Hector Freytes
Giuseppe Sergioli

Heyting κ-Frames

Abstract. In the framework of algebras with infinitary operations, the equational theory of $\bigvee_\kappa$-complete Heyting algebras or Heyting κ-frames is studied. A Hilbert style calculus algebraizable in this class is formulated. Based on the infinitary structure of Heyting κ-frames, an equational type completeness theorem related to the $\langle \bigvee, \wedge, \rightarrow, 0\rangle$-structure of frames is also obtained.

Keywords: $\bigvee_\kappa$-complete Heyting algebras, Frames, Infinitary operations.

Mathematics Subject Classification: 08A65, 06D20, 06D22.

Introduction

The notion of frame is a lattice theoretic approach to the open subsets of a topological space when finite intersections and arbitrary unions of open sets are considered. This idea of regarding frames as generalized topological spaces was studied by several authors [3, 11, 16–18, 24]. Frames define a category, denoted by **Frm**, whose objects are complete lattices satisfying the distributive law $x \wedge \bigvee S = \bigvee\{x \wedge s : s \in S\}$ and whose arrows, called *frame-homomorphisms*, are lattice order homomorphisms preserving arbitrary supremum. Charles Ehresmann called these lattices *local lattices* but it was Clifford Dowker who first introduced the terminology *frame* to refer to these mathematical objects. The lattice ordered structure of a frame A implicitly defines an implication "$\rightarrow$" given by $a \rightarrow b = \bigvee\{x \in A : x \wedge a \leq b\}$ which is the residuum of the meet. In this way, frames are naturally endowed with a complete Heyting algebra structure. Further, frames and complete Heyting algebras determine the same class of lattices. However, a frame-homomorphism is not, in general, a Heyting homomorphism because it does not necessarily preserve the implication.[1]

[1] There are special cases of frame-homomorphisms preserving implication and all meets. They are called *open* since these correspond to open continuous functions between topological spaces.

Presented by **Francesco Paoli**; *Received* January 30, 2023

Studia Logica
https://doi.org/10.1007/s11225-023-10072-3

Published online: 07 October 2023

This work is motivated by the equational theory related to the underlying $\langle \bigvee, \wedge, \rightarrow, 0\rangle$-Heyting structure of frames defining a category $\mathcal{H}_\infty$ whose arrows are $\langle \bigvee, \wedge, \rightarrow, 0\rangle$-preserving maps.

For infinitary structures, the condition of equational definability is usually treated in a categorical way (see for example [17, § II] and [26, § 1]) and, in the particular case of $\mathcal{H}_\infty$, this approach becomes very useful in topology. However, this categorical framework may not be the most suitable from the algebraic logic viewpoint. One of the crucial logical problem with respect to the infinitary algebraic structures is that we deal with proper class of operations giving rise to languages of algebras, or term algebras, that are proper classes too. A structure of this type, involving a proper class of operations, is difficult to approach with the usual universal algebra techniques. Despite this fact, in the particular case of $\mathcal{H}_\infty$, the problem mentioned above can be indirectly dealt from infinity structures whose language of algebras can be described by a set. Indeed: We first note that $\bigvee$ is not an operation. By fixing an infinite cardinal number κ, the supremum of the sets of cardinality κ, denoted by $\bigvee_\kappa$, defines an infinitary operation of arity κ. Then, an equational presentation for $\mathcal{H}_\infty$ can be formulated taking into account the proper class of operations $\langle (\bigvee_\kappa)_{\omega \leq \kappa \in \mathbf{Card}}, \wedge, \rightarrow, 0\rangle$ where **Card** is the class of cardinal numbers [26, p. 69]. If, for each cardinal number $\kappa \geq \omega$, the reduct $\langle \bigvee_\kappa, \wedge, \rightarrow, 0\rangle$ is considered, then it is an algebra with an infinitary operation defined by a proper set of operations giving the possibility to define a language of algebras described by a set. In each frame, the $\langle \bigvee_\kappa, \wedge, \rightarrow, 0\rangle$-reduct is a Heyting algebra admitting κ-joins or, equivalently, a κ-frame [22, 23] in which the Heyting residuum of $\wedge$ is defined. These kinds of κ-frames define a category $\mathcal{H}_\kappa$ whose arrows are $\langle \bigvee_\kappa, \wedge, \rightarrow, 0\rangle$-preserving maps. In order to emphasize the preservation of these operations by homomorphisms, we refer to the objects of $\mathcal{H}_\kappa$ as *Heyting κ-frames*.

Thus, in addition to the interest in itself of the Heyting κ-frames, these algebras will allow to describe the equational theory of $\mathcal{H}_\infty$. In this perspective, the first aim of this paper is to study the equational theory of Heyting κ-frames in the framework of algebras with infinitary operations. Secondly, we will study the basic properties of the class $\mathcal{H}_\kappa$ adapting several techniques of the universal algebra. Among these topics we will focus on congruences, direct indecomposability, a Glivenko type theorem, amalgamation and injective objects. The subclass of subdirect irreducible Heyting κ-frames is also characterized since the equational completeness of $\mathcal{H}_\kappa$ with respect to the mentioned subclass of algebras can be proved in this particular infinitary structure.

