the biological distribution of the chemical signals (see Remark 2.1).

5.2.2 Presentation of the Theorems

This work finds its motivations in the observation that there is no automatic connection between the occurrence of blow-up for solutions to model (5.2.1) in the $L^\infty(\Omega)$-norm and that in $L^p(\Omega)$-norm ($p > 1$). Indeed, for a bounded domain Ω, it is seen that

$$
\|u(\cdot,t)\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{p}} \|u(\cdot,t)\|_{L^\infty(\Omega)},
$$

so that unboundedness in $L^p(\Omega)$-norm implies that in $L^\infty(\Omega)$-norm, but oppositely $\int_\Omega u^p$ might even remain bounded in a neighborhood of $T_{\max}$ when $\max_\Omega u$ uncontrollably increases at some finite time $T_{\max}$.

In light of this, in order to bridge the gap between the analysis of the blow-up time $T_{\max}$ in the two different mentioned norms, we aim at

(i) detecting suitable $L^p(\Omega)$ spaces, for certain p depending on n, m_1, m_2, m_3, α and β, such that given unbounded solutions also blow up in the associated $L^p(\Omega)$-norms;

(ii) providing lower bounds for the blow-up time of the aforementioned solutions in these $L^p(\Omega)$-norms.

In order to deal with issues (i) and (ii), we fix the following relations, determining some precise interplay involving constants defining system (5.2.1):

Assumptions 5.14. *Let $n \in \mathbb{N}$, $m_1, m_2, m_3 \in \mathbb{R}$ and $\alpha, \beta, \chi, \xi, \lambda, \mu > 0$ and $k > 1$ be such that*

($\mathcal{H}_1$) $m_2 + \alpha > m_1 + \frac{2}{n}$.

Moreover, for $\beta > 0$, let either

($\mathcal{H}_2$) $m_2 + \alpha > \max\{1, m_3 + \beta\}$ or ($\mathcal{H}_3$) $m_2 + \alpha \geq m_3 + \beta$ and $m_1 > 1 - \frac{2}{n}$,

whereas, for $\beta \in (0, 1]$, let either

($\mathcal{H}_4$) $m_3 \leq 1$ and $m_1 > 1 - \frac{2}{n}$ or ($\mathcal{H}_5$) $m_2 + \alpha \geq m_3$ and $m_1 > 1 - \frac{2}{n}$.

With the support of the above position, and as far as the analysis of (i) is concerned, in the spirit of [Fre18, Theorem 2.2] we will prove this first result, dealing with properties of *given unbounded solutions* to model (5.2.1). Specifically, the proof is based on the analysis of the functional $\varphi(t) = \frac{1}{p}\int_\Omega (u+1)^p$, defined for local solutions to problem (5.2.1) on $(0, T_{\max})$. We will show that if for some p sufficiently large $\varphi(t)$ is uniformly bounded in time, then it also is for any arbitrarily large $p > 1$, so contrasting with the unboundedness of u itself.

Theorem 5.15. *Let Ω be a bounded and smooth domain of $\mathbb{R}^n$, and condition ($\mathcal{H}_1$) as well as one of ($\mathcal{H}_2$),($\mathcal{H}_3$),($\mathcal{H}_4$), ($\mathcal{H}_5$) in Assumptions 5.14 hold true. Moreover, for f_1 and f_2 complying with* (5.2.3)*, let (u, v, w) be a solution to problem* (5.2.1) *which blows-up at some finite time $T_{\max}$, in the sense that*

$$\limsup_{t \to T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = +\infty. \tag{5.2.4}$$

Then, for any $\mathfrak{p} > \frac{n}{2}(m_2 - m_1 + \alpha)$, we also have

$$\limsup_{t \to T_{\max}} \|u(\cdot, t)\|_{L^{\mathfrak{p}}(\Omega)} = +\infty.$$

Successively aim (ii) is achieved by establishing for the same functional $\varphi(t)$ a first order differential inequality (ODI) as introduced in Section 4.4.

Theorem 5.16. *Let the hypotheses of Theorem 5.15 be satisfied. Then there exist $\bar{p} > 1$ and positive constants $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$, as well as $\gamma > \delta > 1$, such that the blow-up time $T_{\max}$ complies with both the implicit estimate*

$$T_{\max} \geq \int_{\varphi(0)}^{\infty} \frac{d\tau}{\mathcal{A}\tau^\gamma + \mathcal{B}\tau^\delta + \mathcal{C}}, \tag{5.2.5}$$

and the explicit one

$$T_{\max} \geq \frac{\varphi(0)^{1-\gamma}}{\mathcal{D}(\gamma - 1)},$$

where $\varphi(0) = \dfrac{1}{p}\displaystyle\int_\Omega (u_0 + 1)^p$, for any $p \geq \bar{p}$.

Figure 5.3: Some infimum of $\mathfrak{p}$ and $\bar{p}$, taken from their definitions in Lemma 5.17. From top to bottom:

- Under assumption $(\mathcal{H}_3)$, and $n=1$, $m_1=\frac{81}{50}$, $m_2=-\frac{149}{100}$, $m_3=\frac{33}{20}$, $\alpha=\frac{587}{100}$ and $\beta=\frac{63}{25}$, we have $\mathfrak{p}=\frac{n}{2}\left(m_2-m_1+\alpha\right)=\frac{69}{50}$ and $\bar{p}=m_3(n+2)(n+1)=\frac{99}{10}$;
- Under the assumption $(\mathcal{H}_2)$, and $n=5$, $m_1=-\frac{3}{50}$, $m_2=\frac{6}{5}$, $m_3=-\frac{143}{100}$, $\alpha=\frac{163}{100}$ and $\beta=\frac{169}{100}$, we have $\mathfrak{p}=\frac{n}{2}\left(m_2-m_1+\alpha\right)=\frac{289}{40}$ and $\bar{p}=m_2(n+2)(n+1)=\frac{252}{5}$;
- Under the assumption $(\mathcal{H}_2)$, and $n=4$, $m_1=-\frac{187}{100}$, $m_2=-\frac{89}{100}$, $m_3=-\frac{181}{100}$, $\alpha=\frac{353}{100}$ and $\beta=\frac{19}{50}$, we have $\mathfrak{p}=\frac{n}{2}\left(m_2-m_1+\alpha\right)=\frac{451}{50}$ and $\bar{p}=n\alpha=\frac{353}{25}$;
- Under the assumption $(\mathcal{H}_3)$, and $n=4$, $m_1=\frac{47}{50}$, $m_2=\frac{6}{25}$, $m_3=-\frac{63}{50}$, $\alpha=\frac{179}{100}$ and $\beta=\frac{119}{50}$, we have $\mathfrak{p}=\frac{n}{2}\left(m_2-m_1+\alpha\right)=\frac{109}{50}$ and $\bar{p}=n\beta=\frac{238}{25}$.

5.2.3 Miscellaneous and general comments

On the parameters $\mathfrak{p}$ and $\bar{p}$

We herein want to spend some words on the role of the parameters $\mathfrak{p}$ and $\bar{p}$ appearing in Theorems 5.15 and 5.16. In particular, as we will observe in Lemma 5.17 below, $\mathfrak{p}$ and $\bar{p}$ depend on n, m_1, m_2, m_3, α and β, and Figure 5.3 collects some of their values on the p axis. In the specific, for a blowing-up solution to (5.2.1), we will have that $\int_\Omega u^p$ is bounded on $(0, T_{\max})$ for $p=1$, while it blows up for $p \geq \mathfrak{p}$; in general, the behaviour of $\int_\Omega u^p$ on $(0, T_{\max})$ for $p \in (1, \mathfrak{p})$ is unknown. On the other hand, an estimate for $T_{\max}$ is given in terms of $\frac{1}{p}\int_\Omega (u_0+1)^p$, for $p \geq \bar{p}$, but not when $p \in (\mathfrak{p}, \bar{p})$.

On the automatic applicability of Theorems 5.15 and 5.16 in some related chemotaxis contexts

We can observe that Theorems 5.15 and 5.16 can also be used in models close to (5.2.1), for which unbounded solutions can be detected; in particular, for attraction-repulsion (linear and nonlinear) models with or without logistic and general production laws ([WZZ23, LL21]), or for only attraction ones ([WL21, BFL21, Win18a, Tan22, CMTY21, YMXD21]).

5.2.4 Necessary parameters

An application of Proposition 4.8 with $\rho = 1$, leads to

$$\int_\Omega u \leq M := \max\{M_0, C\} \quad \text{for all } t \in [0, T_{\max}), \tag{5.2.6}$$

where $M_0 = \displaystyle\int_\Omega u_0(x)\,dx$ and $C := \left(\frac{\lambda}{\mu}|\Omega|^{k-1}\right)^{\frac{1}{k-1}}$.

Let us continue with the following technical

Lemma 5.17. *Let $n \in \mathbb{N}$, $m_1, m_2, m_3, \alpha, \beta$ be as in Assumptions 5.14. Moreover, for $\mathfrak{p} > \frac{n}{2}(m_2 - m_1 + \alpha)$ let*

$$\bar{p} > \max\left\{\mathfrak{p},\, 1 - m_1(n+2),\, 1 - m_2,\, 1 - m_3,\, 2 - m_1 - \tfrac{2}{n},\, n\alpha,\, n\beta,\, m_2(n+2)(n+1),\, m_3(n+2)(n+1)\right\},$$

$$\sigma := \frac{2(p + m_2 + \alpha - 1)}{p + m_1 - 1}, \quad \hat{\sigma} := \frac{2p}{p + m_1 - 1},$$

$$\gamma := \frac{\frac{m_1 - m_2 - \alpha}{2} + \frac{p}{n} + \frac{m_2 + \alpha - 1}{n}}{\frac{m_1 - m_2 - \alpha}{2} + \frac{p}{n}}, \quad \delta := \frac{p + m_2 + \alpha - 1}{p}.$$

Then for all $p \geq \bar{p}$ these relations hold

$$p > \frac{n}{2}(1 - m_1), \tag{5.2.7a}$$

$$0 < \theta := \frac{\frac{p+m_1-1}{2\mathfrak{p}} - \frac{p+m_1-1}{2(p+m_2+\alpha-1)}}{\frac{p+m_1-1}{2\mathfrak{p}} + \frac{1}{n} - \frac{1}{2}} < 1, \tag{5.2.7b}$$

$$0 < \frac{\sigma\theta}{2} < 1, \tag{5.2.7c}$$

$$0 < \hat{\theta} := \frac{\frac{p+m_1-1}{2} - \frac{p+m_1-1}{2p}}{\frac{p+m_1-1}{2} + \frac{1}{n} - \frac{1}{2}} < 1, \tag{5.2.7d}$$

$$0 < \frac{\hat{\sigma}\hat{\theta}}{2} < 1, \tag{5.2.7e}$$

$$0 < \frac{p + m_3 - 1}{p + m_2 + \alpha - 1} < 1, \tag{5.2.7f}$$

$$0 < \frac{\beta}{p + m_2 + \alpha - 1} < 1, \tag{5.2.7g}$$

$$\frac{p + m_3 - 1}{p + m_2 + \alpha - 1} + \frac{\beta}{p + m_2 + \alpha - 1} < 1, \tag{5.2.7h}$$

$$0 < \frac{p}{p + m_2 + \alpha - 1} < 1, \tag{5.2.7i}$$

$$\gamma > \delta > 1, \tag{5.2.7j}$$

$$0 < \frac{\sigma\bar{\theta}}{2} < 1, \tag{5.2.7k}$$

$$0 < \frac{p + m_3 - 1}{p} < 1, \tag{5.2.7l}$$

$$0 < \bar{\theta} := \frac{\frac{p+m_1-1}{2p} - \frac{p+m_1-1}{2(p+m_2+\alpha-1)}}{\frac{p+m_1-1}{2p} + \frac{1}{n} - \frac{1}{2}} < 1. \tag{5.2.7m}$$

Proof. To show our relations, we will need $p > 1 - m_1$ and $p > 1 + \beta - m_2 - \alpha$. As to the first inequality, due to $(\mathcal{H}_1)$ and the restriction on $\mathfrak{p}$, we have that $p \geq \bar{p} > \mathfrak{p} > 1 \geq 1 - m_1$ for $m_1 \geq 0$, whilst for $m_1 < 0$ it suddenly derives from $p \geq \bar{p} > 1 - m_1(n+2)$. For the second lower bound, assumptions $(\mathcal{H}_2)$ or $(\mathcal{H}_3)$, together with the definition of $\bar{p}$, give $p \geq \bar{p} > 1 - m_3 > 1 + \beta - m_2 - \alpha$. Besides, $m_2 + \alpha > 1$ is automatically true through $(\mathcal{H}_2)$, or alternatively by means of assumption $(\mathcal{H}_1)$ in conjunction with one among $(\mathcal{H}_3)$, $(\mathcal{H}_4)$ or $(\mathcal{H}_5)$. The same condition $m_2 + \alpha > 1$, $p \geq \bar{p}$ and the restriction on $\mathfrak{p}$ and $p > 1 - m_1$ ensure relations (5.2.7a), (5.2.7b), (5.2.7c), (5.2.7m) and (5.2.7k). Moreover, from (5.2.7a), $p > 1 - m_1$ and $p > 2 - m_1 - \frac{2}{n}$, we obtain (5.2.7d) and by using also $m_1 > 1 - \frac{2}{n}$, we get (5.2.7e). On the other hand, restrictions (5.2.7f), (5.2.7g), (5.2.7h), (5.2.7i) and (5.2.7l) come from the definition of $\bar{p}$ in conjunction with $m_2 + \alpha > m_3$, $p > 1 + \beta - m_2 - \alpha$, $m_2 + \alpha > m_3 + \beta$ and $m_2 + \alpha > 1$, respectively. Finally, from $(\mathcal{H}_1)$ it follows that $m_2 + \alpha > m_1$, which combined with $m_2 + \alpha > 1$ gives (5.2.7j). □

Remark 5.18. *For reasons which will be exploited later on, and precisely in Lemma 5.19, it appears important to point out that assumptions $(\mathcal{H}_3)$, $(\mathcal{H}_4)$ and $(\mathcal{H}_5)$ imply that the ratios* (5.2.7f), (5.2.7h) *and* (5.2.7l) *in Lemma 5.17 can be also taken equal to 1.*

5.2.5 A priori estimates and proof of Theorems 5.15 and 5.16

In this section we will use $(\mathcal{H}_1)$—$(\mathcal{H}_5)$.

Lemma 5.19. *Under the hypotheses of Lemma 5.17, let $p = \bar{p}$ and $\mathfrak{p}$ be any of the constants therein defined. If (u, v, w) is a classical solution to problem* (5.2.1)*, $u \in L^\infty((0, T_{\max}); L^{\mathfrak{p}}(\Omega))$ and $\varphi(t)$ is the energy function*

$$\varphi(t) := \frac{1}{p}\int_\Omega (u+1)^p \quad \text{on } (0, T_{\max}),$$

then there exist c_1, c_2 such that

$$\varphi'(t) \leq -c_1 \int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_2 \quad \text{for all } t \in (0, T_{\max}). \tag{5.2.8}$$

Proof. Let us differentiate the functional $\varphi(t) = \frac{1}{p}\int_\Omega (u+1)^p$. Using the first equation of (5.2.1) and the divergence theorem we have for every $t \in (0, T_{\max})$

$$\begin{aligned}
\varphi'(t) =& \int_\Omega (u+1)^{p-1} u_t \\
=& \int_\Omega (u+1)^{p-1} \nabla \cdot ((u+1)^{m_1-1}\nabla u) - \chi \int_\Omega (u+1)^{p-1}\nabla \cdot (u(u+1)^{m_2-1}\nabla v) \\
&+ \xi \int_\Omega (u+1)^{p-1}\nabla \cdot (u(u+1)^{m_3-1}\nabla w) + \lambda \int_\Omega (u+1)^{p-1} u - \mu \int_\Omega (u+1)^{p-1}u^k \\
=& -(p-1)\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2 + (p-1)\chi \int_\Omega u(u+1)^{p+m_2-3}\nabla u \cdot \nabla v \\
&-(p-1)\xi \int_\Omega u(u+1)^{p+m_3-3}\nabla u \cdot \nabla w + \lambda \int_\Omega (u+1)^{p-1}u - \mu \int_\Omega (u+1)^{p-1}u^k.
\end{aligned}$$

For $j \in \{m_2, m_3\}$, we now define

$$F_j(u) = \int_0^u \hat{u}(\hat{u}+1)^{p+j-3} d\hat{u},$$

so observing that

$$0 \leq F_j(u) \leq \frac{1}{p+j-1}\left[(u+1)^{p+j-1} - 1\right]. \tag{5.2.9}$$

By considering the definition of $F_j(u)$ above again the divergence theorem, we have

$$\begin{aligned}
\varphi'(t) \leq& -c_3 \int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_4 \int_\Omega \nabla F_{m_2}(u)\cdot \nabla v - c_5 \int_\Omega \nabla F_{m_3}(u) \cdot \nabla w \\
&+ c_6 \int_\Omega (u+1)^p - c_7 \int_\Omega (u+1)^{p-1}u^k \\
=& -c_3 \int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 - c_4 \int_\Omega F_{m_2}(u)\Delta v + c_5 \int_\Omega F_{m_3}(u)\Delta w \\
&+ c_6 \int_\Omega (u+1)^p - c_7 \int_\Omega (u+1)^{p-1}u^k \quad \text{for every } t \in (0, T_{\max}).
\end{aligned} \tag{5.2.10}$$

Let us now specify how each of the constrains in Assumptions 5.14 takes part in our computation. First, from $(\mathcal{H}_1)$ we have $\mathfrak{p} > 1$; this makes meaningful our assumption $u \in L^\infty((0, T_{\max}); L^{\mathfrak{p}}(\Omega))$. (Recall that $u \in L^\infty((0, T_{\max}); L^1(\Omega))$ is always met by (5.2.6).) Now, by exploiting $(\mathcal{H}_2)$, the second and third equation of (5.2.1) and (5.2.9), we can see that from (5.2.10), if we neglect the nonpositive terms we get

$$\begin{aligned}
\varphi'(t) \leq& -c_3 \int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_8 \int_\Omega (u+1)^{p+m_2+\alpha-1} \\
&+ c_9 \int_\Omega (u+1)^{p+m_3-1}\int_\Omega (u+1)^\beta + c_6 \int_\Omega (u+1)^p \quad \text{on } (0, T_{\max}).
\end{aligned} \tag{5.2.11}$$

As to the third term, by using twice Hölder's inequality (recall (5.2.7f) and (5.2.7g)), we obtain, for every $t \in (0, T_{\max})$,

$$\begin{split} c_9 \int_\Omega (u+1)^{p+m_3-1} \int_\Omega (u+1)^\beta \leq & c_{10} \Big(\int_\Omega (u+1)^{p+m_2+\alpha-1} \Big)^{\frac{p+m_3-1}{p+m_2+\alpha-1}} \times \\ & \Big(\int_\Omega (u+1)^{p+m_2+\alpha-1} \Big)^{\frac{\beta}{p+m_2+\alpha-1}}. \end{split} \tag{5.2.12}$$

By virtue of (5.2.7h) and (3.1.6), we have that

$$c_9 \int_\Omega (u+1)^{p+m_3-1} \int_\Omega (u+1)^\beta \leq \int_\Omega (u+1)^{p+m_2+\alpha-1} + c_{11} \quad \text{for all } t \in (0, T_{\max}). \tag{5.2.13}$$

Through (5.2.7i), an application of Young's inequality provides

$$c_6 \int_\Omega (u+1)^p \leq \int_\Omega (u+1)^{p+m_2+\alpha-1} + c_{12} \quad \text{on } (0, T_{\max}). \tag{5.2.14}$$

To control the term $\int_\Omega (u+1)^{p+m_2+\alpha-1}$, we invoke the Gagliardo–Nirenberg and Young's inequalities, so to bound the mentioned integral with $\int_\Omega |\nabla (u+1)^{\frac{p+m_1-1}{2}}|^2$. More exactly by relying on relations (5.2.7b) and (5.2.7c), boundedness of $\int_\Omega (u+1)^{\mathfrak{p}}$ and of $\int_\Omega u$ provide for any $L_1 > 0$

$$\begin{split} L_1 \int_\Omega (u+1)^{p+m_2+\alpha-1} = & L_1 \Big\| (u+1)^{\frac{p+m_1-1}{2}} \Big\|^{\frac{2(p+m_2+\alpha-1)}{p+m_1-1}}_{L^{\frac{2(p+m_2+\alpha-1)}{p+m_1-1}}(\Omega)} \\ \leq & c_{13} \Bigg(\Big\| \nabla (u+1)^{\frac{p+m_1-1}{2}} \Big\|^{\sigma\theta}_{L^2(\Omega)} \Big\| (u+1)^{\frac{p+m_1-1}{2}} \Big\|^{\sigma(1-\theta)}_{L^{\frac{2\mathfrak{p}}{p+m_1-1}}(\Omega)} \\ & + \Big\| (u+1)^{\frac{p+m_1-1}{2}} \Big\|^{\sigma}_{L^{\frac{2}{p+m_1-1}}(\Omega)} \Bigg) \\ \leq & c_{14} \Big(\int_\Omega \Big| \nabla (u+1)^{\frac{p+m_1-1}{2}} \Big|^2 \Big)^{\frac{\sigma\theta}{2}} + c_{15} \\ \leq & \frac{c_3}{2} \int_\Omega \Big| \nabla (u+1)^{\frac{p+m_1-1}{2}} \Big|^2 + c_{16} \quad \text{for all } t \in (0, T_{\max}). \end{split} \tag{5.2.15}$$

(We underline that herein we have used the elementary inequality

$$C_1(\tau)(A^\tau + B^\tau) \leq (A+B)^\tau \leq C_2(\tau)(A^\tau + B^\tau) \quad \text{for all } A, B \geq 0, \tau > 0 \text{ and proper } C_1, C_2 > 0, \tag{5.2.16}$$

which might tacitly be used in the next lines.) Putting together (5.2.11), (5.2.13), (5.2.14), (5.2.15), we have the claim.

If assumption $(\mathcal{H}_3)$ is complied, bound (5.2.14) can be replaced for any $L_2 > 0$ with

$$\begin{split} L_2 \int_\Omega (u+1)^p = & L_2 \Big\| (u+1)^{\frac{p+m_1-1}{2}} \Big\|^{\frac{2p}{p+m_1-1}}_{L^{\frac{2p}{p+m_1-1}}(\Omega)} \\ \leq & c_{17} \Bigg(\Big\| \nabla (u+1)^{\frac{p+m_1-1}{2}} \Big\|^{\hat\sigma\hat\theta}_{L^2(\Omega)} \Big\| (u+1)^{\frac{p+m_1-1}{2}} \Big\|^{\hat\sigma(1-\hat\theta)}_{L^{\frac{2}{p+m_1-1}}(\Omega)} \\ & + \Big\| (u+1)^{\frac{p+m_1-1}{2}} \Big\|^{\hat\sigma}_{L^{\frac{2}{p+m_1-1}}(\Omega)} \Bigg) \\ \leq & c_{17} \Big(\int_\Omega \Big| \nabla (u+1)^{\frac{p+m_1-1}{2}} \Big|^2 \Big)^{\frac{\hat\sigma\hat\theta}{2}} + c_{18} \\ \leq & \frac{c_3}{4} \int_\Omega \Big| \nabla (u+1)^{\frac{p+m_1-1}{2}} \Big|^2 + c_{19} \quad \text{on } (0, T_{\max}), \end{split} \tag{5.2.17}$$

where in this last step we used Young's inequality in conjunction with (5.2.7d) and (5.2.7e). Moreover, by taking $m_2 + \alpha = m_3 + \beta$ in $(\mathcal{H}_3)$ (recall Remark 5.18), in estimate (5.2.12) the powers at the right-hand side have 1 as sum; subsequently, up to constants, it becomes (5.2.13). So, by considering (5.2.11), (5.2.13), (5.2.15) and (5.2.17) we have the claim when either ($\mathcal{H}_1$) and ($\mathcal{H}_2$) or ($\mathcal{H}_1$) and ($\mathcal{H}_3$) are used.

Let us show how achieving the same conclusion by applying either ($\mathcal{H}_1$) and ($\mathcal{H}_4$) or ($\mathcal{H}_1$) and ($\mathcal{H}_5$). The main idea is alternatively treating in relation (5.2.11) the term

$$\int_\Omega (u+1)^{p+m_3-1} \int_\Omega (u+1)^\beta \quad \text{on } (0, T_{\max}).$$

More specifically, if $\beta \in (0,1]$ and we take into account the boundedness of $\int_\Omega u$ on $(0, T_{\max})$, we have that

$$\int_\Omega (u+1)^{p+m_3-1} \int_\Omega (u+1)^\beta \le c_{20} \int_\Omega (u+1)^{p+m_3-1} \quad \text{for all } t \in (0, T_{\max}). \tag{5.2.18}$$

In turn, by relying respectively on assumption ($\mathcal{H}_4$) or ($\mathcal{H}_5$) (both with strict inequality), by means of Young's inequality, thanks to (5.2.7l) and (5.2.7f), it is also possible to see that

$$c_{20} \int_\Omega (u+1)^{p+m_3-1} \le \int_\Omega (u+1)^p + c_{21} \quad \text{on } (0, T_{\max}), \tag{5.2.19}$$

or

$$c_{20} \int_\Omega (u+1)^{p+m_3-1} \le \int_\Omega (u+1)^{p+m_2-1+\alpha} + c_{22} \quad \text{for all } t \in (0, T_{\max}). \tag{5.2.20}$$

As before, bounds (5.2.11), (5.2.18), (5.2.19), or alternatively (5.2.20), (5.2.17) and (5.2.15), lead to the same conclusion. (For the limit cases in ($\mathcal{H}_4$) or ($\mathcal{H}_5$), namely $m_3 = 1$ or $m_2 + \alpha = m_3$, by relying again on Remark 5.18, we can directly exploit estimate (5.2.18), without using Young's inequality, and conclude as above.) □

The next step consists in ensuring some time independent estimate of u in the $L^{\bar{p}}(\Omega)$-norm.

Lemma 5.20. *Let Lemma 5.19 be true. Then* $u \in L^\infty((0, T_{\max}); L^{\bar{p}}(\Omega))$.

Proof. With a view to inequality (5.2.8), as already done in (5.2.17), a further application of the Gagliardo–Nirenberg inequality, supported by (5.2.7d), leads also thanks to (5.2.16) to

$$\int_\Omega (u+1)^p \le c_1 \left(\left\| \nabla (u+1)^{\frac{p+m_1-1}{2}} \right\|_{L^2(\Omega)}^{\hat{\sigma}\hat{\theta}} + 1 \right) \le c_2 \left(\int_\Omega \left| \nabla (u+1)^{\frac{p+m_1-1}{2}} \right|^2 + 1 \right)^{\frac{\hat{\sigma}\hat{\theta}}{2}} \quad \text{on } (0, T_{\max}),$$

or, equivalently

$$-c_3 \int_\Omega \left| \nabla (u+1)^{\frac{p+m_1-1}{2}} \right|^2 \le - \left(\int_\Omega (u+1)^p \right)^{\frac{2}{\hat{\sigma}\hat{\theta}}} + c_3 \quad \text{for every } t \in (0, T_{\max}). \tag{5.2.21}$$

Finally, by using relations (5.2.8) and (5.2.21), we arrive at this initial problem

$$\begin{cases} \varphi'(t) \le c_4 - c_5 \varphi(t)^{\frac{2}{\hat{\sigma}\hat{\theta}}} & \text{for every } t \in (0, T_{\max}), \\ \varphi(0) = \dfrac{1}{p} \displaystyle\int_\Omega (u_0+1)^p, \end{cases}$$

which ensures that $\int_\Omega u^{\bar{p}} = \int_\Omega u^p \le \int_\Omega (u+1)^p \le p \max \left\{ \varphi(0), \left(\frac{c_4}{c_5}\right)^{\frac{\hat{\sigma}\hat{\theta}}{2}} \right\}$ for all $t < T_{\max}$ thanks to Lemma 3.13. □

By taking advantage from the previous lemma, let us show the uniform-in-time boundedness of u.

Lemma 5.21. *Under the hypotheses of Lemma 5.20,* $u \in L^\infty((0, T_{\max}); L^\infty(\Omega))$.

