

Classical dynamics on curved Snyder space

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Abstract

We study the classical dynamics of a particle in nonrelativistic Snyder-de Sitter space. We show that for spherically symmetric systems, parametrizing the solutions in terms of an auxiliary time variable, which is a function only of the physical time and of the energy and angular momentum of the particles, one can reduce the problem to the equivalent one in classical mechanics. We also discuss a relativistic extension of these results, and a generalization to the case in which the algebra is realized in flat space.

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1 Introduction

In recent times, the hypothesis that spacetime has a noncommutative structure is becoming rather popular and has been investigated from several perspectives, leading to a variety of models [1]. Among these proposals, the Snyder model [2] has the remarkable property of leaving the Lorentz invariance unaltered. In its classical limit the noncommutativity emerges in dimensions $N \geq 2$ from the nontrivial Poisson brackets of the position coordinates, that depend on a scale β , usually identified with the inverse of the Planck mass. Because of the presence of this fundamental scale, the Snyder model can be interpreted as an example of doubly special relativity [3]. As such, it can be realized in a curved momentum space [4]. Special relativity is recovered in the limit $\beta \rightarrow 0$. Several investigations on its properties have been carried out in the last years [5, 6, 7, 8, 9], and also a generalization interpolating with κ -Minkowski space has been proposed [10].

More recently, the proposal was advanced to extend the Snyder model to spacetimes of constant curvature, by introducing a new fundamental constant α , whose square is proportional to the inverse of the cosmological constant [11]. This extension was motivated by arguments coming from loop quantum gravity [11]. The model (originally named triply special relativity because it contains the three fundamental scales c , α and β) presents an interesting duality between position and momentum space, both being realized as spaces of constant curvature [12]. For this reason we shall refer to it as Snyder-de Sitter (SdS) model, although in this paper we study its Euclidean version. Its properties have been investigated in a number of papers [12, 13].

In particular, the one-particle nonrelativistic classical and quantum dynamics were studied in [12]. In that paper, the investigation of the classical dynamics was limited to one dimension. Explicit solutions were found for the free particle and the harmonic potential, and the qualitative difference between models with opposite sign of the coupling constants was highlighted.

In the present paper, we extend the discussion of the classical dynamics to generic dimensions. This generalization is especially interesting because the noncommutativity of the coordinates is not effective in one dimension. From our investigation will result however that the radial equation of motion in higher dimensions does not substantially differ from the one-dimensional equation. Moreover, for spherically symmetric potentials it is always possible to find the solutions in terms of those of classical mechanics, using an energy-dependent rescaling of the time parameter. Remarkable differences arise instead in the angular sector. In this case improving terms must be added to the potential if one wants to extend the above property to the angular variables.

We also give a relativistic extension of the dynamics, in the hypothesis that the time components of the position and momentum vectors obey canonical Poisson brackets with the other variables.

The plan of the paper is the following: in sect. 2 we review the classical version of the nonrelativistic SdS model. In sect. 3 we recall the geometry of spaces of constant curvature, introducing projective coordinates. In sect. 4 we present the Hamiltonian formalism for our model and in sect. 5 determine the 2-dimensional free particle motion in polar coordinates. Sect. 6 is dedicated to the 2-dimensional harmonic oscillator, and sect. 7 to the Kepler problem. In sect. 8 we briefly discuss some generalizations to higher dimensions and to

relativistic dynamics. Finally, in sect. 9 we consider the case in which the SdS algebra is realized in a flat background.

2 The model

Let us consider the N -dimensional nonrelativistic SdS model. This is defined on a $2N$ -dimensional phase space with position coordinates x_i and momentum coordinates p_i ($i = 1, \dots, N$), with fundamental Poisson brackets

$$\begin{aligned} \{x_i, x_j\} &= \beta^2 J_{ij}, & \{p_i, p_j\} &= \alpha^2 J_{ij}, \\ \{x_i, p_j\} &= \delta_{ij} + \alpha^2 x_i x_j + \beta^2 p_i p_j + 2\alpha\beta p_i x_j, \end{aligned} \quad (1)$$

where $J_{ij} = x_i p_j - x_j p_i$ are the generators of the group of rotations, and α and β are two coupling constants with dimension of inverse length and inverse mass, respectively¹. Of course, these Poisson brackets obey the Jacobi identity and therefore define a consistent symplectic structure on phase space.

According to these definitions, the J_{ij} obey the usual Poisson brackets of the generators of the N -dimensional rotation group, while x_i and p_i transform as N -vectors under the same group. The rotational invariance is therefore preserved. However, the translation invariance, generated by the p_i , is realized in a nonlinear way on position space [5, 6], as is evident from (1). In the limit $\alpha \rightarrow 0$, the algebra (1) reduces to that of the ordinary nonrelativistic Snyder model, while for $\beta \rightarrow 0$ it is the symmetry algebra of a sphere endowed with projective coordinates.