The paper is structured as follows. In Section 1 we recall some basic notions about algebras with infinitary operations. In Section 2 Heyting κ-frames are introduced as an equational class of algebras. We devote Section 3 to expose the theory of filters and congruences for Heyting κ-frames and $\mathcal{H}_\infty$-algebras. Conditions for the validity of the congruence extension property (CEP) are also introduced. In Section 4 the direct idecomposability is studied. In Section 5 a Glivenko type theorem for Heyting κ-frames is established. The Gödel negative translation of terms in the language of $\mathcal{H}_\omega$ is specially treated. Moreover a kind of amalgamation property, related to regular elements of Heyting κ-frames, is given. In Section 6 injective objects of $\mathcal{H}_\kappa$ and $\mathcal{H}_\infty$ are characterized. In Section 7 an infinitary Hilbert-style calculus is introduced obtaining a strong completeness theorem for the class $\mathcal{H}_\kappa$. In Section 8 an equational completeness theorem for Heyting κ-frames is established. Based on the equational theory of Heyting κ-frames, in Section 9, an equational type completeness theorem for $\mathcal{H}_\infty$ is formulated.

In Section 10 a relation between Heyting κ-frames and algebraic structures, whose study is motivated by the intuitionistic predicate logic, is established. Finally, in Section 11, some concluding remarks and topics to be studied about the structure of Heyting κ-frames are suggested.

1. Basic Notions

The first development on abstract algebras with infinitary operations has been formulated by Słomiński [31]. In the mentioned article it can be seen that many results on classical universal algebra can be generalized to classes of algebras admitting infinitary operations. In this generalized framework, properties related to the cardinality of the algebraic structures become relevant. We denote by **ON** the class of ordinal numbers. Since a cardinal is an ordinal number α such that no ordinal smaller than α is equipotent to α, then $\mathbf{Card} \subseteq \mathbf{ON}$ as classes of sets. For each set A we use the notation $|A|$ to indicate the cardinal number of A. The following proposition provides two useful results about cardinal arithmetic.

Proposition 1.1. *Let κ, μ be cardinals. Suppose that at least one of them is infinite. Then*

1. $\kappa + \mu = \max\{\kappa, \mu\}$.
2. *If we assume that neither κ nor μ are equal to* 0 *then* $\kappa\mu = \max\{\kappa, \mu\}$.

■

An infinite cardinal κ is said to be *regular* iff for each family of cardinal numbers $(\lambda_i)_{i\in I}$ such that $\lambda_i < \kappa$ and $|I| < \kappa$, then $\sum_{i\in I}\lambda_i < \kappa$. An infinite cardinal that is not regular is called *singular*. The successor cardinal κ^+ of κ is the least cardinal $> \kappa$.

PROPOSITION 1.2. *Let κ be an infinite cardinal. Then:*

1. *κ is a limit ordinal;*
2. *κ^+ is a regular cardinal;*
3. *if $(\alpha_i)_{i\in\kappa}$ is a set of ordinal numbers such that for each $i \in \kappa$, $\alpha_i < \kappa^+$ then $\bigcup_{i\in\kappa}\alpha_i$ is an ordinal number such that $\bigcup_{i\in\kappa}\alpha_i < \kappa^+$.*

PROOF. (1), (2) They are well known properties about cardinals numbers. (3) It is also well known that the union of a nonempty set of ordinals numbers is an ordinal number too. Thus, $\bigcup_{i\in\kappa}\alpha_i$ is an ordinal number. Suppose that $\kappa^+ \leq \bigcup_{i\in\kappa}\alpha_i$. Since $|\alpha_i| < \kappa^+$, $\kappa < \kappa^+$ and κ^+ is a regular cardinal, by item 2, $\kappa^+ \leq |\bigcup_{i\in\kappa}\alpha_i| \leq \sum_{i\in\kappa}|\alpha_i| < \kappa^+$ which is a contradiction. Hence our claim. ■

In what follows we introduce and adapt from [31] some basic notions about algebras with infinitary operations.

Let A be a non-empty set and κ be a cardinal number. If f is a function with domain A^κ and $\overline{a} = (a_i)_{i\in\kappa} \in A^\kappa$ then, $f(\overline{a})$ represents the value $f(a_0, a_1, \ldots a_i, \ldots)$ where $i \in \kappa$. A *type* is a set τ of operation symbols where cardinal numbers represent the arities. If $\varphi \in \tau$ then, $arity(\varphi)$ denotes the arity of φ. We denote by τ_0 the subset of τ given by $\tau_0 = \{\varphi \in \tau : arity(\varphi) = 0\}$ i.e. the set of *constant operation symbols*. Let τ be a type. An algebra of type τ is a pair $\langle A, \tau^A\rangle$ where A is a non-empty set and $\tau^A = \{\varphi^A : \varphi \in \tau\}$ is a set of operations on A such that each element $\varphi^A \in \tau^A$ has the form $\varphi^A : A^{arity(\varphi)} \to A$. An algebra A is *trivial* iff it has one element only.

