Sunspot in Endogenous Growth Two-Sector Models

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Abstract

In this paper we consider a class of endogenous growth two-sector models.The system possesses stochastic characteristics which arise from indeterminate equilibrium and cycles (Hopf cycles, closed to the steady state) (see Chiappori and Guesnerie, 1991, Benhabib, Nishimura, and Shigoka, 2006, Slobodyan 2009). As applications of our analysis, we compare the dynamical behaviour of the following well-known models: the Lucas Model, the Modified Romer Model, the natural resource model (see Bella 2010). For each of them, we show the existence of indeterminacy and sunspot equilibria close to a Hopf cycle. We show that the stochastic approach suggests a way out to the poverty enviroment trap only for the natural disposal resource model.

Keywords: multiple steady states, sunspots, indeterminacy, Hopf Bifurcations,

JEL classification: C61, C62, E32

1 Introduction

The problem of the indeterminacy and sunspot equilibrium in economic financial models has been analysed by many authors in recent times: among them, Nishimura, Shigoka, Yano (2006), and Benhabib, Nishimura, Shigoka (2008), Slobodyan (2009).

Many papers reported the occurrence of the stochastic behaviour (sunspots) also in presence of individual optimization, self-fulfilling expectations and compensations of competitive markets. We remember that a phenomenon is called sunspot when the fundamental characteristics of an economy are deterministic but the economic agents believe nevertheless that equilibrium dynamics is affected by random factors apparently irrelevant to the fundamental characteristics (Nishimura, Shigoka, Yano 2006).

In this paper, we consider the mechanism that leads to the existence of sunspots close to the indeterminate equilibrium (the Hopf orbit) in a class of

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endogenous growth two-sector-models with externality. Nishimura, and Shigoka (2006) consider the reduced form of the Lucas and Romer models, as a continous deterministic three-dimensional non linear differential system, with one pre-determined variable (a combination of the state variables) and two non-predetermined variables (related with the control variables). They constructed a stationary sunspot equilibrium near the stable Hopf cycle that emerges from the unique equilibrium point adding a Wiener variable to the non-predetermined variables, in their formulation, the cycle represents a compact solution of the stochastic process associated with the deterministic model.

Benhabib, Nishimura, Shigoka (2008) prove the existence of a sunspot equilibrium that comes from a Hopf cycle or a homoclinic orbit in a continuous time model of economic growth with positive externalities and with variable capacity utilization; the model has one only predetermined variable (the state variable) and one non-predetermined variable (the control variable). In their model, the positive externality produces the existence of multiple equilibria. Through dynamical analysis they show that the equilibrium is globally indeterminate in the periodic orbit; moreover, there exists a sunspot equilibrium with a support located in the bounded region enclosed by either a homoclinic orbit or a periodic orbit, such that each sample path does not converge to any specific point and continues to fluctuate without decaying asymptotically. As in the previous model, the stochastic formulation comes from adding a white noise to the non-predetermined variable.

Slobodyan (1999) treated the indeterminacy in a deterministic continuous-time model with infinitely lived agents, one predetermined variable (the state variable) and one non-predetermined variable (the control variable); the model is characterized by increasing social returns to scale due to externality in the production function of which the agents are assumed to be unaware. There are two steady states: one has zero capital and zero consumption (the origin), while the other is characterized by positive levels of both capital and consumption. For some parameter values, both steady states are indeterminate, and the whole state space is separated into two regions of attraction of the steady states. The region of attraction in the origin can be regarded as a development trap. Also in this case, the indeterminacy allows for the existence of sunspot equilibria related with the non-predetermined variable. Slobodyan (2009) studied the possibility of "rescuing" an economy from a development trap through sunspot-driven self-fulfilling expectations.

In this work, we construct sunspot equilibria in a deterministic general class of endogenous growth two sector models with externalities in the line of Mulligan and Sala-i-Martin (1993), Venturi (2014) and we compare the dynamical situations arising from the different applications: the Lucas model, the Romer model and a resource disposable endogenous growth two-sector model (Bella 2010).

Following Mulligan and Sala-i-Martin (1993), we put the endogeneous growth two-sector model in the reduced form; it means that we consider a three dimensional deterministic continous non linear differential system, whose solution is called a Balanced Growth Path (BGP) if it entails a set of functions of time solving the optimal control problem, that is all variables grow at a constant rate.

Starting from a three-dimensional standard reduced deterministic model that admits stable cycles (with one predetermined variable and two non-predetermined variables), the model can be reformulated adding a stochastic term to the non-predetermined variables (as a white noise) and transforming it into a stochastic model that leads to indeterminacy, multiple steady stades and bifurcations. If for a given endogenous growth model, there exists a continuum of equilibria in a small neighborhood of a BGP that continues to stay in this neighborhood, it is said that equilibrium is locally indeterminate. If there exists a continuum of equilibria outside a small neighborhood of a BGP, it is said that equilibrium is globally indeterminate (i.e. Hopf cycle or homoclinic orbit; see Mattana, Nishimura, Shigoka 2008).

Following Slobodyan 2009, in our formulation, the stochastic approch suggests a way out from the cycle trap only for disposable resource application (the model has multiple equilibrium points); in the other examples (Lucas, Romer) the model has one only equilibrium point and we have no way out when the initial conditions start inside the stable bounded cycle: this is the so-called poverty development trap.

The paper is organized as follows. The second Section analyze the general economic model.and introduces stochastic dynamics related with the existence of sunspots in the economic model. The third Section compares Lucas, Romer and the Resource model (Bella 2010). In the last we show the results and the economic implications of existence of sunspot equilibrium in a natural resource system with externalities.

2 The Economic General Model

We consider the deterministic economic general model (Mulligan, Sala-i-Martin 1993, Venturi 2014) that deals with the maximization of an objective function

$$\underset{c(t),\ u(t)}{Max} \int_0^\infty U(c)e^{-\rho t}dt \tag{2.1}$$

$$\begin{aligned}
subject\ to &\ : \\
\dot{k} &= A((r(t)^{\alpha_h} u(t)^{\alpha_u})(\nu(t)^{\alpha_\nu} k(t)^{\alpha_k})\hat{r}(t)^{\alpha_{\hat{r}}} k(t)^{\alpha_{\hat{k}}} - \tau_k k(t) - c(t) \\
\dot{r} &= B((r(t)^{\beta_r}(1-u(t)^{\beta_u}))((1-\nu(t)^{\beta_\nu} k(t)^{\beta_k})\hat{r}(t)^{\beta_{\hat{r}}} k(t)^{\beta_{\hat{k}}} - \tau_r r(t) \\
k(0) &= k_0 \\
r(0) &= r_0
\end{aligned}$$

where

$$U(c) = \frac{c^{1-\sigma}-1}{1-\sigma} \tag{2.2}$$

is a standard utility function, c is per-capita consumption, ρ is a positive discount factor and σ is the inverse of the intertemporal elasticity of substitution.

The constraints are two equations related with the growth process of the analyzed economic system .