An alternative possibility is that the sign in front of the coupling constants be reversed, and therefore

$$\begin{aligned} \{x_i, x_j\} &= -\beta^2 J_{ij}, & \{p_i, p_j\} &= -\alpha^2 J_{ij}, \\ \{x_i, p_j\} &= \delta_{ij} - \alpha^2 x_i x_j - \beta^2 p_i p_j - 2\alpha\beta p_i x_j, \end{aligned} \quad (2)$$

This can be seen as an analytical continuation of the previous model for $\alpha \rightarrow i\alpha$ and $\beta \rightarrow i\beta$, and was called anti-Snyder-de Sitter (aSdS) model in [12]. Notice that it is not possible to make the analytic continuation of only one of the coupling constants. Moreover, in this case the bound $(\alpha x_k + \beta p_k)^2 < 1$ must be imposed in phase space [12], in analogy with other models derived from doubly special relativity. The limit $\alpha = 0$ corresponds to the nonrelativistic anti-Snyder model in flat space [9], while $\beta = 0$ is the symmetry algebra of an hyperbolic space.

As is evident from the previous discussion, the two models have rather different physical properties [12]. Here, we discuss in detail the first case, and only give some hints on the second.

3 Geometry

In order to better understand the behavior of the trajectories in space, it is useful to discuss the geometry of the model [7]. In particular, the algebra generated by the J_{ij} and p_i is

¹We adopt units in which $c = G = 1$.

$SO(N+1)$ for $\alpha^2 > 0$ or $SO(N,1)$ for $\alpha^2 < 0$ and describes the spatial symmetries of an N -sphere or pseudosphere, respectively.

The metric of the N -sphere can be obtained by embedding it in an $(N+1)$ -dimensional Euclidean space, with coordinates X_α , with $\alpha = 1, \dots, N+1$, on which is imposed the constraint $X_\alpha^2 = 1/\alpha^2$. If one chooses projective coordinates $x_i = X_i/(\alpha X_{N+1})$ to parametrize the sphere, its metric takes the form

$$g_{ij} = \frac{(1 + \alpha^2 x^2)\delta_{ij} - \alpha^2 x_i x_j}{(1 + \alpha^2 x^2)^2}, \quad (3)$$

with inverse

$$g^{ij} = (1 + \alpha^2 x^2)(\delta^{ij} + \alpha^2 x^i x^j). \quad (4)$$

As is well known, a single parametrization cannot be regular on the whole sphere. In particular, in these coordinates the north and south poles $X_{N+1} = 0$ correspond to $x_i \rightarrow \infty$.

It is easy to see that the symmetry group of this metric is $SO(N+1)$, generated by the rotation generators J_{ij} , and the translation generators $p_i = J_{i,N+1}$ satisfying the algebra (1) with $\beta = 0$. The algebra admits the Casimir operator $p^2 + \frac{\alpha^2}{2} J^2$, with $J^2 = J_{ij}J^{ij}$.

When $\alpha^2 < 0$, the relevant manifold is an N -dimensional pseudosphere, that can be constructed as an hyperboloid of equation $X_i^2 - X_{N+1}^2 = -1/\alpha^2$ in an $(N+1)$ -dimensional Minkowski space, with timelike coordinate X_{N+1} . Choosing projective coordinates as before, one obtains the metric

$$g_{ij} = \frac{(1 - \alpha^2 x^2)\delta_{ij} + \alpha^2 x_i x_j}{(1 - \alpha^2 x^2)^2}, \quad (5)$$

which is the analytic continuation of (3) for $\alpha \rightarrow i\alpha$, and whose inverse is

$$g^{ij} = (1 - \alpha^2 x^2)(\delta^{ij} - \alpha^2 x^i x^j). \quad (6)$$

Now one must impose $x^2 \leq 1/\alpha^2$, and the points for which $x^2 = 1/\alpha^2$ correspond to infinity of the hyperboloid.

In this case the symmetry group of the metric is $SO(N,1)$, generated by the rotation generators J_{ij} and the translation generators p_i , that satisfy the algebra (2) with $\beta = 0$. Also this algebra admits a Casimir operator $p^2 - \frac{\alpha^2}{2} J^2$.

It is interesting to observe that an analogous construction can be done for the momentum space [4]. In that case the symmetries are generated by the J_{ij} and the positions x_i , and satisfy the algebra (1) or (2) with $\alpha = 0$. Combining with spatial symmetries, it is easy to see that the phase space is invariant under $SO(N+2)$ for SdS, or $SO(N+1,1)$ for aSdS. Also these symmetries are realized nonlinearly.