Let A and B be two algebras of type τ. We say that B is a *subalgebra* of A iff $B \subseteq A$ and for every $\varphi \in \tau$, φ^B is φ^A restricted to B. Let $S \subseteq A$. If there exists a smallest subalgebra of A that contains S then it is called the *subalgebra of A generated by S*. A function $f : A \to B$ is said to be a *τ-homomorphism* iff for each operation symbol $\varphi \in \tau$ with arity κ and for each indexed subset $(a_i)_{i\in\kappa}$ of A, then $f(\varphi^A(a_0, a_1, \ldots)) = \varphi^B(f(a_0), f(a_1), \ldots)$.

An equivalence relation θ on A is a *τ-congruence* iff θ satisfies the following *compatibility property*: for each operation symbol $\varphi \in \tau$ of arity κ and indexed sets $\overline{a} = (a_i)_{i\in\kappa}$, $\overline{b} = (b_i)_{i\in\kappa}$ of elements of A, if $(a_i, b_i) \in \theta$ for each $i \in \kappa$ then $(\varphi(\overline{a}), \varphi(\overline{b})) \in \theta$. It is clear that the *diagonal relation* Δ on A and A^2, denoted by ∇, are τ-congruences. The set of all τ-congruences on A is denoted by $Con_\tau(A)$ and $\langle Con_\tau(A), \subseteq\rangle$ is a complete lattice. A is *simple* iff $Con_\tau(A) = \{\Delta, \nabla\}$. If $\theta \in Con_\tau(A)$ then the *quotient algebra* of A by θ is the algebra whose universe is the quotient set $A/_\theta$ and whose operations satisfy $\varphi^{A/_\theta}(a_0/_\theta, a_1/_\theta, \ldots) = \varphi^A(a_0, a_1, \ldots)/_\theta$, where $\varphi \in \tau$ has arity κ and $(a_i)_{i\in\kappa} \in A^\kappa$. Note that $A/_\theta$ is a τ-algebra and the natural map $p_\theta : A \to A/_\theta$ is a surjective τ-homomorphism. If $f : A \to B$ is a τ-homomorphism then $Ker(f) = \{(a,b) \in A^2 : f(a) = f(b)\}$ is a τ-congruence.

The *direct product* of a set of algebras $(A_i)_{i\in I}$ of type τ, denoted by $\prod_{i\in I} A_i$, is the algebra of type τ obtained by endowing the set-theoretical Cartesian product with the operation of type τ, defined pointwise. For each $j \in I$ the j^{th}-projection π_j is a τ-homomorphism onto A_j. The algebra A is *directly indecomposable* iff A is not τ-isomorphic to a direct product of two non trivial algebras of type τ.

A class $\mathcal{A}$ of algebras of type τ is said to be a *variety* iff it is closed with respect to direct products, subalgebras and homomorphic images.

Let τ be a type of algebras, $\varphi \in \tau$ and $\bar{t} = (t_i)_{i\in arity(\varphi)}$ be a $arity(\varphi)$-tuple of symbols. Then the expresion $\varphi(\bar{t})$ is a syntactic abbreviation of $\varphi(t_o, t_1, \ldots, t_i, \ldots)$.

Let X be a set, called *set of variables*, τ be a type of algebras and we always assume that $X \cap \tau = \emptyset$. We say that a set T is (X,τ)-*closed* iff $X \cup \tau_0 \subseteq T$ and $\varphi(\bar{t}) \in T$ whenever $\varphi \in \tau - \tau_0$ and $\bar{t} \in T^{arity(\varphi)}$. In order to establish the existence of (X,τ)-*closed* sets we first introduce the following family, ordered by ordinal numbers:

$$X_0 = X \cup \tau_0, \tag{1}$$
$$X_\gamma = \bigcup_{j\in\gamma} X_j \cup \bigcup_{\varphi\in\tau} \{\varphi(\bar{t}) : \bar{t} \text{ is a } arity(\varphi)\text{-tuple taken in } \textstyle\bigcup_{j\in\gamma} X_j\}.$$

Proposition 1.3. *Let X be a set of variables, τ be a type of algebras and $(X_i)_i$ be the family introduced in Eq.* (1)*. Let us consider the cardinal number*

$$\gamma_\tau = \min\{\kappa \in \mathbf{Card} : \forall\varphi \in \tau, arity(\varphi) \leq \kappa\}.$$

Then the set $T_{\gamma_\tau^+} = \bigcup_{i\in\gamma_\tau^+} X_i$ is (X,τ)-closed.

Proof. We first note that γ_τ exists since **Card** is a well order class. By definition of $T_{\gamma_\tau^+}$, it is immediate that $X \cup \tau_0 \subseteq T_{\gamma_\tau^+}$. Let $\varphi \in \tau - \tau_0$ and $\bar{t} = (t_i)_{i\in arity(\varphi)} \in T_{\gamma_\tau^+}^{arity(\varphi)}$. Then, for each $i \in arity(\varphi)$, there exists an ordinal number $\alpha_i < \gamma_\tau^+$ such that $t_i \in X_{\alpha_i}$. If $\alpha = \bigcup_{i\in arity(\varphi)} \alpha_i$ then $t_i \in X_\alpha$ for each $i \in arity(\varphi)$ and, by Proposition 1.2-3, α is an ordinal number satisfying $\alpha < \gamma_\tau^+$. By Proposition 1.2-2, $\alpha + 1 < \gamma_\tau^+$ where $\alpha + 1$

is the successor ordinal of α. Therefore $\varphi(\bar{t}) \in X_{\alpha+1} \subseteq T_{\gamma_\tau^+}$ and $T_{\gamma_\tau^+}$ is (X,τ)-closed. ■

The well ordering of **Card** and the Proposition 1.3 allow us to introduce the following definition.