Notation is as follows c is per-capita consumption, k is a physical capital and r could be the human capital (see the Lucas model 1998), the knowledge (the Romer model, 199) or a natural resource (see Bella, 2010). Individuals have a fixed endowment of time, normalized to unity at each point in time, which is allocated to physical and to the other capital sector (respectively: human, knowledge or natural resource), $\alpha_k$and $\alpha_r$being the private share of physical and the other capital sector in the output sector, β_k and β_r being the corresponding shares share in the second sector, u and v are the fraction of aggrega other sector and physical capital used in the final output sector at instant t (and conversely, $(1-u)$ and $(1-v)$ are the fractions used in the second sector), A and B are the level of the technology in each sector, τ is a discount factor, $\alpha_{\hat{k}}$ is a positive externality parameter in the production of physical capital, $\alpha_{\hat{r}}$ is a positive externality parameter in the production of the second sector

The equalities $\alpha_k + \alpha_r = 1$ and $\beta_k + \beta_r = 1$ ensure that there are constant returns to scale at the private level. At the social level, however, there may be increasing, constant or decreasing returns depending on the signs of the externality parameters.

All other parameters $\pi = (\alpha_k, \alpha_{\hat{k}}, \alpha_r, \alpha_{\hat{r}}, \beta_k, \beta_{\hat{k}}, \beta_r, \beta_{\hat{r}}, \sigma, \gamma, \delta, \rho)$ live inside the following set $\Pi \subset (0, 1)\times(0, 1)\times(0, 1)\times(0,1)\times(0,1)\times(0,1)\times(0,1)\times(0, 1)\times \mathrm{R}^4_+$.

The representative agent's problem (1.1)-(1.2) is solved by

defining the current value Hamiltonian.

$$\begin{aligned} H \;=\; & \frac{c^{1-\sigma}-1}{1-\sigma} + \lambda_1 (A((r(t)^{\alpha_r} u(t)^{\alpha_u})(\nu(t)^{\alpha_\nu} k(t)^{\alpha_k})\hat{r}(t)^{\alpha_{\hat{r}}} k(t)^{\alpha_{\hat{k}}} - \tau_k k(t) - c(t)) \\ & + \lambda_2 (B((r(t)^{\beta_r}(1-u(t)^{\beta_u}))((1-\nu(t)^{\beta_\nu} k(t)^{\beta_k})\hat{r}(t)^{\beta_{\hat{r}}} k(t)^{\beta_{\hat{k}}} - \tau_r r(t)) \qquad (2 \end{aligned}$$

where λ_1 and λ_2 are co-state variables which can be interpreted as shadow prices of the accumulation. The solution candidate comes from the first-order necessary conditions (for an interior solution) obtained by means of the Pontryagin Maximum Principle with the usual transversality condition

$$\lim_{t\to\infty} \left[e^{-\rho t} \left(\lambda_1 k + \lambda_2 r \right) \right] = 0 \qquad (2.4)$$

We consider only the competitive equilibrium solution (as well known, it follows from the presence of the externality that the competitive solution differs from the planner's solution[1]).

[1] The planner's solution involves a choice of k, r, c, u, and r_a which maximizes the control optimal model (2.1) and to $r = r_a$ for all t.

In the other hand the path for r coincides with the given path r_a in the competive solution then the system is in equilibrium (see Lucas 1990, Mattana and Venturi, 1999).

The equilibrium solution taking r_a as exogenously determined.

After eliminating $v(t)$ the rest of the first order conditions and accumulation constraints entail four first order non linear differential equations in four variables: two controls (c and u) and two states (k and r). The solution of this autonomous system is called a Balanced Growth Path (BGP) if it entails a set of functions of time solving the optimal control problem (2.1)-(2.4) such that k, r and c grow at a constant rate and u is constant.

With a change of variable. in standard way, (since k, r and c grow at a constant rate and u is a constant in the BGP), we transforme a system of four first orderordinary differential equations in c, u, k and r into a system of three first order ordinary differential equations with two non-predetermined variables (the control variables) and one predetermined (a linear combination of the state variables)

Setting $A = B = 1$ and

$$x_1 = kr^{\frac{\alpha_{\hat{r}}}{(\alpha_{\hat{r}} - 1)}} \quad ; \quad x_2 = u; \quad x_3 = \frac{c}{k} \tag{2.5}$$

we get:

$$\begin{aligned} \dot{x}_1 &= \phi_1(x_1, x_2, x_3) \\ \dot{x}_2 &= \phi_2(x_1, x_2, x_3) \\ \dot{x}_3 &= \phi_3(x_1, x_2, x_3) \end{aligned} \tag{2.6}$$

in vectorial form

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3)^T = \phi_i(x_1, x_2, x_3) \tag{2.7}$$

where the $\phi_i \in R^3$, are countinuos and derivable complicated nonlinear functions,which depend of the parameters $(\alpha_k, \alpha_{\hat{k}}, \alpha_r, \alpha_{\hat{r}}, \beta_k, \beta_{\hat{k}}, \beta_r, \beta_{\hat{r}}, \sigma, \gamma, \delta, \rho)$ of the model, and $\phi_i : UxR^3 \longrightarrow R^3$ with $U \subset R$, an open subset, and $i = 1, 2, 3$.

3 The Emergence of a Hopf Orbit in the general Model.

As well known a stationary (equilibrium) point of system (2.6) is any solution of

$$\begin{aligned} \dot{x}_1 &= \phi_1(x_1, x_2, x_3) = 0 \\ \dot{x}_2 &= \phi_2(x_1, x_2, x_3) = 0 \\ \dot{x}_3 &= \phi_3(x_1, x_2, x_3) = 0 \end{aligned} \tag{3.1}$$

Assuming the existence, at least of one solution, at some point $P^*(x_1^*, x_2^*, x_3^*)$ the local dynamical properties of (2.6) are described in terms of the Jacobian (see P. Mattana, and B. Venturi (1999), U. Neri and B. Venturi (2007) matrix of (2.6), $J(P)$,with $J(P^*) = J^*$ for brevity.

Lemma 1 *Let $\pi \in \hat{\Pi} \subseteq \Pi$ be. In $\hat{\Pi}$ where is at least one value $\sigma = \sigma_c$, (σ is the bifurcation parameter*[2] *) such that the Jacobian matrix J^* has a pair of purely imaginary roots and a real root different from zero.*

Proof. By Routh-Hurvitz's criterion we can state that J^* can have one (real) eigenvalue $\lambda_1 = r$ and two complex conjugate roots $\lambda_{2/3} = p \pm qi$ whose real parts can be either positive or negative. The real part of the two complex conjugate roots is a countinous function (a four order polinomial) $G(\sigma)$:

$$G(\sigma) = -B(J^*)Tr(J^*) + Det(J^*) \tag{3.2}$$

that change sign *in* $\hat{\Pi}$ when the parameter σ is varied. ■

In fact, since the real parts of the complex conjugate roots vary continuously with respect to σ, there must exist at least one value $\sigma{=}\sigma_c$ such that $\mathrm{G}(\sigma) = 0$. When this occurs, by Vieta's theorem, J^* has a simple pair of purely imaginary eigenvalues.