Proof. We are going to use Lemma 4.6. We know that u classically solves in $\Omega \times (0, T_{\max})$ problem (A.1) for

$$D(x,t,u) = (u+1)^{m_1-1}, \quad f(x,t) = \chi u(u+1)^{m_2-1}\nabla v - \xi u(u+1)^{m_3-1}\nabla w, \quad g(x,t) = \lambda u - \mu u^k.$$

In particular, also taking into account the boundary condition on v and w, we can see that (A.2)–(A.5) are met. On the other hand, for any $\lambda, \mu > 0$ and $k > 1$, it holds that $\lambda u - \mu u^k$ has a positive maximum L at $u_M = \left(\frac{\lambda}{k\mu}\right)^{\frac{1}{k-1}}$, so that from $g(x,t) \leq L$ in $\Omega \times (0, T_{\max})$ the second inclusion of (A.6) is accomplished for any choice of q_2. As to (A.7) and (A.8), let us first define the quantities

$$l_1(q_1) = 1 - m_1\frac{(n+1)q_1 - (n+2)}{q_1 - (n+2)}, \quad l_2(q_2) = 1 - m_1\frac{1}{1 - \frac{nq_2}{(n+2)(q_2-1)}} \quad \text{and} \quad l_3 = \frac{n}{2}(1-m_1).$$

Recalling the definition and properties of $p = \bar{p}$, we have $p > 1 - m_1(n+2)$ and henceforth for any $m_1 \in \mathbb{R}$, it holds $1 - m_1(n+2) \geq l_1((n+2)(n+1))$ and for all $q_2 \leq n+1$ we also have $1 - m_1(n+2) \geq l_2(q_2)$. Subsequently, (A.7) (i.e. $u \in L^\infty((0, T_{\max}); L^p(\Omega))$, that is $\int_\Omega u^p \leq c_1$) is obviously accomplished and (A.8) is fulfilled, thanks to (5.2.7a), for $q_1 = (n+2)(n+1)$, $\frac{n+2}{2} < q_2 \leq n+1$. Let us dedicate to the first inclusion of (A.6). Starting from the gained bound of u, let us exploit elliptic regularity results applied to the second and third equation of system (5.2.1); in particular, we only analyze $-\Delta v = f_1(u) - \frac{1}{|\Omega|}\int_\Omega f_1(u)$, being the case for w equivalent. Let us observe that from relation (5.2.3) on f_1, and $p = \bar{p} > n\alpha$ naturally implying $(1+u)^\alpha \leq (1+u)^p$, we have for all $t \in (0, T_{\max})$

$$\begin{aligned}\int_\Omega |f_1(u) - \frac{1}{|\Omega|}\int_\Omega f_1(u)|^{\frac{p}{\alpha}} \leq & c_2\int_\Omega (1+u)^p + c_3\int_\Omega \left(\int_\Omega (1+u)^\alpha\right)^{\frac{p}{\alpha}} \\ \leq & c_2\int_\Omega (1+u)^p + c_3\int_\Omega \left(\int_\Omega (1+u)^p\right)^{\frac{p}{\alpha}} \leq c_4,\end{aligned}$$

which gives $f_1(u) - \frac{1}{|\Omega|}\int_\Omega f_1(u) \in L^\infty((0, T_{\max}); L^{\frac{p}{\alpha}}(\Omega))$, in turn $v \in L^\infty((0, T_{\max}); W^{2,\frac{p}{\alpha}}(\Omega))$ and, hence, through the Sobolev embeddings $\nabla v \in L^\infty((0, T_{\max}); W^{1,\frac{p}{\alpha}}(\Omega)) \hookrightarrow L^\infty((0, T_{\max}); L^\infty(\Omega))$. Consequently, thanks to the Hölder inequality (recall that from Lemma 5.17 one has that $p > m_2(n+2)(n+1)$), by using the uniform-in-time boundedness of u in $L^{\bar{p}}(\Omega)$ we can write on $(0, T_{\max})$

$$\begin{aligned}\int_\Omega |u(u+1)^{m_2-1}\nabla v|^{(n+2)(n+1)} \leq & \|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}^{(n+2)(n+1)} |\Omega|^{\frac{p-m_2(n+2)(n+1)}{p}} \left(\int_\Omega (u+1)^p\right)^{\frac{m_2(n+2)(n+1)}{p}} \\ \leq & c_5.\end{aligned}$$

Reasoning in a similar way on the third equation of (5.2.1), we have $\nabla w \in L^\infty((0, T_{\max}); L^\infty(\Omega))$,

$$\int_\Omega |u(u+1)^{m_3-1}\nabla w|^{(n+2)(n+1)} \leq c_6 \quad \text{for all } t \in (0, T_{\max}),$$

and as a consequence

$$f = \chi u(u+1)^{m_2-1}\nabla v - \xi u(u+1)^{m_3-1}\nabla w \in L^\infty((0, T_{\max}); L^{(n+2)(n+1)}(\Omega)).$$

Since all the hypotheses of Lemma 4.6 are fulfilled, we have the claim. □

Now we are in a position to prove our first results.

Proof of Theorem 5.15:

Let (u, v, w) be a given blow-up solution at some finite time $T_{\max}$ to problem (5.2.1). If u was not unbounded in some $L^{\mathfrak{p}}(\Omega)$-norm, Lemma 5.21 would imply the uniform-in-time boundedness of u, contradicting hypothesis (5.2.4). □

Proof of Theorem 5.16:

By making use of (5.2.13) and (5.2.14) (up to a constant) or altogether (5.2.18), (5.2.19), (5.2.20) and again (5.2.14), we observe that bound (5.2.11) can essentially be reorganized as

$$\varphi'(t) \leq -c_1 \int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_2 \int_\Omega (u+1)^{p+m_2+\alpha-1} + c_3 \quad \text{on } (0, T_{\max}). \tag{5.2.22}$$

Moreover, by recalling (5.2.7m) and (5.2.7k), we can derive again through the Gagliardo–Nirenberg and Young's inequalities this bound

$$\begin{aligned} c_2 \int_\Omega (u+1)^{p+m_2+\alpha-1} =& c_2 \left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|^{\frac{2(p+m_2+\alpha-1)}{p+m_1-1}}_{L^{\frac{2(p+m_2+\alpha-1)}{p+m_1-1}}(\Omega)} \\ \leq& c_4 \left(\left\|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right\|^{\sigma\bar\theta}_{L^2(\Omega)} \left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|^{\sigma(1-\bar\theta)}_{L^{\frac{2p}{p+m_1-1}}(\Omega)} \right. \\ &\left. + \left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|^{\sigma}_{L^{\frac{2p}{p+m_1-1}}(\Omega)} \right) \\ \leq& c_5 \left(\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 \right)^{\frac{\sigma\bar\theta}{2}} \left(\int_\Omega (u+1)^p \right)^{\frac{p+m_1-1}{2p}\sigma(1-\bar\theta)} \\ &+ c_6 \left(\int_\Omega (u+1)^p \right)^{\frac{p+m_1-1}{2p}\sigma} \\ \leq& c_1 \int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_7 \left(\int_\Omega (u+1)^p \right)^{\gamma} \\ &+ c_8 \left(\int_\Omega (u+1)^p \right)^{\delta} \quad \text{on } (0, T_{\max}). \end{aligned} \tag{5.2.23}$$

Finally, by inserting relation (5.2.23) into (5.2.22), we obtain for proper positive $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$

$$\varphi'(t) \leq \mathcal{A}\varphi^\gamma(t) + \mathcal{B}\varphi^\delta(t) + \mathcal{C} \quad \text{on } (0, T_{\max}). \tag{5.2.24}$$

Since $p = \bar p > \mathfrak{p}$, from Theorem 5.15 we know that $\limsup_{t\to T_{\max}} \frac{1}{p}\int_\Omega (u+1)^p = \infty$. On the other hand, since $\varphi(t)$ satisfies relation (5.2.24) for any $0 < t < T_{\max}$, the function $\Psi(\xi) = \mathcal{A}\xi^\gamma + \mathcal{B}\xi^\delta + \mathcal{C}$ obeys the Osgood criterion (4.4.1). Thereafter, by integrating (5.2.24) between 0 and $T_{\max}$, we obtain estimate (5.2.5), and the first conclusion is achieved.

As to the derivation of the explicit expression for the lower bound T, let us reduce (5.2.24) as follows: from $|\Omega| \leq \int_\Omega (u+1)^p = p\varphi(t)$, we can estimate $\mathcal{C}$ in relation (5.2.24) as

$$\mathcal{C} \leq p\frac{\mathcal{C}}{|\Omega|}\varphi(t) =: \bar{\mathcal{C}}\varphi(t),$$

so that (5.2.24) can be rewritten in this form:

$$\varphi'(t) \leq \mathcal{A}\varphi^\gamma(t) + \mathcal{B}\varphi^\delta(t) + \bar{\mathcal{C}}\varphi(t) \quad \text{on } (0, T_{\max}). \tag{5.2.25}$$

Now, since φ blows up at finite time $T_{\max}$, there exists a time $t_1 \in [0, T_{\max})$ such that

$$\varphi(t) \geq \varphi(0) \quad \text{for all } t \geq t_1.$$

From $\gamma > \delta > 1$ (recall (5.2.7j)), we can estimate the second and third term on the right-hand side of (5.2.25) by means of φ^γ:

$$\varphi^\delta(t) \leq \varphi(0)^{\delta-\gamma}\varphi^\gamma(t) \quad \text{and} \quad \varphi(t) \leq \varphi(0)^{1-\gamma}\varphi^\gamma(t) \quad \text{for all } t \geq t_1. \tag{5.2.26}$$

By plugging expressions (5.2.26) into (5.2.25) we obtain for

$$\mathcal{D} = \mathcal{A} + \mathcal{B}\varphi(0)^{\delta-\gamma} + \bar{\mathcal{C}}\varphi(0)^{1-\gamma},$$

$$\varphi'(t) \leq \mathcal{D}\varphi^{\gamma}(t) \quad \text{for all } \; t \geq t_1, \tag{5.2.27}$$

so that an integration of (5.2.27) on $(t_1, T_{\max})$ yields this explicit lower bound for $T_{\max}$:

$$T = \frac{\varphi(0)^{1-\gamma}}{\mathcal{D}(\gamma - 1)} = \int_{\varphi(0)}^{\infty} \frac{d\tau}{\mathcal{D}\tau^{\gamma}} \leq \int_{t_1}^{T_{\max}} d\tau \leq \int_{0}^{T_{\max}} d\tau = T_{\max}.$$

□

5.3 [CDFF24] Uniform-in-time boundedness in a class of local and nonlocal nonlinear attraction-repulsion chemotaxis models with logistics

5.3.1 The local and nonlocal models

Our attention is directed to these two systems, one of local type (indicated with $(\mathcal{L})$) and the other of nonlocal nature, $(\mathcal{N})$:

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v + \xi u(u+1)^{m_3-1}\nabla w\big) & \\ \qquad +\lambda u - \mu u^r & \text{in } \Omega \times (0, T_{\max}), \\ \tau v_t = \Delta v - \beta v + f_1(u) & \text{in } \Omega \times (0, T_{\max}), \\ \tau w_t = \Delta w - \delta w + f_2(u) & \text{in } \Omega \times (0, T_{\max}), \\ u_\nu = v_\nu = w_\nu = 0 & \text{on } \partial\Omega \times (0, T_{\max}), \\ u(x,0) = u_0(x), \tau v(x,0) = \tau v_0(x), \tau w(x,0) = \tau w_0(x) & x \in \bar{\Omega}, \end{cases} \tag{$\mathcal{L}$}$$

and

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v + \xi u(u+1)^{m_3-1}\nabla w\big) & \\ \qquad +\lambda u - \mu u^r & \text{in } \Omega \times (0, T_{\max}), \\ 0 = \Delta v - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega f_1(u) + f_1(u) & \text{in } \Omega \times (0, T_{\max}), \\ 0 = \Delta w - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega f_2(u) + f_2(u) & \text{in } \Omega \times (0, T_{\max}), \\ u_\nu = v_\nu = w_\nu = 0 & \text{on } \partial\Omega \times (0, T_{\max}), \\ u(x,0) = u_0(x) & x \in \bar{\Omega}, \\ \displaystyle\int_\Omega v(x,t)\,dx = \int_\Omega w(x,t)\,dx = 0 & \text{for all } t \in (0, T_{\max}). \end{cases} \tag{$\mathcal{N}$}$$

Remark 5.22. *In problem $(\mathcal{L})$, the equation $\tau v_t = \Delta v - \beta v + f_1(u)$ describes the local interaction between the involved quantities at each spatial point x; contrarily, in $(\mathcal{N})$ the nonlocal term $\frac{1}{|\Omega|}\int_\Omega f_1(u)$ appears, and it takes into account the entire distribution of u all over the domain Ω. (Naturally, the same observation can be done for the equations related to w.)*

Additionally, let us clarify that in the nonlocal model, v and w stand for the deviations of the chemoattractant and the chemorepellent (see Remark 2.1). As a consequence, v and w change sign.

Consistently with the above problems, this section is contextualized in the framework of a series of results dealing with local and nonlocal, and linear and nonlinear, attraction-repulsion chemotaxis systems; in particular this research improves and extends already known analyses in the literature (we will give more details in the sequel).

More specifically, as far as we know, for such discussed fully nonlinear versions, the literature is rather poor; nevertheless, issues dealing with boundedness and blow-up of solutions have been addressed. More precisely, for problem $(\mathcal{L})$ we mention the works [RL20] and [CY22], where in the first the authors ensure the boundedness of solutions, for both $\tau = 0$ and $\tau = 1$; in the second boundedness and blow-up analyses, in this case for the simplified parabolic-elliptic-elliptic version, are discussed.

5.3.2 Presentation of the main theorems

In order to present our claims, some positions have to be previously fixed. In particular, here we assume that sources f_1, f_2 comply with (4.1.1) and are such that, for $\alpha, l, k > 0$ and $0 < \gamma_0 \le \gamma_1$,

$$0 \le f(s) \le \alpha(s+1)^k \quad \text{and} \quad \gamma_0(s+1)^l \le g(s) \le \gamma_1(s+1)^l. \tag{5.3.1}$$

Additionally, we will frequently invoke these

Assumptions 5.23. *For $n \in \mathbb{N}$, $m_1, m_2, m_3 \in \mathbb{R}$, $k, l > 0$ and $r > 1$ let us set:*

($\mathcal{A}_1$) $m_2 + k < m_3 + l$;

($\mathcal{A}_2$) $m_2 + k < r$;

($\mathcal{A}_3$) $m_2 + k < m_1 + \frac{2}{n}$;

($\mathcal{A}_4$) $m_3 + l < r$;

($\mathcal{A}_5$) $m_3 + l < m_1 + \frac{2}{n}$.

The above preparations allow us to state the following two theorems.

Theorem 5.24. *For $\tau = 0$, let f_1, f_2 comply with hypotheses in* (5.3.1), $\lambda, \mu, \chi, \xi, \beta, \delta > 0$. *Additionally, let one among ($\mathcal{A}_1$), ($\mathcal{A}_2$), ($\mathcal{A}_3$) in Assumptions 5.23 hold true. Then, the solution to problems* ($\mathcal{L}$) *and* ($\mathcal{N}$) *is global and bounded.*

Theorem 5.25. *For $\tau = 1$, let f_1, f_2 comply with hypotheses in* (5.3.1), $\lambda, \mu, \chi, \xi, \beta, \delta > 0$. *Additionally, let one among ($\mathcal{A}_2$) and ($\mathcal{A}_3$) jointly with one between ($\mathcal{A}_4$) and ($\mathcal{A}_5$) in Assumptions 5.23 hold true. Then the solution to problem* ($\mathcal{L}$) *is global and bounded.*

As anticipated in the introductory part, let us specify in which sense our results improve what known so far in the literature.

Remark 5.26. (Comparisons with nonlocal problem ($\mathcal{N}$)) *Let us make these observations.*

- *Nonlinear diffusion and sensitivities ($m_1, m_2, m_3 \in \mathbb{R}$): in [JJL23, Theorem 1.1] boundedness for problem ($\mathcal{N}$) is achieved for each of the following cases:*

  i) $m_3 + l < m_2 + k + 1$ *and* $m_2 + k < m_1 + \frac{2}{n}$;
  ii) $m_2 + k < m_3 + l < m_1 + \frac{2}{n}$;
  iii) $\max\{m_2 + k, m_3 + l\} < r < m_1 + \frac{2}{n}$.

  It is seen that assumption ($\mathcal{A}_1$) in Theorem 5.24 is sharper with respect to ii), and ($\mathcal{A}_2$) with respect to iii) and, finally, ($\mathcal{A}_3$) with respect to i).

- *Linear diffusion and sensitivities ($m_1 = m_2 = m_3 = 1$): in Theorem 5.4 boundedness for problem ($\mathcal{N}$) is established, among other situations, whenever:*

  (a) $k < l$;
  (b) $k = l \in (0, \frac{2}{n})$ *and* $\Theta_0 := \chi\alpha - \xi\gamma_0 \geq 0$.

  Evidently, if from the one hand (a) is recovered from ($\mathcal{A}_1$), on the other hand, also in this case some milder conclusions can be observed: in the specific ($\mathcal{A}_3$) improves (b) and provides a further situation where $l > 0$ and $k \in \left(0, \frac{2}{n}\right)$.

Remark 5.27. (Comparisons with local problem ($\mathcal{L}$)) *Even for the local problem, the present analysis gives some more precise insight in the context of attraction-repulsion chemotaxis mechanisms.*

- *Nonlinear diffusion and sensitivities ($m_1, m_2, m_3 \in \mathbb{R}$):*

  - $\tau = 0$*: In [CY22, Theorem 3.1 and Theorem 3.5], where the authors consider the problem with linear productions ($k = l = 1$), globality of solutions is carried out for the case $m_2 \leq m_3$, so leaving room for the case $m_2 > m_3$; in particular, for either $m_2 < m_3$ or $m_2 = m_3$ and $\Theta_0 = \chi\alpha - \xi\gamma_0 < 0$ (naturally, for $k = l = 1$ we have $\gamma_0 = \gamma_1$), boundedness is, respectively, achieved. Indeed, assumptions ($\mathcal{A}_2$) and ($\mathcal{A}_3$) of Theorem 5.24 can be used independently by the relation between m_2 and m_3.*
    Furthermore, Theorem 5.24 extends [RL20, Theorem 1.3], where only the case $m_1 \in \mathbb{R}$, $m_2 = m_3 = 1$ and $k, l \geq 1$ is addressed; particularly assumption ($\mathcal{A}_3$) is less restricted than [RL20, Theorem 1.3, (i)], since it requires the extra restriction $1 < r < k + 1$ and ($\mathcal{A}_1$) provides a further condition. Oppositely, [RL20, Theorem 1.3, (ii)] coincides with ($\mathcal{A}_2$), whereas [RL20, Theorem 1.3, (iii)] covers the limit case $r = k + 1$, not studied in our work.

▷ $\tau = 1$*: In [RL20, Theorem 1.1] boundedness of solutions is established, inter alia, if*

$$\max\{m_2 + k, m_3 + l\} < m_1 + \frac{2}{n} \quad \textit{or} \quad \max\{m_2 + k, m_3 + l\} < r \tag{5.3.2}$$

are fulfilled. In this way, (($\mathcal{A}_2$), ($\mathcal{A}_4$)) and (($\mathcal{A}_3$), ($\mathcal{A}_5$)) of Theorem 5.25 coincide with (5.3.2)*, but (($\mathcal{A}_2$), ($\mathcal{A}_5$)) and (($\mathcal{A}_3$), ($\mathcal{A}_4$)) provide a further scenario towards boundedness.*

- *Linear diffusion and sensitivities ($m_1 = m_2 = m_3 = 1$): For $\tau = 0$ if we compare Theorem 5.24 with Theorem 5.2, not only the same conclusion given in the second item of Remark 5.26 still applies but even the second hypothesis in Theorem 5.2 is turned into a weaker condition. For the fully parabolic case, assumption ($\mathcal{A}_3$) and ($\mathcal{A}_5$) of Theorem 5.25 imply $l, k \in (0, \frac{2}{n})$, so improving Theorem 5.3.*

5.3.3 Adjusting parameters

The following lemma defines crucial parameters; its proof is based on some of the relations fixed in Assumptions 5.23.

Lemma 5.28. *Let n, k, l, m_1, m_2, m_3 be as in Assumptions 5.23 and let relations ($\mathcal{A}_3$) and ($\mathcal{A}_5$) be valid. Then, there exists $\bar{p} > 1$ such that, for all $p > \bar{p}$, $q > 1$ and*

$$\theta(p) := \frac{\frac{p+m_1-1}{2} - \frac{p+m_1-1}{2(p+m_2+k-1)}}{\frac{p+m_1-1}{2} - \frac{1}{2} + \frac{1}{n}}, \quad \sigma(p) := \frac{2(p+m_2+k-1)}{p+m_1-1}, \quad \theta_1(p) := \frac{\frac{p+m_1-1}{2} - \frac{p+m_1-1}{2(p+m_3+l-1)}}{\frac{p+m_1-1}{2} - \frac{1}{2} + \frac{1}{n}},$$

$$\sigma_1(p) := \frac{2(p+m_3+l-1)}{p+m_1-1} \quad \theta_2(p) := \frac{\frac{p+m_1-1}{2} - \frac{p+m_1-1}{2l}}{\frac{p+m_1-1}{2} - \frac{1}{2} + \frac{1}{n}},$$

$$\theta_3(p) := \frac{\frac{p+m_1-1}{2} - \frac{p+m_1-1}{2p}}{\frac{p+m_1-1}{2} - \frac{1}{2} + \frac{1}{n}}, \qquad \theta_4(p) := \frac{\frac{p+m_1-1}{2} - \frac{p+m_1-1}{2q}}{\frac{p+m_1-1}{2} - \frac{1}{2} + \frac{1}{n}}, \qquad \sigma_2(p) := \frac{2(p+q)}{p+m_1-1},$$

these relations hold

$$0 < \theta < 1, \tag{5.3.3a}$$
$$0 < \frac{\sigma\theta}{2} < 1, \tag{5.3.3b}$$
$$0 < \theta_1 < 1, \tag{5.3.3c}$$
$$0 < \frac{\sigma_1\theta_1}{2} < 1, \tag{5.3.3d}$$
$$0 < \theta_2 < 1, \tag{5.3.3e}$$
$$0 < \frac{\sigma_1\theta_2}{2} < 1, \tag{5.3.3f}$$
$$0 < \theta_4 < 1, \tag{5.3.3g}$$
$$0 < \frac{\sigma_2\theta_4}{2} < 1, \tag{5.3.3h}$$
$$0 < \theta_3 < 1, \tag{5.3.3i}$$

where (5.3.3e) *and* (5.3.3f) *are valid only for $l > 1$.*

Proof. First let us consider the function $\theta(p)$. Since $\lim_{p\to\infty} \theta(p) = 1$, we have that $\theta(p)$ is definitively positive; on the other hand, $\theta(p)$ increases for p sufficiently large so that 1 is an upper bound. Therefore, (5.3.3a) is proved. Relations (5.3.3c), (5.3.3i), (5.3.3b) and (5.3.3d) follow by means of analogous arguments, once for (5.3.3b) and (5.3.3d) conditions ($\mathcal{A}_3$) and ($\mathcal{A}_5$) are respectively taken into account. Let us consider now the functions $\theta_2(p)$ and $\sigma_1(p)$. Relations (5.3.3e) and (5.3.3f) are consequence of

$$\lim_{p\to\infty} \theta_2(p) = \lim_{p\to\infty} \frac{\sigma_1(p)\theta_2(p)}{2} = 1 - \frac{1}{l} \in (0,1), \quad \text{for } l > 1.$$

In the same way, relations (5.3.3g) and (5.3.3h) are given for every $q > 1$.

From all of the above, once the parameters of our problem and any $q > 1$ are fixed, it is possible to find $\bar{p} > 1$ (depending itself on the parameters and also on q) such that the previous relations are complied for every $p > \bar{p}$. □

For our purposes and for reasons which will be clear later on, the value of $p > \bar{p} > 1$ derived in the above result can be taken arbitrarily large.

Lemma 5.29. *The function u is such that*

$$\int_\Omega u \leq M := \max\left\{\int_\Omega u_0(x)\,dx, \left(\frac{\lambda}{\mu}|\Omega|^{r-1}\right)^{\frac{1}{r-1}}\right\} \quad \text{for all } t \in [0, T_{\max}).$$

Proof. Proposition 4.8 with $\rho = 1$ and $k = r$ gives the bound. □

5.3.4 The extension criterion

The aim is to prove the next Lemma.

Lemma 5.30. *If $u \in L^\infty((0, T_{\max}), L^p(\Omega))$ for all $p > 1$, then $u \in L^\infty((0, \infty), L^\infty(\Omega))$.*

Proof. Thanks to Proposition 4.4, considering also Remark 4.5, it is enough to show that $u \in L^\infty((0, T_{\max}), L^\infty(\Omega))$. Naturally $u = u(x,t)$, with $(x,t) \in \Omega \times (0, T_{\max})$, also classically solves problem (A.1) in Lemma 4.6 for

$$D(x,t,u) = (u+1)^{m_1-1}, \quad f(x,t) = -\chi u(u+1)^{m_2-1}\nabla v + \xi u(u+1)^{m_3-1}\nabla w, \quad g(x,t) = \lambda u - \mu u^r.$$

Specifically, by making use of the Neumann boundary conditions, we can see that (A.2)–(A.5) are complied with. On the other hand, for any $\lambda, \mu > 0$ and $r > 1$, it holds that $\lambda u - \mu u^r$ has a positive maximum L at $u_M = \left(\frac{\lambda}{r\mu}\right)^{\frac{1}{r-1}}$, so that from $g(x,t) \leq L$ in $\Omega \times (0, T_{max})$ the second requirement in (A.6) is verified for any $q_2 > 1$. Since, by hypotheses, $u \in L^\infty((0, T_{\max}); L^p(\Omega))$ for every $p > 1$, we have that $\int_\Omega u^p$ is uniformly bounded on $(0, T_{\max})$ for p arbitrary large (without relabeling it) and henceforth conditions (A.7) and (A.8) are fulfilled. As to the first assumption of (A.6), if $\tau = 0$ in problem $(\mathcal{L})$, from the obtained inclusion $u \in L^\infty((0, T_{\max}); L^p(\Omega))$ we have $f_1(u) \in L^\infty((0, T_{\max}); L^p(\Omega))$, and in turn relation (3.3.3) provides $\nabla v \in L^\infty((0, T_{\max}); L^\infty(\Omega))$ (where $\psi = f_1$), and similarly $\nabla w \in L^\infty((0, T_{\max}); L^\infty(\Omega))$ (where $\psi = f_2$). For $\tau = 1$ we directly invoke (3.3.6) (where $g = f_1$ or $g = f_2$); in both cases we have that, for any $q_1 > 1$, $f \in L^\infty((0, T_{\max}); L^{q_1}(\Omega))$. As a consequence of what explained, we have the claim by virtue of Lemma 4.6. □

5.3.5 Some a priori estimates

This part is dedicated to deriving some uniform-in-time estimates: in fact, by applying Lemma 5.30 we can conclude that the solution is global and bounded. The forthcoming lemma holds for models $(\mathcal{L})$ and $(\mathcal{N})$.

Lemma 5.31. *Let the hypotheses of Lemma 5.28 be valid and let $\bar{p} > 1$ be the value therein found. Then, for every $\hat{c}, \varepsilon_1, \varepsilon_2 > 0$ and for every $p > \bar{p}$, there exist some constants $c_1, c_2, c_3, c_4, c_5, c_6$ such that on $(0, T_{\max})$*

$$\hat{c}\int_\Omega (u+1)^{p+m_2+k-1} \leq \frac{p-1}{(p+m_1-1)^2}\int_\Omega \left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_1 \quad \text{provided } (\mathcal{A}_3), \tag{5.3.4a}$$

$$\hat{c}\int_\Omega (u+1)^{p+m_3+l-1} \leq \frac{p-1}{(p+m_1-1)^2}\int_\Omega \left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_2 \quad \text{provided } (\mathcal{A}_5), \tag{5.3.4b}$$

$$\begin{aligned}\hat{c}\int_\Omega (u+1)^{p+m_3-1}\int_\Omega (u+1)^l \leq{}& \varepsilon_1\int_\Omega (u+1)^{p+m_3+l-1} \\ &+ \frac{p-1}{(p+m_1-1)^2}\int_\Omega \left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_3\end{aligned} \tag{5.3.4c}$$

$$\hat{c}\int_\Omega (u+1)^{p+m_3-1}w \leq \varepsilon_2\int_\Omega (u+1)^{p+m_3+l-1} + \frac{p-1}{(p+m_1-1)^2}\int_\Omega \left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_4 \tag{5.3.4d}$$

$$-\int_\Omega \left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2 \leq c_5 - c_6\left(\int_\Omega (u+1)^p\right)^{\frac{p+m_1-1}{p\theta_3}} \tag{5.3.4e}$$

Proof. Let us show (5.3.4a). Under assumption ($\mathcal{A}_3$), taking into account (5.3.3a), (5.3.3b) and the boundedness of the mass (Lemma 5.29), we can derive through the Gagliardo–Nirenberg, used in its less common version proved in [LL16a, Lemma 2.3], and Young's inequalities this bound

$$
\begin{aligned}
\hat{c}\int_\Omega (u+1)^{p+m_2+k-1} =&\hat{c}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2(p+m_2+k-1)}{p+m_1-1}}(\Omega)}^{\frac{2(p+m_2+k-1)}{p+m_1-1}}\\
\leq& c_7\left(\left\|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^2(\Omega)}^{\theta}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}^{1-\theta}\right.\\
&\left.+\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}\right)^{\sigma}\\
\leq& c_8\left(\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2\right)^{\frac{\sigma\theta}{2}}+c_9\\
\leq&\frac{p-1}{(p+m_1-1)^2}\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2+c_1\quad\text{on }(0,T_{\max}).
\end{aligned}
$$

(We point out that we have made use of (3.1.5) which we will employ without mentioning it if not necessary.)

Similarly to what we have done before, by supposing ($\mathcal{A}_5$), estimate (5.3.4b) is obtained by applying in this case (5.3.3c) and (5.3.3d), so entailing

$$
\begin{aligned}
\hat{c}\int_\Omega (u+1)^{p+m_3+l-1} =&\hat{c}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2(p+m_3+l-1)}{p+m_1-1}}(\Omega)}^{\frac{2(p+m_3+l-1)}{p+m_1-1}}\\
\leq& c_{10}\left(\left\|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^2(\Omega)}^{\theta_1}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}^{1-\theta_1}\right.\\
&\left.+\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}\right)^{\sigma_1}\\
\leq& c_{11}\left(\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2\right)^{\frac{\sigma_1\theta_1}{2}}+c_{12}\\
\leq&\frac{p-1}{(p+m_1-1)^2}\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2+c_2\quad\text{for all } t\in(0,T_{\max}).
\end{aligned}
$$

In order to prove (5.3.4c) we distinguish the cases $l\leq 1$ and $l>1$. If $0<l\leq 1$ the boundedness of the mass, given in Lemma 5.29, and Young's inequality yield

$$
\hat{c}\int_\Omega (u+1)^{p+m_3-1}\int_\Omega (u+1)^l\leq c_{13}\int_\Omega (u+1)^{p+m_3-1}\leq\varepsilon_1\int_\Omega (u+1)^{p+m_3+l-1}+c_3\quad\text{on }(0,T_{\max}).
$$

On the other hand, if $l>1$ we first have to introduce $q>\max\{l,m_3+l-1\}$. Subsequently, by applying twice the Hölder inequality, we get on $(0,T_{\max})$, by relying on (3.1.6)

$$
\begin{aligned}
\hat{c}\int_\Omega (u+1)^{p+m_3-1}\int_\Omega (u+1)^l&\leq c_{14}\left[\int_\Omega (u+1)^{p+m_3+l-1}\right]^{\frac{p+m_3-1}{p+m_3+l-1}}\left[\left(\int_\Omega (u+1)^q\right)^{\frac{p}{q}+1}\right]^{\frac{l}{p+q}}\\
&\leq\varepsilon_1\int_\Omega (u+1)^{p+m_3+l-1}+\varepsilon_1\left(\int_\Omega (u+1)^q\right)^{\frac{p}{q}+1}+c_{15}.
\end{aligned}
\tag{5.3.5}
$$

Now we focus on the second integral of the right-hand side of (5.3.5). By exploiting (5.3.3g), (5.3.3h) and

Lemma 5.29, a combination of the Gagliardo–Nirenberg and Young's inequalities gives

$$
\begin{aligned}
\varepsilon_1\left(\int_\Omega (u+1)^q\right)^{\frac{p}{q}+1} =& \varepsilon_1\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2q}{p+m_1-1}}(\Omega)}^{\frac{2(p+q)}{p+m_1-1}}\\
\leq & c_{16}\left(\left\|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^2(\Omega)}^{\theta_4}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}^{1-\theta_4}\right.\\
&\left.+\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}\right)^{\sigma_2}\\
\leq & c_{17}\left(\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2\right)^{\frac{\sigma_2\theta_4}{2}} + c_{18}\\
\leq & \frac{p-1}{(p+m_1-1)^2}\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_3 \quad \text{on } (0,T_{\max}),
\end{aligned}
$$

which plugged in the previous one concludes the proof.