DEFINITION 1.4. Let X be a set of variables, τ be a type of algebras and $(X_i)_i$ be the family introduced in Eq. (1). Let us consider the cardinal number

$$\gamma_{X,\tau} = \min\left\{\kappa \in \mathbf{Card} : \bigcup_{i\in\kappa} X_i \text{ is } (X,\tau)\text{-closed}\right\}. \tag{2}$$

Then we define the set of τ-terms over X as

$$Term_\tau(X) = \bigcup_{i\in\gamma_{X,\tau}} X_i.$$

If $t \in Term_\tau(X)$ we use the notation $t(\overline{x})$ to indicate that the variables occurring in t are among $\overline{x} = \{x_i\}_i$. We can introduce a notion of complexity of terms as a function of the form $Comp : Term_\tau(X) \to \gamma_{X,\tau}$ such that:

$$\begin{aligned} &Comp(t) = 0 \quad \text{iff } t \in X_0, \\ &Comp(t) = \gamma \ \text{ iff } t \in X_\gamma \text{ and } t \notin X_i \text{ for all } i < \gamma. \end{aligned} \tag{3}$$

Let us notice that $Term_\tau(X)$ has a natural algebraic structure of type τ given by $\langle Term_\tau(X), \tau^{Term_\tau(X)}\rangle$ where for each $\varphi \in \tau$

$$Term_\tau(X)^{arity(\varphi)} \ni \bar{t} \mapsto \varphi^{Term_\tau(X)}(\bar{t}) = \varphi(\bar{t}).$$

The algebra $\langle Term_\tau(X), \tau^{Term_\tau(X)}\rangle$ is called the *absolutely free algebra of type τ over X*.

Let A be an algebra of type τ and X be a non-empty set of variables. A *valuation in A* is a function $v : X \to A$. By transfinite induction on $Comp(t)$, any valuation v in A can be uniquely extended to a τ-homomorphism $\widehat{v} : Term_\tau(X) \to A$. In view of this and, when there is no possible ambiguity, we adopt the following identification $\widehat{v} \cong \widehat{v}/_X = v$.

Let $t(\overline{x}) \in Term_\tau(X)$ where $\overline{x} = \{x_i\}_i$ and $v : Term_\tau(X) \to A$ be a valuation. Then, we use the notation $t^A(\overline{a})$, where $\overline{a} = \{a_i\}_i \subseteq A$, to indicate the value $v(t)$ where $v(x_i) = a_i$. An *equation* of type τ over X is an expression of the form $t = s$ where $t, s \in Term_\tau(X)$. An algebra A of type τ satisfies an equation $t = s$, indistinctly abbreviated as $A \models t = s$ or $\models_A t = s$, iff for each valuation $v : X \to A$, $v(t) = v(s)$. Let $\mathcal{A}$ be a class of algebras of type τ. Then, the equation $t = s$ is satisfied in $\mathcal{A}$, indistinctly abbreviated as $\mathcal{A} \models t = s$ or $\models_{\mathcal{A}} t = s$, iff for each $A \in \mathcal{A}$, $A \models t = s$.

Let E be a class of equations of type τ. We denote by $Alg_\tau(E)$ the class of algebras of type τ satisfying the equations in E. $\mathcal{A}$ is said to be *equationally definable* iff there exists a class E of equations of type τ such that $\mathcal{A} = Alg_\tau(E)$.

If $\mathcal{A}$ is an equationally definable class of algebras of type τ and $t = s$ is an equation such that $t, s \in Term_\tau(X)$ then, for the sake of simplicity, we denote by $\mathcal{A} \cup \{t = s\}$ the equational subclass of algebras of $\mathcal{A}$ satisfying the equation $t = s$.

Let $\mathcal{A}$ be a category whose objects are algebras of type τ and whose arrows, called $\mathcal{A}$-homomorphism, are τ-homomorphisms between algebras of $\mathcal{A}$. An $\mathcal{A}$-homomorphism f is said to be a *monomorphism* iff $f \circ g = f \circ h$ implies $g = h$ for any $\mathcal{A}$-homomorphism g, h. A *V-formation* in the category $\mathcal{A}$ is a scheme of $\mathcal{A}$-monomorphisms of the form $B \overset{i}{\hookleftarrow} A \overset{j}{\hookrightarrow} C$. An *amalgam* of this *V-formation* is a scheme of $\mathcal{A}$-monomorphisms of the form $C \overset{k}{\hookrightarrow} D \overset{h}{\hookleftarrow} B$ such that $hi = kj$. The amalgam is *strong* if, in addition, $Imag(k) \cap Imag(h) = Imag(kj) = Imag(hi)$. The category $\mathcal{A}$ has the (strong) amalgamation property iff every V-formation in $\mathcal{A}$ can be (strongly) amalgamated. An algebra $A \in \mathcal{A}$ is said to be *injective object* iff for every $\mathcal{A}$-monomorphism $f : B \to C$ and every $\mathcal{A}$-homomorphism $g : B \to A$ there exists a $\mathcal{A}$-homomorphism $h : C \to A$ such that $hf = g$. An algebra A in $\mathcal{A}$ is said to be *free over* $\mathcal{A}$ iff there exists a subset $S \subseteq A$ generating A and, if $B \in \mathcal{A}$ and $f : S \to B$ is a function then there exists a uniquely determined $\mathcal{A}$-homomorphism $g : A \to B$ such that $f = g/_S$. More precisely, we refer to the algebra A as a *free algebra on* X *generator*.