4 Dynamics

In [12] the classical dynamics of the nonrelativistic SdS model was investigated in one dimension. The solutions for a free particle and a harmonic oscillator were found. Here we generalize these solutions to N dimensions, and show that for spherically symmetric potentials the problem can be reduced to the study of a radial function, whose equation of motion

can be reduced to the classical one by a suitable coordinate-dependent reparametrization of time.

Due to the nontrivial symplectic structure, it is convenient to write down the dynamics in Hamiltonian form. The first step is the choice of a Hamiltonian. This choice is an additional input to the theory, independent of the symplectic structure, and must be consistent with the identification of x and p as the physical position and momentum. However, this requirement does not fix the form of the Hamiltonian uniquely.

For the free particle, the most trivial choice would be

$$H = \frac{p^2}{2m}. \quad (7)$$

However, in a curved space it is more appropriate to define the Hamiltonian as

$$H = \frac{1}{2m} g^{ij} p_i p_j. \quad (8)$$

This is also equivalent to choosing the Casimir operator of the spacetime symmetry group $SO(N+1)$ as Hamiltonian. In particular, in our case one has

$$H = \frac{1}{2m} \left[(1 + \alpha^2 x^2) p^2 - \alpha^2 (x \cdot p)^2 \right] = \frac{1}{2m} \left(p^2 + \frac{\alpha^2}{2} J^2 \right). \quad (9)$$

In principle one may add to the Hamiltonian further correction terms proportional to β , in order to improve its invariance under the symmetries of phase space. For example, in [6, 7] considerations based on the momentum space invariance suggested the definition of a Hamiltonian $H = p^2/(1 + \beta^2 p^2)$ for the Snyder model in flat space, with straightforward generalization to curved space [7]. Since this choice does not affect the main results of this paper we will use the simpler forms (7) or (9) adopted by most authors [11, 13, 8], which is closer in spirit to Snyder's original idea of deforming only the Heisenberg commutation relations. An analogous choice has also been adopted for more general models of nonrelativistic dynamics on noncommutative space [14].

5 Free particle dynamics in two dimensions

In two dimensions, the free particle dynamics can be investigated adopting polar coordinates. One defines

$$\begin{aligned} \rho &= \sqrt{x_1^2 + x_2^2}, & \theta &= \arctan \frac{x_2}{x_1}, \\ p_\rho &= \frac{x_1 p_1 + x_2 p_2}{\sqrt{x_1^2 + x_2^2}}, & p_\theta &\equiv J_{12} = x_1 p_2 - x_2 p_1. \end{aligned} \quad (10)$$

The nonvanishing Poisson brackets for these coordinates read (notice that the transformations (10) are not symplectic)

$$\{\rho, \theta\} = \beta^2 \frac{p_\theta}{\rho}, \quad \{\theta, p_\rho\} = \beta^2 \frac{p_\rho p_\theta}{\rho^2} + 2\alpha\beta \frac{p_\theta}{\rho},$$

$$\{\rho, p_\rho\} = 1 + \alpha^2 \rho^2 + \beta^2 \left(p_\rho^2 + \frac{p_\theta^2}{\rho^2} \right) + 2\alpha\beta\rho p_\rho, \quad \{\theta, p_\theta\} = 1. \quad (11)$$

Since ρ and p_ρ transform as scalars, their Poisson brackets with p_θ vanish.

In these coordinates, the Hamiltonian (9) reads

$$H = \frac{1}{2m} \left(p_\rho^2 + \frac{p_\theta^2}{\rho^2} + \alpha^2 p_\theta^2 \right). \quad (12)$$

Using (11) one can then write down the Hamilton equations, that take a very simple form, namely

$$\begin{aligned} \dot{\rho} &= \frac{\Delta}{m} p_\rho, & \dot{p}_\rho &= \frac{\Delta}{m} \frac{p_\theta^2}{\rho^3}, \\ \dot{\theta} &= \frac{\Delta}{m} \frac{p_\theta}{\rho^2}, & \dot{p}_\theta &= 0, \end{aligned} \quad (13)$$

with

$$\Delta = 1 + \alpha^2 \rho^2 + \beta^2 \left(p_\rho^2 + \frac{p_\theta^2}{\rho^2} \right) + 2\alpha\beta\rho p_\rho. \quad (14)$$

The form of these equations is exactly the same as in the classical case, except for the factor Δ . One can however get rid of it by defining an auxiliary time variable τ such that

$$d\tau = \Delta dt. \quad (15)$$

In terms of the variable τ the equations are the same as in classical mechanics, and hence have the same solution. In particular, the angular momentum p_θ is conserved.