The solutions of characteristic polinomial
$-\lambda^3 + Tr(J^*)\lambda^2 - B(J^*)\lambda + Det(J^*) = 0.$
for $\sigma{=}\sigma_c$ became:
$\lambda_1 = Tr(J^*)$ and $\lambda_{2/3} = \pm\sqrt[2]{B(J)}i$
cvd.

Lemma 2 *If $\pi \in \hat{\Pi}$ the derivative of the real part of the complex conjugate eigenvalues with respect to σ, evaluated at $\sigma{=}\sigma_c$, is always different from zero.*

Proof. We have only to verify that the following derivative $\frac{d}{d\sigma}G(\sigma) \neq 0$ is different from zero in $\hat{\Pi}$. ■

Theorem 1 *The system (2.6) undergoes at hopf bifurcations in $\hat{\Pi}$ for $\sigma{=}\sigma_c$.*
Proof. *It follows directly from lemma 1 and lemma 2 that the assumptions of Hopf Bifurcations Theorem are satisfied.* ■

As well known the study of the stability of the emerging orbits on the center manifold[3], can be performed by calculating the sign of a coefficient q depending on second and third order derivatives of the non-linear part of the system written in normal form.

If $q > 0$ ($q < 0$) then the closed orbits Hopf-bifurcating from the steady state $P_c^*(x_1^*, x_2^*, x_3^*)$ are attracting (super-critical) (repelling (sub-critical)) on the center manifold.

[2] The dynamical characteristic of the Jacobian matrix evalueted in the the steady state point suggest which bifurcation parameter choosen.

[3] It is a manifold associated with the complex conjugate roots with real part zero

4 Stochastic Dynamic

We build a stochastic system that has sunspot equilibria as solutions, the costruction is very similar to that reported in T. Shigoka (1994) and J. Benhabib, K. Nishimura, and T. Shigoka (2006), J. Benhabib, K. Nishimura and Y. Mitra, (2008).

Our system (2.7) includes one predetermined variable x_1 and two non-predetermined variables x_2 and x_3, in vectorial form :

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3)^T = \phi_i \ (x_1, x_2, x_3) \tag{4.1}$$

with $\phi_i : UxR^3 \longrightarrow R^3$, $U \subset R$, $i = 1, 2, 3$.

If the deterministic dynamic given by (2.7) satisfies the hypothesis of the theorem 1, in other words the parameters set of the model belongs to $\overset{\wedge}{\Pi}$, then (2.7) has a period solution Γ [4].

The equilibrium is globally indeterminate in the interior of the bounded region enclosed by Γ.

We remember that a probability space is a triple $(\Omega,\ B_{R^3},\ P_{R^3})$ where: Ω denotes the space of events, B is the set of possible outcomes of a random process; B is a family of subsets of Ω that, from a mathematical point of view, represents a σ-algebra[5].

The σ-algebra can be interpreted as information (on the properties of the events)[6].

We add a"noise" (a Wiener process) in the equations related with the control variables of the optimal choice problem.

Let $s_t(\omega) = (\omega, t)$ be a random variable irrelevant to fundamental characteristic of the optimal economy, it means that doesn't affect preferences , technology and endowment (i.e. sunspot). We assume that a set of sunspot variable $\{s_t(\omega)\}_{t\ \geq 0}$ is generated by a two-state continuous-time Markov process with stationary transition probabilities and that $s_t : \Omega \longrightarrow \{1, 2\}$ for each $t \geq 0$. Let $[\{s_t(\omega)\}_{t \geq 0}, (\Omega,\ B_{R^3},\ P_{R^3})]$ be a continuous time stochastic process [7], where $\omega \in \Omega$, B_Ω is a σ-field in Ω, and P_Ω is a probability measure.The probability space is a complete measure space and the stochastic process is separable.

Let $(R^3_{+++},\ B_{R^3_{+++}},\ P_{R^3_{+++}})$ be a probability space on the open subset R^3_{+++} of R^3 where $B_{R^3_{+++}}$ denotes the Borel σ-field in R^3_{++}. Let (Φ, B, P) be the product probability space of $(R^3_{+++},\ B_{R^3_{+++}},\ P_{R^3_{+++}})$ and $(\Omega,\ B_\Omega,\ P_\Omega)$, that is $(R^3_{+++} \times \Omega,\ B_{R^3_{+++}} \times B_\Omega,\ P_{R^3_{+++}} \times P_\Omega)$. Let $(\Phi,\ B^*,\ P^*)$ be the completion of

[4] The Hopf cycle is an invariant set inside a two dimensional manifold: the center manifold.

[5] A σ - algebra differs from an algebra, for the property that the union of infinite elements of the family must belong to the set.

[6] The smallest σ- algebra that can be built with subsets of real numbers is represented by open intervals that are called Borel sets and indicated with B. So, it can be concluded that a probability space is a triple $(\Omega,\ F,\ Pr)$ where Ω denotes the space of events and F a family of subsets of Ω.

[7] A stochastic process is a family of random variables

(Φ, B, P). Let $(x_0^1,\ x_0^2, x_0^3)$ be the value of our model at the time $t = 0$. We denote a point $(x_0^1,\ x_0^2, x_0^3, \omega)$ in Φ as φ: in other words, $\varphi = (x_0^1,\ x_0^2, x_0^3, \omega)$. Let $B_t = B(x_0^1, x_0^2, x_0^3, s_s, s \leq t)$ the smallest σ-field of φ respect to which $(x_0^1,$ $x_0^2, x_0^3)$ $s_s, s \leq t$ are measurable.

Let $B_t^* = B^*(x_0^1,\ x_0^2, x_0^3, s_s, s \leq t)$ be the σ-field of φ sets which are either B_t sets or which differ from B_t sets by sets of probability zero.

Let E_t the conditional expectation operator relative to B_t^*.

The following equation is a first order condition of some intertemporal optimization problem with market equilibrium conditions incorporated:

$$(\dot{x}_1, E_t(d\dot{x}_2/dt), E_t(d\dot{x}_3/dt) = \phi_i(x_1(\varphi), x_2(\varphi), x_3(\varphi)) \tag{4.2}$$

where $(x_0^1(\varphi), x_0^2(\varphi), x_0^3(\varphi)) = (x_0^1,\ x_0^2, x_0^3)$ and $\frac{dx_{1t}}{dt}$, $\frac{dx_{2t}}{dt}$, $\frac{dx_{3t}}{dt}$ are defined as:

$$\frac{dx_t^i}{dt} = \lim_{h \to 0^+} \frac{(x_{t+h}^i - x_t^i)}{h} (i = 1, 2, 3)$$

if the limit exists.