THEOREM 1.5. *Let* $\mathcal{A}$ *be an equationally definable category of algebras of the same type. Then:*

1. *Monomorphisms in* $\mathcal{A}$ *are exactly injective homomorphisms* [26, § 1].
2. $\mathcal{A}$ *is a variety* [31, § 7]. ∎

2. The Equational Theory of Heyting κ-Frames

In order to introduce and study our main algebraic structure, i.e. Heyting κ-frames, we first need some basic notions about Heyting algebras. A *Heyting algebra* is an algebra $\langle A, \vee, \wedge, \to, 0\rangle$ of type $\langle 2, 2, 2, 0\rangle$ satisfying the following equations:

H1. $\langle A, \vee, \wedge, 0\rangle$ is a lattice with universal lower bound 0,

H2. $x \wedge y = x \wedge (x \to y)$,

H3. $x \wedge (y \rightarrow z) = x \wedge ((x \wedge y) \rightarrow (x \wedge z))$,

H4. $z \wedge ((x \wedge y) \rightarrow x) = z$.

We denote by $\mathcal{H}$ the variety of Heyting algebras. In accordance with the usual $\mathcal{H}$-algebraic operations we define $\neg x = x \rightarrow 0$ and $1 = \neg 0$. Let us recall that if A is a Heyting algebra then the lattice order is given by $x \leq y$ iff $1 = x \rightarrow y$ and the reduct $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice. It is well known that in each Heyting algebra H, $a \rightarrow b = \bigvee\{x \in H : x \wedge a = b\}$. *Boolean algebras* are Heyting algebras where $x \rightarrow y = \neg x \vee y$. Moreover the variety of Boolean algebras $\mathcal{B}$ can be indistinctly defined as

$$\mathcal{B} = \mathcal{H} \cup \{x \rightarrow y = \neg x \vee y\} = \mathcal{H} \cup \{\neg\neg x = x\} = \mathcal{H} \cup \{\neg x \vee x = 1\}. \quad (4)$$

Let A be a totally ordered set with first element 0 and last element 1. If we consider the natural lattice order structure $\langle A, \vee, \wedge, 0, 1\rangle$ then

$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise} \end{cases} \quad (5)$$

is the unique operation making $\langle A, \vee, \wedge, \rightarrow, 0\rangle$ a Heyting algebra. These algebras are called *totally ordered Heyting algebras*.

PROPOSITION 2.1. *In each Heyting algebra A the following relations are satisfied:*

1. $x \wedge y \leq z$ *iff* $x \leq y \rightarrow z$,
2. *if* $x \leq y$ *then,* $t \rightarrow x \leq t \rightarrow y$ *and* $y \rightarrow t \leq x \rightarrow t$,
3. $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$,
4. $(x \wedge y) \rightarrow z = x \rightarrow (y \rightarrow z)$,
5. $x \vee t \leq (x \rightarrow t) \rightarrow t$,
6. *if* $x \leq y \rightarrow t$ *then* $(x \rightarrow t) \rightarrow t \leq y \rightarrow t$,
7. $\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y$,
8. $\neg(x \rightarrow y) = \neg\neg x \wedge \neg y$,
9. $\neg\neg(x \rightarrow y) = \neg\neg x \rightarrow \neg\neg y$.

Moreover if $\bigvee_{i \in I} x_i$ *exists in* A *then:*

10. $x \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I}(x \wedge x_i)$,
11. $(\bigvee_{i \in I} x_i) \rightarrow x = \bigwedge_{i \in I}(x_i \rightarrow x)$,
12. $\neg\neg \bigvee_{i \in I} x_i = \neg\neg \bigvee_{i \in I} \neg\neg x_i = \neg \bigwedge_{i \in I} \neg x_i$.

PROOF. We only prove items 5 and 6 because the other items are well known results about Heyting algebras (see [2, §IX] and [15, §IV]). (5) By H2 we have that $x \wedge (x \to t) = x \wedge t \leq t$. Then, by item 1, $x \leq (x \to t) \to t$. We also note that $t \wedge (x \to b) \leq t$ and, by item 1, $t \leq (x \to t) \to t$. It proves that $x \vee t \leq (x \to t) \to t$. (6) Suppose that $x \leq y \to t$. By item 1, $y \leq x \to t$. Then, by item 2, $(x \to t) \to t \leq y \to t$. ■

The condition expressed in Proposition 2.1-1 says that Heyting algebras are residuated lattices in the sense of [15] and the operation $\to$ is also called the *residuum* of $\wedge$.