The solution of the SdS model can therefore simply be obtained by integrating (15) from the classical solution, which can in turn be derived from the conservation of the Hamiltonian. Calling E the value of the Hamiltonian and l that of the angular momentum p_θ , one has

$$2E = \frac{1}{m} \left(p_\rho^2 + \frac{l^2}{\rho^2} + \alpha^2 l^2 \right). \quad (16)$$

Substituting $p_\rho = m d\rho/d\tau$,

$$\frac{d\rho}{d\tau} = \frac{1}{m} \sqrt{2mE' - \frac{l^2}{\rho^2}}, \quad (17)$$

where $E' = E - \alpha^2 l^2/2m$. Integrating, one obtains²

$$\rho^2 = \frac{1}{m} \left(2E'\tau^2 + \frac{l^2}{2E'} \right), \quad p_\rho^2 = \frac{8mE'^3\tau^2}{4E'^2\tau^2 + l^2}. \quad (18)$$

Hence

$$\Delta = 1 + 2\beta^2 mE' + \frac{\alpha^2}{m} \left(2E'\tau^2 + \frac{l^2}{2E'} \right) + 4\alpha\beta E'\tau, \quad (19)$$

²All solutions we discuss admit an arbitrary choice of the origin of time. For simplicity we shall always set to zero this integration constant.

and then

$$t = \int \frac{d\tau}{\Delta} = \frac{1}{\lambda} \arctan \left[\frac{2\alpha E'}{\lambda} \left(\frac{\alpha}{m} \tau + \beta \right) \right], \quad (20)$$

where

$$\lambda = \alpha \sqrt{\frac{2}{m} \left(E' + \frac{\alpha^2 l^2}{2m} \right)} = \alpha \sqrt{\frac{2E}{m}}. \quad (21)$$

Inverting,

$$\tau = \frac{m}{\alpha} \left(\frac{\lambda}{2\alpha E'} \tan \lambda t - \beta \right), \quad (22)$$

and substituting back in (18) one has

$$\rho^2 = \frac{1}{\alpha^2} \left(\sqrt{\frac{E}{E'}} \tan \lambda t - \beta \sqrt{2mE'} \right)^2 + \frac{l^2}{2mE'}. \quad (23)$$

The radial coordinate ρ diverges when the argument of the tangent is $\pm \frac{\pi}{2}$. However, these points are simply the poles of the 2-sphere. In fact, the proper distance along a radial trajectory is given by

$$s = \int ds = \int \frac{d\rho}{1 + \alpha^2 \rho^2} = \frac{\arctan \alpha \rho}{\alpha}, \quad (24)$$

which is always finite for finite t . For $l = 0$, the solution has the same form as in one dimension [12].

The solution for the angle θ can be obtained in a analogous way. One has

$$\theta = \int \frac{l}{\rho^2} d\tau = \arctan \frac{2E'\tau}{l}, \quad (25)$$

with τ given by (22).

A similar calculation can be done in the aSdS case. The results can be more easily obtained by analytic continuation of the previous ones. In particular, the solution (18) is still valid, but now $E' = E + \alpha^2 l^2$ and

$$\tau = \frac{m}{\alpha} \left(\frac{\lambda}{2\alpha E'} \tanh \lambda t - \beta \right), \quad (26)$$

It is easy to check that the bound $(\alpha x_k + \beta p_k)^2 < 1$ is satisfied.

6 Harmonic oscillator in two dimensions

We consider now the simplest example of an interacting particle, given by the harmonic oscillator. Starting from the Hamiltonian

$$H = \frac{1}{2m} \left(p^2 + \frac{\alpha^2}{2} J^2 \right) + \frac{m\omega_0^2}{2} x^2, \quad (27)$$

and going to polar coordinates, one has

$$H = \frac{1}{2m} \left(p_\rho^2 + \frac{p_\theta^2}{\rho^2} + \alpha^2 p_\theta^2 \right) + \frac{m\omega_0^2}{2} \rho^2, \quad (28)$$

and hence

$$\begin{aligned} \dot{\rho} &= \frac{\Delta}{m} p_\rho, & \dot{p}_\rho &= \frac{\Delta}{m} \left(\frac{l^2}{\rho^3} + \omega_0^2 \rho \right), \\ \dot{\theta} &= \frac{\Delta}{m} \frac{p_\theta}{\rho^2} - \beta^2 m \omega_0^2 p_\theta, & \dot{p}_\theta &= 0, \end{aligned} \quad (29)$$

where Δ is given by (14). The form of the radial equations is still the same as in the classical case, except for the factor Δ , while the angular equation has a more general form. In the following we shall show how to modify the Hamiltonian in order to have an analogous equation also for the angular variable. For the moment, we proceed to solve the radial equation by the substitution $dt = \Delta d\tau$.