Definition 2 *Suppose that* $\{(x_{1t}(\varphi), x_{2t}(\varphi), x_{3t}(\varphi))\}_{t\ \geq 0}$ *is a solution of the stochastic differential equation (4.2) with* $(x_{1t}(\varphi), x_{2t}(\varphi), x_{3t}(\varphi)) \in R_{+++}^3$ *. If for any pair* $(t > s \geq 0)$,$(x_{1t}(\varphi), x_{2t}(\varphi), x_{3t}(\varphi))$ *is* B_t^**-measurable but non* B_s^**-measurable it constitutes a sunspot equilibrium.*

Theorem 3 *If the deterministic system (4.1) has a Hopf solution or a homoclinic orbit (a cycle), then a sunspot equilibrium (SE) is a solution of the stochastic process* $\{(x_0^1(\varphi), x_0^2(\varphi), x_0^3(\varphi))\}_{t\ \geq 0}$ *with a compact support.*
Proof. *It follows directly from the definition of sunspot and the construction of the stochastic system.* ■

4.1 The Lucas Model

As an application of the general model we indicated above, we consider now the Lucas model. In the original optimal control model, the state variables are: k, the physical capital and $r = h$, the human capital; the control variables are: u, the non-leisure time and c, the consumption.

The deterministic reduced form of this model is given by:

$$\begin{aligned}
\dot{x}_1 &= Ax_1^{\beta} x_2^{1-\beta} + \tfrac{\delta(1-\beta+\gamma)}{\beta}(1 - x_2)x_1 - x_3 x_1 \\
\dot{x}_2 &= \tfrac{\delta(\beta-\gamma)}{\beta} x_2^2 + \tfrac{\delta(1-\beta+\gamma)}{\beta} x_2 - x_3 x_2 \\
\dot{x}_3 &= -\tfrac{\rho}{\sigma} x_3 + A\tfrac{\beta-\sigma}{\sigma} x_1^{\beta-1} x_2^{1-\beta} x_3 + x_3^2
\end{aligned} \tag{4.3}$$

where:$x_1 = h^{\left(\frac{1-\beta+\gamma}{\beta-1}\right)} k$; $x_2 = u$; $x_3 = \frac{c}{k}$; $\gamma = \alpha_{\hat{h}}$. x_1 is a pre-determined variable (it is a combination of the state variables), while x_3 and x_2 are the non pre-determined variables.

The three dimensional deterministic system (4.3) undergoes a stable Hopf bifurcation in a parameter set (see Mattana Venturi 1999; Nishimura Shigoka Yano 2006). In line with Nishimura Shigoka Yano, we can re-write the model in the form of a stochastic system and we can apply the theorem 3. Then the system has a compact sunspot equilibrium as a solution of the stochastic process.

The model has one only equilibrium point and there is no way out when the initial conditions start inside the stable bounded cycle or very close to the boundary.

4.2 The Modified Romer Model

We consider now the modified Romer model. In the original optimal control model, the state variables are: k, the physical capital and $r = A$,where A is the level of knowledge currently available, the human capital (Romer 1990, Slobodyan 2007); the control variables are: H_Y, is the human capital, the skilled labour employed in the final sector; c, the consumption.

The deterministic reduced form of this model is given by:

$$\begin{aligned}
\dot{x}_1 &= x_1^{\gamma} x_2^{\alpha} - \tfrac{\zeta-\gamma}{1-\gamma}\delta(1-x_2)x_1 - x_3 x_1 \qquad (4.3)\\
\dot{x}_2 &= \tfrac{\gamma(\zeta-\gamma)}{\zeta(1-\alpha)} x_1^{\gamma-1} x_2^{\alpha+1} + \tfrac{\delta(\zeta-\gamma-1)}{1-\alpha} x_2 - \tfrac{\delta}{1-\alpha}(1-\zeta+\gamma+\tfrac{\gamma}{\zeta\alpha}(\zeta-\gamma))x_2^2 - \tfrac{\gamma}{1-\alpha} x_2 x_3\\
\dot{x}_3 &= x_3^2 + (\tfrac{\gamma^2}{\sigma\zeta} - 1)x_1^{\gamma-1} x_2^{\alpha} x_3 - \tfrac{\rho}{\sigma} x_3
\end{aligned}$$

where:$x_1 = A^{\left(\frac{\alpha+\beta+\gamma}{\alpha+\beta}\right)} k$; $x_2 = H_Y$; $x_3 = \frac{c}{k}$; $\gamma = \alpha_{\hat{h}}$.

x_1 is a pre-determined variable (it is a combination of the state variables), while x_3 and x_2 are the non pre-determined variables; the parameters $\alpha+\beta+\gamma = 1$. and $\zeta \geq 1$ is a parameter that captures the degree of complementarity between the inputs (the case $\zeta = 1$ corresponds to non complementarity).

The three dimensional deterministic system (4.3) undergoes a stable Hopf bifurcation in a parameter set (see Mattana Venturi 1999; Nishimura Shigoka Yano 2006). In line with Nishimura Shigoka Yano, we can re-write the model in the form of a stochastic system and we can apply the theorem 3. Then the system has a compact sunspot equilibrium as a solution of the stochastic process.

The model has one only equilibrium point and there is no way out when the initial conditions start inside the stable bounded cycle or very close to the boundary.

4.3 The Natural Resource System

Our natural disposal resource system (5.3) includes two non-predetermined variables $x_1$and x_2 and one, predetermined variable x_3.:

$$\begin{aligned}\dot{x}_1 &= (-\tfrac{\rho}{\sigma})x_1 + (\tfrac{\beta-\sigma}{\sigma})x_1 - x_1^2 \\ \dot{x}_2 &= (\tfrac{\gamma\delta}{\beta})(1-x_2)x_2 + x_1x_2 \\ \dot{x}_3 &= (\tfrac{\gamma\delta}{\beta})((1-x_2)x_3 + (\beta-1)x_3^2\end{aligned} \tag{4.4}$$

where $x_1 = \frac{c}{k}; x_2 = nr; x_3 = \frac{y}{k}$.

We can build a stochastic process considering the following equation:

$$(E_t(d\dot{x}_1/dt), E_t(d\dot{x}_2/dt), \dot{x}_3) = \phi_i(x_1(\varphi), x_2(\varphi), x_3(\varphi)) \tag{4.5}$$

where $(x_0^1(\varphi), x_0^2(\varphi), x_0^3(\varphi)) = (x_0^1, x_0^2, x_0^3)$ and $\frac{dx_{1t}}{dt}, \frac{dx_{2t}}{dt}, \frac{dx_{3t}}{dt}$ are defined as $\frac{dx_t^i}{dt} = \lim\limits_{h\to 0^+} \frac{(x_{t+h}^i - x_t^i)}{h}$ $(i = 1, 2, 3)$ if the limit exists. It can be demonstrated that the theorem 3 applies also to this model

The deterministic equilibrium dynamic (5.3) has a family of periodic orbits Γ_{σ_c} emerging from one steady state , with Γ_{σ_c} in the center manifold (a two-dimensional invariant manifold in R^3_{+++}). For some set of parameters in the model (see Bella 2010), there exists a sunspot equilibrium whose support is located in the bounded region enclosed by the periodic orbit Γ_{σ_c} . Each sample path of the sunspot equilibrium does not converge to any specific point and continues to fluctuate without decaying asymptotically.