As to (5.3.4d), again by means of the Young inequality and relation (3.3.4), recalling the definition of f_2 in (5.3.1) we can write on $(0,T_{\max})$

$$
\begin{aligned}
\hat{c}\int_\Omega (u+1)^{p+m_3-1}w &\leq \rho\int_\Omega (u+1)^{p+m_3+l-1} + c_{19}\int_\Omega w^{\frac{p+m_3-1}{l}+1}\\
&\leq \rho\int_\Omega (u+1)^{p+m_3+l-1} + \rho\int_\Omega f_2(u)^{\frac{p+m_3+l-1}{l}} + C_\rho\left(\int_\Omega f_2(u)\right)^{\frac{p+m_3+l-1}{l}}\\
&\leq \varepsilon_2\int_\Omega (u+1)^{p+m_3+l-1} + c_{20}\left(\int_\Omega (u+1)^l\right)^{\frac{p+m_3+l-1}{l}}
\end{aligned}
\tag{5.3.6}
$$

For $l\leq 1$, due to Lemma 5.29 the last integral of the right-hand side of (5.3.6) is bounded, and we have

$$
\hat{c}\int_\Omega (u+1)^{p+m_3-1}w \leq \varepsilon_2\int_\Omega (u+1)^{p+m_3+l-1} + c_4 \quad \text{on } (0,T_{\max}).
$$

Instead, for $l>1$, by virtue of (5.3.3e), Lemma 5.29 and (5.3.3f), the Gagliardo–Nirenberg and Young's inequalities imply

$$
\begin{aligned}
c_{20}\left(\int_\Omega (u+1)^l\right)^{\frac{p+m_3+l-1}{l}} =& c_{20}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2l}{p+m_1-1}}(\Omega)}^{\sigma_1}\\
\leq & c_{21}\left(\left\|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^2(\Omega)}^{\theta_2}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}^{1-\theta_2}\right.\\
&\left.+\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}\right)^{\sigma_1}\\
\leq & c_{22}\left(\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2\right)^{\frac{\sigma_1\theta_2}{2}} + c_{23}\\
\leq & \frac{p-1}{(p+m_1-1)^2}\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_4 \quad \text{on } (0,T_{\max}),
\end{aligned}
$$

and we conclude by plugging the bound above into (5.3.6).

Finally, let us prove (5.3.4e). Taking into account (5.3.3i) and Lemma 5.29, a further application of the Gagliardo–Nirenberg inequality leads to

$$
\begin{aligned}
\int_\Omega (u+1)^p =& \left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|^{\frac{2p}{p+m_1-1}}_{L^{\frac{2p}{p+m_1-1}}(\Omega)}\\
\leq & c_{24}\left(\left\|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right\|^{\theta_3}_{L^2(\Omega)}\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|^{1-\theta_3}_{L^{\frac{2}{p+m_1-1}}(\Omega)}\right.\\
&\left.+\left\|(u+1)^{\frac{p+m_1-1}{2}}\right\|_{L^{\frac{2}{p+m_1-1}}(\Omega)}\right)^{\frac{2p}{p+m_1-1}}\\
\leq & c_{25}\left(\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2\right)^{\frac{p\theta_3}{p+m_1-1}}+c_{26}\\
\leq & c_{27}\left(\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2+1\right)^{\frac{p\theta_3}{p+m_1-1}} \quad \text{on } (0,T_{\max}),
\end{aligned}
$$

obtaining the claim after basic manipulations in the previous estimate. □

Lemma 5.32. *For the value of $\bar p>1$ found in Lemma 5.28, let for $p>\bar p$ define the functional*

$$\varphi(t):=\frac{1}{p}\int_\Omega (u+1)^p,$$

and for $j\in\{m_2,m_3\}$, let

$$F_j(u):=\int_0^u \hat u(\hat u+1)^{p+j-3}d\hat u. \tag{5.3.7}$$

Then, for all $t\in(0,T_{\max})$ and some c_{28} and c_{29}, it holds

$$
\begin{aligned}
\varphi'(t)\leq & -\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2-(p-1)\chi\int_\Omega F_{m_2}(u)\Delta v\\
&+(p-1)\xi\int_\Omega F_{m_3}(u)\Delta w-c_{28}\int_\Omega (u+1)^{p+r-1}+c_{29}.
\end{aligned}
\tag{5.3.8}
$$

Proof. By differentiating the energy functional, we have on $(0,T_{\max})$,

$$
\begin{aligned}
\varphi'(t)=&\int_\Omega (u+1)^{p-1}u_t\\
=&\int_\Omega (u+1)^{p-1}\nabla\cdot\big((u+1)^{m_1-1}\nabla u\big)-\chi\int_\Omega (u+1)^{p-1}\nabla\cdot\big(u(u+1)^{m_2-1}\nabla v\big)\\
&+\xi\int_\Omega (u+1)^{p-1}\nabla\cdot\big(u(u+1)^{m_3-1}\nabla w\big)+\lambda\int_\Omega (u+1)^{p-1}u-\mu\int_\Omega (u+1)^{p-1}u^r.
\end{aligned}
$$

Due to the divergence theorem, the three first integrals on the right-hand side make that the above identity becomes, for all $t\in(0,T_{\max})$,

$$
\begin{aligned}
\varphi'(t)=&-(p-1)\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2+(p-1)\chi\int_\Omega u(u+1)^{p+m_2-3}\nabla u\cdot\nabla v\\
&-(p-1)\xi\int_\Omega u(u+1)^{p+m_3-3}\nabla u\cdot\nabla w+\lambda\int_\Omega (u+1)^{p-1}u-\mu\int_\Omega (u+1)^{p-1}u^r.
\end{aligned}
$$

By recalling (5.3.7), we rewrite the previous expression as

$$
\begin{aligned}
\varphi'(t)=&-\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega\left|\nabla(u+1)^{\frac{p+m_1-1}{2}}\right|^2+(p-1)\chi\int_\Omega\nabla F_{m_2}(u)\cdot\nabla v\\
&-(p-1)\xi\int_\Omega\nabla F_{m_3}(u)\cdot\nabla w+\lambda\int_\Omega (u+1)^{p-1}u-\mu\int_\Omega (u+1)^{p-1}u^r \quad \text{on } (0,T_{\max}),
\end{aligned}
$$

and again the divergence theorem provides

$$\begin{aligned}\varphi'(t) = &-\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 - (p-1)\chi\int_\Omega F_{m_2}(u)\Delta v\\ &+ (p-1)\xi\int_\Omega F_{m_3}(u)\Delta w + \lambda\int_\Omega (u+1)^{p-1}u - \mu\int_\Omega (u+1)^{p-1}u^r, \quad \text{on } (0,T_{\max}).\end{aligned}$$

The inequality

$$\frac{1}{2^r}\int_\Omega (u+1)^{p-1+r} \le \int_\Omega (u+1)^{p-1}u^r + \int_\Omega (u+1)^{p-1} \quad \text{on } (0,T_{\max}),$$

justified by inequality (3.1.5), leads to the further bound

$$\begin{aligned}\varphi'(t) \le &-\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 - (p-1)\chi\int_\Omega F_{m_2}(u)\Delta v\\ &+ (p-1)\xi\int_\Omega F_{m_3}(u)\Delta w + \lambda\int_\Omega (u+1)^{p} + \mu\int_\Omega (u+1)^{p-1}\\ &-\frac{\mu}{2^r}\int_\Omega (u+1)^{p-1+r} \quad \text{for all } t\in(0,T_{\max}).\end{aligned} \tag{5.3.9}$$

Finally, a double application of Young's inequality (recalling $r>1$) gives

$$\lambda\int_\Omega (u+1)^p + \mu\int_\Omega (u+1)^{p-1} \le \frac{\mu}{2^{r+1}}\int_\Omega (u+1)^{p-1+r} + c_{30} \quad \text{on } (0,T_{\max}),$$

which, in conjunction with inequality (5.3.9), leads to (5.3.8). ☐

From now on we will distinguish the analyses for the local problem and the nonlocal one.

Study of the local problem $(\mathcal{L})$

Let us separate the elliptic and parabolic cases, i.e., $\tau=0$ and $\tau=1$, respectively.
The parabolic-elliptic case: $\tau=0$

Lemma 5.33. *For the value of $\bar{p}>1$ found in Lemma 5.28, let conditions* (5.3.1)*, and one among* $(\mathcal{A}_1)$*,* $(\mathcal{A}_2)$ *and* $(\mathcal{A}_3)$ *be satisfied. Then,* $u\in L^\infty((0,T_{\max}),L^p(\Omega))$ *for all* $p>\bar{p}$*.*

Proof. The second and third equations in $(\mathcal{L})$ allow the substitution of Δv and Δw in (5.3.8), obtaining on $(0,T_{\max})$, once nonpositive terms are dropped,

$$\begin{aligned}\varphi'(t) \le &-\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 - (p-1)\chi\int_\Omega F_{m_2}(u)(\beta v - f_1(u))\\ &+ (p-1)\xi\int_\Omega F_{m_3}(u)(\delta w - f_2(u)) \quad - c_{28}\int_\Omega (u+1)^{p+r-1} + c_{29}\\ \le &-\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + (p-1)\chi\int_\Omega F_{m_2}(u)f_1(u)\\ &+ c_{31}\int_\Omega F_{m_3}(u)w - (p-1)\xi\int_\Omega F_{m_3}(u)f_2(u) \quad - c_{28}\int_\Omega (u+1)^{p+r-1} + c_{29}.\end{aligned} \tag{5.3.10}$$

With some calculations, by using the definition in (5.3.7), we find that

$$0 \le \frac{1}{p+j-1}u^{p+j-1} \le F_j(u) \le \frac{1}{p+j-1}\big((u+1)^{p+j-1}-1\big). \tag{5.3.11}$$

By considering (5.3.1), (5.3.10), and (5.3.11), we have

$$\begin{aligned}
\varphi'(t) \leq & -\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_{32}\int_\Omega (u+1)^{p+m_2+k-1}\\
&+c_{33}\int_\Omega (u+1)^{p+m_3-1}w - c_{34}\int_\Omega u^{p+m_3-1}(u+1)^l - c_{28}\int_\Omega (u+1)^{p-1+r} + c_{29}\\
\leq & -\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_{32}\int_\Omega (u+1)^{p+m_2+k-1}\\
&+c_{33}\int_\Omega (u+1)^{p+m_3-1}w - c_{35}\int_\Omega (u+1)^{p+m_3+l-1} + c_{34}\int_\Omega (u+1)^l\\
&- c_{28}\int_\Omega (u+1)^{p-1+r} + c_{29} \quad \text{for all } t\in(0,T_{\max}),
\end{aligned} \tag{5.3.12}$$

where we have rearranged the term proportional to $-\int_\Omega u^{p+m_3-1}(u+1)^l$ through (3.1.5). Let us deal with the two terms $\int_\Omega (u+1)^l$ and $\int_\Omega (u+1)^{p+m_3-1}w$. For every $l>0$, we have thanks to the Young inequality that for all $\hat{c}>0$

$$\hat{c}\int_\Omega (u+1)^l \leq \frac{c_{28}}{2}\int_\Omega (u+1)^{p-1+r} + c_{36} \quad \text{on } (0,T_{\max}), \tag{5.3.13}$$

(for $l\leq 1$ the boundedness of the mass (see again Lemma 5.29) would provide a sharper estimate), whereas through (5.3.4d) applied with $\varepsilon_2 < c_{35}$, from (5.3.12) we derive

$$\begin{aligned}
\varphi'(t) \leq & -c_{37}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_{32}\int_\Omega (u+1)^{p+m_2+k-1} - c_{38}\int_\Omega (u+1)^{p+m_3+l-1}\\
&-\frac{c_{28}}{2}\int_\Omega (u+1)^{p-1+r} + c_{39} \quad \text{for all } t\in(0,T_{\max}).
\end{aligned} \tag{5.3.14}$$

Now we use assumption $(\mathcal{A}_1)$ to write by Young's inequality

$$c_{32}\int_\Omega (u+1)^{p+m_2+k-1} \leq c_{38}\int_\Omega (u+1)^{p+m_3+l-1} + c_{40} \quad \text{on } (0,T_{\max}),$$

which, introduced into (5.3.14), provides after neglecting the term from the logistic source

$$\varphi'(t) \leq -c_{37}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_{41} \quad \text{for all } t\in(0,T_{\max}).$$

The same estimate, up to constants, is obtained with assumption $(\mathcal{A}_2)$ by controlling $\int_\Omega (u+1)^{p+m_2+k-1}$ with $\int_\Omega (u+1)^{p-1+r}$, invoking the Young inequality, or relying on assumption $(\mathcal{A}_3)$ by exploiting (5.3.4a).

Finally, inequality (5.3.4e) yields the initial problem

$$\begin{cases}
\varphi'(t) \leq c_{42} - c_{43}\varphi(t)^{\frac{p+m_1-1}{p\theta_3}} & \text{on } (0,T_{\max}),\\
\varphi(0) = \frac{1}{p}\int_\Omega (u_0+1)^p.
\end{cases}$$

Now, Lemma 3.13 ensures $\varphi(t) \leq \max\left\{\varphi(0), \left(\frac{c_{42}}{c_{43}}\right)^{\frac{p\theta_3}{p+m_1-1}}\right\}$ for all $t\in(0,T_{\max})$. □

The parabolic-parabolic case: $\tau = 1$

Lemma 5.34. *For the value of $\bar{p}>1$ found in Lemma 5.28, let conditions* (5.3.1)*, and one among* $(\mathcal{A}_2)$*,* $(\mathcal{A}_3)$*,* $(\mathcal{A}_4)$ *and* $(\mathcal{A}_5)$ *be valid. Then,* $u\in L^\infty((0,T_{\max}),L^p(\Omega))$ *for all* $p>\bar{p}$*.*

Proof. We start dealing with bounds for the terms $\int_\Omega F_{m_2}(u)\Delta v$ and $\int_\Omega F_{m_3}(u)\Delta w$. Relations in (5.3.11) and Young's inequality provide

$$\begin{aligned}-(p-1)\chi\int_\Omega F_{m_2}(u)\Delta v \leq&(p-1)\chi\int_\Omega F_{m_2}(u)|\Delta v| \leq c_{44}\int_\Omega (u+1)^{p+m_2-1}|\Delta v|\\ &\leq \int_\Omega (u+1)^{p+m_2+k-1} + c_{45}\int_\Omega |\Delta v|^{\frac{p+m_2-1+k}{k}} \quad \text{on } (0,T_{\max}),\end{aligned} \tag{5.3.15}$$

and analogously

$$(p-1)\xi\int_\Omega F_{m_3}(u)\Delta w \leq \int_\Omega (u+1)^{p+m_3+l-1} + c_{46}\int_\Omega |\Delta w|^{\frac{p+m_3-1+l}{l}}. \tag{5.3.16}$$

For $z=v$, $\eta=\beta$, $\psi=f_1(u)$ (and using (5.3.1)) and $q=\frac{p+m_2+k-1}{k}$, inequality (3.3.5) allows us to write

$$\int_0^t e^s\int_\Omega |\Delta v|^{\frac{p+m_2+k-1}{k}}\,ds \leq C_P\left(1+\int_0^t e^s\int_\Omega (u+1)^{p+m_2+k-1}\,ds\right) \quad \text{on } (0,T_{\max}), \tag{5.3.17}$$

and, reasoning similarly for the third equation in model $(\mathcal{L})$,

$$\int_0^t e^s\int_\Omega |\Delta w|^{\frac{p+m_3+l-1}{l}}\,ds \leq \tilde{C}_P\left(1+\int_0^t e^s\int_\Omega (u+1)^{p+m_3+l-1}\,ds\right) \quad \text{for all } t\in(0,T_{\max}). \tag{5.3.18}$$

By substituting (5.3.15) and (5.3.16) into (5.3.8), we obtain

$$\begin{aligned}\varphi'(t) \leq &-\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + \int_\Omega (u+1)^{p+m_2+k-1}\\ &+ c_{45}\int_\Omega |\Delta v|^{\frac{p+m_2+k-1}{k}} + \int_\Omega (u+1)^{p+m_3+l-1}\\ &+ c_{46}\int_\Omega |\Delta w|^{\frac{p+m_3+l-1}{l}} - c_{28}\int_\Omega (u+1)^{p+r-1} + c_{29} \quad \text{on } (0,T_{\max}).\end{aligned} \tag{5.3.19}$$

Now we add to both sides of (5.3.19) the term $\varphi(t)$, and successively we multiply by e^t. Since $e^t\varphi'(t)+e^t\varphi(t)=\frac{d}{dt}(e^t\varphi(t))$, an integration over $(0,t)$ provides

$$\begin{aligned}e^t\varphi(t) \leq&\varphi(0) + \int_0^t e^s\left(-\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + \int_\Omega (u+1)^{p+m_2+k-1}\right.\\ &+ c_{45}\int_\Omega |\Delta v|^{\frac{p+m_2+k-1}{k}} + \int_\Omega (u+1)^{p+m_3+l-1} + c_{46}\int_\Omega |\Delta w|^{\frac{p+m_3+l-1}{l}}\\ &\left.+\frac{1}{p}\int_\Omega (u+1)^p - c_{28}\int_\Omega (u+1)^{p+r-1} + c_{29}\right)\,ds \quad \text{for all } t\in(0,T_{\max}).\end{aligned} \tag{5.3.20}$$

Relations (5.3.17) and (5.3.18), in conjunction with Young's inequality (recall $r>1$) entail on $(0,T_{\max})$ once inserted into (5.3.20)

$$\begin{aligned}\frac{e^t}{p}\int_\Omega u^p \leq e^t\varphi(t) \leq&\varphi(0) + \int_0^t e^s\left(-\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2\right.\\ &+ c_{47}\int_\Omega (u+1)^{p+m_2+k-1} + c_{48}\int_\Omega (u+1)^{p+m_3+l-1}\\ &\left.-\frac{c_{28}}{2}\int_\Omega (u+1)^{p+r-1} + c_{49}\right)\,ds.\end{aligned} \tag{5.3.21}$$

At this stage, analogously to what we have done in the previous lemma, if restriction $(\mathcal{A}_2)$ is supported with one between $(\mathcal{A}_4)$ and $(\mathcal{A}_5)$, Young's inequality and (5.3.4b) make that bound (5.3.21) is turned into

$$e^t\int_\Omega u^p \leq c_{50} + c_{51}(e^t-1) \quad \text{for all } t\in(0,T_{\max}). \tag{5.3.22}$$

The same procedure applies if we take into account ($\mathcal{A}_3$), implying (5.3.4a), with either ($\mathcal{A}_4$) (by means of Young's inequality) or ($\mathcal{A}_5$), ensuring (5.3.4b).

Henceforth, also in this case inequality (5.3.22) is derived and as a consequence, in each of the above situations, we conclude that

$$\int_\Omega u^p \leq c_{52} \quad \text{on } (0, T_{\max}).$$

□

Study of the nonlocal problem ($\mathcal{N}$)

Lemma 5.35. *Let the hypotheses of Lemma 5.33 be valid. Then, $u \in L^\infty((0,T_{\max}), L^p(\Omega))$ for all $p > \bar{p}$.*

Proof. Starting from relation (5.3.8), the second and the third equations in model ($\mathcal{N}$) lead to

$$\begin{aligned}
\varphi'(t) \leq & -\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + (p-1)\chi \int_\Omega F_{m_2}(u) f_1(u) \\
& + \frac{(p-1)\xi}{|\Omega|}\int_\Omega F_{m_3}(u)\int_\Omega f_2(u) - (p-1)\xi \int_\Omega F_{m_3}(u) f_2(u) - c_{28}\int_\Omega (u+1)^{p-1+r} + c_{29} \\
\leq & -\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_{53}\int_\Omega (u+1)^{p+m_2+k-1} \\
& + c_{54}\int_\Omega (u+1)^{p+m_3-1}\int_\Omega (u+1)^l - c_{55}\int_\Omega u^{p+m_3-1}(u+1)^l - c_{28}\int_\Omega (u+1)^{p-1+r} + c_{29},
\end{aligned}$$

for all $t \in (0, T_{\max})$, where we have taken in mind the properties of F_j and f_1 and f_2 fixed in (5.3.1) and (5.3.11), and dropped nonpositive terms. In turn, thanks again to (3.1.5), we have

$$\begin{aligned}
\varphi'(t) \leq & -\frac{4(p-1)}{(p+m_1-1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_{53}\int_\Omega (u+1)^{p+m_2+k-1} \\
& + c_{54}\int_\Omega (u+1)^{p+m_3-1}\int_\Omega (u+1)^l - c_{56}\int_\Omega (u+1)^{p+m_3+l-1} \\
& + c_{57}\int_\Omega (u+1)^l - c_{28}\int_\Omega (u+1)^{p-1+r} + c_{29} \quad \text{on } (0, T_{\max}).
\end{aligned}$$

With the aid of estimate (5.3.13), by exploiting (5.3.4c) with $\varepsilon_1 < c_{56}$, we obtain

$$\begin{aligned}
\varphi'(t) \leq & -c_{11}\int_\Omega \left|\nabla (u+1)^{\frac{p+m_1-1}{2}}\right|^2 + c_{53}\int_\Omega (u+1)^{p+m_2+k-1} \\
& - c_{58}\int_\Omega (u+1)^{p+m_3+l-1} - \frac{c_{28}}{2}\int_\Omega (u+1)^{p-1+r} + c_{59} \quad \text{on } (0, T_{\max}).
\end{aligned}$$

Since up to constants the previous estimate coincides with (5.3.14), we can follow the same arguments of Lemma 5.33 to conclude. □

5.3.6 Proof of Theorems 5.24 and 5.25

For $\tau = 0$, we use respectively Lemma 5.33 and Lemma 5.35, supported by Lemma 5.30; in this way Theorem 5.24 is established. Theorem 5.25 is evidently obtained by invoking Lemma 5.34.

5.4 [LASCV25] Dissipative gradient nonlinearities prevent δ-formations in local and nonlocal attraction-repulsion chemotaxis models

5.4.1 Presentation of the models and of the main results

We herein are interested to address questions connected to local-well posedness and global solvability of the following *local* and *nonlocal* attraction-repulsion chemotaxis models with gradient-dependent sources:

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v + \xi u(u+1)^{m_3-1}\nabla w\big) & \\ \qquad +\lambda u^\rho - \mu u^k - c|\nabla u|^\gamma & \text{in } \Omega\times(0,T_{\max}),\\ \tau v_t = \Delta v - v + f_1(u) & \text{in } \Omega\times(0,T_{\max}),\\ \tau w_t = \Delta w - w + f_2(u) & \text{in } \Omega\times(0,T_{\max}),\\ u_\nu = v_\nu = w_\nu = 0 & \text{on } \partial\Omega\times(0,T_{\max}),\\ u(x,0)=u_0(x), \tau v(x,0) = \tau v_0(x), \tau w(x,0)=\tau w_0(x) & x\in\bar\Omega, \end{cases} \tag{$\mathcal{L}_\tau$}$$

and

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v + \xi u(u+1)^{m_3-1}\nabla w\big) & \\ \qquad +\lambda u^\rho - \mu u^k - c|\nabla u|^\gamma & \text{in } \Omega\times(0,T_{\max}),\\ 0 = \Delta v - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega f_1(u) + f_1(u) & \text{in } \Omega\times(0,T_{\max}),\\ 0 = \Delta w - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega f_2(u) + f_2(u) & \text{in } \Omega\times(0,T_{\max}),\\ u_\nu = v_\nu = w_\nu = 0 & \text{on } \partial\Omega\times(0,T_{\max}),\\ u(x,0)=u_0(x) & x\in\bar\Omega,\\ \displaystyle\int_\Omega v(x,t)\,dx = \int_\Omega w(x,t)\,dx = 0 & \text{on } (0,T_{\max}). \end{cases} \tag{$\mathcal{NL}$}$$

Moreover, we might require that for proper $\alpha, \beta, k_1, k_3 \geq k_2 > 0$,

$$f_1(s) \leq k_1(s+1)^\alpha \quad \text{and} \quad k_2(s+1)^\beta \leq f_2(s) \leq k_3(s+1)^\beta. \tag{5.4.1}$$

Finally, for $\tau \in \{0,1\}$, $m_1, m_2, m_3 \in \mathbb{R}$ and $\alpha, \beta, \chi, \xi, \lambda, \mu, c > 0$, $1 \leq \rho < k$ let

$$\max\left\{1, \frac{n}{n+1}(m_2+\alpha), \tau\frac{n}{n+1}(m_3+\beta)\right\} < \gamma \leq 2. \tag{$\mathcal{G}_\tau$}$$

We will prove the following result.

Theorem 5.36. *For $\tau \in \{0,1\}$, let f_1, f_2 comply with hypotheses in* (5.4.1)*. Additionally, let $\chi, \xi, \lambda, \mu, c > 0$, and $m_1, m_2, m_3 \in \mathbb{R}$, $k > \rho \leq 1$, $\gamma \geq 1$ such that condition* ($\mathcal{G}_\tau$) *hold true. Then problems* ($\mathcal{L}_\tau$) *and* ($\mathcal{NL}$) *admit a global and bounded solution.*

Let us give some comments.

Remark 5.37 (Comparison with [ILV24])**.** *Let us analyze our research in the spirit of [ILV24].*

- *For $w \equiv 0$, models ($\mathcal{L}_\tau$) generalize those studied in [ILV24]. In particular, for $m_1 = m_2 = \alpha = 1$, problem ($\mathcal{L}_\tau$) is reduced to [ILV24, (2)] and condition ($\mathcal{G}_\tau$) reads $\frac{2n}{n+1} < \gamma \leq 2$; henceforth, [ILV24, Theorem 1.2] is a particular case of Theorem 5.36.*
- *Even though the local-well posedness is established in [ILV24, Theorem 1.1], for the fully nonlinear case some adjustments are required; additionally, conversely to [ILV24, Theorem 1.1], Theorem 4.1 gives all the details concerning the uniqueness of classical solutions.*

- *[ILV24, Remark 1.5] leaves open the question whether for some small* $\gamma \in [1, \frac{2n}{n+1}]$ *there may exist unbounded solutions to problem* $(\mathcal{L}_\tau)$*, naturally for* $w \equiv 0, m_1 = m_2 = \alpha = 1$*. Through some numerical simulations, we herein give evidences supporting some possible scenarios in such a direction.*
- *In [ILV24] the model involves a larger class of sources than those herein utilized, and whose general expression is* $h(u, \nabla u) = f(u) - g(\nabla u)$*, being* f *and* g *function with certain growth properties.*

5.4.2 Necessary parameters

We will make use of this result, which we present without any extra comment. We also underline that ε indicates an *arbitrary* positive real number, and the multiplication by another constant and the sum with another *arbitrary* constant are not performed, and the final result is for commodity as well labeled with ε.

Lemma 5.38. *Let* $n \in \mathbb{N}$*,* $m_2, m_3 \in \mathbb{R}$*,* $\gamma, \alpha, \beta > 0$*, and the lower bound of the relation* $(\mathcal{G}_\tau)$ *be fixed. Then, for all* $q > 1$ *there exists* $\bar{p} > 1$ *such that for all* $p > \bar{p}$ *and*

$$\theta(p) := \frac{\frac{p+\gamma-1}{\gamma} - \frac{p+\gamma-1}{p\gamma}}{\frac{p+\gamma-1}{\gamma} - \frac{1}{\gamma} + \frac{1}{n}}, \qquad \sigma(p) := \frac{p\gamma}{p+\gamma-1}, \qquad \bar{\theta}(p) := \frac{\frac{p+\gamma-1}{\gamma} - \frac{p+\gamma-1}{\gamma(p+m_2+\alpha-1)}}{\frac{p+\gamma-1}{\gamma} - \frac{1}{\gamma} + \frac{1}{n}},$$

$$\bar{\sigma}(p) := \frac{\gamma(p+m_2+\alpha-1)}{p+\gamma-1}, \qquad \hat{\theta}(p) := \frac{\frac{p+\gamma-1}{\gamma} - \frac{p+\gamma-1}{\gamma(p+m_3+\beta-1)}}{\frac{p+\gamma-1}{\gamma} - \frac{1}{\gamma} + \frac{1}{n}}, \qquad \hat{\sigma}(p) := \frac{\gamma(p+m_3+\beta-1)}{p+\gamma-1},$$

$$\tilde{\theta}(p) := \frac{\frac{p+\gamma-1}{\gamma} - \frac{p+\gamma-1}{\gamma\beta}}{\frac{p+\gamma-1}{\gamma} - \frac{1}{\gamma} + \frac{1}{n}}, \qquad \underline{\theta}(p) := \frac{\frac{p+\gamma-1}{\gamma} - \frac{p+\gamma-1}{\gamma q}}{\frac{p+\gamma-1}{\gamma} - \frac{1}{\gamma} + \frac{1}{n}}, \qquad \underline{\sigma}(p) := \frac{\gamma(p+q)}{p+\gamma-1},$$

these inequalities hold:

$$0 < \theta < 1, \quad (5.4.2a) \qquad 0 < \frac{\sigma\theta}{\gamma} < 1. \quad (5.4.2b) \qquad 0 < \bar{\theta} < 1, \quad (5.4.2c) \qquad 0 < \frac{\bar{\sigma}\bar{\theta}}{\gamma} < 1. \quad (5.4.2d)$$

$$0 < \hat{\theta} < 1, \quad (5.4.2e) \qquad 0 < \frac{\hat{\sigma}\hat{\theta}}{\gamma} < 1. \quad (5.4.2f) \qquad 0 < \tilde{\theta} < 1, \quad (5.4.2g) \qquad 0 < \frac{\hat{\sigma}\tilde{\theta}}{\gamma} < 1. \quad (5.4.2h)$$

$$0 < \underline{\theta} < 1, \quad (5.4.2i) \qquad 0 < \frac{\underline{\sigma}\,\underline{\theta}}{\gamma} < 1. \quad (5.4.2j)$$

In the specific (5.4.2g) *and* (5.4.2h) *are satisfied under the stronger assumption* $\beta > 1$*.*

Proof. By computing $\theta'(p)$, we observe that $\theta'(p) > 0$ for p sufficiently large, so that $\theta(p)$ is definitely increasing. Moreover, since $\lim_{p\to+\infty} \theta(p) = 1$, relation (5.4.2a) remains valid for some large p. Similarly we can obtain (5.4.2b) (for which we use $\gamma > 1$), (5.4.2c), (5.4.2e), whereas for (5.4.2d) and (5.4.2f) we exploit, respectively, $\gamma > \frac{n}{n+1}(m_2 + \alpha)$ and $\gamma > \frac{n}{n+1}(m_3 + \beta)$. As to (5.4.2i) and (5.4.2j), we observe that

$$\lim_{p\to+\infty} \underline{\theta}(p) = \lim_{p\to+\infty} \frac{\underline{\sigma}(p)\underline{\theta}(p)}{\gamma} = 1 - \frac{1}{q} \in (0,1).$$

On the other hand, for $\beta > 1$

$$\lim_{p\to+\infty} \tilde{\theta}(p) = \lim_{p\to+\infty} \frac{\hat{\sigma}(p)\tilde{\theta}(p)}{\gamma} = 1 - \frac{1}{\beta} \in (0,1),$$

so that (5.4.2g) and (5.4.2h) are fulfilled. From the above reasoning, we can pick some $\bar{p} > 1$ such that for all $p > \bar{p}$ the claim is given. □

We also remind that $u \in L^\infty((0,T_{\max}),L^1(\Omega))$ in the sense that

$$\int_\Omega u \le M := \max\left\{\int_\Omega u_0(x)\,dx, \left(\frac{\lambda}{\mu}|\Omega|^{k-\rho}\right)^{\frac{1}{k-\rho}}\right\} \quad \text{on } [0,T_{\max}), \tag{5.4.3}$$

as it is proved in Proposition 4.8.