Let A be a Heyting algebra. An element $a \in A$ is called *regular* iff $\neg\neg a = a$ and it is called *dense* iff $\neg a = 0$. The set of regular elements of A will be denoted by $Reg(A)$ and the set of dense elements will be denoted by $Ds(A)$. For each $x \in A$ it is well known that $\neg\neg x \in Reg(A)$, $x_d = \neg\neg x \to x \in Ds(A)$ and x can be expresed as

$$x = \neg\neg x \wedge x_d. \tag{6}$$

The following proposition provides some useful properties about dense and regular elements.

PROPOSITION 2.2. *Let A be a Heyting algebra and $x \in A$. Then:*

1. *if $r \in Reg(A)$ and $y \leq x \to r$ then $\neg\neg y \leq x \to r$,*
2. *if $r \in Reg(A)$ then $x \to r = \neg\neg x \to r$,*
3. *if $x \to y \in Ds(A)$ then, $x \to \neg\neg y = 1$ and $x \to y = x \to y_d$,*
4. *if $\bigvee_{i \in I} x_i$ exists and $(x_i \to y)_{i \in I} \subseteq Ds(A)$ then $\bigwedge_{i \in I}(x_i \to y) \in Ds(A)$.*

PROOF. (1) If $r \in Reg(A)$ and $y \leq x \to r$ then, by Proposition 2.1-9, $\neg\neg y \leq \neg\neg(x \to r) = \neg\neg x \to \neg\neg r = \neg\neg x \to r$. By Proposition 2.1-2 $\neg\neg x \to r \leq x \to r$ because $x \leq \neg\neg x$. Hence $\neg\neg y \leq x \to r$.

(2) Since $x \leq \neg\neg x$, by Proposition 2.1-2, $\neg\neg x \to r \leq x \to r$. For the other inequality, $\neg\neg x \wedge (x \to r) \leq \neg\neg x \wedge \neg\neg(x \to r) = \neg\neg x \wedge (\neg\neg x \to \neg\neg r) = \neg\neg x \wedge (\neg\neg x \to r) = \neg\neg x \wedge r \leq r$. Then, by Proposition 2.1-1, $x \to r \leq \neg\neg x \to r$.

(3) By item 2 and Proposition 2.1-3 we have that $x \to y = x \to (\neg\neg y \wedge y_d) = (x \to \neg\neg y) \wedge (x \to y_d) \leq x \to \neg\neg y$. Since $\neg\neg y \in Reg(A)$ and $x \to y \in Ds(A)$, by item 1, we have $1 = \neg\neg(x \to y) \leq x \to \neg\neg y$ and then $x \to y = x \to y_d$.

(4) Let us suppose that $\bigvee_{i \in I} x_i$ exists and let $(x_i \to y)_{i \in I}$ be a subset of $Ds(A)$. By item 3, for each $i \in I$, $x_i \to y = x_i \to y_d$. Thus, by Proposition 2.1-(8 and 11),

$$\neg \bigwedge_{i\in I}(x_i \to y) = \neg \bigwedge_{i\in I}(x_i \to y_d) = \neg((\bigvee_{i\in I} x_i) \to y_d) = \neg\neg \bigvee_{i\in I} x_i \wedge \neg y_d$$
$$= \neg\neg \bigvee_{i\in I} x_i \wedge 0 = 0.$$

Hence, $\bigwedge_{i\in I}(x_i \to y) \in Ds(A)$. ■

Let us notice that if $\mathcal{A}$ is a class of algebras with an underlying Heyting structure then these algebras are residuated too and a characterization of equations of $\mathcal{A}$ is given by

$$\mathcal{A} \models t = s \quad iff \quad \mathcal{A} \models (t \to s) \wedge (s \to t) = 1. \tag{7}$$

Therefore we can safely assume that all $\mathcal{A}$-equations are of the form $t = 1$.

A *complete Heyting algebra* is a Heyting algebra which is complete as a lattice.

THEOREM 2.3. [2] *Each Heyting algebra A can be embedded into a complete Heyting algebra preserving all existing joins and meets in A.* ■

We now introduce the class of Heyting κ-frames.

DEFINITION 2.4. Let κ be a cardinal number such that $\omega \leq \kappa$. A *Heyting κ-frame* is an algebra $\langle A, \bigsqcup, \wedge, \to, 0\rangle$ of type $\langle \kappa, 2, 2, 0\rangle$ such that, upon defining $x \vee y = \bigsqcup(x, y, 0, 0, \cdots)$ and $\bigsqcup_{i\in\kappa} x_i = \bigsqcup(x_0, x_1, \ldots, x_i, \ldots)$ for each string $(x_i)_{i\in\kappa} \in A^\kappa$, satisfies the following.

κ1. $\langle A, \vee, \wedge, \to, 0\rangle$ is a Heyting algebra,

κ2. $1 = x_i \to \bigsqcup_{i\in\kappa} x_i$,

κ3. $y \wedge \bigsqcup_{i\in\kappa} x_i = \bigsqcup_{i\in\kappa}(y \wedge x_i)$.

We denote by $\mathcal{H}_\kappa$ the category whose objects are Heyting κ-frames and whose arrows are functions preserving the operations $\bigsqcup, \to, \wedge, 0$. These arrows are called *$\mathcal{H}_\kappa$-homomorphisms*. In the same way we denote by $\mathcal{H}_\infty$ the category whose objects are frames (or indistinctly complete Heyting algebras) and whose arrows are $\langle \bigvee, \to, \wedge, 0\rangle$-preserving functions.