For some parameter value, due to pessimistic self-fulfilling expectations, sunspot equilibria exist in some neighbourhood of a steady-state. If the periodic orbit emerging from a steady state is super-critical , there is no way out (Slobodyan, 2007). If the periodic solution is repelling, then there is a possibility of a way out of the orbit (in fact the growth rate of economy μ becames positive for low level of the externality γ) and the optimal path can reach another steady state. Such situation can be understood as a poverty or development trap.

5 Conclusions

In the applicative examples here proposed, we analyzed different aspects.

In Lucas model, there are two main causes of endogeneous growth: the first is the accumulation of human capital: in other words, this endogeneous growth is due to the fact that the factors determining human capital accumulation remain unchanged; the second is the presence of the externality: it is not necessary to have endogenous growth but it works as an incentive to accumulate human capital and not to let it decrease as time passes by .

In his models, Romer (1986) describes capital that has decreasing returns to scale on the microeconomic level but increasing returns on the macroeconomic level, due to spillovers; so it predicts positive sustained per capita growth. On the other hand, Romer (1990) puts knowledge (and the technological change that is its fruit) at the heart of economic growth: it provides the incentive for capital accumulation and accounts for much of the increase in output per hour worked. In this line, authors like Sala-i-Martin (1997) have shown that the investment share is a robust variable in explaining economic growth.

In the model of natural resource, the endogeneous growth and the chance to escape the development trap are strictly bounded to the expectations: positive or negative outlook can crucially determinate the growth of the economic system.

This positive and statistically significant effect of investment on the growth rate of countries suggests that investment not only affects the stock of physical capital but also increases intangible capital (for example, knowledge) in a way such that the social return to investment is larger than the private return.

For real-world economies, it's important to underline that, in case of endogenous growth, the balanced growth rate crucially depends on the marginal product of physical capital, which varies positively with the stock of knowledge capital. Thus, when the level of knowledge capital plays its important role, economies with a large stock of knowledge may compensate for a large stock of physical capital. Consequently, the growth rate will be higher in those countries in which the stock of knowledge capital is relatively large: this fact can explain high growth rates of Germany and Japan after World War II, for example.

More generally, economies may be both globally and locally indeterminate. Global indeterminacy refers to the balanced growth rate that is obtained in the long run and states that the initial value of consumption crucially determines to which BGP the economy converges and, thus, the long-run balanced growth rate.

Moreover, local indeterminacy around the BGP with the lower growth rate, can be observed if the parameter constellation is such that the trace of the Jacobian matrix is smaller than zero, so that both eigenvalues have negative real parts. If in that situation a certain parameter is varied, two purely imaginary eigenvalues may be observed that generate a Hopf bifurcation, which leads to stable limit cycles.

Some other authors have studies both development and poverty traps: it is indicated that poverty traps and indeterminacy in macroeconomic models may be caused by the same set of reasons, like externalities or increasing returns to scale. Among many, Slobodyan (1999) tried to understand, in this framework, how important sunspot-driven fluctuations could be for the economy's escape from the poverty trap: for a chosen level of the noise intensity (approximately 14% SD of the log consumption), the probability of escaping the trap is not negligible, only when the initial condition is very close to the trap boundary. The set of those initial conditions is not very large and is restricted to initial level of consumption, within 85% of the level necessary to put the system right on the boundary between the poverty trap and the region of attraction of the positive steady state.

The probability of escape, as expected, increases as expectations become more optimistic: for very optimistic expectations (i.e., initial consumption very close to the boundary) absolute majority of escapes happens very fast. So the economy that starts with a very low initial capital and very pessimistic expectations of future interest rates and wages gets trapped. It will probably continue the downward spiral (the change from "pessimistic" level of consumption to the "optimistic" one may constitute hundreds and thousands percent of the "pessimistic" level). The escape happens if it chosen a random variable with bounded support, as the sunspot variable. The sunspot variable has a natural interpretation of a change in perceived present discounted wealth.

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6 Appendix

The application 3 (natural resource model)

1. The reduction form

$$x = \frac{c}{k}, q = nr, m = \frac{y}{k}$$

$\frac{\dot{k}}{k} = Ak^{\beta-1}\left(nr\right)^{1-\beta} r_a^{\gamma} - c$

$\frac{\dot{r}}{r} = \delta(1-n)$

$\frac{Y'}{y} = \delta(1-n)$

$\frac{n'}{n} =$

$y{=}Ak^{\beta}\left(nr\right)^{1-\beta} r_a^{\gamma}$

$\frac{d}{dt}\log y = \frac{d}{dt}(\log Ak^{\beta}\left(nr\right)^{1-\beta} r_a^{\gamma}) =$

$=A\beta\frac{k^{\beta-1}(t)}{k^{\beta}(t)}k'(t)+(1-\beta)\frac{n^{1-\beta-1}(t)}{n^{1-\beta}}n'(t)+(1-\beta)\frac{r^{1-\beta-1}(t)}{r^{1-\beta}}r'(t)+\gamma\frac{r_a^{\gamma-1}(t)}{r_a^{\gamma}}r_a'(t) =$

$=A\beta\frac{k'(t)}{k(t)} + (1-\beta)\frac{n'(t)}{n(t)} + (1-\beta)\frac{r'(t)}{r(t)} + \gamma\frac{r_a'(t)}{r_a(t)}$

$\log m = \log\frac{y}{k} = \log y - \log k$

$\frac{d}{dt}(\log m) = \frac{d}{dt}(\log y - \log k) = \frac{y'(t)}{y(t)} - \frac{k'(t)}{k(t)}$

$\frac{d}{dt}(\frac{c(t)}{k(t)}) = \frac{c'(t)k(t)-c(t)k'(t)}{k(t)^2} = \frac{c'(t)k(t)-c(t)k'(t)}{k(t)} = \frac{c'(t)}{k(t)} - \frac{c(t)}{k(t)}\frac{k'(t)}{k(t)}$

$\frac{d}{dt}(nr) = n'(t)r(t) + n(t)r'(t) =$

$$\begin{aligned} \frac{d}{dt}(\frac{c(t)}{k(t)}) &= \frac{c'(t)k(t)-c(t)k'(t)}{k(t)^2} = \frac{c'(t)k(t)-c(t)k'(t)}{k(t)^2} \quad \text{(A.1)} \\ &= \frac{c'(t)}{k(t)} - \frac{c(t)}{k(t)}\frac{k'(t)}{k(t)} = \frac{c(t)}{k(t)}[\frac{c'(t)}{c(t)} - \frac{k'(t)}{k(t)}] \quad (1) \end{aligned}$$

$$\frac{d}{dt}(\frac{y(t)}{k(t)}) = \frac{y'(t)k(t)-y(t)k'(t)}{k(t)^2} = \frac{y'(t)k(t)-y(t)k'(t)}{k(t)} = \frac{y'(t)}{k(t)} - \frac{y(t)}{k(t)}\frac{k'(t)}{k(t)} \quad \text{(A.2)}$$

$G(\sigma) = -B(J^*)Tr(J^*) + Det(J^*)$

$\rho = 0.002$ $\delta = 0.04$

$\sigma^* = 0.66$ $\sigma^{**} = 0.975$

$\beta = 0.66$

$\gamma = 2$

$\sigma^* = \beta$ $\sigma^* = \frac{\beta\gamma - \rho(1-\beta)}{\beta\gamma}$

$\mu = -\frac{\rho}{\sigma - \beta\gamma + \beta^2}$.