5.4.3 Extension criterion

Lemma 5.39. *Let $f_1(u)$ and $f_2(u)$ comply with* (5.4.1) *and $\gamma \in [1,2]$. If $u \in L^\infty((0,T_{\max});L^p(\Omega))$ for every $p > 1$, then $u \in L^\infty((0,\infty);L^\infty(\Omega))$.*

Proof. With the aid of Proposition 4.4, it is sufficient to show that $u \in L^\infty((0,T_{\max});L^\infty(\Omega))$. First we observe that for any $\lambda,\mu,c>0$, and $k>\rho\ge 1$, $\gamma \ge 1$, the function $u \mapsto \lambda u^\rho - \mu u^k - c|\nabla u|^\gamma$, for $u \ge 0$, is bounded from above by a positive constant L; henceforth, we can write $g(x,t) := \lambda u^\rho - \mu u^k - c|\nabla u|^\gamma \le L$ in $\Omega \times (0,T_{\max})$. Subsequently, we notice that u classically solves solves problem (A.1) in Lemma 4.6 with

$$D(x,t,u) = (u+1)^{m_1-1}, \quad f(x,t) = -\chi u(u+1)^{m_2-1}\nabla v + \xi u(u+1)^{m_3-1}\nabla w, \quad g(x,t) = L.$$

In particular, this in conjunction with the Neumann boundary conditions on v and w imply that hypotheses (A.2)–(A.5) are met, whereas the second inclusion of (A.6) holds for any choice of $q_2 > 1$. Since we are assuming p arbitrarily large, and $u \in L^\infty((0,T_{\max});L^p(\Omega))$, conditions (A.7) and (A.8) are automatically complied. Let us now deal with the first inclusion of (A.6). If $\tau = 0$ in problem $(\mathcal{L}_\tau)$ (or directly considering model $(\mathcal{NL})$), from $u \in L^\infty((0,T_{\max});L^p(\Omega))$ we have $f_1(u) \in L^\infty((0,T_{\max});L^p(\Omega))$ so that (3.3.3) imply $\nabla v \in L^\infty((0,T_{\max});L^\infty(\Omega))$ (where $\psi = f_1$). Since the same inclusion holds for ∇w (by applying again (3.3.3) with $\psi = f_2$), we have that, for any $q_1 > 1$,

$$f = -\chi u(u+1)^{m_2-1}\nabla v + \xi u(u+1)^{m_3-1}\nabla w \in L^\infty((0,T_{\max});L^{q_1}(\Omega)).$$

If $\tau = 1$, we can reason in the same way invoking (3.3.6). So we have the claim exactly by virtue of Lemma 4.6. □

5.4.4 Derivation of sufficient a priori estimates. Proof of Theorem 5.36

The following two lemmas will be used later on. (The appearing value $\bar{p}$ is taken from Lemma 5.38.)

Lemma 5.40. *Under the hypotheses of Lemma 5.38, for every ε we have that for all $p > \bar{p}$*

$$\int_\Omega (u+1)^p \le \varepsilon \int_\Omega \left|\nabla (u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_1 \quad \textit{on } (0,T_{\max}), \tag{5.4.4}$$

$$-\int_\Omega \left|\nabla (u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma \le c_2 - c_3\left(\int_\Omega (u+1)^p\right)^{\frac{\gamma}{\sigma\theta}} \quad \textit{on } (0,T_{\max}), \tag{5.4.5}$$

$$\int_\Omega (u+1)^{p+m_2+\alpha-1} \le \varepsilon \int_\Omega \left|\nabla (u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_4 \quad \textit{on } (0,T_{\max}), \tag{5.4.6}$$

$$\int_\Omega (u+1)^{p+m_3+\beta-1} \le \varepsilon \int_\Omega \left|\nabla (u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_5 \quad \textit{on } (0,T_{\max}), \tag{5.4.7}$$

$$\int_\Omega (u+1)^{p+m_3-1} w \le \varepsilon \int_\Omega (u+1)^{p+m_3+\beta-1} + \varepsilon \int_\Omega \left|\nabla (u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_6 \quad \textit{on } (0,T_{\max}), \tag{5.4.8}$$

$$\begin{aligned}\int_\Omega (u+1)^{p+m_3-1} \int_\Omega (u+1)^\beta \le{}& \varepsilon \int_\Omega (u+1)^{p+m_3+\beta-1} \\ &+ \varepsilon \int_\Omega \left|\nabla (u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_7 \quad \textit{on } (0,T_{\max}).\end{aligned} \tag{5.4.9}$$

Proof. In order to prove relation (5.4.4), we use the Gagliardo–Nirenberg inequality (see Proposition 3.5) in conjunction with (5.4.2a) and (5.4.3), obtaining

$$\begin{aligned}\int_\Omega (u+1)^p =&\left\|(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^{\frac{p\gamma}{p+\gamma-1}}(\Omega)}^{\frac{p\gamma}{p+\gamma-1}}\\ \leq& c_8\left(\left\|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^\gamma(\Omega)}^{\theta}\left\|(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^{\frac{\gamma}{p+\gamma-1}}(\Omega)}^{1-\theta}+\left\|(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^{\frac{\gamma}{p+\gamma-1}}(\Omega)}\right)^{\sigma}\\ \leq& c_9\left(\int_\Omega\left|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma\right)^{\frac{\sigma\theta}{\gamma}}+c_{10}\leq\varepsilon\int_\Omega\left|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma+c_1\quad\text{on }(0,T_{\max}),\end{aligned}$$

where in the second-last step we employed relation (3.1.5), and in the last one Young's inequality, justified by relation (5.4.2b).

As to bound (5.4.5) we refrain by using the Young inequality in the previous derivation: subsequently we have

$$\int_\Omega (u+1)^p =\left\|(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^{\frac{p\gamma}{p+\gamma-1}}(\Omega)}^{\frac{p\gamma}{p+\gamma-1}}\leq c_9\left(\int_\Omega\left|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma\right)^{\frac{\sigma\theta}{\gamma}}+c_{10}\quad\text{on }(0,T_{\max}).$$

and we conclude again by relying on (3.1.5).

On the other hand, relation (5.4.6) is achieved again by invoking the Gagliardo–Nirenberg inequality, (5.4.2c), the boundedness of the mass (i.e., (5.4.3)), Young's inequality and (5.4.2d): specifically we have

$$\begin{aligned}\int_\Omega (u+1)^{p+m_2+\alpha-1} =&\left\|(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^{\frac{\gamma(p+m_2+\alpha-1)}{p+\gamma-1}}(\Omega)}^{\frac{\gamma(p+m_2+\alpha-1)}{p+\gamma-1}}\\ \leq& c_{11}\left(\left\|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^\gamma(\Omega)}^{\bar\theta}\left\|(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^{\frac{\gamma}{p+\gamma-1}}(\Omega)}^{1-\bar\theta}\right.\\ &\left.+\left\|(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^{\frac{\gamma}{p+\gamma-1}}(\Omega)}\right)^{\bar\sigma}\\ \leq& c_{12}\left(\int_\Omega\left|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma\right)^{\frac{\bar\sigma\bar\theta}{\gamma}}+c_{13}\\ \leq&\varepsilon\int_\Omega\left|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma+c_4\quad\text{on }(0,T_{\max}).\end{aligned}$$

Moreover, in this circumstance relations (5.4.2e) and (5.4.2f) give

$$\begin{aligned}\int_\Omega (u+1)^{p+m_3+\beta-1} =&\left\|(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^{\frac{\gamma(p+m_3+\beta-1)}{p+\gamma-1}}(\Omega)}^{\frac{\gamma(p+m_3+\beta-1)}{p+\gamma-1}}\\ \leq& c_{14}\left(\left\|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^\gamma(\Omega)}^{\hat\theta}\left\|(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^{\frac{\gamma}{p+\gamma-1}}(\Omega)}^{1-\hat\theta}\right.\\ &\left.+\left\|(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^{\frac{\gamma}{p+\gamma-1}}(\Omega)}\right)^{\hat\sigma}\\ \leq& c_{15}\left(\int_\Omega\left|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma\right)^{\frac{\hat\sigma\hat\theta}{\gamma}}+c_{16}\\ \leq&\varepsilon\int_\Omega\left|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma+c_5\quad\text{for all }t\in(0,T_{\max}).\end{aligned}$$

We also note, by using Young's inequality that

$$\int_\Omega (u+1)^{p+m_3-1}w\leq\varepsilon\int_\Omega(u+1)^{p+m_3+\beta-1}+c_{17}\int_\Omega w^{\frac{p+m_3+\beta-1}{\beta}}\quad\text{on }(0,T_{\max}),$$

so that from (3.3.4) and (5.4.1)

$$\begin{aligned}\int_\Omega (u+1)^{p+m_3-1} w \leq& \varepsilon \int_\Omega (u+1)^{p+m_3+\beta-1} + \varepsilon \int_\Omega f_2(u)^{\frac{p+m_3+\beta-1}{\beta}}\\ &+ c_{18}\Big(\int_\Omega f_2(u)\Big)^{\frac{p+m_3+\beta-1}{\beta}}\\ \leq& \varepsilon \int_\Omega (u+1)^{p+m_3+\beta-1} + c_{19}\Big(\int_\Omega (u+1)^\beta\Big)^{\frac{p+m_3+\beta-1}{\beta}},\end{aligned} \tag{5.4.10}$$

on $(0, T_{\max})$, where $c_{19} = c_{19}(C_\varepsilon)$. Now, in order to estimate $c_{19}\big(\int_\Omega (u+1)^\beta\big)^{\frac{p+m_3+\beta-1}{\beta}}$, we obtain that if $\beta \leq 1$, relation (5.4.3) implies

$$c_{19}\Big(\int_\Omega (u+1)^\beta\Big)^{\frac{p+m_3+\beta-1}{\beta}} \leq c_6 \quad \text{on } (0, T_{\max}).$$

Conversely, for $\beta > 1$, by properly using the Gagliardo–Nirenberg and Young inequalities, relations (5.4.2g) and (5.4.2h), and (5.4.3), we can write

$$\begin{aligned}c_{19}\Big(\int_\Omega (u+1)^\beta\Big)^{\frac{p+m_3+\beta-1}{\beta}} =& c_{19}\Big\|(u+1)^{\frac{p+\gamma-1}{\gamma}}\Big\|_{L^{\frac{\beta\gamma}{p+\gamma-1}}(\Omega)}^{\frac{\gamma(p+m_3+\beta-1)}{p+\gamma-1}}\\ \leq& c_{20}\left(\Big\|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\Big\|_{L^\gamma(\Omega)}^{\tilde\theta}\Big\|(u+1)^{\frac{p+\gamma-1}{\gamma}}\Big\|_{L^{\frac{\gamma}{p+\gamma-1}}(\Omega)}^{1-\tilde\theta}\right.\\ &\left.+\Big\|(u+1)^{\frac{p+\gamma-1}{\gamma}}\Big\|_{L^{\frac{\gamma}{p+\gamma-1}}(\Omega)}\right)^{\hat\sigma}\\ \leq& c_{21}\Big(\int_\Omega\Big|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\Big|^\gamma\Big)^{\frac{\hat\sigma\tilde\theta}{\gamma}} + c_{22}\\ \leq& \varepsilon\int_\Omega\Big|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\Big|^\gamma + c_6 \quad \text{on } (0, T_{\max}).\end{aligned}$$

Coming back to bound (5.4.10), the two derivations above established show that for $\beta > 0$ estimate (5.4.8) is valid.

We similarly operate to derive (5.4.9): if $\beta \leq 1$, it is directly obtained by using the boundedness of the mass (i.e., (5.4.3)) and then Young's inequality so to arrive at

$$\int_\Omega (u+1)^{p+m_3-1}\int_\Omega (u+1)^\beta \leq c_{23}\int_\Omega (u+1)^{p+m_3-1} \leq \varepsilon\int_\Omega (u+1)^{p+m_3+\beta-1} + c_7 \quad \text{on } (0, T_{\max}).$$

If $\beta > 1$, we fix $q > \max\{\beta, m_3+\beta-1\}$ and we get through the Hölder inequality and (3.1.6) that

$$\begin{aligned}\int_\Omega (u+1)^{p+m_3-1}\int_\Omega (u+1)^\beta \leq& c_{24}\Big[\int_\Omega (u+1)^{p+m_3+\beta-1}\Big]^{\frac{p+m_3-1}{p+m_3+\beta-1}}\left[\Big(\int_\Omega (u+1)^q\Big)^{\frac{p}{q}+1}\right]^{\frac{\beta}{p+q}}\\ \leq& \varepsilon\int_\Omega (u+1)^{p+m_3+\beta-1} + \Big(\int_\Omega (u+1)^q\Big)^{\frac{p}{q}+1} + c_{25} \quad \text{on } (0, T_{\max}).\end{aligned} \tag{5.4.11}$$

On the other hand, by virtue of $u \in L^\infty((0, T_{\max}); L^1(\Omega))$ and relations (5.4.2i) and (5.4.2j) we can write on $(0, T_{\max})$

$$\begin{aligned}\left(\int_\Omega (u+1)^q\right)^{\frac{p}{q}+1} =&\left\|(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^{\frac{\gamma q}{p+\gamma-1}}(\Omega)}^{\frac{\gamma(p+q)}{p+\gamma-1}}\\ \leq& c_{26}\left(\left\|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^\gamma(\Omega)}^{\theta}\left\|(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^{\frac{\gamma}{p+\gamma-1}}(\Omega)}^{1-\theta}\right.\\ &\left.+\left\|(u+1)^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^{\frac{\gamma}{p+\gamma-1}}(\Omega)}\right)^{\sigma}\\ \leq& c_{27}\left(\int_\Omega\left|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma\right)^{\frac{\sigma\theta}{\gamma}}+c_{28}\leq \varepsilon\int_\Omega\left|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma+c_{29}.\end{aligned} \tag{5.4.12}$$

By inserting (5.4.12) into (5.4.11) also for $\beta > 1$ bound (5.4.9) is achieved. □

In the previous result, we did not use that (u, v, w) is a solution of the models we are considering. The forthcoming lemma, indeed, is valid for solutions to both problems $(\mathcal{L}_\tau)$ and $(\mathcal{NL})$.

Lemma 5.41. *Let the same hypotheses of Lemma 5.38 be given. Then on* $(0, T_{\max})$

$$\varphi'(t) \leq -c_{30}\int_\Omega F_{m_2}(u)\Delta v + c_{31}\int_\Omega F_{m_3}(u)\Delta w - c_{32}\int_\Omega\left|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_{33}, \tag{5.4.13}$$

where $\varphi(t) = \frac{1}{p}\int_\Omega (u+1)^p$ *and, for* $j \in \{m_2, m_3\}$

$$F_j(u) := \int_0^u \hat{u}(\hat{u}+1)^{p+j-3}d\hat{u}. \tag{5.4.14}$$

Proof. Let us differentiate the functional $\varphi(t) = \frac{1}{p}\int_\Omega (u+1)^p$. Using the first equation of $(\mathcal{L}_\tau)$ and the divergence theorem, we have for every $t \in (0, T_{\max})$

$$\begin{aligned}\varphi'(t) =& \int_\Omega (u+1)^{p-1}u_t\\ =&\int_\Omega (u+1)^{p-1}\nabla\cdot((u+1)^{m_1-1}\nabla u) - \chi\int_\Omega (u+1)^{p-1}\nabla\cdot(u(u+1)^{m_2-1}\nabla v)\\ &+\xi\int_\Omega (u+1)^{p-1}\nabla\cdot(u(u+1)^{m_3-1}\nabla w) + \lambda\int_\Omega (u+1)^{p-1}u^\rho - \mu\int_\Omega (u+1)^{p-1}u^k\\ &- c\int_\Omega (u+1)^{p-1}|\nabla u|^\gamma\\ \leq& -(p-1)\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2 + \chi(p-1)\int_\Omega u(u+1)^{p+m_2-3}\nabla u\cdot\nabla v\\ &-\xi(p-1)\int_\Omega u(u+1)^{p+m_3-3}\nabla u\cdot\nabla w + \lambda\int_\Omega (u+1)^{p+\rho-1}\\ &-\mu\int_\Omega (u+1)^{p-1}u^k - c\left(\frac{p-1+\gamma}{\gamma}\right)^{-\gamma}\int_\Omega\left|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma.\end{aligned}$$

By considering the definition of $F_j(u)$ in (5.4.14), we have $\nabla F_j(u) = u(u+1)^{p+j-3}\nabla u$, so that again the divergence theorem yields on $(0, T_{\max})$

$$\begin{aligned}\varphi'(t) \leq& -c_{30}\int_\Omega F_{m_2}(u)\Delta v + c_{31}\int_\Omega F_{m_3}(u)\Delta w + \lambda\int_\Omega (u+1)^{p+\rho-1}\\ &-\mu\int_\Omega (u+1)^{p-1}u^k - c_{32}\int_\Omega\left|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma.\end{aligned}$$

Inverting relation (3.1.5) we also have

$$-\mu\int_\Omega (u+1)^{p-1}u^k \leq -\frac{\mu}{2^{k-1}}\int_\Omega (u+1)^{p+k-1} + \mu\int_\Omega (u+1)^{p-1} \quad \text{for all } t < T_{\max},$$

and consequently

$$\begin{aligned}\varphi'(t) \leq & -c_{30}\int_\Omega F_{m_2}(u)\Delta v + c_{31}\int_\Omega F_{m_3}(u)\Delta w + \lambda\int_\Omega (u+1)^{p+\rho-1} - c_{34}\int_\Omega (u+1)^{p+k-1} \\ & + \mu\int_\Omega (u+1)^{p-1} - c_{32}\int_\Omega \left|\nabla (u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma \quad \text{on } (0,T_{\max}).\end{aligned} \tag{5.4.15}$$

Moreover, since $k > \rho \geq 1$, we can invoke Young's inequality to observe that

$$\lambda\int_\Omega (u+1)^{p+\rho-1} \leq \frac{c_{34}}{2}\int_\Omega (u+1)^{p+k-1} + c_{35} \quad \text{on } (0,T_{\max}), \tag{5.4.16}$$

and

$$\mu\int_\Omega (u+1)^{p-1} \leq \frac{c_{34}}{2}\int_\Omega (u+1)^{p+k-1} + c_{36} \quad \text{on } (0,T_{\max}). \tag{5.4.17}$$

Finally, by using bounds (5.4.16) and (5.4.17) into relation (5.4.15) we obtain what claimed. □

The local model: analysis of problem ($\mathcal{L}_\tau$)

In this section we will give the proof of Theorem 5.36, by deriving the crucial uniform-in-time boundedness of the solution u in some $L^p(\Omega)$-norm. We will separate the cases $\tau = 0$ and $\tau = 1$.

Supported by assumptions (5.4.1), we observe that from relation (5.4.14) we have

$$\frac{u^{p+j-1}}{p+j-1} \leq F_j(u) \leq \frac{1}{p+j-1}\left[(u+1)^{p+j-1} - 1\right]. \tag{5.4.18}$$

We will make use of this property.
The parabolic-elliptic case: problem ($\mathcal{L}_0$)
Let us start with the case $\tau = 0$.

Lemma 5.42. *Let the hypotheses of Lemma 5.38 be given. Then for all $p > \bar{p}$ we have that $u \in L^\infty((0,T_{\max}); L^p(\Omega))$.*

Proof. By taking into account (5.4.13) and (5.4.18), by using the second and third equations of ($\mathcal{L}_\tau$) we obtain on $(0,T_{\max})$

$$\begin{aligned}\varphi'(t) \leq & c_{37}\int_\Omega (u+1)^{p+m_2+\alpha-1} + c_{38}\int_\Omega (u+1)^{p+m_3-1}w - c_{39}\int_\Omega u^{p+m_3-1}f_2(u) \\ & - c_{32}\int_\Omega \left|\nabla (u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_{33}.\end{aligned}$$

Successively, by exploiting (3.1.5) and (5.4.1), we get on $(0,T_{\max})$

$$\begin{aligned}\varphi'(t) \leq & c_{37}\int_\Omega (u+1)^{p+m_2+\alpha-1} + c_{38}\int_\Omega (u+1)^{p+m_3-1}w - c_{40}\int_\Omega (u+1)^{p+m_3+\beta-1} + c_{41}\int_\Omega (u+1)^\beta \\ & - c_{32}\int_\Omega \left|\nabla (u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_{33},\end{aligned}$$

and by manipulating $c_{41}\int_\Omega (u+1)^\beta$ through the Young inequality we have on $(0,T_{\max})$

$$\begin{aligned}\varphi'(t) \leq & c_{37}\int_\Omega (u+1)^{p+m_2+\alpha-1} + c_{38}\int_\Omega (u+1)^{p+m_3-1}w - c_{42}\int_\Omega (u+1)^{p+m_3+\beta-1} \\ & - c_{32}\int_\Omega \left|\nabla (u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_{43}.\end{aligned}$$

In this position, by invoking (5.4.6) and (5.4.8) for proper small ε we have

$$\varphi'(t) \leq -c_{44}\int_\Omega \left|\nabla (u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_{45} \quad \text{on } (0,T_{\max}),$$

so that an application of (5.4.5) yields

$$\begin{cases} \varphi'(t) \le c_{46} - c_{47}\varphi(t)^{\frac{\gamma}{\sigma\theta}} & \text{for all } t \in (0, T_{\max}), \\ \varphi(0) = \dfrac{1}{p}\displaystyle\int_\Omega (u_0+1)^p. \end{cases} \tag{5.4.19}$$

Lemma 3.13 infers $\varphi(t) \le \max\left\{\varphi(0), \left(\frac{c_{46}}{c_{47}}\right)^{\frac{\sigma\theta}{\gamma}}\right\}$ for all $0 < t < T_{\max}$, i.e., $u \in L^\infty((0, T_{\max}); L^p(\Omega))$. □

The parabolic-parabolic case: problem $(\mathcal{L}_1)$
Let us continue the analysis of model $(\mathcal{L}_\tau)$ when $\tau = 1$. We will make use of the parabolic regularity result given in Lemma 3.12.

Lemma 5.43. *Let the hypotheses of Lemma 5.38 be given. Moreover let $f_1(u)$ and $f_2(u)$ comply with (5.4.1). Then for all $p > \bar{p}$ we have that $u \in L^\infty((0, T_{\max}); L^p(\Omega))$.*

Proof. In the claim of Lemma 5.41, let us estimate the integrals involving Δv and Δw by invoking relation (5.4.18) and Young's inequality. We have for all $t \in (0, T_{\max})$

$$c_{30}\int_\Omega F_{m_2}(u)|\Delta v| \le c_{48}\int_\Omega (u+1)^{p+m_2-1}|\Delta v| \le \int_\Omega (u+1)^{p+m_2+\alpha-1} + c_{49}\int_\Omega |\Delta v|^{\frac{p+m_2+\alpha-1}{\alpha}} \tag{5.4.20}$$

and, similarly,

$$c_{31}\int_\Omega F_{m_3}(u)|\Delta w| \le \int_\Omega (u+1)^{p+m_3+\beta-1} + c_{50}\int_\Omega |\Delta w|^{\frac{p+m_3+\beta-1}{\beta}}. \tag{5.4.21}$$

By plugging into bound (5.4.13) estimates (5.4.20) and (5.4.21), we obtain

$$\begin{split} \varphi'(t) \le& \int_\Omega (u+1)^{p+m_2+\alpha-1} + c_{49}\int_\Omega |\Delta v|^{\frac{p+m_2+\alpha-1}{\alpha}} + \int_\Omega (u+1)^{p+m_3+\beta-1} \\ &+ c_{50}\int_\Omega |\Delta w|^{\frac{p+m_3+\beta-1}{\beta}} - c_{32}\int_\Omega \left|\nabla (u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_{33} \quad \text{for all } t \in (0, T_{\max}). \end{split} \tag{5.4.22}$$

With relation (3.3.5) in our hands, by considering $z = v$, $g = f_1(u)$ (and using in addition (5.4.1)) and $q = \frac{p+m_2+\alpha-1}{\alpha}$, we derive

$$\int_0^t e^s \int_\Omega |\Delta v|^{\frac{p+m_2+\alpha-1}{\alpha}}\, ds \le c_{51} + c_{51}\int_0^t e^s \int_\Omega (u+1)^{p+m_2+\alpha-1}\, ds \quad \text{for all } t \in (0, T_{\max}), \tag{5.4.23}$$

and, similarly for the equation of w,

$$\int_0^t e^s \int_\Omega |\Delta w|^{\frac{p+m_3+\beta-1}{\beta}}\, ds \le c_{52} + c_{52}\int_0^t e^s \int_\Omega (u+1)^{p+m_3+\beta-1}\, ds \quad \text{for all } t \in (0, T_{\max}), \tag{5.4.24}$$

with $c_{51} = c_{51}(C_P)$, $c_{52} = c_{52}(C_P)$. In turn, relation (5.4.22) can be also rephrased, on $(0, T_{\max})$, as

$$\begin{split} \varphi'(t) + \varphi(t) \le& \int_\Omega (u+1)^{p+m_2+\alpha-1} + c_{49}\int_\Omega |\Delta v|^{\frac{p+m_2+\alpha-1}{\alpha}} + \int_\Omega (u+1)^{p+m_3+\beta-1} \\ &+ c_{50}\int_\Omega |\Delta w|^{\frac{p+m_3+\beta-1}{\beta}} + \frac{1}{p}\int_\Omega (u+1)^p - c_{32}\int_\Omega \left|\nabla (u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_{33} \end{split}$$

and by solving this differential inequality we have on $(0, T_{\max})$,

$$\begin{split} e^t\varphi(t) \le& \varphi(0) + \int_0^t e^s\Bigg(\int_\Omega (u+1)^{p+m_2+\alpha-1} + c_{49}\int_\Omega |\Delta v|^{\frac{p+m_2+\alpha-1}{\alpha}} + \int_\Omega (u+1)^{p+m_3+\beta-1} \\ &+ c_{50}\int_\Omega |\Delta w|^{\frac{p+m_3+\beta-1}{\beta}} + \frac{1}{p}\int_\Omega (u+1)^p - c_{32}\int_\Omega \left|\nabla (u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_{33}\Bigg)\, ds. \end{split}$$

Now relations (5.4.23) and (5.4.24) entail on $(0, T_{\max})$

$$\begin{aligned}\frac{1}{p}e^t\int_\Omega u^p \leq & \frac{1}{p}e^t\int_\Omega (u+1)^p = e^t\varphi(t)\\ & \leq c_{53} + \int_0^t e^s\Big(c_{54}\int_\Omega (u+1)^{p+m_2+\alpha-1} + c_{55}\int_\Omega (u+1)^{p+m_3+\beta-1}\\ & \quad + \frac{1}{p}\int_\Omega (u+1)^p - c_{32}\int_\Omega \Big|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\Big|^\gamma + c_{33}\Big)\, ds.\end{aligned}$$

We now use (5.4.4), (5.4.6) and (5.4.7), with small enough ε, in a such way that the gradient term in the above inequality may control the remaining three integrals so to arrive at

$$e^t\int_\Omega u^p \leq c_{56} + c_{57}(e^t - 1) \quad \text{on } (0, T_{\max}), \text{ or equivalently } \int_\Omega u^p \leq c_{58} \quad \text{for all } t \in (0, T_{\max}).$$

□

The nonlocal model: analysis of problem ($\mathcal{NL}$)

Let us now turn our attention to problem ($\mathcal{NL}$), for which the goal is establishing for u the same inclusion obtained in Lemma 5.42 as well.