The classical solution can be obtained from the conservation of the Hamiltonian. With the same notations as in the previous section,

$$2E = \frac{1}{m} \left(p_\rho^2 + \frac{l^2}{\rho^2} + \alpha^2 l^2 \right) + m\omega_0^2 \rho^2, \quad (30)$$

and therefore

$$\frac{d\rho}{d\tau} = \frac{1}{m} \sqrt{2mE' - \frac{l^2}{\rho^2} - m^2\omega_0^2\rho^2}. \quad (31)$$

Integrating, one gets

$$\rho^2 = \frac{1}{m\omega_0^2} \left[E' - \sqrt{E'^2 - \omega_0^2 l^2} \cos 2\omega_0 \tau \right], \quad p_\rho^2 = \frac{E'^2 - \omega_0^2 l^2}{\omega_0^2 \rho^2} \sin^2 2\omega_0 \tau, \quad (32)$$

and hence

$$\begin{aligned} \Delta &= 1 + \left(m\beta^2 + \frac{\alpha^2}{m\omega_0^2} \right) E' + \left(m\beta^2 - \frac{\alpha^2}{m\omega_0^2} \right) \sqrt{E'^2 - \omega_0^2 l^2} \cos 2\omega_0 \tau \\ &\quad + \frac{2\alpha\beta}{\omega_0} \sqrt{E'^2 - \omega_0^2 l^2} \sin 2\omega_0 \tau. \end{aligned} \quad (33)$$

From (15), after some straightforward algebraic manipulations, one obtains

$$\tan \omega_0 \tau = \frac{\kappa \tan \omega t - \frac{2\alpha\beta}{\omega_0} \sqrt{E'^2 - \omega_0^2 l^2}}{1 + \left(\beta^2 m + \frac{\alpha^2}{m\omega_0^2} \right) E' - \left(\beta^2 m - \frac{\alpha^2}{m\omega_0^2} \right) \sqrt{E'^2 - \omega_0^2 l^2}} \quad (34)$$

where

$$\kappa = \sqrt{1 + 2E' \left(\beta^2 m + \frac{\alpha^2}{m\omega_0^2} \right) + \omega_0^2 l^2 \left(\beta^2 m + \frac{\alpha^2}{m\omega_0^2} \right)^2}, \quad \omega = \kappa \omega_0 \quad (35)$$

After substituting in (32) one trivially gets the solution. Since it is very cumbersome, we do not report its explicit form here. When $l = 0$, (32) reproduces the one-dimensional solution

[12]. The solution is periodic in time, with frequency ω that depends on the energy and increases with it.

The equation for θ is rather complicated. However, it can be simplified if one adopts a modified Hamiltonian, as suggested in [12] for the quantum problem. This is obtained by adding a term $\frac{\beta^2}{4}m\omega_0^2J^2$ to the original Hamiltonian, getting

$$H = \frac{1}{2m} \left(p^2 + \frac{\alpha^2}{2} J^2 \right) + \frac{m\omega_0^2}{2} \left(x^2 + \frac{\beta^2}{2} J^2 \right). \quad (36)$$

The new term renders the Hamiltonian invariant under the full $SO(3)$ symmetry of momentum space and therefore enhances its symmetry properties. In particular, the potential has now vanishing Poisson brackets with the x_i . In the quantum case, the addition of this term permits to preserve the degeneracy of the energy with respect to angular momentum [12]. The additional term only affects the Hamilton equation for the angle θ , that now takes the simple form

$$\dot{\theta} = \frac{\Delta}{m} \frac{p_\theta}{\rho^2} \quad (37)$$

The solutions can then be obtained as above introducing the auxiliary variable τ . The solution (32), (34) of the radial equation is unchanged, except for the substitution $E' \rightarrow E'' = E' - \beta^2 m \omega_0^2 l^2 / 2$. The solution of (37) is instead

$$\theta = \arctan \left[\frac{E'' + \sqrt{E''^2 - \omega_0^2 l^2}}{\omega_0 l} \tan \omega_0 \tau \right], \quad (38)$$

with $\tan \omega_0 \tau$ still given by (34).

We also notice that if one writes down the equation for the trajectories ensuing from the Hamiltonian (36), $d\rho/d\theta = \dot{\rho}/\dot{\theta}$, the Δ factors cancel out and one recovers the classical equation. Hence the trajectories are still ellipses, although they are covered with a velocity different from the classical one.

In the aSdS case, the solution can simply be obtained by analytic continuation of (34) and (35) to negative values of α^2 and β^2 . The frequency of oscillation now decreases when the energy increases. It can be shown that the bound $(\alpha x_k + \beta p_k)^2 < 1$ is satisfied.