The Jacobian matrix associated with the system (4.4) is given by

$$J(P) = \begin{bmatrix} \frac{1}{\sigma}\left(-\rho + 2\sigma x_1 + (\beta-\sigma)x_3\right) & 0 & \frac{1}{\sigma}x_1\left(\beta-\sigma\right) \\ -x_2 & \frac{1}{\beta}\left(\gamma\delta(1-2x_2) + \beta x_1\right) & 0 \\ 0 & \frac{1}{\beta}\gamma\delta x_3 & \frac{1}{\beta}\gamma\delta\left(1-x_2\right) + 2(\beta-1)\,x_3 \end{bmatrix} \quad \text{(A.3)}$$

The invariant of the Jacobian matrix $J(P)$ a evaluated in $P^*(x_1^*, x_2^*.x_3^*)$: $J(P) = J^*$ are:

Let us consider the Jacobian matrix of the system (2.6) in steady states

Trace: $Tr(J) = x_1 - 3\frac{\gamma\delta}{\beta}x_2 + \left(\frac{\beta}{\sigma} + 2\beta - 3\right)x_3 + 2\frac{\gamma\delta}{\beta} - \frac{\rho}{\sigma}$;

Determinant

$Det(J^*) = \left(-\frac{\rho}{\sigma} + \frac{\beta-\sigma}{\sigma}x_3 + 2x_1\right)\cdot\left(\frac{\gamma\delta}{\beta}(1-2x_2) - x_1\right)\cdot\left(2(\beta-1)x_3 + \frac{\gamma\delta}{\beta}(1-x_2)\right) +$
$x_2\left(\frac{\gamma\delta}{\beta}x_3 \cdot \frac{\beta-\sigma}{\sigma}x_1\left(-\frac{\rho}{\sigma} + \frac{\beta-\sigma}{\sigma}x_3 + 2x_1\right)\right)$;

The sum of the principals minors of $J(P)$.

$B(J^*) = \left(\frac{\gamma\delta}{\beta}(1-2x_2) - x_1\right)\cdot\left[2(\beta-1)x_3 + \frac{\gamma\delta}{\beta}(1-x_2)\right] + \left(-\frac{\rho}{\sigma} + \frac{\beta-\sigma}{\sigma}x_3 + 2x_1\right)\left[2(\beta-1)x_3\right.$
$\left(-\frac{\rho}{\sigma} + \frac{\beta-\sigma}{\sigma}x_3 + 2x_1\right)\cdot\left(\frac{\gamma\delta}{\beta}(1-2x_2) - x_1\right)$;

The expression of the characteristic polynomial of J*in term of Jacobian invariants is given by:

$-\lambda^3 + Tr(J)\lambda^2 - B(J)\lambda + Det(J) = 0.$

A stationary (equilibrium) point of the system is any solution of

$$\begin{array}{r} (\frac{\rho}{\sigma})x_1 + (\frac{\alpha-\sigma}{\sigma})x_1 - x_1^2 = 0 \\ (\frac{\gamma\delta}{\sigma})(1-x_2)x_2 + x_1x_2 = 0 \\ -(\frac{\gamma\delta}{\sigma})((1-x_2)x_3 + (\alpha-1)x_3^2 = 0 \end{array} \tag{A.4}$$

We found eight steady states values:

1)$P_1^*(0,0,0)$;

2)$P_2^*(0,0,(\frac{\gamma\delta}{\beta(1-\beta)}))$;

3)$P_3^*(0,1,0)$ (It is a double solution);

4)$P_4^*(\frac{\rho}{\sigma},0,0)$;

5)$P_5^*(\frac{1}{\sigma}[\rho - \frac{\gamma\delta(\beta-\sigma)}{\beta(1-\beta)}],0,(\frac{\gamma\delta}{\beta(1-\beta)}))$;

6)$P_6^*(\frac{\rho}{\sigma},1-\frac{(\beta\rho)}{\sigma\gamma\delta},0)$;

7)$P_7^*(\frac{\rho(1-\beta)}{\beta(1-\sigma)},1-\frac{\rho(1-\beta)}{\gamma\delta(1-\sigma)},\frac{\rho}{\beta(1-\sigma)})$.

It is well-known that many theoretical result relating to the system depend upon the eigenvalues of the Jacobian matrix evaluated at the stationary point P_i^* with $i = 1,2,3,4,5,6,7$ in some values of the parameters.

The local bifurcation analysis permits to determine the structurally unstable solution of the model.

6.1 Hopf Bifurcations

Theorem 4 ***Lemma 3*** *There is a parameter $\sigma = \sigma_c$ such that in the steady state P_7^* the Jacobian Matrix $J(P_7^*)$ possesses two complex conjugated roots with real part equal to zero and the real root different from zero.*

Proof. *We consider the function:*

$$G(\sigma) = -B(J_{P_7^*})Tr(J_{P_7^*}) + Det(J_{P_7^*}) \tag{A.5}$$

We evalute the invariant elements of the Jacobian and we get:
$Tr(J_{P_7^*}) = \frac{\rho(1-\beta)}{\beta(1-\sigma)} - \frac{\gamma\delta}{\beta}$;
$B(J_{P_7^*}) = -\left(\frac{\rho(1-\beta)}{\beta(1-\sigma)}\right)$;
$Det(J_{P_7^*}) = \frac{\rho^2\gamma\delta(1-\beta)}{\beta^2\sigma(1-\sigma)}\left(1 - \frac{\rho(1-\beta)}{\gamma\delta(1-\sigma)}\right)$;
We determine two solutions in which the function $G(\sigma)$ vanishes: $\sigma_c^* = \beta$ and $\sigma_c^{**} = \frac{\gamma\delta-\rho(1-\beta)}{\gamma\delta}$. ■

Lemma 4 *The derivative of the real part of the complex conjugate eigenvalues of $(J_{P_7^*})$ evaluated at $\sigma = \sigma_c^*$ and $\sigma = \sigma_c^{**}$, is always different from zero.*

The proof follows from direct calculation.

Theorem 5 *A family of Hopf bifurcations emerges around the steady state P_7^* of the system (5.3), for $\sigma = \sigma_c^*$ and $\sigma = \sigma_c^{**}$.*
Proof. *It follows directly from the assumptions of the Hopf bifurcation Theorem. Q.E.D.* ■

We consider some numerical simulations: in particular, we consider the following two sets of parameters (Bella, 2010):
a)
$\rho = 0.002$
$\beta = 0.66$
$\gamma = 2$
$\delta = 0.04$
$\sigma_c^* = 0.66$
b)
$\rho = 0.002$
$\beta = 0.66$
$\gamma = 2$
$\delta = 0.04$
$\sigma_c^{**} = 0.975$
We evaluate the growth rate of an economy (appendix 1) and we get $\mu = -\frac{\rho}{\sigma-\beta\gamma+\beta^2}$.
In a first simulation, we fix $\gamma = 2$, $\beta = 0.66$.
We consider μ as a function of σ :

$\mu = f(\sigma) = -\frac{0.002}{\sigma - 2 \cdot 0.66 + 0.66^2}$.