Lemma 5.44. *Let the hypotheses of Lemma 5.38 be fixed. Moreover let $f_1(u)$ and $f_2(u)$ comply with* (5.4.1)*. Then for all $p > \bar{p}$ we have that $u \in L^\infty((0, T_{\max}); L^p(\Omega))$.*

Proof. Let us exploit again relation (5.4.13); we use the second and third equations of model ($\mathcal{NL}$) so that, by neglecting nonpositive terms, we have on $(0, T_{\max})$ and taking in mind restrictions in (5.4.1)

$$\begin{aligned}\varphi'(t) \leq & -c_{30}\int_\Omega F_{m_2}(u)\Delta v + c_{31}\int_\Omega F_{m_3}(u)\Delta w - c_{32}\int_\Omega \Big|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\Big|^\gamma + c_{33}\\ \leq & c_{30}\int_\Omega F_{m_2}(u)f_1(u) + c_{59}\int_\Omega F_{m_3}(u)\int_\Omega f_2(u) - c_{31}\int_\Omega F_{m_3}(u)f_2(u)\\ & - c_{32}\int_\Omega \Big|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\Big|^\gamma + c_{33}\\ \leq & c_{60}\int_\Omega (u+1)^{p+m_2+\alpha-1} + c_{61}\int_\Omega (u+1)^{p+m_3-1}\int_\Omega (u+1)^\beta - c_{62}\int_\Omega u^{p+m_3-1}(u+1)^\beta\\ & - c_{32}\int_\Omega \Big|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\Big|^\gamma + c_{33}.\end{aligned}$$

Similarly to what done previously, by using relation (3.1.5) and the Young inequality on the third term of the right-hand side of the now obtained estimate, we have

$$\begin{aligned}\varphi'(t) \leq & c_{60}\int_\Omega (u+1)^{p+m_2+\alpha-1} + c_{61}\int_\Omega (u+1)^{p+m_3-1}\int_\Omega (u+1)^\beta - c_{63}\int_\Omega (u+1)^{p+m_3+\beta-1}\\ & - c_{32}\int_\Omega \Big|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\Big|^\gamma + c_{64}.\end{aligned}$$

By recalling relations (5.4.6) and (5.4.9), we again have for proper ε and up to constants problem (5.4.19), and we conclude. □

5.4.5 Numerical simulations

Numerical approximation and tools

The numerical approximation of chemotaxis models is far to be straightforward, especially when dealing with blow-up situations, which are very challenging to reproduce in the discrete framework. In fact, any positive and mass-bounded discrete solution, in the sense that the $\|\cdot\|_{L^1(\Omega)}$ is bounded, will never show a proper blow-up phenomenon as it is also bounded in norm $\|\cdot\|_{L^\infty(\Omega)}$ due to the equivalence of the norms in a discrete, finite-dimensional, space. Moreover, these chemotactic collapse situations lead to the appearance of very steep gradients which may produce severe nonphysical spurious oscillations as shown in [CK08, GSRG21]. In this work these difficulties can even be strengthen due to the damping gradient term introduced in systems $(\mathcal{L}_\tau)$ and $(\mathcal{NL})$ that may tend to infinity when a chemotactic collapse occurs.

In order to overcome these inconveniences and provide an accurate and meaningful numerical approximation for the solution of the related mechanism, we are inspired by the approach in [ASGGRG23], where the classical Keller–Segel model is rewritten as a gradient flow. In particular, regarding the spatial approximation, we use an upwind discontinuous Galerkin (DG) method [DPE11] with piecewise constant polynomials defined on a triangular/tetrahedral mesh of the (2-dimensional/3-dimensional) domain Ω with size h. This is combined with a proper implicit-explicit approximation in time, defined on an equispaced partition of the temporal domain with time step Δt. By means of this method, we can obtain a positive approximating solution of the problem which prevents spurious oscillations from appearing. Moreover, the resulting scheme is linear, and it decouples the equations for the chemical signals, v and w, which are computed first in each time step, from the equation for the cell density, u, computed afterwards in each time step.

As to the employed software, we have used the Python interface of the open source library FEniCSx [SDRW22].

The general setting and a blow-up criterion. Simulating model $(\mathcal{L}_0)$

1. We confine our numerical studies to the parabolic-elliptic-elliptic version of problem $(\mathcal{L}_\tau)$, since we expect the results to be similar when $\tau = 1$ and for model $(\mathcal{NL})$. (Some tests in similar attraction-repulsion contexts for the case $\tau = 1$ can be found, for instance, in [FRV23].)

2. We define $f_1(u) = u^\alpha$, $f_2(u) = u^\beta$ and we assume that all the parameters are set to 1, but $k = 1.1$, unless otherwise specified. For the sake of clarity, we also indicate in the figures any different value of the parameters with respect to those already fixed.

3. As to the detection of blow-up phenomena, since an accurate discrete solution idealizing such scenario will exhibit a mass accumulation in small regions of the domain, leading to an exponential growth in a certain time interval, the following *blow-up criterion* appears consistent (see [SX20, BBGS23, ASGGRG23] for details):

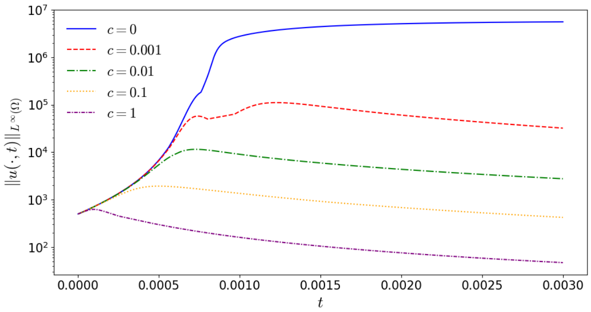
   Criterion 5.45 (Blow-up criterion)**.** *Blow-up occurs when the norm $\|\cdot\|_{L^\infty(\Omega)}$ of the approximated solution stabilizes over time. An estimate of the blow-up time $T_{\max}$ is the approximated value of the instant of time around which the stabilization exhibits.*

Numerical simulations for the linear and only attraction model

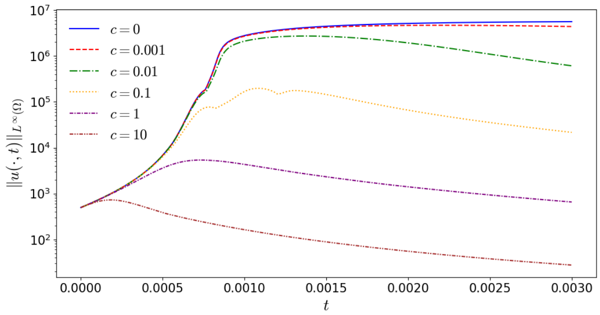
In this first example, we consider a 3D version ($n = 3$) of model $(\mathcal{L}_\tau)$ defined in the spatial domain $\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$ with the following initial condition

$$u_0(x, y, z) := 500e^{-35(x^2+y^2+z^2)},$$

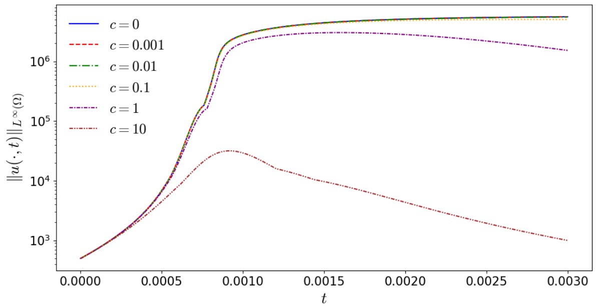
plotted in Figure 5.4a. Moreover, we take $\chi = 5$ and $\xi = 0$ (no repulsion) and we use a mesh of size $h \approx 4.4 \cdot 10^{-2}$ and a time step $\Delta t = 10^{-5}$. In Figure 5.4 the approximation obtained with $c = 0$ is plotted at different time steps. We can observe that a blow-up phenomenon seems to appear (indeed, it is observable in Figures 5.4b and 5.4c how the maximum of the solution increases in the center of the sphere, eventually achieving the value 10^5), according with the analytical results given in [Win18b] – notice that $k = 1.1 < 7/6 \approx 1.167$ so the condition in [Win18b, (1.4), Theorem 1.1] is satisfied. This blow-up phenomenon (interpretable in the sense of Criterion 5.45 in the next figures) can be prevented using appropriate damping gradient terms, as established in Theorem 5.36 (generalization of [ILV24, Theorem 1.2]). As a matter of fact, we show in Figure 5.5 the evolution of the maximum of the approximations for different choices of c and γ. Notice that the maximum shown in Figure 5.5 ($c = 0$) achieves even higher values, up to the order 10^6, than those exhibited in Figure 5.4 where a continuous, smoother, projection of the actual discontinuous approximated solution had been plotted. As expected, blow-up is prevented for



(a) $\gamma = 1.75$



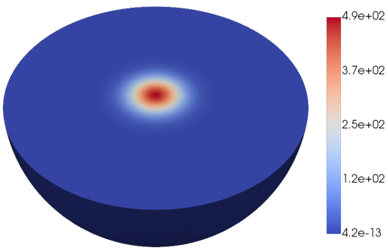
(b) $\gamma = 1.4$



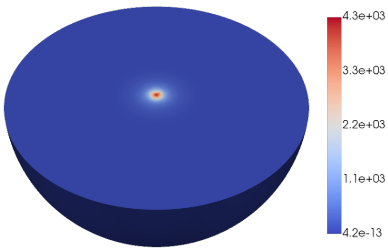
(c) $\gamma = 1.1$

Figure 5.5: Maximum of the approximation of u over time for different values of c and γ ($\chi = 5$, $\xi = 0$).

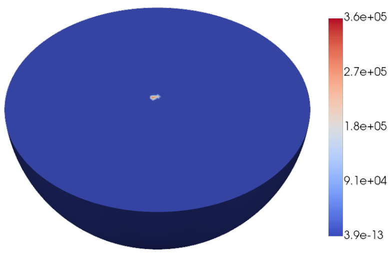
whatever choice of $c > 0$ (even for small values like $c = 10^{-3}$) if γ satisfies the bound $\mathcal{G}_\tau$, i.e., $1.5 < \gamma \leq 2$ (see Figure 5.5a). However, in case that γ does not comply with $\mathcal{G}_\tau$, we may require a big enough value of c to prevent the gathering phenomena. This value of c, apparently breaking down the tendency toward chemotactic collapse, increases as long as γ moves away from the value $\gamma^* = 1.5$ for which the equality holds in $\mathcal{G}_\tau$ (see Figures 5.5b and 5.5c).



(a) $t = 0$.



(b) $t = 5 \cdot 10^{-4}$.



(c) $t = 3 \cdot 10^{-3}$.

Figure 5.4: Approximation of the solution u at different time steps ($c = 0$, $\chi = 5$, $\xi = 0$).

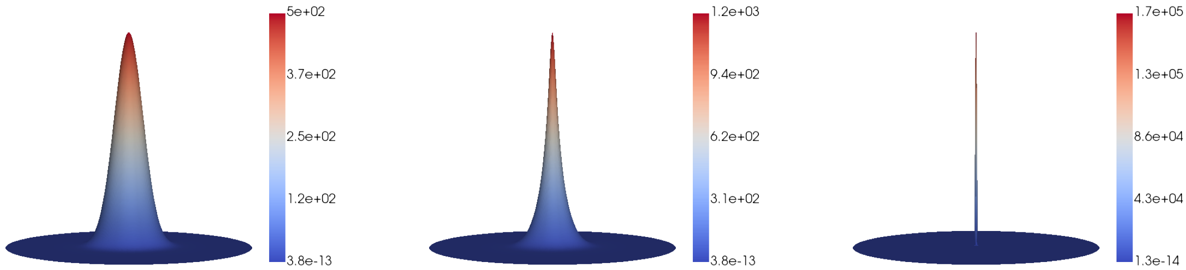
Numerical simulations for the fully nonlinear attraction-repulsion model

Motivated by Section 5.2, let us analyze model ($\mathcal{L}_\tau$) under the assumption that

$$m_2 = m_3, m_1 \in \mathbb{R}, \alpha > \beta \quad \text{and} \quad \begin{cases} m_2 + \alpha > \max\left\{m_1 + \frac{2}{n}k, k\right\} & \text{if } m_1 \geq 0, \\ m_2 + \alpha > \max\left\{\frac{2}{n}k, k\right\} & \text{if } m_1 < 0. \end{cases} \tag{5.4.25}$$

In particular, we focus on the 2D case ($n = 2$) in the domain $\Omega = \{(x, y) \colon x^2 + y^2 < 1\}$. We set $\alpha = 1.5$, $\xi = \chi = 1$, and we define the following initial condition

$$u_0(x, y) \coloneqq 500e^{-35(x^2+y^2)},$$



(a) $t = 0$. (b) $t = 5 \cdot 10^{-5}$. (c) $t = 3 \cdot 10^{-4}$.

Figure 5.6: Approximation of the solution u at different time steps ($m_1 = 1$, $c = 0$, $\alpha = 1.5$).

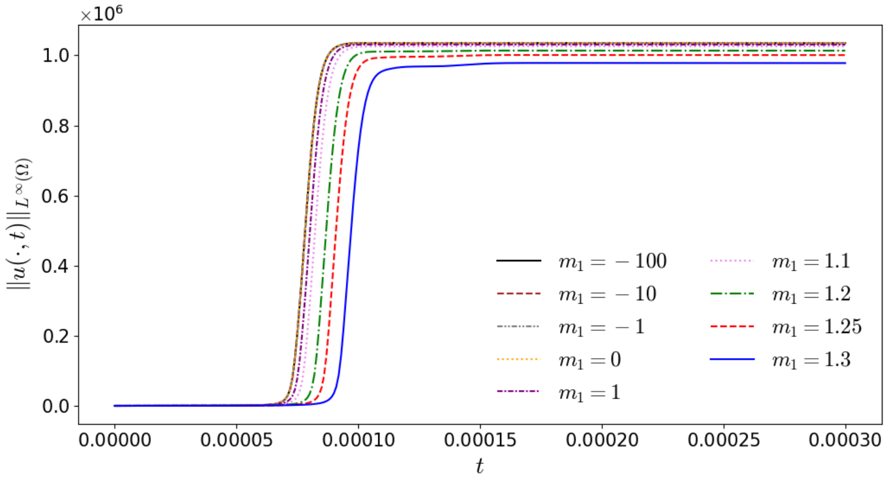


Figure 5.7: Maximum of the approximation of u for different values of m_1 ($c = 0$, $\alpha = 1.5$).

plotted in Figure 5.6a. Also, we take a spatial and a temporal partition of sizes $h \approx 1.37 \cdot 10^{-2}$ and $\Delta t = 10^{-6}$.

First, in Figure 5.6 we plot the evolution of u over time without the damping gradient term, i.e., $c = 0$. We observe that due to the choice of the parameters and the initial condition, it seems to occur a blow-up phenomenon (notable in Figures 5.6b and 5.6c), in accordance with the results in [CFV23a, Theorem 3] under restriction (5.4.25). In fact, if we use a nonlinear diffusion term moving the value of m_1 we can observe in Figure 5.7 that we still obtain a chemotactic collapse but with times of explosion. Consistently to the real phenomenon, one may notice that the blow-up time increases (accordingly to Criterion 5.45) with m_1, the diffusion coefficient working against coalescence effects, specially for large values. Moreover, as in the previous example, if we introduce the damping gradient nonlinearities using a strictly positive small value for c, such as $c = 10^{-3}$, the blow-up phenomenon is avoided if we stick to the bounds on γ given in $\mathcal{G}_\tau$, i.e., $1.67 \approx 5/3 < \gamma \leq 2$. However, this singularity seems to remain for a small enough value of c if γ does not satisfy $\mathcal{G}_\tau$; see Figure 5.8.

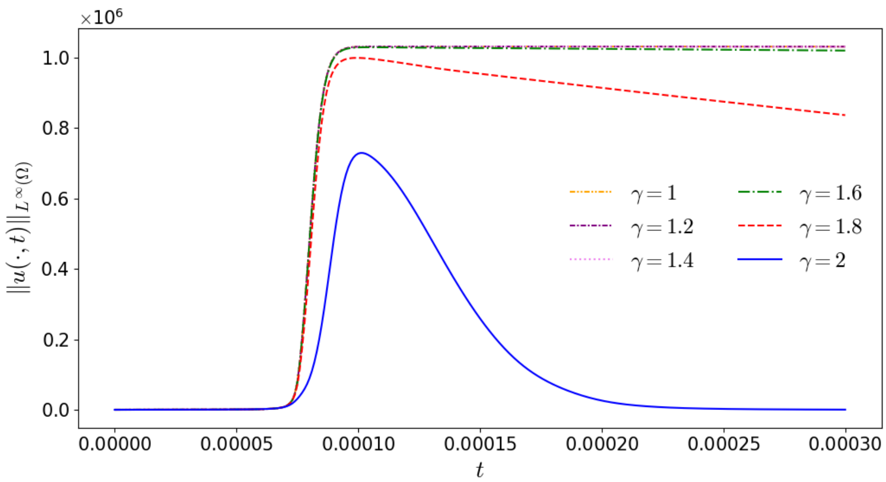


Figure 5.8: Maximum of the approximation of u for different values of γ ($m_1 = 1$, $c = 10^{-3}$, $\alpha = 1.5$).

Chapter 6

Advances in the framework of only attraction mechanisms

As previously anticipated, this chapter examines three problems in which only an attractive effect is present: in the first two, the chemical signal v is consumed by the population u, while in the third, an indirect interaction between u and v arises due to the presence of a third substance. As established earlier, the solution exists and is unique for each of the problems under consideration.

6.1 [Col25] Boundedness criteria for a chemotaxis consumption model with gradient nonlinearities

6.1.1 Presentation of the model and of the main result

The idea is to consider a consumption model with a dampening logistic that depends also on ∇u:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \lambda u - \mu u^2 - c|\nabla u|^\gamma, & \text{in } \Omega \times (0, T_{\max}),\\ v_t = \Delta v - uv, & \text{in } \Omega \times (0, T_{\max}),\\ u_\nu = v_\nu = 0, & \text{on } \partial\Omega \times (0, T_{\max}),\\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{6.1.1}$$

For $p > 1$ and $\eta > 0$, we define the functions

$$K_1(p,n) := \frac{p}{p+1}\left(\frac{8(4p^2+n)}{p(p+1)}\right)^{\frac{1}{p}}\left(\frac{p(p-1)}{2}\right)^{\frac{p+1}{p}}, \tag{6.1.2}$$

and

$$K_2(p,n,\eta) := \frac{2p^{\frac{p+1}{2}}(p+n+\eta-1)^{\frac{p+1}{2}}}{p+1}\left(\frac{8(4p^2+n)(p-1)}{p(p+1)}\right)^{\frac{p-1}{2}}. \tag{6.1.3}$$

Moreover, let $M := \max\left\{\int_\Omega u_0, \frac{\lambda}{\mu}|\Omega|\right\}$. We enunciate now the main result of this section:

Theorem 6.1. *Let $\lambda, \mu, c, \chi > 0$. There exists $\mathcal{C} > 0$ such that whenever*

($\mathcal{A}_1$) $\dfrac{2n}{n+1} < \gamma \leq 2$;

or

($\mathcal{A}_2$) $\gamma = \dfrac{2n}{n+1}$ *and*

$$c\frac{n}{2}\left(\frac{2n}{n^2+3n-2}\right)^{\frac{2n}{n+1}}\mathcal{C}M^{-\frac{2}{n+1}} + \mu\frac{n}{2} > K_1\left(\frac{n}{2}, n\right)\|v_0\|^{\frac{4}{n}}_{L^\infty(\Omega)}\chi^{\frac{2}{n}(n+2)} + K_2\left(\frac{n}{2}, n, 0\right)\|v_0\|^n_{L^\infty(\Omega)},$$

the solution to problem (6.1.1) *is global and uniformly bounded.*

Remark 6.2. *As introduced in Remark 4.7, the parameter* $p = \frac{n}{2}$ *can be selected in condition* $(\mathcal{A}_2)$ *by employing a continuity argument with respect to p. This approach is justified because the proof of Theorem 6.1 is founded on a boundedness criterion for which this value of p is optimal (see Lemma 6.7).*

Remark 6.3. *Let us dedicate to the case* $\gamma = \frac{2n}{n+1}$ *and rewrite now the inequality in condition* $(\mathcal{A}_2)$ *as*

$$c\mathcal{F}M^{-\frac{2}{n+1}} + \mu\frac{n}{2} > \mathcal{K}, \tag{6.1.4}$$

where

$$\mathcal{F} = \frac{n}{2}\left(\frac{2n}{n^2+3n-2}\right)^{\frac{2n}{n+1}}\mathcal{C} \quad \text{and} \quad \mathcal{K} = K_1\left(\frac{n}{2}, n\right)\|v_0\|_{L^\infty(\Omega)}^{\frac{4}{n}}\chi^{\frac{2}{n}(n+2)} + K_2\left(\frac{n}{2}, n, 0\right)\|v_0\|_{L^\infty(\Omega)}^{n}.$$

We can observe what follows:

$(\mathcal{B}_1)$ *Fix* $c = 0$*. We can write* (6.1.4) *as* $\mu = \bar{\mu} > \frac{2}{n}\mathcal{K}$*, and this is another way to see the result contained in [LW17], where the authors use* $p = n$*, so the condition for the boundedness contained in that work is a special case of this one.*

$(\mathcal{B}_2)$ *In the case* $\mu < \frac{2}{n}\mathcal{K}$ *and* $c > 0$*, we have two different possibilities:*

- *if* $M = \int_\Omega u_0$*, we obtain*

$$c\mathcal{F}\left(\int_\Omega u_0\right)^{-\frac{2}{n+1}} + \frac{n}{2}\mu > \mathcal{K}, \quad \text{that leads to} \quad \int_\Omega u_0 < \left(\frac{c\mathcal{F}}{\mathcal{K} - \frac{n}{2}\mu}\right)^{\frac{n+1}{2}},$$

  so we have a condition on the initial mass;

- *if* $M = \frac{\lambda}{\mu}|\Omega|$ *we have*

$$\mathcal{E}\mu^{\frac{2}{n+1}} + \frac{n}{2}\mu > \mathcal{K} \quad \text{where} \quad \mathcal{E} = \frac{c\mathcal{F}}{(\lambda|\Omega|)^{\frac{2}{n+1}}}, \tag{6.1.5}$$

  and so there exists $\mu < \bar{\mu}$ *such that* (6.1.5) *is satisfied.*

$(\mathcal{B}_3)$ *In the situation where* $\mu = \frac{2}{n}\mathcal{K}$ *and* $c > 0$*, relation* (6.1.4) *is simply true.*

6.1.2 Some fundamental parameters

The next Lemma provides some technical inequalities which will be used later on.

Lemma 6.4. *Let* $n \in \mathbb{N}$ *and the lower bound of condition* $(\mathcal{A}_1)$ *be fixed. Then for every* $p > 1$ *and*

$$\theta(p,\gamma) = \frac{\frac{p+\gamma-1}{\gamma}\left(1 - \frac{1}{p+1}\right)}{\frac{p+\gamma-1}{\gamma} + \frac{1}{n} - \frac{1}{\gamma}} \qquad \sigma(p,\gamma) = \frac{\gamma(p+1)}{p+\gamma-1} \qquad \bar{\theta}(p) = \frac{\frac{p}{2} - \frac{1}{2}}{\frac{p}{2} - \frac{1}{2} + \frac{1}{n}},$$

$$\hat{\theta} = \theta\left(p, \frac{2n}{n+1}\right) \qquad \hat{\sigma} = \sigma\left(p, \frac{2n}{n+1}\right)$$

these inequalities hold

$$0 < \theta < 1, \tag{6.1.6a}$$

$$0 < \frac{\theta\sigma}{\gamma} < 1, \tag{6.1.6b}$$

$$0 < \bar{\theta} < 1, \tag{6.1.6c}$$

$$\frac{\hat{\sigma}\hat{\theta}(n+1)}{2n} = 1. \tag{6.1.6d}$$

Proof. First of all, we can see that (6.1.6a),(6.1.6c) and the lower bound of (6.1.6b) are true for every $p > 1$. We need $\gamma > \frac{2n}{n+1}$ in order to prove the upper bound of (6.1.6b). $\square$

6.1.3 Properties of the solution

Thanks to Theorem 4.1 we know that the problem admits a unique solution.

Lemma 6.5. *Let $\lambda, \mu, c, \chi > 0$ and $\gamma \geq 1$. Then*

$$\int_\Omega u \leq M := \max\left\{\int_\Omega u_0, \frac{\lambda}{\mu}|\Omega|\right\} \quad \text{on } (0, T_{\max}), \tag{6.1.7}$$

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)} \quad \text{for all } t \in (0, T_{\max}), \tag{6.1.8}$$

$$\|v(\cdot,t)\|_{L^1(\Omega)} \leq \|v_0\|_{L^1(\Omega)} \quad \text{for all } t \in (0, T_{\max}). \tag{6.1.9}$$

Proof. We can apply Proposition 4.8 with $\rho = 1$ and $k = 2$ and derive (6.1.7). Moreover, (6.1.8) is a consequence of Proposition 3.9 applied to the second equation. Ultimately, in order to prove (6.1.9), we observe that

$$\frac{d}{dt}\int_\Omega v = \int_\Omega v_t = -\int_\Omega uv \leq 0 \quad \text{for all } t \in (0, T_{\max}).$$

□

Lemma 6.6. *Suppose there exists $\tau \in (0, \min\{1, T_{\max}\})$ and $p \in [1, n]$ such that $\|u(\cdot,t)\|_{L^p(\Omega)} \leq c_1$ for all $t \in (\tau, T_{\max})$. Then for every $\delta \in (0, T_{\max} - \tau)$ and for all $q < \frac{np}{n-p}$ there exists c_2 such that*

$$\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \leq c_2 \quad \text{for every } t \in (\tau + \delta, T_{\max}).$$

Proof. Let $q < \frac{np}{n-p}$ and $\beta > \frac{1}{2}$ such that $\beta + \frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) < 1$. Now we write the second equation as $v_t = -(A_q + 1)v + v(1-u)$ and we use the representation formula to affirm that

$$v(\cdot,t) = e^{-(t-\tau)}v(\cdot,\tau) + \int_\tau^t e^{-(t-s)(A_q+1)}v(\cdot,s)(1 - u(\cdot,s))\,ds \quad \text{for all } t \in [\tau, T_{\max}).$$

By applying $(A_q + 1)^\beta$ to both sides and using (3.5.4) and (6.1.9) we have

$$\begin{aligned}
\left\|(A_q+1)^\beta v(\cdot,t)\right\|_{L^q(\Omega)} \leq& c_3(t-\tau)^{-\beta-\frac{n}{2}\left(1-\frac{1}{q}\right)}\|v(\cdot,\tau)\|_{L^1(\Omega)} \\
&+ c_4\int_\tau^t (t-s)^{-\beta-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}e^{-\hat{\mu}(t-s)}\|v(\cdot,s)(1-u(\cdot,s))\|_{L^p(\Omega)}\,ds \\
\leq& c_3\delta^{-\beta-\frac{n}{2}\left(1-\frac{1}{q}\right)}\|v(\cdot,0)\|_{L^1(\Omega)} \\
&+ c_4\|v(\cdot,0)\|_{L^\infty(\Omega)}\int_\tau^t (t-s)^{-\beta-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}e^{-\hat{\mu}(t-s)}\left(|\Omega|^{\frac{1}{p}} + \|u(\cdot,s)\|_{L^p(\Omega)}\right)ds \\
\leq& c_5 + c_6\int_0^\infty z^{-\beta-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}e^{-\hat{\mu}z}\,dz \leq c_7 \quad \text{on } (0, T_{\max}).
\end{aligned}$$

Finally, (3.5.3) gives the claim. □

The next Lemma is the equivalent of Lemma 4.6 in the linear case when the objective is to obtain an optimal value for p.

Lemma 6.7. *Let $T_{\max} \in (0, \infty]$, $\frac{n}{2} < p < n$, and*

$$\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \leq c_2 \quad \text{for every } t \in (\tau + \delta, T_{\max}). \tag{6.1.10}$$

for all $q < \frac{np}{n-p}$. If $u \in L^\infty((0, T_{\max}), L^p(\Omega))$, then $u \in L^\infty((0, T_{\max}), L^\infty(\Omega))$.

Proof. We write $u(\cdot,t)$ using the variation-of-constants formula for all $t \in (0,T_{\max})$, $t_0 = \max\{0, t-1\}$ and $m = \frac{1}{|\Omega|}\int_\Omega u(\cdot,t_0)$

$$\begin{aligned} u(\cdot,t) - m = & e^{(t-t_0)\Delta}(u(\cdot,t_0) - m) - \int_{t_0}^t e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s))\,ds \\ & + \int_{t_0}^t e^{(t-s)\Delta}\big(\lambda u(\cdot,s) - \mu u^2(\cdot,s) - c|\nabla u(\cdot,s)|^\gamma\big)\,ds. \end{aligned} \tag{6.1.11}$$

Focusing on the logistic term, we can say that

$$\lambda u - \mu u^2 - c|\nabla u|^\gamma \le \lambda u - \mu u^2 \le \sup_{\xi>0}(\lambda\xi - \mu\xi^2) \coloneqq c_8,$$

and, in conjunction with the positivity of the heat semigroup, we have

$$\int_{t_0}^t e^{(t-s)\Delta}\big(\lambda u(\cdot,s) - \mu u^2(\cdot,s) - c|\nabla u(\cdot,s)|^\gamma\big)\,ds \le c_8(t-t_0) \le c_8. \tag{6.1.12}$$

Consequently, if $t \le 1$, which is equivalent to $t_0 = 0$, Proposition 3.9 ensures that

$$\left\| e^{(t-t_0)\Delta}(u(\cdot,t_0) - m)\right\|_{L^\infty(\Omega)} \le \|u_0 - m\|_{L^\infty(\Omega)}. \tag{6.1.13}$$

Instead, if $t > 1$, that implies $t - t_0 = 1$, we can use (3.5.1) with $f = u(\cdot,t_0) - m$ and the boundedness of the mass so to write

$$\left\| e^{(t-t_0)\Delta}(u(\cdot,t_0) - m)\right\|_{L^\infty(\Omega)} \le C_f\big(1 + (t-t_0)^{-\frac{n}{2}}\big)\|u(\cdot,t_0) - m\|_{L^1(\Omega)} \le c_9. \tag{6.1.14}$$

Moreover, for all $r \in (n,q)$, by applying (3.5.2), we know that

$$\left\| \int_{t_0}^t e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s))\,ds\right\|_{L^\infty(\Omega)} \le c_{10}\int_{t_0}^t \Big(1 + (t-s)^{-\frac{1}{2}-\frac{n}{2r}}\Big)\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^r(\Omega)}\,ds. \tag{6.1.15}$$

We now define

$$A(t') = \sup_{t\in(0,t')}\|u(\cdot,t)\|_{L^\infty(\Omega)},$$

so that we can find

$$\begin{aligned} \|u(\cdot,s)\nabla v(\cdot,s)\|_{L^r(\Omega)} \le & \|u(\cdot,s)\|_{L^{\frac{rq}{q-r}}(\Omega)}\|\nabla v(\cdot,s)\|_{L^q(\Omega)} \\ \le & \|u(\cdot,t)\|_{L^\infty(\Omega)}^{1-\frac{p(q-r)}{rq}}\|u(\cdot,t)\|_{L^p(\Omega)}^{\frac{p(q-r)}{rq}}\|\nabla v(\cdot,s)\|_{L^q(\Omega)} \\ \le & c_{11}A(t')^{1-\frac{p(q-r)}{rq}} = c_{11}A^l(t'), \end{aligned} \tag{6.1.16}$$

with $l < 1$, where in the last step we used hypothesis (6.1.10). By recalling (6.1.11), (6.1.12), (6.1.13), (6.1.14), (6.1.15) and (6.1.16) we have $A(t') \le c_{12}\big(A^l(t') + 1\big)$, that implies, thanks to Lemma 3.3, $\|u(\cdot,t)\|_{L^\infty(\Omega)} \le A(t') \le \max\left\{1, (2c_{12})^{\frac{1}{1-l}}\right\} = c_{13}$ for all $t \in (0,t')$ and, in particular, c_{13} does not depend on the time. □

Lemma 6.8. *If $u \in L^\infty((0,T_{\max}); L^p(\Omega))$ for $\frac{n}{2} < p < n$, then $u \in L^\infty((0,\infty); L^\infty(\Omega))$.*

Proof. Lemmata 6.6 and 6.7 ensure that $u \in L^\infty((0,T_{\max}); L^\infty(\Omega))$. The conclusion is given by Proposition 4.4. □

So, in order to prove that the solution is bounded, we need to prove that $u \in L^\infty((0,T_{\max}); L^p(\Omega))$.