7 The Kepler problem

Several generalizations of the previous results are possible. Of course, using polar coordinates is especially useful for the study of spherically symmetric potentials. In particular, given the Hamiltonian

$$H = \frac{1}{2m} \left(p_\rho^2 + \frac{p_\theta^2}{\rho^2} + \alpha^2 p_\theta^2 \right) + V(\rho), \quad (39)$$

the relation $\dot{\rho} = \Delta p_\rho / m$ still holds, and one can define as before an auxiliary variable τ in terms of which the system obeys the classical radial equation of motion. Once the classical solution is known, one can obtain $\tau(t)$ and hence the solution of the radial SdS equation, by integrating (15).

The solution of the angular equations instead is in general difficult, unless one improves the symmetry of the potential by the substitution $V(\rho) \rightarrow V(\sqrt{\rho^2 + \beta^2 J^2/2})$. An example is given by the Kepler problem. For a potential $k/\sqrt{\rho^2 + \beta^2 J^2/2}$, the calculations can be done as explained above, and in particular the orbits maintain their classical shape, like for the harmonic oscillator. Only the period of the orbits is changed and can be evaluated performing the integral of $\Delta^{-1}d\tau$ over an orbit.

Of course solutions can be found also if one chooses the traditional potential $V = -k/\rho$, but are much more involved. Let us consider the equation of the orbits in this case. Since the angular momentum is conserved, the problem can be reduced to the motion on a plane. In polar coordinates, the equations of motion are

$$\begin{aligned}\dot{\rho} &= \frac{\Delta}{m} p_\rho, & \dot{p}_\rho &= \frac{\Delta}{m} \left(\frac{p_\theta^2}{\rho^3} - \frac{k}{\rho^2} \right), \\ \dot{\theta} &= \frac{\Delta}{m} \frac{p_\theta}{\rho^2} - \beta^2 \frac{k p_\theta}{\rho^3}, & \dot{p}_\theta &= 0,\end{aligned}\tag{40}$$

where Δ is given as usual by (14). It is apparent that the angular momentum component $p_\theta = l$ is conserved. Moreover, from the conservation of the Hamiltonian follows that

$$\dot{\rho}^2 = \frac{\Delta^2}{m^2} \left(2mE' + \frac{2km}{\rho} - \frac{l^2}{\rho^2} \right),\tag{41}$$

where $E' = E - \alpha^2 l^2/2m$. Since at leading orders the parameter α is not relevant in the following calculations, we set it to 0. Then,

$$\left(\frac{1}{\rho^2} \frac{d\rho}{d\theta} \right)^2 = \frac{2mE + 2km/\rho - l^2/\rho^2}{l^2(1 - \beta^2 km/\Delta\rho)^2}.\tag{42}$$

Defining $u = 1/\rho$ and expanding at first order in β^2 ,

$$\left(\frac{du}{d\theta} \right)^2 \approx \left(\frac{2mE'}{l^2} + \frac{2km}{l^2} u - u^2 \right) (1 + 2\beta^2 km u).\tag{43}$$

Deriving with respect to θ , eq. (43) becomes

$$\frac{d^2 u}{d\theta^2} = \frac{km}{l^2} (1 + 2\beta^2 m E') - \left(1 - 4\beta^2 \frac{k^2 m^2}{l^2} \right) u - 3\beta^2 km u^2,\tag{44}$$

and can be solved perturbatively in β^2 , expanding u as $u = u_0 + \beta^2 u_1 + \dots$. At order 0, the equation admits the well-known solution

$$u_0 = \frac{mk}{l^2} (1 + e_0 \cos \theta),\tag{45}$$

where $e_0 = \sqrt{1 + 2El^2/mk^2}$ is the eccentricity of the orbit. Substituting in (44), one obtains for the first-order correction

$$u_1 = -\frac{m^2 k}{l^2} \left[E + \frac{k^2 m}{2l^2} + \frac{k^2 m e_0}{l^2} \theta \sin \theta - \frac{k^2 m e_0^2}{2l^2} \cos 2\theta \right],\tag{46}$$

and hence

$$u \approx \frac{mk}{l^2} \left[1 - \beta^2 m^2 \left(\frac{E}{m} + \frac{k^2}{2l^2} \right) + e_0 \cos \left(1 + \beta^2 m^2 \frac{k^2}{l^2} \right) \theta + \beta^2 m^2 \frac{k^2 e_0^2}{2l^2} \cos 2\theta \right]. \quad (47)$$

It follows that a perihelion shift $\delta\theta = -2\pi\beta^2 m^2 k^2 / l^2$ arises. A similar result was obtained in [15] using a different approach.