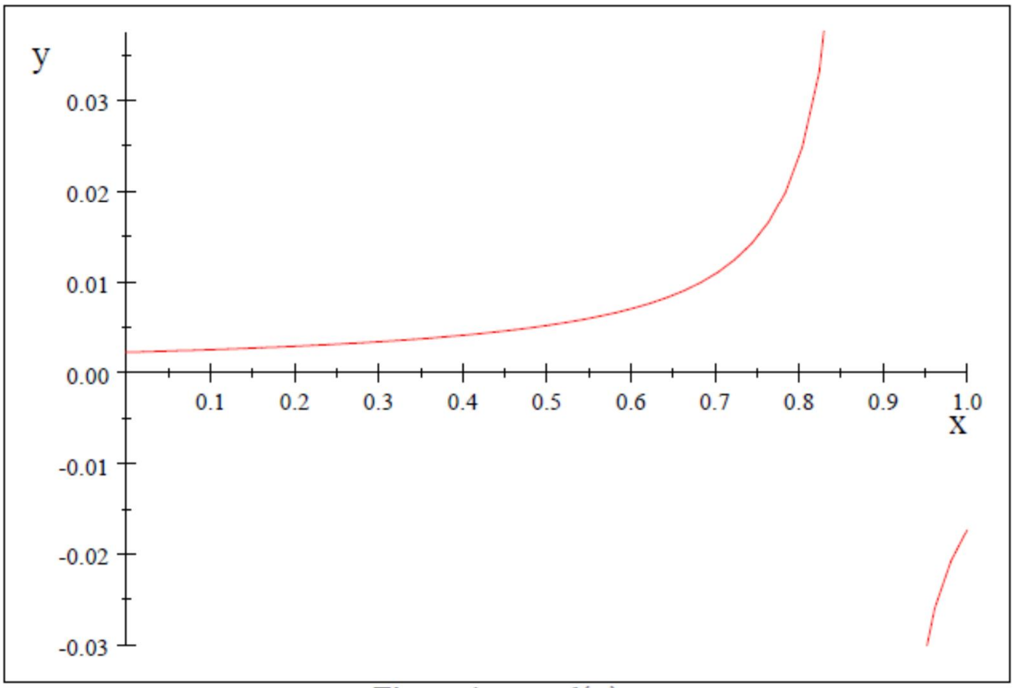

Figure 1. $\mu = f(\sigma)$

We find that the growth rate is positive for $\sigma < 0.8844$.

For the first set of parameters we have:

$\mu_c^* = 0.0089$

For the second set of parameters it is:

$\mu_c^{**} = -0.022$

In another simulation we consider μ^* as a function of β :fixing the value of $\sigma_c^* = 0.66$

$\mu = f(\beta) = -\frac{0.002}{0.66 - 2\beta + \beta^2}$

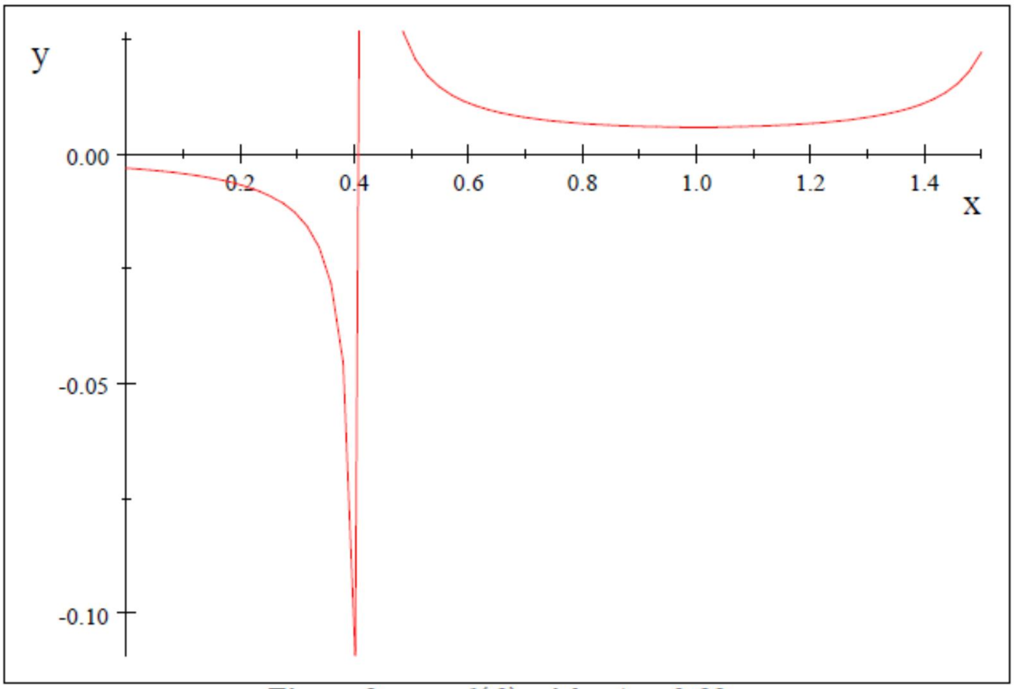

Figure 2. $\mu = f(\beta)$ with $\sigma_c^* = 0.66$

This time, growth rate is positive for $0.41 < \beta < 1.58$.
A third simulation has been conducted fixing the value of $\sigma_c^{**} = 0.975$,