In order to do that, we begin with some technical inequalities which we will use in the next section and from now on, the proof will follow [LW17].

Lemma 6.9. *Let $\lambda, \mu, c, \chi > 0$ and $\gamma \geq 1$. Then there exists $C > 0$ such that*

$$\int_\Omega |\nabla v(\cdot,t)|^2 \leq C \quad \text{for all } t \in (0, T_{\max}). \tag{6.1.17}$$

Proof. The reader can find this proof in [LW17, Lemma 3.4]. □

Lemma 6.10. *Under the hypotheses of Lemma 6.4, we have that for every $p > 1$ and for all $\varepsilon, \hat{c} > 0$ there exists $c_{14}, c_{15}, c_{16}, c_{17}$ such that these inequalities hold*

$$(\lambda p + 1)\int_\Omega u^p \leq \frac{p-1}{p}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 + c_{14} \quad \text{on } (0, T_{\max}), \tag{6.1.18}$$

$$\int_\Omega |\nabla v|^{2p} \leq \frac{p}{4}\int_\Omega |\nabla v|^{2p-2}\left|D^2 v\right|^2 + c_{15} \quad \text{on } (0, T_{\max}), \tag{6.1.19}$$

$$\hat{c}\int_\Omega u^{p+1} \leq \varepsilon \int_\Omega \left|\nabla u^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_{16} \quad \text{on } (0, T_{\max}), \tag{6.1.20}$$

$$\hat{c}\int_\Omega u^{p+1} \leq \hat{c}(2C_{GN})^{\hat{\sigma}} M^{\frac{2}{n+1}}\int_\Omega \left|\nabla u^{\frac{p(n+1)+n-1}{2n}}\right|^{\frac{2n}{n+1}} + c_{17} \quad \text{on } (0, T_{\max}), \tag{6.1.21}$$

$$\begin{aligned}\frac{p(p-1)}{2}\chi^2\int_\Omega u^p|\nabla v|^2 \leq& \frac{p}{4}\int_\Omega |\nabla v|^{2p-2}\left|D^2 v\right|^2 \\ &+ K_1(p,n)\|v_0\|^{\frac{2}{p}}_{L^\infty(\Omega)}\chi^{\frac{2(p+1)}{p}}\int_\Omega u^{p+1} \quad \text{on } (0, T_{\max}),\end{aligned} \tag{6.1.22}$$

$$\tilde{c}\int_\Omega u^2|\nabla v|^{2p-2} \leq \frac{p}{4}\int_\Omega |\nabla v|^{2p-2}\left|D^2 v\right|^2 + K_2(p,n,\eta)\|v_0\|^{2p}_{L^\infty(\Omega)}\int_\Omega u^{p+1} \quad \text{on } (0, T_{\max}), \tag{6.1.23}$$

where $\tilde{c} = p(p+n-1+\eta)\|v_0\|^2_{L^\infty(\Omega)}$ and $K_1(p,n)$ and $K_2(p,n,\eta)$ are defined in (6.1.2) *and* (6.1.3).

Proof. In order to prove (6.1.18) we make use of the Gagliardo–Nirenberg and Young inequalities in conjunction with (6.1.6c) and (6.1.7):

$$\begin{aligned}(\lambda p + 1)\int_\Omega u^p = (\lambda p+1)\left\|u^{\frac{p}{2}}\right\|^2_{L^2(\Omega)} \leq& c_{18}\left(\left\|\nabla u^{\frac{p}{2}}\right\|^{\bar{\theta}}_{L^2(\Omega)}\left\|u^{\frac{p}{2}}\right\|^{1-\bar{\theta}}_{L^{\frac{2}{p}}(\Omega)} + \left\|u^{\frac{p}{2}}\right\|_{L^{\frac{2}{p}}(\Omega)}\right)^2 \\ \leq& c_{19}\left(\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2\right)^{\bar{\theta}} + c_{20} \\ \leq& \frac{p-1}{p}\int_\Omega \left|\nabla u^{\frac{p}{2}}\right|^2 + c_{14} \quad \text{on } (0, T_{\max}).\end{aligned}$$

For (6.1.19), we operate in same way but recalling (6.1.17):

$$\begin{aligned}\int_\Omega |\nabla v|^{2p} = \int_\Omega (|\nabla v|^p)^2 = \||\nabla v|^p\|^2_{L^2(\Omega)} \leq c_{21}\left(\|\nabla|\nabla v|^p\|^{\bar{\theta}}_{L^2(\Omega)}\||\nabla v|^p\|^{1-\bar{\theta}}_{L^{\frac{2}{p}}(\Omega)} + \||\nabla v|^p\|_{L^{\frac{2}{p}}(\Omega)}\right)^2 \\ \leq \frac{1}{4p}\int_\Omega |\nabla|\nabla v|^p|^2 + c_{22} \leq \frac{p}{4}\int_\Omega |\nabla v|^{2p-2}\left|D^2 v\right|^2 + c_{15} \quad \text{on } (0, T_{\max}).\end{aligned}$$

Again, with the same inequalities but recalling (6.1.6a), (6.1.6b) and, again, (6.1.7), we are able to prove relation (6.1.20):

$$\begin{aligned}\hat{c}\int_\Omega u^{p+1} =& \hat{c}\left\|u^{\frac{p+\gamma-1}{\gamma}}\right\|^\sigma_{L^\sigma(\Omega)} \\ \leq& \hat{c}C^\sigma_{GN}\left(\left\|\nabla u^{\frac{p+\gamma-1}{\gamma}}\right\|^\theta_{L^\gamma(\Omega)}\left\|u^{\frac{p+\gamma-1}{\gamma}}\right\|^{1-\theta}_{L^{\frac{\gamma}{p+\gamma-1}}(\Omega)} + \left\|u^{\frac{p+\gamma-1}{\gamma}}\right\|_{L^{\frac{\gamma}{p+\gamma-1}}(\Omega)}\right)^\sigma \\ \leq& c_{23}\left(\int_\Omega \left|\nabla u^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma\right)^{\frac{\sigma\theta}{\gamma}} + c_{24} \leq \varepsilon\int_\Omega \left|\nabla u^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_{16} \quad \text{on } (0, T_{\max}).\end{aligned}$$

If $\gamma = \frac{2n}{n+1}$ then (6.1.6d) is verified, so we cannot apply Young's inequality:

$$\begin{aligned}
\hat{c}\int_\Omega u^{p+1} =&\hat{c}\Big\|u^{\frac{p(n+1)+n-1}{2n}}\Big\|_{L^{\hat{\sigma}}(\Omega)}^{\hat{\sigma}}\\
\leq&\hat{c}C_{GN}^{\hat{\sigma}}\left(\Big\|\nabla u^{\frac{p(n+1)+n-1}{2n}}\Big\|_{L^{\frac{2n}{n+1}}(\Omega)}^{\hat{\theta}}\Big\|u^{\frac{p(n+1)+n-1}{2n}}\Big\|_{L^{\frac{2n}{p(n+1)+n-1}}(\Omega)}^{1-\hat{\theta}}\right.\\
&\left.+\Big\|u^{\frac{p(n+1)+n-1}{2n}}\Big\|_{L^{\frac{2n}{p(n+1)+n-1}}(\Omega)}\right)^{\hat{\sigma}}\\
\leq&\hat{c}(2C_{GN})^{\hat{\sigma}}M^{\frac{2}{n+1}}\int_\Omega\Big|\nabla u^{\frac{p(n+1)+n-1}{2n}}\Big|^{\frac{2n}{n+1}}+c_{17}\quad\text{on } (0,T_{\max}).
\end{aligned}$$

In order to obtain bound (6.1.22), firstly we use Young's inequality, where we explicitly calculate $C_Y(\varepsilon,\alpha)$ as in (3.1.3), and then (3.2.3) so to have

$$\begin{aligned}
\frac{p(p-1)}{2}\chi^2\int_\Omega u^p|\nabla v|^2 \leq&\varepsilon_1\int_\Omega|\nabla v|^{2(p+1)}+\frac{p}{p+1}(\varepsilon_1(p+1))^{-\frac{1}{p}}\left[\frac{p(p-1)}{2}\chi^2\right]^{\frac{p+1}{p}}\int_\Omega u^{p+1}\\
\leq&\varepsilon_1 2(4p^2+n)\|v_0\|_{L^\infty(\Omega)}^2\int_\Omega|\nabla v|^{2p-2}\big|D^2v\big|^2\\
&+\frac{p}{p+1}(\varepsilon_1(p+1))^{-\frac{1}{p}}\left[\frac{p(p-1)}{2}\chi^2\right]^{\frac{p+1}{p}}\int_\Omega u^{p+1}\\
=&\frac{p}{4}\int_\Omega|\nabla v|^{2p-2}\big|D^2v\big|^2+K_1(p,n)\|v_0\|_{L^\infty(\Omega)}^{\frac{2}{p}}\chi^{\frac{2(p+1)}{p}}\int_\Omega u^{p+1}\quad\text{on } (0,T_{\max}),
\end{aligned}$$

where in the last step we choose $\varepsilon_1 = \frac{p}{8(4p^2+n)\|v_0\|_{L^\infty(\Omega)}^2}$. As to bound (6.1.23) we have

$$\begin{aligned}
\hat{c}\int_\Omega u^2|\nabla v|^{2p-2} \leq&\varepsilon_1\int_\Omega|\nabla v|^{2(p+1)}+\frac{2}{p+1}(\varepsilon_1\frac{p+1}{p-1})^{-\frac{p-1}{2}}\hat{c}^{\frac{p+1}{2}}\int_\Omega u^{p+1}\\
\leq&\varepsilon_1 2(4p^2+n)\|v_0\|_{L^\infty(\Omega)}^2\int_\Omega|\nabla v|^{2p-2}\big|D^2v\big|^2+\frac{2}{p+1}(\varepsilon_1\frac{p+1}{p-1})^{-\frac{p-1}{2}}\hat{c}^{\frac{p+1}{2}}\int_\Omega u^{p+1}\\
=&\frac{p}{4}\int_\Omega|\nabla v|^{2p-2}\big|D^2v\big|^2+K_2(p,n,\eta)\|v_0\|_{L^\infty(\Omega)}^{2p}\int_\Omega u^{p+1}\quad\text{on } (0,T_{\max}),
\end{aligned}$$

with the same value of ε_1 that we used in the previous proof. □

6.1.4 Conditions for the existence of bounded classical solutions

The goal is to reach a differential inequality of the type $\varphi(t)+\varphi'(t)\leq C$ for some positive C and, to do that, we begin with the following two Lemmas, which will be used together in the proof of Lemma 6.13.

Lemma 6.11. *Let $\lambda,\mu,c,\chi>0$ and $\gamma>1$. Then, for any $p\in[1,\infty)$ the following holds*

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^p \leq&-\frac{2(p-1)}{p}\int_\Omega\Big|\nabla u^{\frac{p}{2}}\Big|^2+\frac{p(p-1)}{2}\chi^2\int_\Omega u^p|\nabla v|^2+\lambda p\int_\Omega u^p\\
&-\mu p\int_\Omega u^{p+1}-cp\Big(\frac{p+\gamma-1}{\gamma}\Big)^{-\gamma}\int_\Omega\Big|\nabla u^{\frac{p+\gamma-1}{\gamma}}\Big|^\gamma,
\end{aligned}\tag{6.1.24}$$

for every $t\in(0,T_{\max})$.

Proof. We calculate the time derivative of the functional $\int_\Omega u^p$ and we apply Young's inequality, so to have, on $(0, T_{\max})$,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^p =& -p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 + \chi p(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v\\
&+\lambda p\int_\Omega u^p - \mu p\int_\Omega u^{p+1} - cp\int_\Omega u^{p-1}|\nabla u|^\gamma\\
\leq& -\frac{2(p-1)}{p}\int_\Omega\left|\nabla u^{\frac{p}{2}}\right|^2 + \frac{\chi^2 p(p-1)}{2}\int_\Omega u^p|\nabla v|^2\\
&+\lambda p\int_\Omega u^p - \mu p\int_\Omega u^{p+1} - cp\left(\frac{p+\gamma-1}{\gamma}\right)^{-\gamma}\int_\Omega\left|\nabla u^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma.
\end{aligned}$$

□

Lemma 6.12. *Let $\lambda, \mu, c, \chi > 0$ and $\gamma \geq 1$. Then there exists $\eta > 0$ such that, for any $p \in [1, \infty)$ this inequality is satisfied for all $t \in (0, T_{\max})$*

$$\frac{d}{dt}\int_\Omega|\nabla v|^{2p} + p\int_\Omega|\nabla v|^{2p-2}\left|D^2 v\right|^2 \leq p(p+n-1+\eta)\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega u^2|\nabla v|^{2p-2} + c_{25}. \tag{6.1.25}$$

Proof. This proof is contained in [LW17, Lemma 4.2]. □

Lemma 6.13. *Let $\lambda, \mu, c, \chi > 0$. If $\gamma > \frac{2n}{n+1}$ or $\gamma = \frac{2n}{n+1}$ and*

$$\begin{aligned}
cp\left(\frac{2n}{(n+1)p+n-1}\right)^{\frac{2n}{n+1}}(2C_{GN})^{-\hat{\sigma}}M^{-\frac{2}{n+1}} + \mu p \geq& K_1(p,n)\|v_0\|^{\frac{2}{p}}_{L^\infty(\Omega)}\chi^{\frac{2(p+1)}{p}}\\
&+K_2(p,n,\eta)\|v_0\|^{2p}_{L^\infty(\Omega)}
\end{aligned} \tag{6.1.26}$$

then

$$\int_\Omega u^p + \int_\Omega|\nabla v|^{2p} \leq C \quad \textit{for all } t \in (0, T_{\max}).$$

Proof. Let $\varphi(t) = \int_\Omega u^p + \int_\Omega|\nabla v|^{2p}$, then, by using (6.1.18), (6.1.19), (6.1.22) and (6.1.23) we have

$$\begin{aligned}
\varphi'(t)+\varphi(t) \leq& -\frac{2(p-1)}{p}\int_\Omega\left|\nabla u^{\frac{p}{2}}\right|^2 + \frac{p(p-1)}{2}\chi^2\int_\Omega u^p|\nabla v|^2 + (\lambda p+1)\int_\Omega u^p - \mu p\int_\Omega u^{p+1}\\
&-cp\left(\frac{p+\gamma-1}{\gamma}\right)^{-\gamma}\int_\Omega\left|\nabla u^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma - p\int_\Omega|\nabla v|^{2p-2}\left|D^2v\right|^2\\
&+p(p+n-1+\eta)\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega u^2|\nabla v|^{2p-2} + \int_\Omega|\nabla v|^{2p} + c_{25}\\
\leq& -\frac{p-1}{p}\int_\Omega\left|\nabla u^{\frac{p}{2}}\right|^2 - \frac{p}{4}\int_\Omega|\nabla v|^{2p-2}\left|D^2v\right|^2 - cp\left(\frac{p+\gamma-1}{\gamma}\right)^{-\gamma}\int_\Omega\left|\nabla u^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma\\
&+\left(K_1(p,n)\|v_0\|^{\frac{2}{p}}_{L^\infty(\Omega)}\chi^{\frac{2(p+1)}{p}} + K_2(p,n,\eta)\|v_0\|^{p-1}_{L^\infty(\Omega)} - \mu p\right)\int_\Omega u^{p+1} + c_{26}\\
\leq&\left(K_1(p,n)\|v_0\|^{\frac{2}{p}}_{L^\infty(\Omega)}\chi^{\frac{2(p+1)}{p}} + K_2(p,n,\eta)\|v_0\|^{p-1}_{L^\infty(\Omega)} - \mu p\right)\int_\Omega u^{p+1}\\
&-cp\left(\frac{p+\gamma-1}{\gamma}\right)^{-\gamma}\int_\Omega\left|\nabla u^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_{26}
\end{aligned}$$

If $\gamma > \frac{2n}{n+1}$ we can apply (6.1.20) with ε small enough. If, instead, $\gamma = \frac{2n}{n+1}$ we apply (6.1.21) and use the condition (6.1.26). In both cases we arrive at

$$\varphi'(t)+\varphi(t) \leq C$$

and an application of Lemma 3.13 ensures the thesis. □

6.1.4.1 Proof of Theorem 6.1

If $(\mathcal{A}_1)$ is valid, Lemmata 6.13 and 6.8 ensure the thesis. Instead, if $(\mathcal{A}_2)$ is satisfied, since $K_1(p,n), K_2(p,n,\eta), C_{GN}$ and $\hat{\sigma}$ are continuous functions of the vaiables p and η, we can say that there exist $p > \frac{n}{2}$ and $\eta > 0$ such that the condition (6.1.26) is true, and again, both Lemmas give us the conclusion (see Remark 6.2).

6.2 [CV25] Boundedness in a nonlinear chemotaxis-consumption model with gradient terms

6.2.1 Main result

This section is dedicated to the following problem

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v\big) & \\ \qquad +\lambda u - \mu u^2 - c|\nabla u|^\gamma & \text{in } \Omega \times (0, T_{\max}),\\ v_t = \Delta v - uv & \text{in } \Omega \times (0, T_{\max}),\\ u_\nu = v_\nu = 0 & \text{on } \partial\Omega \times (0, T_{\max}),\\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) & x \in \bar{\Omega}. \end{cases} \tag{6.2.1}$$

The above model generalizes those in (6.1.1) and in (2.2.3); in particular system (6.1.1) is obtained by imposing $m_1 = m_2 = 1$ in (6.2.1) and the system studied in item iii) is recovered for $c = 0$. We will make some considerations in this regard after having claimed our main result.

6.2.1.1 Claim of the main result

We now present the central result.

Theorem 6.14. *Let $\lambda, \mu, c, \chi > 0$, $m_1, m_2 \in \mathbb{R}$. Then provided either*

($\mathcal{A}_1$) $\max\left\{\frac{2n}{n+1}, \frac{n}{n+1}(2m_2 - m_1 + 1)\right\} < \gamma \leq 2$

or

($\mathcal{A}_2$) $\max\left\{1, \frac{n}{n+1}(2m_2 - m_1 + 1)\right\} < \gamma \leq \frac{2n}{n+1}$ *and* $\mu > \frac{2}{p_0}\mathcal{K}\|\chi v_0\|^{2p_0}_{L^\infty(\Omega)}$,

for some existing positive constants $p_0 = p_0(m_1, m_2, n, \gamma)$ and $\mathcal{K} = \mathcal{K}(p_0)$, the solution to problem (6.2.1) is global and bounded.

Remark 6.15. *Let us make these comments.*

1. *If in problem (6.2.1) we set $m_1 = m_2 = 1$, relations ($\mathcal{A}_1$) and ($\mathcal{A}_2$) recover what was established in Section 6.1.*

2. *The presence of the gradient nonlinearity in problem (6.2.1), i.e. $c > 0$, allows one to obtain boundedness even for strong attraction effect (i.e., m_2 large), provided ($\mathcal{A}_1$) is satisfied. This situation is different from the case where $c = 0$; in this case $m_2 < \frac{m_1+1}{2}$ and μ large enough are required (see [MV18]).*

The remainder of the work is organized as follows. In §6.2.2, we provide some preparatory and well-known preliminaries. Section §6.2.3 focuses on deriving some crucial properties related to the u- and v-components. Very importantly, in the same section, we will give a general boundedness result for parabolic equations with gradient nonlinearities. Then, in §6.2.4, we will control by means of a priori estimates the energy functional

$$\varphi(t) := \int_\Omega (u+1)^p + \chi^{2p}\int_\Omega |\nabla v|^{2p} \quad \text{on} \quad (0, T_{\max}),$$

defined, for $p > 1$, in terms of the local solution. Subsequently, using a comparison principle, we provide a time-independent bound for $\varphi(t)$, which, in turn, gives bounds for u in $L^p(\Omega)$ and ∇v in $L^{2p}(\Omega)$. Finally, we deduce that the solution is actually bounded.

6.2.2 Preliminaries

The following are the technical and general results that will be referenced throughout the work.

Lemma 6.16. *Let $n \in \mathbb{N}$ and*

$$\gamma > \max\left\{\frac{2n}{n+1}, \frac{n}{n+1}(2m_2 - m_1 + 1)\right\},$$

be satisfied. Then, there exists $p_1 = p_1(\gamma, n, m_1, m_2) > 1$ such that for every $p > p_1$ and

$$\bar{\theta}(p,\gamma) = \frac{\frac{p+\gamma-1}{\gamma}\left(1 - \frac{1}{p+1}\right)}{\frac{p+\gamma-1}{\gamma} + \frac{1}{n} - \frac{1}{\gamma}}, \qquad \bar{\sigma}(p,\gamma) = \frac{\gamma(p+1)}{p+\gamma-1}, \qquad \tilde{\theta}(p) = \frac{\frac{p}{2} - \frac{1}{2}}{\frac{p}{2} - \frac{1}{2} + \frac{1}{n}},$$

$$\hat{\theta}(p) = \frac{\frac{p+\gamma-1}{\gamma}\left(1 - \frac{p}{(p+1)(p+2m_2-m_1-1)}\right)}{\frac{p+\gamma-1}{\gamma} + \frac{1}{n} - \frac{1}{\gamma}}, \qquad \hat{\sigma}(p) = \frac{\gamma(p+1)(p+2m_2-m_1-1)}{p(p+\gamma-1)}.$$

these inequalities are complied:

$$0 < \bar{\theta} < 1, \quad (6.2.2a) \qquad \frac{\bar{\sigma}\bar{\theta}}{\gamma} < 1. \quad (6.2.2b) \qquad 0 < \hat{\theta} < 1, \quad (6.2.2c)$$

$$\frac{\hat{\sigma}\hat{\theta}}{\gamma} < 1. \quad (6.2.2d) \qquad 0 < \tilde{\theta} < 1. \quad (6.2.2e)$$

Proof. The proof of the inequalities is straightforward as long as p is chosen sufficiently large; let us indicate with p_1 such a value. We emphasize that the conditions $\gamma > \frac{2n}{n+1}$ and $\gamma > \frac{n}{n+1}(2m_2 - m_1 + 1)$ are only needed to prove (6.2.2b) and (6.2.2d), respectively. $\square$

6.2.3 Properties of solutions

Lemma 6.17. *Let $\lambda, \mu, c, \chi > 0$, $m_1, m_2 \in \mathbb{R}$ and $\gamma \geq 1$. Then we have*

$$v \leq \|v_0\|_{L^\infty(\Omega)} \quad in \quad \Omega \times (0, T_{max}), \tag{6.2.3}$$

$$\int_\Omega u \leq \max\left\{\int_\Omega u_0(x)\,dx, \frac{\lambda}{\mu}|\Omega|\right\} \quad on\ (0, T_{\max}), \tag{6.2.4}$$

and

$$\sup_{t\in(0,T_{\max})} \int_\Omega |\nabla v|^2 < \infty. \tag{6.2.5}$$

Proof. Relation (6.2.3) is a consequence of Proposition 3.9; instead, bound (6.2.4) is derived by using Proposition 4.8 with $\rho = 1$ and $k = 2$. Finally, relation (6.2.5) is achieved in [LW17, Lemma 3.4], with the support of (6.2.4) itself. $\square$

6.2.3.1 Guaranteeing boundedness through estimates in Lebesgue spaces

We will utilize the following result.

Lemma 6.18 (Boundedness criterion)**.** *Let $\gamma \in [1, 2]$. There is $p_2 = p_2(\gamma, n, m_1, m_2) > 1$ such that if $u \in L^\infty((0, T_{\max}); L^{p_2}(\Omega))$ then $u \in L^\infty((0, \infty); L^\infty(\Omega))$.*

Proof. First, we observe that for any $\lambda, \mu, c > 0$ and $\gamma \geq 1$, the function $u \mapsto \lambda u - \mu u^2 - c|\nabla u|^\gamma$, for $u \geq 0$, is bounded from above by a positive constant L; henceforth, we can write

$$\lambda u - \mu u^2 - c|\nabla u|^\gamma \leq L \quad \text{in} \quad \Omega \times (0, T_{\max}).$$

Subsequently, it is seen that u classically solves problem (A.1) in Lemma 4.6 with

$$D(x,t,u)=(u+1)^{m_1-1},\quad f(x,t)=-\chi u(u+1)^{m_2-1}\nabla v,\quad g(x,t)=L.$$

Thereafter, by choosing an appropriately large $p_2>1$ such that from the hypothesis $u\in L^\infty((0,T_{\max});L^{p_2}(\Omega))$ relation (3.3.6) applied to the equation $v_t=\Delta v-uv$ give the necessary regularity for ∇v, we have that (A.2)–(A.8) are all satisfied, and henceforth we have, by applying Lemma 4.6, that actually $u\in L^\infty((0,T_{\max});L^\infty(\Omega))$. Finally, Proposition 4.4 gives the conclusion. □

6.2.4 Uniform bounds and proof of Theorem 6.14

To effectively apply Lemma 6.18, we first need to derive some a priori time-independent estimates. These estimates will provide crucial bounds for u and v, essential for the correct application of the lemma mentioned. To this end, we will study for any $p>1$ the time evolution of the functional

$$\varphi(t):=\int_\Omega (u+1)^p+\chi^{2p}\int_\Omega|\nabla v|^{2p}\quad\text{for all}\quad t\in(0,T_{\max}),\tag{6.2.6}$$

the goal being to establish its boundedness. This will be achieved by deriving a differential problem of the type

$$\begin{cases}\varphi'(t)+\varphi(t)\le C\quad\text{for all } t\in(0,T_{\max}),\\ \varphi(0)=\int_\Omega(u_0(x)+1)^pdx+\chi^{2p}\int_\Omega|\nabla v_0(x)|^{2p}dx,\end{cases}\tag{6.2.7}$$

for some $C>0$, which, thanks to Lemma 3.13, entails

$$\int_\Omega(u+1)^p+\chi^{2p}\int_\Omega|\nabla v|^{2p}\le\max\{C,\varphi(0)\}\quad\text{for all } t\in(0,T_{\max}).$$

Based on the points discussed up to this point, in the following sections, our objective is to establish an inequality for the function φ as that presented in (6.2.7).

6.2.4.1 A priori estimates

The next computations collect uniform-in-time bounds of some norms of u and v; some related derivations will involve the continuous function

$$K(p,n,\eta):=\frac{2p^{\frac{p+1}{2}}(p+n+\eta-1)^{\frac{p+1}{2}}}{p+1}\left(\frac{8(4p^2+n)(p-1)}{p(p+1)}\right)^{\frac{p-1}{2}},\tag{6.2.8}$$

defined for all $p>1$ and $\eta>0$, as well as

$$K_1=p(p+n-1+\eta)\|v_0\|^2_{L^\infty(\Omega)}\quad\text{and}\quad K_2=K(p,n,\eta)\|v_0\|^{2p}_{L^\infty(\Omega)}.\tag{6.2.9}$$

Lemma 6.19. *Let the hypotheses of Lemma 6.16 be fulfilled, and $p_1>1$ the value therein obtained. Then for every $p>p_1$ and for all $\varepsilon,\hat c>0$ these inequalities hold:*

$$\hat c\int_\Omega(u+1)^{p+1}\le\varepsilon\int_\Omega\left|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma+c_{27}\quad\textit{on } (0,T_{\max}),\tag{6.2.10}$$

$$\hat c\int_\Omega(u+1)^{\frac{(p+1)(p+2m_2-m_1-1)}{p}}\le\varepsilon\int_\Omega\left|\nabla(u+1)^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma+c_{28}\quad\textit{on } (0,T_{\max}),\tag{6.2.11}$$

and

$$\int_\Omega|\nabla v|^{2p}\le\frac{p}{4}\int_\Omega|\nabla v|^{2p-2}\left|D^2v\right|^2+c_{15}\quad\textit{on } (0,T_{\max}).\tag{6.2.12}$$

Proof. In order to prove relation (6.2.10), we use the Gagliardo–Nirenberg inequality (see Proposition 3.5) in conjunction with (6.2.2a) and (6.2.4), obtaining (we set for commodity in the next few lines $\hat{u}=(u+1)$)

$$\begin{aligned}\hat{c}\int_\Omega (u+1)^{p+1} =&\hat{c}\Big\|\hat{u}^{\frac{p+\gamma-1}{\gamma}}\Big\|^{\frac{\gamma(p+1)}{p+\gamma-1}}_{L^{\frac{\gamma(p+1)}{p+\gamma-1}}(\Omega)}\\ \leq& c_{29}\Big(\Big\|\nabla\hat{u}^{\frac{p+\gamma-1}{\gamma}}\Big\|^{\bar{\theta}}_{L^\gamma(\Omega)}\Big\|\hat{u}^{\frac{p+\gamma-1}{\gamma}}\Big\|^{1-\bar{\theta}}_{L^{\frac{\gamma}{p+\gamma-1}}(\Omega)}+\Big\|\hat{u}^{\frac{p+\gamma-1}{\gamma}}\Big\|_{L^{\frac{\gamma}{p+\gamma-1}}(\Omega)}\Big)^{\bar{\sigma}}\\ \leq& c_{30}\Big(\int_\Omega\Big|\nabla\hat{u}^{\frac{p+\gamma-1}{\gamma}}\Big|^\gamma\Big)^{\frac{\bar{\sigma}\bar{\theta}}{\gamma}}+c_{31}\leq\varepsilon\int_\Omega\Big|\nabla\hat{u}^{\frac{p+\gamma-1}{\gamma}}\Big|^\gamma+c_{27}\quad\text{on } (0,T_{\max}),\end{aligned}$$

where in the last estimate we applied Young's inequality, justified by relation (6.2.2b), and (3.1.5). (We will tacitly employ such an algebraic inequality without mentioning it.)