This result may be applied to planetary orbits. In this case, $k = GmM$, with G the Newton constant, M the mass of the sun and m that of the planet. Comparing with the known discrepancy from the prediction of general relativity of the perihelion precession of Mercury, of order 10^{-12} rad/rev [16], an upper limit of order 10^{-52} s/kg can be obtained for β^2 . This should be compared with the value expected by dimensional arguments, $\beta^2 \sim \hbar/c^2 M_{Pl}^2 \sim 10^{-34}$ s/kg, where M_{Pl} is the Planck mass. Thus, if one assumes that Snyder mechanics holds at planetary scales, the parameter β must be nearly nine orders of magnitude less than its natural scale. Equivalently, the length scale of the noncommutativity should be less than 10^{-9} Planck lengths.

It may be interesting to compare these results with the ones obtained in the case of a Lie-algebra noncommutative model [16]. In that case, the noncommutativity can be parametrized by a constant Θ , which, due to the structure of the algebra, has different dimension from β^2 . In fact, it turns out that $\delta\theta = 2\pi\Theta mk/l^3$ and $\Theta \sim 10^{-64} \text{m}^2 \sim 10^7 L_{Pl}^2$ [16]. In this case, the astronomical constraints on the value of the noncommutative parameter are much weaker, but the sign of the correction is opposite to the observed one.

Before concluding that observations do not support Snyder mechanics, it must be observed however that its range of validity is usually assumed to include only systems close to the Planck scale, otherwise paradoxical effect, like the "soccer ball" problem of doubly special relativity [17], may arise.

8 Generalizations

The methods of the previous section can also be generalized to higher dimensions. The equations of motion still coincide with the classical ones if written in terms of the auxiliary variable τ . For example, in cartesian coordinates the free particle Hamiltonian (9) yields the equations

$$\dot{x}_i = \frac{\Delta}{m} p_i, \quad \dot{p}_i = 0, \quad (48)$$

while for the harmonic oscillator with Hamiltonian (36),

$$\dot{x}_i = \frac{\Delta}{m} p_i, \quad \dot{p}_i = -m\omega_0^2 \Delta x_i, \quad (49)$$

Analogously, in polar coordinates the radial equation does not depend on the dimension N , while the angular equations have obvious generalizations, as in classical mechanics.

One can also define a relativistic extension of the model. This is obtained by assuming that the timelike coordinates x_0 and p_0 satisfy canonical Poisson brackets with all the variables. Of course this assumption is in contrast with the postulates of the original Snyder model, in which time has nonvanishing Poisson brackets with the spatial variables, and

whose dynamics will be studied elsewhere [18]. Like in special relativity, the Hamiltonian for a free particle in laboratory coordinates is given by its relativistic energy p_0 , or using the mass shell constraint,

$$H = p_0 = \sqrt{p_i^2 + \frac{\alpha^2}{2} J_{ij}^2 + m^2}. \quad (50)$$

In particular, if one defines an effective mass $m^* = \sqrt{m^2 + \alpha^2 J^2/2}$, adapted to curved spaces (i.e. derived from the Casimir operator of the spacetime symmetry group), the Hamiltonian takes its usual relativistic form. Due to the conservation of energy, H equals a constant \mathcal{E} .

From (50) follow the equations of motion

$$\dot{x}_i = \frac{\Delta p_i}{\mathcal{E}}, \quad \dot{p}_i = 0, \quad (51)$$

where Δ is defined as before. In polar coordinates the equations for the position variables read

$$\dot{\rho} = \frac{\Delta p_\rho}{\mathcal{E}}, \quad \dot{\theta} = \frac{\Delta p_\theta}{\mathcal{E} \rho^2}. \quad (52)$$

It follows that also in the relativistic case the equations can be solved using the techniques developed above.

9 Free particle in flat space

The α -modification of the Poisson brackets might have a different origin from the curvature of spacetime, as in some noncommutative models coupled to magnetic fields [19]. It may therefore be interesting to consider the case in which the brackets (1) are realized in flat space. In this case it is natural to adopt the Hamiltonian (7) instead of (9) for the free particle. Due to (1), this Hamiltonian has nonvanishing Poisson brackets with the momenta p_i , which therefore are not conserved.

To solve this problem it turns out to be convenient to use cartesian coordinates. This also allows one to work in generic dimension N . From the rotational invariance follows that all J_{ij} are conserved. Moreover, conservation of energy implies

$$p^2 = 2mE. \quad (53)$$

The equations for the momenta derived from (7) give

$$\dot{p}_i = \frac{\alpha^2}{2m} J_{ij} p_j. \quad (54)$$

Taking the derivative, one gets after some algebraic manipulations,

$$\ddot{p}_i = -\frac{\alpha^4 J^2}{2m^2} p_i, \quad (55)$$

These equations can be immediately integrated, yielding

$$p_i = \sqrt{2mE} [A_i \sin(\nu t + \phi) - B_i \cos(\nu t + \phi)], \quad (56)$$

where $\nu = \alpha^2 J / \sqrt{2m}$, with $J = \sqrt{J_{ij}^2}$, and the constants A_i and B_i can be determined in terms of the J_{ij} from (54) and (53). It follows that in the absence of the term $\alpha^2 J^2 / 2$ in the Hamiltonian, the momenta rotate with frequency ν .