$\mu = f(\beta) = -\frac{0.002}{0.975 - 2\beta + \beta^2}$

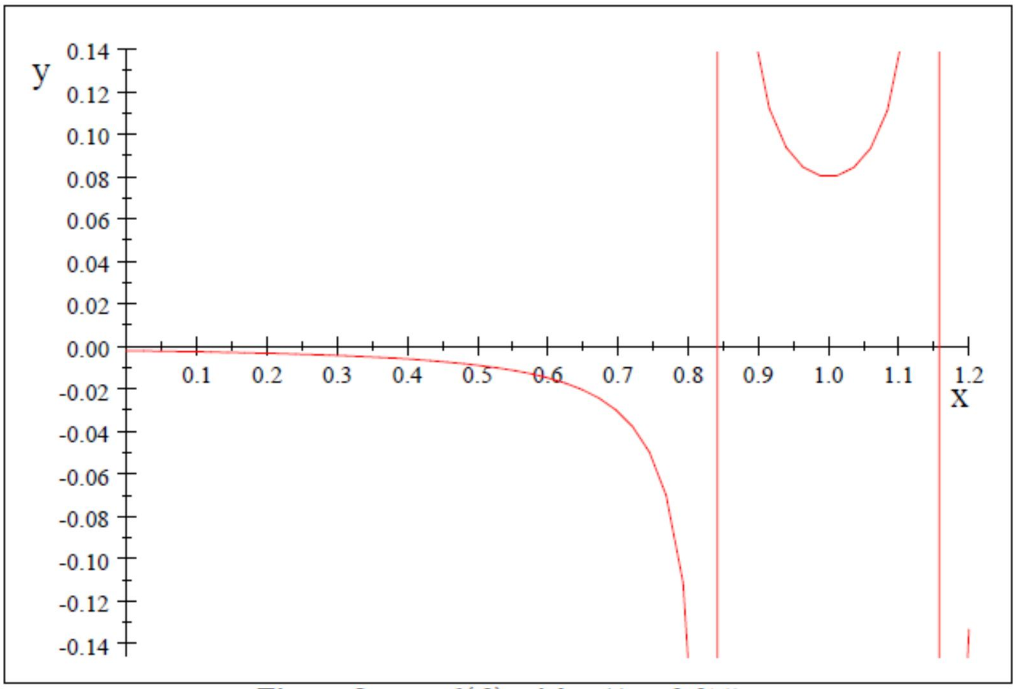

Figure 3. $\mu = f(\beta)$ with $\sigma_c^{**} = 0.975$

The growth rate is positive for $0.84 < \beta < 1.15$.

Another simulation shows the function of the growth rate as function of β and σ :

$\mu = f(\sigma, \beta) = \frac{0.002}{2\beta - \beta^2 + \sigma}$; the graphic is:

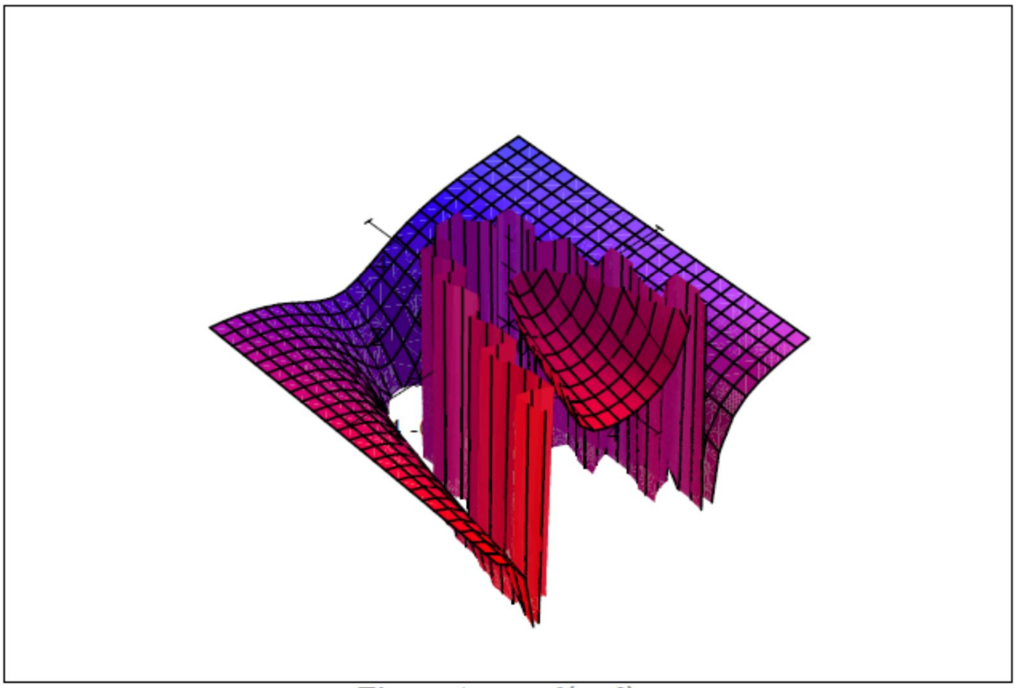

Figure 4. $\mu = f(\sigma, \beta)$

The last simulation shows the function of the growth rate as function of γ and σ :

$\mu = f(\sigma, \gamma) = \frac{0.002}{0.66\gamma - \sigma - 0.4356}$; the graphic is:

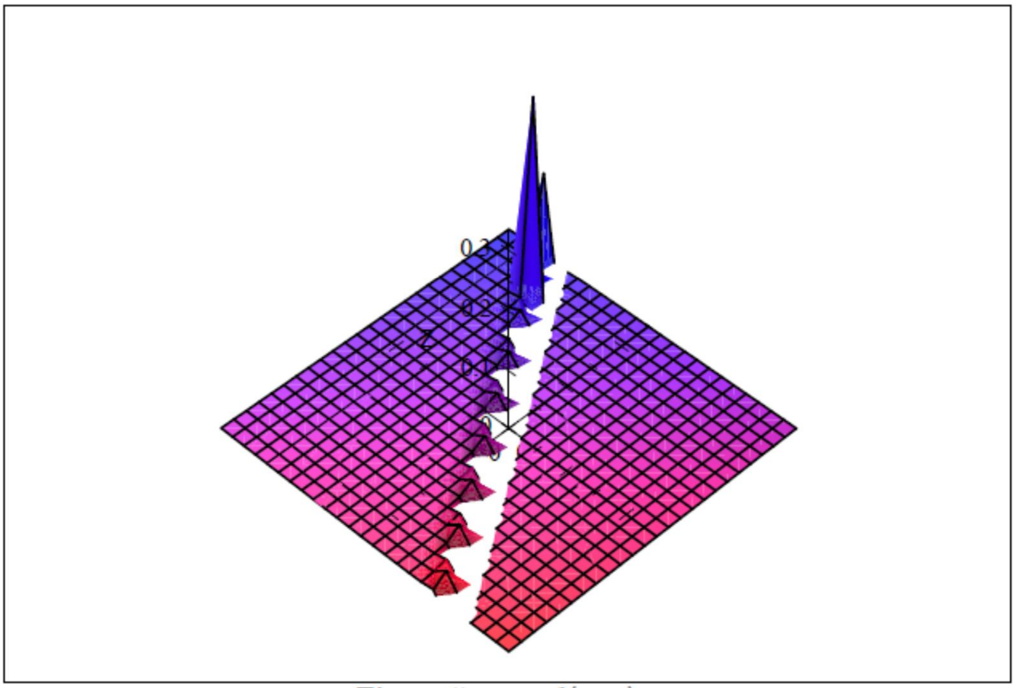

Figure 5. $\mu = f(\sigma, \gamma)$

The calculation of the stability coefficient q (Bella, 2010) gives some results for our parameter sets; for the set $a)$:

$\rho = 0.002$
$\beta = 0.66$
$\gamma = 2$
$\delta = 0.04$
$\sigma^* = 0.66$

we have $q = -2.40 \cdot 10^{12} < 0$; it means that bifurcation is super-critical, the steady state is unstable and the periodic orbits are attracting on the center manifold.

For the set $b)$:

$\rho = 0.002$
$\beta = 0.66$
$\gamma = 2$
$\delta = 0.04$
$\sigma^* = 0.975$

we have $q = 8.37 \cdot 10^{14} > 0$; it means that bifurcation is sub-critical and the periodic orbits start repelling.