In this work several results on Heyting κ-frames are extended to $\mathcal{H}_\infty$. In order to do this and for the sake of clarity we adopt the following convention: we use the notation $\omega \leq \kappa \leq \infty$ to indicate that a certain property will be studied in the category $\mathcal{H}_\infty$ and in the category of Heyting κ-frames $\mathcal{H}_\kappa$ ($\kappa \neq \infty$) for each infinity cardinal number. Further, when there is no ambiguity, we identify the category $\mathcal{H}_\kappa$ with the class of their objects for $\omega \leq \kappa \leq \infty$.

For a set of variables X we denote by $Term_\kappa(X)$ the set of terms built up from X and the operation symbols $\langle \bigsqcup_\kappa, \wedge, \rightarrow, 0\rangle$ that define the language of $\mathcal{H}_\kappa$ on X variables. Since $\mathcal{H}_\kappa$ is equationally definable, by Theorem 1.5, the monomorphisms in $\mathcal{H}_\kappa$ are exactly injective $\mathcal{H}_\kappa$-homomorphisms and it is a variety.

The next proposition shows that the equational system introduced in Definition 2.4 axiomatizes the class of Heyting algebras admitting κ-joins.

Proposition 2.5. *Let κ be a cardinal number such that $\omega \leq \kappa$.*

1. *If A is a Heyting κ-frame then A is a Heyting algebra having κ-joins where $\bigvee_{i\in\kappa} x_i = \bigsqcup_{i\in\kappa} x_i$.*
2. *Let A be a Heyting algebra having κ-joins. If we define the κ-operation $\bigsqcup : A^\kappa \rightarrow A$ such that $\bigsqcup_{i\in\kappa} x_i = \bigvee_{i\in\kappa} x_i$ then $\langle A, \bigsqcup, \wedge, \rightarrow, 0\rangle$ is a Heyting κ-frame.*
3. *$A \in \mathcal{H}_\infty$ iff $\langle A, \bigvee_{|A|}, \wedge, \rightarrow, 0\rangle$ is a Heyting $|A|$-frame iff for each $\kappa \geq |A|$, $\langle A, \bigvee_\kappa, \wedge, \rightarrow, 0\rangle$ is a Heyting κ-frame.*
4. *$f : A \rightarrow B$ is a $\mathcal{H}_\infty$-homomorphism iff f is a $\mathcal{H}_{|A|}$-homomorphism iff for each $\kappa \geq |A|$, f is a $\mathcal{H}_\kappa$-homomorphism.*

Proof. (1) Suppose that A is a Heyting κ-frame and let $(x_i)_{i\in\kappa}$ be a family in A. By axiom $\kappa 2$, $\bigsqcup_{i\in\kappa} x_i$ is an upper bound of the family $(x_i)_{i\in\kappa}$. Let t be an upper bound of the family $(x_i)_{i\in\kappa}$. By axiom $\kappa 3$, $\bigsqcup_{i\in\kappa} x_i = \bigsqcup_{i\in\kappa}(t \wedge x_i) = t \wedge \bigsqcup_{i\in\kappa} x_i \leq t$. Thus $\bigvee_{i\in\kappa} x_i = \bigsqcup_{i\in\kappa} x_i$ and A is a Heyting algebra having κ-joins.

(2) Let us suppose that A is a Heyting algebra having κ-joins and we define the operation $\bigsqcup_{i\in\kappa} x_i = \bigvee_{i\in\kappa} x_i$. Note that $\kappa 2$ is immediately satisfied and , by Proposition 2.1-10, $\kappa 3$ also holds. Hence, $\langle A, \bigsqcup, \wedge, \rightarrow, 0\rangle$ is a Heyting κ-frame.

(3) If $A \in \mathcal{H}_\infty$ then the supremum $\bigvee_{|A|}$ trivially satisfies $\kappa 2$ and $\kappa 3$. Thus, $\langle A, \bigvee_{|A|}, \wedge, \rightarrow, 0\rangle$ is a Heyting $|A|$-frame. In this case, by a simple argument of cardinality, for each cardinal number $\kappa \geq |A|$ we have that $\bigvee_{|A|} = \bigvee_\kappa$. Hence $\langle A, \bigvee_\kappa, \wedge, \rightarrow, 0\rangle$ is a Heyting κ-frame. By a simple argument of cardinality again, the converse is immediate.

(4) Immediately follows by item (3). ∎

Remark 2.6. By the above proposition we can see that if A is a Heyting κ-frame then the reduct $\langle A, \bigsqcup_{i\in\kappa}, \wedge\rangle$ is a κ-frame [22].

Proposition 2.7. *Let A be a Heyting κ-frame and let $(x_i)_{i\in\kappa} \subseteq A$. Then:*

1. *The infimum $\bigwedge_{i\in\kappa} \neg\neg x_i$ exists in A,*

2. If $(x_i)_{i\in\kappa} \subseteq Reg(A)$ *then* $\bigwedge_{i\in\kappa} x_i \in Reg(A)$.

Proof. (1) By Proposition 2.1-11, we have $\bigwedge_{i\in\kappa} \neg\neg x_i = \bigwedge_{i\in\kappa}(\neg x_i \to 0) = (\bigvee_{i\in\kappa} \neg x_i) \to 0$. Then $\bigwedge_{i\in\kappa} \neg\neg x_i$ exists in A.