The equations for the positions read

$$m\dot{x}_i = (1 + 2\beta^2 m E + 2\alpha\beta D) p_i + \alpha^2 D x_i \quad (57)$$

where $D = x \cdot p$ and we have used (53). From (54) and (57) follows that

$$\dot{D} = \frac{\alpha^2}{m} D^2 + 4\alpha\beta E D + 2E(1 + 2\beta^2 m E) + \frac{\alpha^2 J^2}{2m} \quad (58)$$

This equation holds independently of the dimension of space. The solution is

$$D = \frac{m\gamma}{\alpha^2} \tan \gamma t - 2mE \frac{\beta}{\alpha} \quad (59)$$

where $\gamma = \frac{\alpha}{m} \sqrt{2mE + \alpha^2 J^2 / 2}$. The x_i can finally be determined algebraically substituting the solutions (56) and (59) into the definition of the J_{ij} .

For example, in two dimensions one gets

$$\begin{aligned} p_1 &= \sqrt{2m\bar{E}} \sin(\nu t + \phi), & x_1 &= \frac{1}{\alpha} \left(\sqrt{\frac{\bar{E}}{E}} \tan \gamma t - \beta \sqrt{2m\bar{E}} \right) \sin(\nu t + \phi) + \frac{l}{\sqrt{2m\bar{E}}} \cos(\nu t + \phi), \\ p_2 &= \sqrt{2m\bar{E}} \cos(\nu t + \phi), & x_2 &= \frac{1}{\alpha} \left(\sqrt{\frac{\bar{E}}{E}} \tan \gamma t - \beta \sqrt{2m\bar{E}} \right) \cos(\nu t + \phi) - \frac{l}{\sqrt{2m\bar{E}}} \sin(\nu t + \phi), \end{aligned}$$

where $\bar{E} = E + \alpha^2 l^2 / 2m$, $l^2 = J^2 / 2$ and ϕ is an integration constant. (Note that the definition (53) of the energy E is different from that of sec. 5). This is a combination of a motion along a straight line with a rotation around the origin. Some care must be taken however because x_i diverges for a finite value of t .

Of course, one can recover the results of sect. 5 using cartesian coordinates as above. In that case the momenta are conserved, and it is easy to see that the solution is obtained from the previous one taking the limit $\nu \rightarrow 0$.

10 Conclusions

We have calculated exact solutions of the radial equation of motion of the nonrelativistic SdS model in generic dimensions. The results have been obtained explicitly for the free particle and the isotropic harmonic oscillator, but can easily be extended to any spherically symmetric potential.

Moreover, we have considered the possibility that the noncanonical Poisson brackets of the SdS model be realized in flat space, as may happen in the presence of a magnetic field [19]. In this case the momenta are not conserved even for a free particle, but vary in a periodic way.

The main result of our investigation is that the radial motion of a particle in a spherically symmetric potential in SdS space obeys the same equations of classical mechanics when

written in terms of an auxiliary time variable τ , which is a function of the real time and depends on the energy and the angular momentum of the particle. The same result can be extended to the angular motion if one adds to the potential an improving term proportional to the square of the angular momentum, which restores the $SO(N+1)$ invariance of momentum space. In absence of the improving term, it is still possible to obtain approximate solutions. As an example, we have discussed the orbits of particles in a Newtonian potential.

One may interpret these results by saying that time flows faster or slower depending on the value of the energy of the particle, as in some doubly special relativity models [3]. Alternatively, one may invoke a sort of renormalization group flow, according to which the effective coupling constants depend on the energy (cf eq. (35)).

From a formal point of view, it seems plausible that this property follows from the existence of a Darboux transformation between the coordinates p_i, q_i, t and canonical coordinates P_i, Q_i, τ in a parametrized (i.e. with time taken as a canonical variable) version of the model. However, we were not able to find its explicit form.

All the results discussed are simply extended to the case of aSdS space, which is obtained by analytic continuation of SdS. The dynamics is similar, except that in aSdS space the positions and momenta coordinates are constrained to satisfy an upper bound [12]. This is especially evident for the free motion.

To conclude, it must be noted that some arbitrariness is present in the definition of the Hamiltonian for aSdS models in dimension higher than one. The choice depends on the symmetries one wishes to preserve and different Hamiltonians may lead to different physical results (see also [12] for a discussion of the quantum problem). Of course, the correct choice depends on the physical setting one is considering.

The results of this paper may also be extended to more general models, with arbitrary symplectic structure obeying suitable physical assumptions. We plan to discuss this topic in a future paper.

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