As to relation (6.2.11), relying on (6.2.2c) and (6.2.2d), similar rearrangements give on $(0,T_{\max})$

$$\begin{aligned}\hat{c}\int_\Omega (u+1)^{\frac{(p+1)(p+2m_2-m_1-1)}{p}} =&\hat{c}\Big\|\hat{u}^{\frac{p+\gamma-1}{\gamma}}\Big\|^{\frac{\gamma(p+1)(p+2m_2-m_1-1)}{p(p+\gamma-1)}}_{L^{\frac{\gamma(p+1)(p+2m_2-m_1-1)}{p(p+\gamma-1)}}(\Omega)}\\ \leq& c_{29}\Big(\Big\|\nabla\hat{u}^{\frac{p+\gamma-1}{\gamma}}\Big\|^{\hat{\theta}}_{L^\gamma(\Omega)}\Big\|\hat{u}^{\frac{p+\gamma-1}{\gamma}}\Big\|^{1-\hat{\theta}}_{L^{\frac{\gamma}{p+\gamma-1}}(\Omega)}+\Big\|\hat{u}^{\frac{p+\gamma-1}{\gamma}}\Big\|_{L^{\frac{\gamma}{p+\gamma-1}}(\Omega)}\Big)^{\hat{\sigma}}\\ \leq& c_{30}\Big(\int_\Omega\Big|\nabla\hat{u}^{\frac{p+\gamma-1}{\gamma}}\Big|^\gamma\Big)^{\frac{\hat{\sigma}\hat{\theta}}{\gamma}}+c_{31}\leq\varepsilon\int_\Omega\Big|\nabla\hat{u}^{\frac{p+\gamma-1}{\gamma}}\Big|^\gamma+c_{28}.\end{aligned}$$

Ultimately, by recalling (6.2.5) and (6.2.2e) we have

$$\begin{aligned}\int_\Omega|\nabla v|^{2p}=&\int_\Omega(|\nabla v|^p)^2=\||\nabla v|^p\|^2_{L^2(\Omega)}\leq c_{21}\Big(\|\nabla|\nabla v|^p\|^{\tilde{\theta}}_{L^2(\Omega)}\||\nabla v|^p\|^{1-\tilde{\theta}}_{L^{\frac{2}{p}}(\Omega)}+\||\nabla v|^p\|_{L^{\frac{2}{p}}(\Omega)}\Big)^2\\ \leq&\frac{1}{4p}\int_\Omega|\nabla|\nabla v|^p|^2+c_{22}\leq\frac{p}{4}\int_\Omega|\nabla v|^{2p-2}\big|D^2v\big|^2+c_{15}\quad\text{on } (0,T_{\max}),\end{aligned}$$

so that also relation (6.2.12) is achieved. □

Lemma 6.20. *For every $p>1$ and for all $\hat{c}>0$ we have*

$$K_1\int_\Omega u^2|\nabla v|^{2p-2}\leq\frac{p}{4}\int_\Omega|\nabla v|^{2p-2}\big|D^2v\big|^2+K_2\int_\Omega u^{p+1}\quad\textit{on } (0,T_{\max}),\tag{6.2.13}$$

$$\begin{aligned}\hat{c}\int_\Omega(u+1)^{p+2m_2-m_1-1}|\nabla v|^2\leq&\frac{p}{4}\chi^{2p}\int_\Omega|\nabla v|^{2p-2}\big|D^2v\big|^2\\ &+c_{32}\int_\Omega(u+1)^{\frac{(p+1)(p+m_2-m_1-1)}{p}}\quad\textit{on } (0,T_{\max}),\end{aligned}\tag{6.2.14}$$

where K_1 and K_2 have been introduced in (6.2.9).

Proof. Let us prove bounds (6.2.13) and (6.2.14). Indeed, an application of Young's inequality, in conjunction with (3.2.3) and (6.2.3), leads on $(0,T_{\max})$ and for any $\varepsilon_1>0$ and $C(\varepsilon_1)=\frac{2K_1}{p+1}\Big(\varepsilon_1\frac{p+1}{K_1(p-1)}\Big)^{\frac{1-p}{2}}$, as in (3.1.3), to

$$\begin{aligned}K_1\int_\Omega u^2|\nabla v|^{2p-2}\leq&\varepsilon_1\int_\Omega|\nabla v|^{2(p+1)}+C(\varepsilon_1)\int_\Omega u^{p+1}\\ \leq&\varepsilon_1 2(4p^2+n)\|v_0\|^2_{L^\infty(\Omega)}\int_\Omega|\nabla v|^{2p-2}\big|D^2v\big|^2+\frac{2}{p+1}(\varepsilon_1\frac{p+1}{p-1})^{-\frac{p-1}{2}}K_1^{\frac{p+1}{2}}\int_\Omega u^{p+1}\\ =&\frac{p}{4}\int_\Omega|\nabla v|^{2p-2}\big|D^2v\big|^2+K_2\int_\Omega u^{p+1},\end{aligned}$$

and for $\varepsilon_1 = \frac{p}{8(4p^2+n)\|v_0\|^2_{L^\infty(\Omega)}}$ relation (6.2.13) is attained. Likewise, we have for all $t \in (0, T_{\max})$,

$$\hat{c}\int_\Omega (u+1)^{p+2m_2-m_1-1}|\nabla v|^2 \le \frac{p}{4}\chi^{2p}\int_\Omega |\nabla v|^{2p-2}\left|D^2 v\right|^2 + c_{32}\int_\Omega (u+1)^{\frac{(p+1)(p+m_2-m_1-1)}{p}},$$

and (6.2.14) is established as well. ☐

Lemma 6.21. *For any $p > 1$ and for every $t \in (0, T_{\max})$ we have*

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u+1)^p \le & c_{33}\int_\Omega (u+1)^{p+2m_2-m_1-1}|\nabla v|^2 + c_{34}\int_\Omega (u+1)^p \\ & - \frac{\mu p}{2}\int_\Omega (u+1)^{p+1} - c_{35}\int_\Omega \left|\nabla u^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_{36}.\end{aligned} \tag{6.2.15}$$

Proof. We compute the time derivative of $\int_\Omega (u+1)^p$ and apply Young's inequality, yielding on $(0, T_{\max})$

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u+1)^p = & -p(p-1)\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2 \\ & + \chi p(p-1)\int_\Omega u(u+1)^{p+m_2-3}\nabla u\cdot\nabla v + \lambda p\int_\Omega u(u+1)^{p-1} \\ & - \mu p\int_\Omega u^2(u+1)^{p-1} - cp\int_\Omega (u+1)^{p-1}|\nabla u|^\gamma.\end{aligned} \tag{6.2.16}$$

In turn from the already mentioned inequality $(u+1)^2 \le 2(u^2+1)$ we have through Young's inequality that on $(0, T_{\max})$ one has

$$\begin{aligned}-\mu p\int_\Omega u^2(u+1)^{p-1} & \le -\frac{\mu p}{2}\int_\Omega (u+1)^{p+1} + \mu p\int_\Omega (u+1)^{p-1} \\ & \le -\frac{\mu p}{2}\int_\Omega (u+1)^{p+1} + c_{37}\int_\Omega (u+1)^p + c_{36},\end{aligned}$$

and also, by the Young inequality again,

$$\begin{aligned}\chi p(p-1)\int_\Omega u(u+1)^{p+m_2-3}\nabla u\cdot\nabla v \le & \frac{p(p-1)}{2}\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2 \\ & + c_{33}\int_\Omega (u+1)^{p+2m_2-m_1-1}|\nabla v|^2.\end{aligned}$$

These last estimates inserted in (6.2.16) give the claim. ☐

Lemma 6.22. *For all $\eta > 0$, $p \in (1, \infty)$ and $t \in (0, T_{\max})$ this inequality is satisfied*

$$\frac{d}{dt}\int_\Omega |\nabla v|^{2p} + p\int_\Omega |\nabla v|^{2p-2}\left|D^2 v\right|^2 \le K_1\int_\Omega u^2|\nabla v|^{2p-2} + c_{25}, \tag{6.2.17}$$

with K_1 given in (6.2.9).

Proof. The proof is contained in [LW17, Lemma 4.2]. ☐

Lemma 6.23. *Let either*

$$\gamma > \max\left\{\frac{2n}{n+1}, \frac{n}{n+1}(2m_2 - m_1 + 1)\right\}, \tag{6.2.18}$$

or, for any $\eta > 0$ and all $p > p_1$ (p_1 being the value detected in Lemma 6.16) and $K(p, n, \eta)$ as in (6.2.8),

$$\gamma > \max\left\{1, \frac{n}{n+1}(2m_2 - m_1 + 1)\right\} \quad \textit{and} \quad \mu > \frac{2}{p}K(p, n, \eta)\|\chi v_0\|^{2p}_{L^\infty(\Omega)}. \tag{6.2.19}$$

Then $u \in L^\infty((0, T_{\max}); L^p(\Omega))$, for all $p > p_1$.

Proof. Let $\varphi(t)$ fixed as in (6.2.6) and recall the definition of K_1 and K_2 in (6.2.9); by exploiting bounds (6.2.15) and (6.2.17), we have that for all $t \in (0, T_{\max})$ one has

$$\begin{aligned}\varphi'(t)+\varphi(t) \leq & c_{33}\int_\Omega (u+1)^{p+2m_2-m_1-1}|\nabla v|^2 + c_{38}\int_\Omega (u+1)^p - \frac{\mu p}{2}\int_\Omega (u+1)^{p+1}\\ & - c_{35}\int_\Omega \left|\nabla u^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma - \chi^{2p}p\int_\Omega |\nabla v|^{2p-2}\left|D^2 v\right|^2\\ & + \chi^{2p}K_1\int_\Omega u^2|\nabla v|^{2p-2} + \chi^{2p}\int_\Omega |\nabla v|^{2p} + c_{39},\end{aligned}$$

which for all $t \in (0, T_{\max})$ and any $\rho > 0$ is simplified, by virtue of (6.2.12), (6.2.13), (6.2.14) and the Young inequality, into

$$\begin{aligned}\varphi'(t)+\varphi(t) \leq & \Big(K_2\chi^{2p} - \frac{\mu p}{2} + \rho\Big)\int_\Omega (u+1)^{p+1}\\ & + c_{32}\int_\Omega (u+1)^{\frac{(p+1)(p+m_2-m_1-1)}{p}} - c_{35}\int_\Omega \left|\nabla u^{\frac{p+\gamma-1}{\gamma}}\right|^\gamma + c_{40}.\end{aligned}$$

In turn, if relation (6.2.18) is satisfied, independently by the sing of $K_2\chi^{2p} - \frac{\mu p}{2} + \rho$, we use (6.2.10) and (6.2.11), with $\varepsilon = \frac{c_{35}}{2}$, so to obtain an absorption problem as (6.2.7). Conversely, under assumption (6.2.19), we can choose ρ sufficiently small so to have $K_2\chi^{2p} - \frac{\mu p}{2} + \rho \leq 0$; subsequently, again (6.2.11) leads to an alike problem as that in (6.2.7). Definitively, we can conclude. □

6.2.4.2 Proof of Theorem 6.14

We have gathered all the essential elements and mathematical tools required to establish our result.

Proof. For p_1 and p_2 the constants from Lemmas 6.16 and 6.18, respectively, let us set $p_0 = \max\{p_1, p_2\}$. (Let us observe that $p_0 = p_0(m_1, m_2, n, \gamma)$.)

If relation $(\mathcal{A}_1)$ is valid, Lemmata 6.23 and 6.18 directly ensure the claim. Conversely, let $\mathcal{K}(p_0) = K(p_0, n, 0)$, being $K(p, n, \eta)$ the function in (6.2.8) and let assumption $(\mathcal{A}_2)$ be satisfied. Since $K(p, n, \eta)$ is a continuous function of the variables p and η, by continuity arguments there exist $p > p_0$ and $\eta > 0$ such that the second condition in (6.2.19) is fulfilled; consequently, the same mentioned lemmas give the conclusion. □

6.3 [GCDFN24] Global existence and lower bounds in a class of tumor-immune cell interactions chemotaxis systems

6.3.1 Aim of the research

Exactly inspired by [HT20], in this work we consider the following initial-boundary value problem that describes the tumor-immune cell interactions given by

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & \text{in } \Omega \times (0, T_{\max}), \\ \tau v_t = \Delta v + \alpha w - \beta v - \gamma u v, & \text{in } \Omega \times (0, T_{\max}), \\ \tau w_t = \Delta w - \delta u w + \mu w(1-w), & \text{in } \Omega \times (0, T_{\max}), \\ u_\nu = v_\nu = w_\nu = 0, & \text{on } \partial\Omega \times (0, T_{\max}), \\ u(x,0) = u_0(x), \quad \tau v(x,0) = \tau v_0(x), \quad \tau w(x,0) = \tau w_0(x), & x \in \overline{\Omega}. \end{cases} \tag{6.3.1}$$

Herein the parameters χ, α, β, γ, δ, μ are assumed to be positive constants.

The system (6.3.1) describes the interaction of two cells with one chemical secreted by the tumor cells. The term $-\chi \nabla \cdot (u \nabla v)$ represents the movement of the lymphocytes towards the chemical signal and assume that there is a natural logistic growth term $\mu w(1-w)$ for the cancer cells with growth coefficient μ. The constant α is the production rate of the chemical by the tumor cells, β is the decay rate of the chemical and γ is the consumption rate of the chemical by the lymphocytes. The parameter δ is known as the destruction rate of the tumor cells by the lymphocytes.
In this Section, we will discuss the following scenarios.

- ▷ In the fully parabolic case (i.e., $\tau = 1$), we prove global existence and boundedness of classical solutions, under smallness conditions involving the parameters and the initial data (see Theorem 6.24).
- ▷ In the parabolic-elliptic-elliptic case (i.e., $\tau = 0$), we establish the same results as above without any restriction on the parameters and the initial data (see Theorem 6.24).
- ▷ In 3D domains, lower bounds for the maximal existence time of blowing-up solutions are derived (see Theorem 6.26).

6.3.2 Presentation of the main results

The main theorems are stated as follows.

Theorem 6.24. *Let $n \geq 3$ and $\tau \in \{0, 1\}$. Then for any initial data $(u_0, \tau v_0, \tau w_0)$ satisfying*

$$\tau \max \left\{ \frac{\alpha}{\beta}, \frac{\alpha}{\beta} \|w_0\|_{L^\infty(\Omega)}, \|v_0\|_{L^\infty(\Omega)} \right\} < \frac{\pi}{\chi} \sqrt{\frac{2}{n}}, \tag{6.3.2}$$

the solution to system (6.3.1) *is uniformly bounded-in-time.*

The reason why in this work we are considering the case $n \geq 3$ is justified by the result for $n = 2$ (see [HT20]), and by the well known theory concerning chemotaxis model defined in one dimensional settings.

Remark 6.25. *Conversely to the case $\tau = 1$, when $\tau = 0$ we see that* (6.3.2) *in Theorem 6.24 is automatically fulfilled, independently by the choice of the data of the problem. This stresses how dealing with simplified elliptic versions of chemotaxis models is, in general, more straightforward than for the fully parabolic ones. In this sense, it appears natural to investigate whether the introduction of damping terms (as for instance logistic sources; see [TW07], [Win10b]) in the equation for u in problem* (6.3.1)*, naturally for $\tau = 1$, enables to relax restriction* (6.3.2).

As to the estimate of the blow-up time for solutions to model system (6.3.1), we will make use of the following functional:

$$\Psi_\tau = \Psi_\tau(t) := \int_\Omega u^2 + \tau \int_\Omega |\nabla v|^4 + \tau \int_\Omega |\nabla w|^2 \quad \text{for all } t \in (0, T_{\max}), \tag{6.3.3}$$

with initial value
$$\Psi_\tau(0)=\int_\Omega u_0^2+\tau\int_\Omega|\nabla v_0|^4+\tau\int_\Omega|\nabla w_0|^2. \tag{6.3.4}$$
For the upcoming analysis, it is important to note that we must consider an additional restriction on the domain.

Theorem 6.26. *Let $n=3$ and $\tau\in\{0,1\}$. If (u,v,w) is a solution that blows-up at finite time $T_{\max}$, in the sense that*
$$\limsup_{t\to T_{\max}}\|u(\cdot,t)\|_{L^\infty(\Omega)}=+\infty,$$
then there exist computable constants $\mathcal{A}_\tau,\mathcal{B}_\tau,\mathcal{C}_\tau$ such that
$$T_{\max}\geq\int_{\Psi_\tau(0)}^{+\infty}\frac{d\Psi_\tau(t)}{\mathcal{A}_\tau\Psi_\tau(t)^3+\mathcal{B}_\tau\Psi_\tau(t)^{\frac{3}{2}}+\mathcal{C}_\tau\Psi_\tau(t)^\tau}, \tag{6.3.5}$$
where Ψ_τ is defined in (6.3.3). *In particular, we can find a computable $\mathcal{D}_\tau>0$ such that this explicit lower bound*
$$T_{\max}\geq\frac{1}{2\mathcal{D}_\tau\Psi_\tau(0)^2} \tag{6.3.6}$$
can be derived.

6.3.3 Properties of the solution and extensibility criterion

Lemma 6.27. *The solution (u,v,w) satisfies*
$$\big\|u(\cdot,t)\big\|_{L^1(\Omega)}=M_1:=\|u_0\|_{L^1(\Omega)},\qquad\forall t\in(0,T_{\max}) \tag{6.3.7}$$
and
$$\big\|w(\cdot,t)\big\|_{L^\infty(\Omega)}\leq M_2:=\max\Big\{1,\tau\|w_0\|_{L^\infty(\Omega)}\Big\},\qquad\forall t\in(0,T_{\max}), \tag{6.3.8}$$
$$\big\|v(\cdot,t)\big\|_{L^\infty(\Omega)}\leq M_3:=\max\Big\{\frac{\alpha}{\beta}M_2,\tau\|v_0\|_{L^\infty(\Omega)}\Big\},\qquad\forall t\in(0,T_{\max}). \tag{6.3.9}$$

Proof. An application of Proposition 4.8 without logistic term implies
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u=0\quad\text{for all } t\in(0,T_{\max}).$$
Therefore, this problem preserves mass for u, in the sense that (6.3.7) holds. Furthermore, relations (6.3.8) and (6.3.9) follow directly from Propositions 3.8 and 3.9. In light of Remark 3.10, when $\tau=0$ the third equation implies that $\underline{w}\equiv0$ and $\overline{w}\equiv1$ serve as subsolution and supersolution, respectively. Consequently, one obtains the bound $0\leq w\leq1$ for all $(x,t)\in\overline{\Omega}\times(0,T_{\max})$. Building on this observation, we note that $\overline{v}\equiv\frac{\alpha}{\beta}M_2$ constitutes a supersolution for the second equation, while $\underline{v}\equiv0$ constitutes a subsolution. The case $\tau=1$ can be treated analogously, although in this setting the initial conditions must also be taken into account. ☐

The following result allows us to establish boundedness of solutions once some uniform-in-time estimates of these in appropriate Lebesgue spaces are derived.

Lemma 6.28. *If $u\in L^\infty((0,T_{\max});L^p(\Omega))$ for every $p>1$, then $u\in L^\infty((0,\infty);L^\infty(\Omega))$.*

Proof. Thanks to Proposition 4.4, it is sufficient to show that $u\in L^\infty((0,T_{\max});L^\infty(\Omega))$. We notice that u classically solves problem (A.1) in Lemma 4.6 with
$$D(x,t,u)=1,\qquad f(x,t)=-\chi u\nabla v,\qquad g(x,t)=0.$$
In particular, this in conjunction with the Neumann boundary conditions on v imply that hypotheses (A.2)–(A.5) are met, whereas the second inclusion of (A.6) holds for any choice of $q_2>1$. Since we are assuming p

arbitrarily large, and $u \in L^\infty((0,T_{\max});L^p(\Omega))$, conditions (A.7) and (A.8) are automatically complied. Let us now deal with the first inclusion of (A.6). If $\tau = 0$ in problem (6.3.1) , from $u \in L^\infty((0,T_{\max});L^p(\Omega))$, (6.3.8) and (6.3.9) in conjunction with (3.3.3), we have $\nabla v \in L^\infty((0,T_{\max});L^\infty(\Omega))$ (where $\psi = w - uv$). This implies that, for any $q_1 > 1$,

$$f = -\chi u \nabla v \in L^\infty((0,T_{\max});L^{q_1}(\Omega)).$$

If $\tau = 1$, we can reason in the same way invoking (3.3.6). So we have the claim exactly by virtue of Lemma 4.6. $\square$

This proved result gives a criterion toward boundedness which will be exactly used to establish such property for solutions to the model we are considering herein. But, the same result is also crucial to ensure that an unbounded solution to such models even blows-up in some energy function. In the specific we have this

Remark 6.29. *Let us observe what follows: For $n = 3$, if (u,v,w) is a solution to model (6.3.1) which blows-up in finite time $T_{\max}$ in the sense that*

$$\limsup_{t\to T_{\max}} \left\| u(\cdot,t) \right\|_{L^\infty(\Omega)} = \infty,$$

then $\lim_{t\to T_{\max}} \Psi_\tau(t) = +\infty$, where Ψ_τ is defined in (6.3.3). Indeed, if $\int_\Omega u^2$ were uniformly bounded-in-time on $(0,T_{\max})$, necessarily in view of Lemma 6.28 we would obtain that $u \in L^\infty((0,\infty);L^\infty(\Omega))$, which is a contradiction.

6.3.4 A useful tool

We will rely on the following result (see the main ideas in [ZT19] and [BK16], inspired by [Tao11]) which will be used to prove Theorem 6.24 in the case $\tau = 1$.

Lemma 6.30. *Let $\epsilon \in (0,1)$ and $p > 1$. Define the function*

$$\phi(x) := e^{\zeta(x)}, \quad 0 \le x \le K,$$

where

$$\zeta(x) := -\frac{l}{2m}x + \frac{\sqrt{4km - l^2}}{2m}\int_0^x \tan\left(\tfrac{\sqrt{4km-l^2}}{2r}s + \arctan\left(\tfrac{l}{\sqrt{4km-l^2}}\right)\right) \mathrm{d}s$$

with $k = (p-1)^2, l = -4(p-1)\epsilon, m = \frac{4}{p}(1+(p-1)\epsilon)$ and $r = \frac{4}{p}(p-1)(1-\epsilon)$. If

$$K < 2\sqrt{\frac{1-\epsilon}{p(1+p\epsilon)}}\left(\frac{\pi}{2} + \arctan\left(\sqrt{\tfrac{p}{(1-\epsilon)(1+p\epsilon)}}\ \epsilon\right)\right), \tag{6.3.10}$$

then the function $\phi(x) \in C^2([0,K])$ is well defined and satisfies the following properties

$$0 \le \phi^{'}(x), \qquad 1 \le \phi(x) \le \phi(K) \tag{6.3.11}$$

and

$$\frac{1}{p}\phi^{''}(x) - \phi^{'}(x) \ge 0. \tag{6.3.12}$$

Moreover we have

$$|(p-1)\phi(x) - 2\phi^{'}(x)| - 2\sqrt{(p-1)(1-\epsilon)\phi(x)\left(\frac{1}{p}\phi''(x) - \phi'(x)\right)} = 0 \tag{6.3.13}$$

for all $0 \le x \le K$.

Proof. First we prove that the function $\phi(x)$ is well defined. For any $p > 1$ and $\epsilon \in (0,1)$, we attain

$$4km - l^2 = \frac{16(p-1)^2}{p}\Big(1 + (p-1)\epsilon - p\epsilon^2\Big) = \frac{16(p-1)^2}{p}(1-\epsilon)(1+p\epsilon) > 0,$$

$$\frac{l}{\sqrt{4km-l^2}} = -\sqrt{\frac{p}{(1-\epsilon)(1+p\epsilon)}}\ \epsilon, \quad \text{and} \quad \frac{\sqrt{4km-l^2}}{2r} = \frac{1}{2}\sqrt{\frac{p(1+p\epsilon)}{1-\epsilon}}.$$

Thus, for all $s \in [0, K]$ it holds

$$\frac{\sqrt{4km-l^2}}{2r}s + \arctan\left(\tfrac{l}{\sqrt{4km-l^2}}\right) \geq -\arctan\left(\sqrt{\tfrac{p}{(1-\epsilon)(1+p\epsilon)}}\ \epsilon\right) > -\frac{\pi}{2}. \tag{6.3.14}$$

Similarly, provided the bound in (6.3.10), we arrive at

$$\frac{\sqrt{4km-l^2}}{2r}s + \arctan\left(\tfrac{l}{\sqrt{4km-l^2}}\right) \leq \frac{1}{2}\sqrt{\frac{p(1+p\epsilon)}{1-\epsilon}}K - \arctan\left(\sqrt{\tfrac{p}{(1-\epsilon)(1+p\epsilon)}}\ \epsilon\right) < \frac{\pi}{2}. \tag{6.3.15}$$

From (6.3.14) and (6.3.15), we conclude that the function $\zeta(x)$ is well defined and bounded. This implies that $\phi(x)$ is well defined positive for all $0 \leq x \leq K$. Next, observing that $\zeta'(x)$ is a monotonic increasing function and $\zeta'(0) = 0$, we attain

$$\begin{aligned}
\phi^{'}(x) =& \phi(x)\zeta^{'}(x)\\
=& \phi(x)\left(-\frac{l}{2m} + \frac{\sqrt{4km-l^2}}{2m}\tan\left(\tfrac{\sqrt{4km-l^2}}{2r}x + \arctan\left(\tfrac{l}{\sqrt{4km-l^2}}\right)\right)\right)\\
\geq& \phi(x)\zeta^{'}(0) \ = \ 0,
\end{aligned}$$

for all $0 \leq x \leq K$. As a consequence, being $\phi^{'}(x)$ continuous and nonnegative, we achieve $1 \leq \phi(x) \leq \phi(K)$. Moreover, recalling (6.3.15), we have $\phi^{'}(x) < \infty$, for all $0 \leq x \leq K$; these last relations prove (6.3.11).

Now we calculate

$$\begin{aligned}
\zeta^{''}(x) =& \frac{4km-l^2}{4mr}\sec^2\left(\tfrac{\sqrt{4km-l^2}}{2r}x + \arctan\left(\tfrac{l}{\sqrt{4km-l^2}}\right)\right)\\
=& \frac{4km-l^2}{4mr}\left(1 + \tan^2\left(\tfrac{\sqrt{4km-l^2}}{2r}x + \arctan\left(\tfrac{l}{\sqrt{4km-l^2}}\right)\right)\right).
\end{aligned} \tag{6.3.16}$$

From $\zeta^{'}(x)$, we arrive at

$$\frac{4m^2}{4km-l^2}\left(\zeta^{'}(x) + \frac{l}{2m}\right)^2 = \tan^2\left(\tfrac{\sqrt{4km-l^2}}{2r}x + \arctan\left(\tfrac{l}{\sqrt{4km-l^2}}\right)\right). \tag{6.3.17}$$

By substituting (6.3.17) in (6.3.16) we get

$$\zeta^{''}(x) = \frac{4km-l^2}{4mr}\left(1 + \frac{4m^2}{4km-l^2}\left(\zeta^{'}(x) + \frac{l}{2m}\right)^2\right) = \frac{1}{r}\Big(m\zeta^{'}(x)^2 + l\zeta^{'}(x) + k\Big). \tag{6.3.18}$$

Next, with the help of (6.3.18), we calculate

$$\begin{aligned}
\frac{1}{p}\phi^{''}(x) - \phi^{'}(x) =& \phi(x)\left(\frac{1}{p}\Big(\zeta^{''}(x) + \zeta^{'}(x)^2\Big) - \zeta^{'}(x)\right)\\
=& \frac{\phi(x)}{p}\left(\left(\frac{m}{r}+1\right)\zeta^{'}(x)^2 + \left(\frac{l}{r} - p\right)\zeta^{'}(x) + \frac{k}{r}\right)\\
=& \frac{\phi(x)}{p}\left(\frac{p}{(p-1)(1-\epsilon)}\zeta^{'}(x)^2 - \frac{p}{1-\epsilon}\zeta^{'}(x) + \frac{p(p-1)}{4(1-\epsilon)}\right)\\
=& \frac{\phi(x)}{4(p-1)(1-\epsilon)}\Big(4\zeta^{'}(x)^2 - 4(p-1)\zeta^{'}(x) + (p-1)^2\Big)\\
=& \frac{1}{4(p-1)(1-\epsilon)}\frac{1}{\phi(x)}\Big((p-1)\phi(x) - 2\phi^{'}(x)\Big)^2.
\end{aligned}$$

This gives (6.3.12) and (6.3.13). $\square$

6.3.5 A priori estimates

The case $\tau = 1$

We will now provide the global boundedness of the solution for the system (6.3.1). The following lemma establishes the $L^p(\Omega)$ estimate of the first component of the solution.

Lemma 6.31. *For all $p > 1$ and $\epsilon \in (0,1)$, let relation* (6.3.10) *hold with $K = \chi M_3$, being M_3 as in* (6.3.9). *Then we have*

$$\left\|u(\cdot,t)\right\|_{L^p(\Omega)} \leq C_1, \qquad \forall\, t \in (0, T_{\max}),$$

for some constant $C_1 > 0$.

Proof. Let us define $v_1 = \chi v$ and recall the definition of ϕ in Lemma 6.30. Multiplying by u^{p-1}, $p > 1$, the first equation of (6.3.1) and integrating over Ω we get

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\phi(v_1) &= \int_\Omega u^{p-1}u_t\phi(v_1) + \frac{1}{p}\int_\Omega u^p\phi^{'}(v_1)v_{1t}\\ &= \int_\Omega u^{p-1}\phi(v_1)\Big(\Delta u - \nabla\cdot(u\nabla v_1)\Big)\\ &\quad+\frac{1}{p}\int_\Omega u^p\phi^{'}(v_1)\Big(\Delta v_1 + \alpha\chi w - \beta v_1 - \gamma u v_1\Big) \quad \text{for all } t \in (0, T_{\max}).\end{aligned}$$

Integration by parts leads to

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\phi(v_1) &= -(p-1)\int_\Omega u^{p-2}\phi(v_1)|\nabla u|^2 - \int_\Omega u^{p-1}\phi^{'}(v_1)\nabla u\cdot\nabla v_1\\ &\quad+(p-1)\int_\Omega u^{p-1}\phi(v_1)\nabla u\cdot\nabla v_1 + \int_\Omega u^p\phi^{'}(v_1)|\nabla v_1|^2 - \int_\Omega u^{p-1}\phi^{'}(v_1)\nabla u\cdot\nabla v_1\\ &\quad-\frac{1}{p}\int_\Omega u^p\phi^{''}(v_1)|\nabla v_1|^2 + \frac{\alpha\chi}{p}\int_\Omega u^p\phi^{'}(v_1)w - \frac{1}{p}\int_\Omega u^p\phi^{'}(v_1)\left(\beta v_1 + \gamma u v_1\right),\end{aligned}$$

for all $t \in (0, T_{\max})$. Combining the terms, one can get on $(0, T_{\max})$

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\phi(v_1)+(p-1)\epsilon\int_\Omega u^{p-2}\phi(v_1)|\nabla u|^2 &= -(p-1)(1-\epsilon)\int_\Omega u^{p-2}\phi(v_1)|\nabla u|^2\\ &\quad+\int_\Omega\Big((p-1)\phi(v_1) - 2\phi^{'}(v_1)\Big)u^{p-1}\nabla u\cdot\nabla v_1 + \frac{\alpha\chi}{p}\int_\Omega u^p\phi^{'}(v_1)w\\ &\quad-\int_\Omega\Big(\frac{1}{p}\phi^{''}(v_1) - \phi^{'}(v_1)\Big)u^p|\nabla v_1|^2 - \frac{1}{p}\int_\Omega u^p\phi^{'}(v_1)\Big(\beta v_1 + \gamma u v_1\Big).\end{aligned} \tag{6.3.19}$$

We have by the Gagliardo-Nirenberg and the Young inequalities, thanks to Lemma 6.27,

$$\begin{aligned}c_1\int_\Omega u^p = c_1\left\|u^{\frac{p}{2}}\right\|^2_{L^2(\Omega)} &\leq c_2\left(\left\|\nabla u^{\frac{p}{2}}\right\|^{2r_0}_{L^2(\Omega)}\left\|u^{\frac{p}{2}}\right\|^{2(1-r_0)}_{L^{\frac{2}{p}}(\Omega)} + \left\|u^{\frac{p}{2}}\right\|^2_{L^{\frac{2}{p}}(\Omega)}\right)\\ &\leq \frac{(p-1)\epsilon}{2}\int_\Omega u^{p-2}|\nabla u|^2 + c_3 \quad \text{for all } t \in (0, T_{\max}),\end{aligned} \tag{6.3.20}$$

where $r_0 = \frac{\frac{p}{2}-\frac{1}{2}}{\frac{p}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1)$, c_3 is a positive constant and some $\epsilon \in (0,1)$.

In view of (6.3.11), we have $\phi(v_1) \geq 1$, so that

$$c_1\int_\Omega u^p \leq \frac{(p-1)\epsilon}{2}\int_\Omega u^{p-2}\phi(v_1)|\nabla u|^2 + c_3, \quad \text{on } (0, T_{\max}). \tag{6.3.21}$$

From (6.3.21), we can estimate on $(0, T_{\max})$, again with Lemma 6.27,

$$\begin{aligned}\frac{\alpha\chi}{p}\int_\Omega u^p\phi^{'}(v_1)w &\leq \frac{\alpha\chi}{p}\left\|w\right\|_{L^\infty(\Omega)}\|\phi'\|_{L^\infty([0,K])}\int_\Omega u^p\\ &\leq \frac{(p-1)\epsilon}{2}\int_\Omega u^{p-2}\phi(v_1)|\nabla u|^2 + c_4.\end{aligned} \tag{6.3.22}$$

By substituting (6.3.22) in (6.3.19) we obtain, for every $t \in (0, T_{\max})$,

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\phi(v_1) + \frac{(p-1)\epsilon}{2}\int_\Omega u^{p-2}\phi(v_1)|\nabla u|^2 \leq\ & -(p-1)(1-\epsilon)\int_\Omega u^{p-2}\phi(v_1)|\nabla u|^2\\ &+\int_\Omega\Big((p-1)\phi(v_1)-2\phi^{'}(v_1)\Big)u^{p-1}\nabla u\cdot\nabla v_1\\ &-\int_\Omega\left(\frac{1}{p}\phi^{''}(v_1)-\phi^{'}(v_1)\right)u^p|\nabla v_1|^2+c_4.\end{aligned} \tag{6.3.23}$$

Again, using (6.3.21), we have

$$\int_\Omega u^p\phi(v_1) \leq \phi\left(\chi\|v\|_{L^\infty(\Omega)}\right)\int_\Omega u^p \leq \frac{(p-1)\epsilon}{2}\int_\Omega u^{p-2}\phi(v_1)|\nabla u|^2 + c_5, \quad \text{on } (0, T_{\max}). \tag{6.3.24}$$

By plugging (6.3.24) in (6.3.23) we can write

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}&\int_\Omega u^p\phi(v_1)+\int_\Omega u^p\phi(v_1)\leq -(p-1)(1-\epsilon)\int_\Omega u^{p-2}\phi(v_1)|\nabla u|^2\\ &+\int_\Omega\Big|(p-1)\phi(v_1)-2\phi^{'}(v_1)\Big|u^{p-1}|\nabla u||\nabla v_1| - \int_\Omega\left(\frac{1}{p}\phi^{''}(v_1)-\phi^{'}(v_1)\right)u^p|\nabla v_1|^2\\ &+c_6\\ &\leq -\int_\Omega\left(\sqrt{(p-1)(1-\epsilon)\phi(v_1)}u^{\frac{p-2}{2}}|\nabla u|\right)^2-\int_\Omega\left(\sqrt{\frac{1}{p}\phi''(v_1)-\phi'(v_1)}u^{\frac{p}{2}}|\nabla v_1|\right)^2\\ &+\int_\Omega\Big|(p-1)\phi(v_1)-2\phi^{'}(v_1)\Big|u^{p-1}|\nabla u||\nabla v_1|+c_6\\ &\leq -\int_\Omega\Bigg[\left(\sqrt{(p-1)(1-\epsilon)\phi(v_1)}u^{\frac{p-2}{2}}|\nabla u|\right)^2+\left(\sqrt{\frac{1}{p}\phi''(v_1)-\phi'(v_1)}u^{\frac{p}{2}}|\nabla v_1|\right)^2\\ &-2\sqrt{(p-1)(1-\epsilon)\phi(v_1)\left(\frac{1}{p}\phi''(v_1)-\phi'(v_1)\right)}u^{\frac{p-2}{2}+\frac{p}{2}}|\nabla u||\nabla v_1|\Bigg]\\ &-\int_\Omega 2\sqrt{(p-1)(1-\epsilon)\phi(v_1)\left(\frac{1}{p}\phi''(v_1)-\phi'(v_1)\right)}u^{\frac{p-2}{2}+\frac{p}{2}}|\nabla u||\nabla v_1|\\ &+\int_\Omega\Big|(p-1)\phi(v_1)-2\phi^{'}(v_1)\Big|u^{p-1}|\nabla u||\nabla v_1|+c_6, \quad \text{for all } t\in(0,T_{\max}).\end{aligned}$$

Collecting the terms in the above inequality, one gets

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}&\int_\Omega u^p\phi(v_1)+\int_\Omega u^p\phi(v_1)\leq\\ &-\int_\Omega\left[\sqrt{(p-1)(1-\epsilon)\phi(v_1)}u^{\frac{p-2}{2}}|\nabla u|-\sqrt{\frac{1}{p}\phi''(v_1)-\phi'(v_1)}u^{\frac{p}{2}}|\nabla v_1|\right]^2\\ &+\int_\Omega\left(\Big|(p-1)\phi(v_1)-2\phi^{'}(v_1)\Big|-2\sqrt{(p-1)(1-\epsilon)\phi(v_1)\left(\frac{1}{p}\phi''(v_1)-\phi'(v_1)\right)}\right)\\ &\times u^{p-1}|\nabla u||\nabla v_1|+c_6\\ &\leq -\int_\Omega\left[\sqrt{(p-1)(1-\epsilon)\phi(v_1)}u^{\frac{p-2}{2}}|\nabla u|-\sqrt{\frac{1}{p}\phi''(v_1)-\phi'(v_1)}u^{\frac{p}{2}}|\nabla v_1|\right]^2\\ &+\int_\Omega\varphi u^{p-1}|\nabla u||\nabla v_1|+c_6, \quad \text{for every } t\in(0,T_{\max}),\end{aligned}$$

where $\varphi = \Big|(p-1)\phi(v_1)-2\phi^{'}(v_1)\Big|-2\sqrt{(p-1)(1-\epsilon)\phi(v_1)\left(\frac{1}{p}\phi''(v_1)-\phi'(v_1)\right)} = 0$ from (6.3.13), for all

$0 \le v_1 \le \chi M_3$. As a consequence, we have that for some $c_7 > 0$

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\phi(v_1) + \int_\Omega u^p\phi(v_1) \le c_7 \quad \text{for all}\, t \in (0, T_{\max}),$$

and Lemma 3.13 allows us to conclude that

$$\int_\Omega u^p \le c_8, \qquad\qquad \text{for all } t \in (0, T_{\max}).$$

This completes the proof. □

The case $\tau = 0$

In this part we will study problem (6.3.1) in the case where the second and third equations are elliptic, i.e. $\tau = 0$. For simplicity we will explicitly make mention to the following model

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v), & \text{in } \Omega\times(0,T_{\max}),\\ 0 = \Delta v + \alpha w - \beta v - \gamma uv, & \text{in } \Omega\times(0,T_{\max}),\\ 0 = \Delta w - \delta uw + \mu w(1-w), & \text{in } \Omega\times(0,T_{\max}),\\ u_\nu = v_\nu = w_\nu = 0, & \text{on } \partial\Omega\times(0,T_{\max}),\\ u(x,0) = u_0(x), & x \in \overline{\Omega}. \end{cases} \tag{6.3.25}$$

In the specific let us establish this result.

Lemma 6.32. *For every $p > 1$ there exists $C_2 > 0$ such that the local solution (u, v, w) of (6.3.25) satisfies*

$$\left\|u(\cdot,t)\right\|_{L^p(\Omega)} \le C_2, \qquad \textit{for all } t \in (0, T_{\max}).$$

Proof. The first step is the differentiation of the functional $\Phi(t) = \int_\Omega u^p$ for which we apply the divergence theorem and obtain

$$\begin{aligned} \Phi'(t) =& p\int_\Omega u^{p-1}\Delta u - p\chi\int_\Omega u^{p-1}\nabla\cdot(u\nabla v)\\ =& -p(p-1)\int_\Omega u^{p-2}\left|\nabla u\right|^2 + p(p-1)\chi\int_\Omega u^{p-1}\nabla u\cdot\nabla v\\ =& -\frac{4(p-1)}{p}\int_\Omega\left|\nabla u^{\frac{p}{2}}\right|^2 - (p-1)\chi\int_\Omega u^p\Delta v \quad \text{on } (0, T_{\max}). \end{aligned}$$

We now make use of the second equation in conjunction with the fact that u, v and w are positive and w is bounded as in (6.3.8), so to write

$$\begin{aligned} \Phi'(t) =& -\frac{4(p-1)}{p}\int_\Omega\left|\nabla u^{\frac{p}{2}}\right|^2 - (p-1)\chi\int_\Omega u^p\left(-\alpha w + \beta v + \gamma uv\right)\\ \le& -\frac{4(p-1)}{p}\int_\Omega\left|\nabla u^{\frac{p}{2}}\right|^2 + \alpha(p-1)\chi M_2\int_\Omega u^p \quad \text{on } (0, T_{\max}). \end{aligned} \tag{6.3.26}$$

Exactly as in (6.3.20), for any $c_1 > 0$ there is $c_2 > 0$ such that

$$c_1\int_\Omega u^p \le \frac{2(p-1)}{p}\int_\Omega|\nabla u^{\frac{p}{2}}|^2 + c_2 \quad \text{on } (0, T_{\max}). \tag{6.3.27}$$

In this way (6.3.26) and (6.3.27) lead to this initial problem:

$$\begin{cases} \Phi'(t) \le 2c_2 - c_1\Phi(t) & \text{on } (0, T_{\max}),\\ \Phi(0) = \displaystyle\int_\Omega u_0^p. \end{cases}$$

Naturally, the above problem is solvable and, again, for Lemma 3.13 we have $\Phi(t) \le \max\left\{\int_\Omega u_0^p, \frac{2c_2}{c_1}\right\}$. □

Proof of Theorem 6.24.

For the case $\tau = 0$, we directly consider Lemmas 6.32 and 6.28. For $\tau = 1$, with the aim of exploiting Lemma 6.31 (and successively again Lemma 6.28), it is sufficient to verify (6.3.10) with $K := \chi M_3 < \sqrt{\frac{2}{n}}\pi$. We can choose

$$\epsilon = \frac{\pi^2 - \frac{n}{2}K^2}{2\left(\pi^2 + (\frac{n}{2})^2 K^2\right)} \in (0,1)$$

such that

$$K = \frac{2}{\sqrt{\frac{n}{2}}}\sqrt{\frac{1-2\epsilon}{1+2\epsilon\frac{n}{2}}}\frac{\pi}{2} < \frac{2}{\sqrt{\frac{n}{2}}}\sqrt{\frac{1-\epsilon}{1+\epsilon\frac{n}{2}}}\frac{\pi}{2},$$

for some $p > \frac{n}{2}$. Then we have

$$K < \frac{2}{\sqrt{p}}\sqrt{\frac{1-\epsilon}{1+p\epsilon}}\frac{\pi}{2} < \frac{2}{\sqrt{p}}\sqrt{\frac{1-\epsilon}{1+p\epsilon}}\left(\frac{\pi}{2} + \arctan\sqrt{\frac{p}{1+(p-1)\epsilon - p\epsilon^2}\epsilon}\right). \quad \square$$

6.3.6 Lower bounds for the maximal existence time of solutions in $\mathbb{R}^3$

This section provides the lower bounds for the maximal existence time of solutions in $\mathbb{R}^3$. In this direction, we have to rely on what follows.

Lemma 6.33. *For $n = 3$ and $\tau \in \{0,1\}$, let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain and let (u, v, w) be a solution to model* (6.3.1)*. Then, there exist computable constants $\mathcal{A}_\tau, \mathcal{B}_\tau, \mathcal{C}_\tau$ such that*

$$\Psi_\tau'(t) \le \mathcal{A}_\tau \Psi_\tau(t)^3 + \mathcal{B}_\tau \Psi_\tau(t)^{\frac{3}{2}} + \mathcal{C}_\tau \Psi_\tau(t)^\tau \quad \text{on } (0, T_{\max}), \tag{6.3.28}$$

where Ψ_τ is defined in (6.3.3).

Proof. We will distinguish the fully parabolic case, i.e., $\tau = 1$, and the parabolic-elliptic-elliptic case, i.e., $\tau = 0$.

The case $\tau = 1$

We have to deal with the following energy function

$$\Psi_1(t) := \int_\Omega u^2 + \int_\Omega |\nabla v|^4 + \int_\Omega |\nabla w|^2 \qquad \text{for all } t \in (0, T_{\max}). \tag{6.3.29}$$

Let us denote the integrals in (6.3.29) by $I_1(t)$, $I_2(t)$, and $I_3(t)$ respectively. Let us then compute their derivatives one at a time.

For the first integral $I_1(t)$ we have thanks to the Young inequality

$$\begin{aligned}\frac{dI_1(t)}{dt} &= 2\int_\Omega u\left[\Delta u - \chi\nabla\cdot(u\nabla v)\right] = -2\int_\Omega |\nabla u|^2 + 2\chi\int_\Omega \nabla u \cdot u\nabla v \\ &\le -\frac{3}{2}\int_\Omega |\nabla u|^2 + 2\chi^2\int_\Omega u^2|\nabla v|^2 \quad \text{on } (0, T_{\max}).\end{aligned} \tag{6.3.30}$$

The Young inequality applied again to $u^2|\nabla v|^2$ and successively the inequality in (3.2.4), with $\varepsilon = \frac{1}{4A_3\chi^2}$ for $\int_\Omega u^3$ and $\varepsilon = \frac{27}{16A_3\chi^2}$ for $\int_\Omega |\nabla v|^6$, provide for all $t \in (0, T_{\max})$

$$\begin{aligned}\frac{dI_1(t)}{dt} &\le -\frac{3}{2}\int_\Omega |\nabla u|^2 + 2\chi^2\int_\Omega u^3 + \frac{8\chi^2}{27}\int_\Omega |\nabla v|^6 \\ &\le -\int_\Omega |\nabla u|^2 + 2A_1\chi^2\left(\int_\Omega u^2\right)^{\frac{3}{2}} + 2^7 A_2 A_3^3 \chi^8\left(\int_\Omega u^2\right)^3 \\ &\quad + \frac{8A_1\chi^2}{27}\left(\int_\Omega |\nabla v|^4\right)^{\frac{3}{2}} + \frac{2^{15}A_2A_3^3\chi^8}{3^{12}}\left(\int_\Omega |\nabla v|^4\right)^3 + 2\int_\Omega |\nabla v|^2|D^2 v|^2.\end{aligned} \tag{6.3.31}$$

In the last step we invoked as well the inequality $|\nabla|\nabla v|^2|^2 \leq 4|\nabla v|^2|D^2 v|^2$, applied to the integral term $\frac{1}{2}\int_\Omega |\nabla|\nabla v|^2|^2$.

With the second term $I_2(t)$ we obtain, once again by relying on the divergence theorem and moreover by virtue of the identity $\Delta(|\nabla v|^2) = 2\nabla v \cdot \nabla \Delta v + 2|D^2 v|^2$, for $t \in (0, T_{\max})$:

$$
\begin{aligned}
\frac{dI_2(t)}{dt} &= 4\int_\Omega |\nabla v|^2 \nabla v \cdot [\nabla \Delta v + \alpha \nabla w - \beta \nabla v - \gamma v \nabla u - \gamma u \nabla v] \\
&= 4\int_\Omega |\nabla v|^2 \nabla v \cdot \nabla \Delta v + 4\alpha \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla w - 4\beta \int_\Omega |\nabla v|^4 \\
&\quad - 4\gamma \int_\Omega |\nabla v|^2 v \nabla v \cdot \nabla u - 4\gamma \int_\Omega |\nabla v|^4 u \\
&= 2\int_\Omega |\nabla v|^2 \Delta(|\nabla v|^2) - 4\int_\Omega |\nabla v|^2 |D^2 v|^2 + 4\alpha \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla w - 4\beta \int_\Omega |\nabla v|^4 \\
&\quad - 4\gamma \int_\Omega |\nabla v|^2 v \nabla v \cdot \nabla u - 4\gamma \int_\Omega |\nabla v|^4 u \\
&= 2\int_{\partial\Omega} |\nabla v|^2 \nabla(|\nabla v|^2) \cdot \nu - 2\int_\Omega |\nabla|\nabla v|^2|^2 - 4\int_\Omega |\nabla v|^2 |D^2 v|^2 \\
&\quad + 4\alpha \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla w - 4\beta \int_\Omega |\nabla v|^4 - 4\gamma \int_\Omega |\nabla v|^2 v \nabla v \cdot \nabla u - 4\gamma \int_\Omega |\nabla v|^4 u.
\end{aligned}
$$

In turn, thanks to the convexity of the domain, the inequality $\nabla(|\nabla v|^2) \cdot \nu \leq 0$ on $\partial\Omega$ proved in [TW12a], after ignoring nonpositive terms, leads to

$$
\begin{aligned}
\frac{dI_2(t)}{dt} &\leq -2\int_\Omega |\nabla|\nabla v|^2|^2 - 4\int_\Omega |\nabla v|^2 |D^2 v|^2 + 4\alpha \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla w \\
&\quad - 4\gamma \int_\Omega |\nabla v|^2 v \nabla v \cdot \nabla u \\
&\leq -2\int_\Omega |\nabla|\nabla v|^2|^2 - 4\int_\Omega |\nabla v|^2 |D^2 v|^2 + 4\left(\alpha + 4(\gamma M_3)^2\right) \int_\Omega |\nabla v|^6 \\
&\quad + \alpha \int_\Omega |\nabla w|^2 + \frac{1}{4}\int_\Omega |\nabla u|^2, \quad \text{with } t \in (0, T_{\max})
\end{aligned}
$$

having exploited the Young inequality, as well as $\|v\|_{L^\infty(\Omega)} \leq M_3$ (recall (6.3.9)).

The inequality in (3.2.4), with the choice $\varepsilon = \frac{5}{8A_3(\alpha + 4(\gamma M_3)^2)}$, conduces to

$$
\begin{aligned}
\frac{dI_2(t)}{dt} &\leq -4\int_\Omega |\nabla v|^2 |D^2 v|^2 + \alpha \int_\Omega |\nabla w|^2 + \frac{1}{4}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega |\nabla|\nabla v|^2|^2 \\
&\quad + 4\left(\alpha + 4(\gamma M_3)^2\right) A_1 \left(\int_\Omega |\nabla v|^4\right)^{\frac{3}{2}} + \frac{2^{11} A_2 A_3^3 \left(\alpha + 4(\gamma M_3)^2\right)^4}{5^3} \left(\int_\Omega |\nabla v|^4\right)^3 \\
&\leq -2\int_\Omega |\nabla v|^2 |D^2 v|^2 + \alpha \int_\Omega |\nabla w|^2 + \frac{1}{4}\int_\Omega |\nabla u|^2 \\
&\quad + 4\left(\alpha + 4(\gamma M_3)^2\right) A_1 \left(\int_\Omega |\nabla v|^4\right)^{\frac{3}{2}} + \frac{2^{11} A_2 A_3^3 \left(\alpha + 4(\gamma M_3)^2\right)^4}{5^3} \left(\int_\Omega |\nabla v|^4\right)^3
\end{aligned}
\tag{6.3.32}
$$

on $(0, T_{\max})$.

With analogous steps, last integral $I_3(t)$ leads, for $t \in (0, T_{\max})$, to

$$\begin{aligned}
\frac{dI_3(t)}{dt} &= 2\int_\Omega \nabla w \cdot [\nabla\Delta w - \delta w \nabla u - \delta u \nabla w + \mu \nabla w - 2\mu w \nabla w] \\
&= 2\int_\Omega \nabla w \cdot \nabla \Delta w - 2\delta \int_\Omega w\nabla w \cdot \nabla u - 2\delta \int_\Omega u|\nabla w|^2 + 2\mu \int_\Omega |\nabla w|^2 \\
&\quad - 4\mu \int_\Omega w|\nabla w|^2 \\
&= \int_\Omega \Delta(|\nabla w|^2) - 2\int_\Omega |D^2 w|^2 - 2\delta \int_\Omega w\nabla w \cdot \nabla u - 2\delta \int_\Omega u|\nabla w|^2 \\
&\quad + 2\mu \int_\Omega |\nabla w|^2 - 4\mu \int_\Omega w|\nabla w|^2 \\
&\le -2\delta \int_\Omega w \nabla w \cdot \nabla u + 2\mu \int_\Omega |\nabla w|^2 \le 2\delta M_2 \int_\Omega |\nabla w||\nabla u| + 2\mu \int_\Omega |\nabla w|^2 \\
&\le 2\left(2(\delta M_2)^2 + \mu\right) \int_\Omega |\nabla w|^2 + \frac{1}{4}\int_\Omega |\nabla u|^2,
\end{aligned} \tag{6.3.33}$$

recalling that $\|w\|_{L^\infty(\Omega)} \le M_2$ and after having used again the Young inequality.

By combining (6.3.31), (6.3.32), and (6.3.33), we can conclude for $t \in (0, T_{\max})$

$$\begin{aligned}
\Psi_1'(t) &\le \left(\alpha + 2\left(2(\delta M_2)^2 + \mu\right)\right) \int_\Omega |\nabla w|^2 + 2A_1\chi^2 \left(\int_\Omega u^2\right)^{\frac{3}{2}} \\
&\quad + 4A_1 \left(\frac{2\chi^2}{27} + \left(\alpha + 4(\gamma M_3)^2\right)\right) \left(\int_\Omega |\nabla v|^4\right)^{\frac{3}{2}} + 2^7 A_2 A_3^3 \chi^8 \left(\int_\Omega u^2\right)^3 \\
&\quad + 2^{11} A_2 A_3^3 \left(\frac{2^4\chi^8}{3^{12}} + \frac{\left(\alpha + 4(\gamma M_3)^2\right)^4}{5^3}\right) \left(\int_\Omega |\nabla v|^4\right)^3 \\
&\le \mathcal{A}_1 \Psi_1(t)^3 + \mathcal{B}_1 \Psi_1(t)^{\frac{3}{2}} + \mathcal{C}_1 \Psi_1(t),
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{A}_1 &:= 2^7 A_2 A_3^3 \chi^8 \max\left\{1\,,\ \frac{2^8}{3^{12}} + \frac{2^4\left(\alpha + 4(\gamma M_3)^2\right)^4}{5^3\chi^8}\right\}, \\
\mathcal{B}_1 &:= 2A_1\chi^2 \max\left\{1\,,\ \frac{4}{27} + \frac{2\left(\alpha + 4(\gamma M_3)^2\right)}{\chi^2}\right\}, \\
\mathcal{C}_1 &:= \alpha + 4(\delta M_2)^2 + 2\mu.
\end{aligned}$$

The case $\tau = 0$

Naturally, this simplification makes that of the three expressions I_1, I_2 and I_3 in (6.3.29) only the first one takes part in the analysis, and in particular the energy function is reduced into

$$\Psi_0(t) := \int_\Omega u^2 \quad \text{for all } t \in (0, T_{\max}).$$

Subsequently, from (6.3.30) we can write, by applying twice the divergence theorem,

$$\frac{dI_1(t)}{dt} = 2\int_\Omega u\,[\Delta u - \chi \nabla \cdot (u\nabla v)] = -2\int_\Omega |\nabla u|^2 - \chi \int_\Omega u^2 \Delta v, \quad \text{on} \quad (0, T_{\max})$$

which thanks to the second equation in system (6.3.1) turns into

$$\begin{aligned}
\Psi_0'(t) &= -2\int_\Omega |\nabla u|^2 + \chi\alpha\int_\Omega u^2 w - \chi\beta\int_\Omega u^2 v - \chi\gamma\int_\Omega u^3 v\\
&\le -2\int_\Omega |\nabla u|^2 + \chi\alpha\int_\Omega u^2 w\\
&\le -2\int_\Omega |\nabla u|^2 + \int_\Omega u^3 + \frac{4}{27\chi\alpha}M_2^3|\Omega| \quad \text{for all } \ t\in(0,T_{\max});
\end{aligned}$$

we observe that in the last step we used the Young inequality and the estimate on w given in (6.3.8).

Since the case $\tau = 1$ has been deeply discussed, we understand that we may omit for this situation some details. In particular, by relying on the inequality (3.2.4), the integral $\int_\Omega u^3$ can be properly estimated in terms of $\int_\Omega |\nabla u|^2$, $\left(\int_\Omega u^2\right)^3$ and $\left(\int_\Omega u^2\right)^{\frac{3}{2}}$. Eventually, we obtain for some $\mathcal{A}_0$, $\mathcal{B}_0$ and $\mathcal{C}_0$ positive this inequality

$$\Psi_0'(t) \le \mathcal{A}_0\Psi_0(t)^3 + \mathcal{B}_0\Psi_0(t)^{\frac{3}{2}} + \mathcal{C}_0 \quad \text{on} \quad (0,T_{\max}).$$ □

Proof of Theorem 6.26.

Integrating (6.3.28) in $(0,T_{\max})$, and taking into account that from Remark 6.29 one has that $\lim_{t\to T_{\max}}\Psi_\tau(t) = +\infty$, we obtain (6.3.5). To derive the explicit lower bound in (6.3.6), in line with Section 5.2, we observe that, since $\lim_{t\to T_{\max}}\Psi_\tau(t) = +\infty$ there exists $t_0\in[0,T_{\max})$ such that $\Psi_\tau(t)\ge\Psi_\tau(0)$ for all $t\in(t_0,T_{\max})$. Subsequently, estimating the terms $\Psi_\tau(t)^{\frac{3}{2}}$ and $\Psi_\tau(t)^\tau$ by means of $\Psi_\tau(t)^3$ as

$$\Psi_\tau(t)^{\frac{3}{2}} \le \Psi_\tau(0)^{-\frac{3}{2}}\Psi_\tau(t)^3 \quad \text{and} \quad \Psi_\tau(t)^\tau \le \Psi_\tau(0)^{\tau-3}\Psi_\tau(t)^3, \ \text{for all } t\in(t_0,T_{\max}),$$

we obtain

$$\begin{aligned}
T_{\max} &\ge \int_{\Psi_\tau(0)}^{+\infty} \frac{d\Psi_\tau(t)}{\mathcal{A}_\tau\Psi_\tau(t)^3 + \mathcal{B}_\tau\Psi_\tau(t)^{\frac{3}{2}} + \mathcal{C}_\tau\Psi_\tau(t)^\tau}\\
&\ge \int_{\Psi_\tau(0)}^{+\infty} \frac{d\Psi_\tau(t)}{\mathcal{D}_\tau\Psi_\tau(t)^3} = \frac{1}{2\mathcal{D}_\tau\Psi_\tau(0)^2},
\end{aligned}$$

where $\mathcal{D}_\tau := \mathcal{A}_\tau + \mathcal{B}_\tau\Psi_\tau(0)^{-\frac{3}{2}} + \mathcal{C}_\tau\Psi_\tau(0)^{\tau-3}$.

Remark 6.34. *We understand, that a similar analysis derived in Theorem 6.26 can be carried out in domains included in $\mathbb{R}^n$ with $n\ge 4$. In this case,* (3.2.4) *cannot be employed and an alternative inequality is required. In this direction, one can invoke some appropriate Sobolev embedding, exactly in the spirit of [AD17].*

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