(2) Since $\neg\neg x_i = x_i$ for each $i \in \kappa$, by item 1, $\bigwedge_{i\in\kappa} x_i$ exists in A. Note that $\neg\neg \bigwedge_{i\in\kappa} x_i \leq \neg\neg x_i = x_i$. Then $\neg\neg \bigwedge_{i\in\kappa} x_i \leq \bigwedge_{i\in\kappa} x_i$. Thus $\bigwedge_{i\in\kappa} x_i \in Reg(A)$. ∎

For each cardinal number κ such that $\omega \leq \kappa \leq \infty$ we denote by $\mathcal{B}_\kappa$ the category of κ-complete Boolean algebras whose arrows are Boolean homomorphisms preserving κ-joins. Let us notice that, by Eq. (4), we have:

$$\mathcal{B}_\kappa = \mathcal{H}_\kappa \cup \{x \to y = \neg x \vee y\} = \mathcal{H}_\kappa \cup \{\neg\neg x = x\} = \mathcal{H}_\kappa \cup \{\neg x \vee x = 1\}. \tag{8}$$

We also denote by $\mathbf{2}_\kappa$ the Boolean algebra of two elements interpreted as a κ-complete structure in $\mathcal{B}_\kappa \subseteq \mathcal{H}_\kappa$ for each cardinal number $\omega \leq \kappa \leq \infty$.

Let A be a Heyting κ-frame and X be a non-empty subset of A. The *sub Heyting κ-frame of A generated by X*, denoted by $G_A(X)$, always exists and it is the intersection of sub Heyting κ-frames of A containing X. Clearly it is the minimum subalgebra of A containing X. The infinitary algebra $G_A(X)$ can be concretely realized in a similar way as in standard universal algebra. Indeed, let us define the following family of subsets of A indexed by ordinals:

$$\begin{aligned}
\mathbb{E}_0(X) &= X \cup \{0\},\\
\mathbb{E}_\alpha(X) &= \bigcup_{i\in\alpha} \mathbb{E}_i(X) \cup \{x \to y : x, y \in \bigcup_{i\in\alpha} \mathbb{E}_i(X)\}\\
&\quad \cup \{x \wedge y : x, y \in \bigcup_{i\in\alpha} \mathbb{E}_i(X)\} \cup \{\bigsqcup_{i\in\kappa} x_i : x_i \in \bigcup_{i\in\alpha} \mathbb{E}_i(X)\}.
\end{aligned} \tag{9}$$

Proposition 2.8. *Let A be a Heyting κ-frame, X be non-empty subset of A and let us consider the family of subsets of A defined in Eq. (9). Then,*

1. $G_A(X) = \bigcup_{\alpha\in\kappa^+} \mathbb{E}_\alpha(X)$,

2. $|G_A(X)| \leq \min\{\kappa^+ \cdot |X|, |A|\}$.

Proof. (1) Let $\mathbb{E} = \bigcup_{\alpha\in\kappa^+} \mathbb{E}_\alpha(X)$. It is immediate to see that $\mathbb{E}$ is Heyting sub algebra of A. We shall prove that $\mathbb{E}$ is closed under $\bigsqcup_{i\in\kappa}$. Let $(x_i)_{i\in\kappa}$ be a subset of $\mathbb{E}$. Then, for each $i \in \kappa$ there exists an ordinal number $\alpha_i < \kappa^+$ such that $x_i \in \mathbb{E}_{\alpha_i}(X)$. Taking into account that κ^+ is a regular cardinal, by Proposition 1.2-3, it follows that $\alpha = \bigcup_{i\in\kappa} \alpha_i < \kappa^+$. Then, the successor ordinal $\alpha + 1 < \kappa^+$ because κ^+ is an ordinal limit. By definition of $\mathbb{E}_{\alpha+1}(X)$, we have that $\bigsqcup_{i\in\kappa} x_i \in \mathbb{E}_{\alpha+1}(X) \subset \mathbb{E}$ and $\mathbb{E}$ is closed under $\bigsqcup_{i\in\kappa}$. It proves that $\mathbb{E}$ is a sub Heyting κ-frame of A. Thus $G_A(X) \subseteq \mathbb{E}$ because $\mathbb{E}$ contains X. Let B be a sub Heyting κ-frame of A containing X. It is not very hard to see that for each $\alpha < \gamma$, if $\mathbb{E}_\alpha(X) \subseteq B$ then $\bigcup_{\alpha\in\gamma} \mathbb{E}_\alpha(X) \subseteq B$. Thus, by transfinite induction, $\mathbb{E} \subseteq B$. It proves that $G_A(X) = \mathbb{E}$ as required.

(2) We first note that $|\mathbb{E}_0(X)| = |X| \leq \kappa^+|X|$. By inductive hypothesis, if $|\mathbb{E}_\alpha(X)| \leq \kappa^+ \cdot |X|$ for each $\alpha \in \kappa^+$ then, by Proposition 1.1-2,

$$|\bigcup_{\alpha\in\kappa^+} \mathbb{E}_\alpha(X)| \leq \sum_{\alpha\in\kappa^+} |\mathbb{E}_\alpha(X)| \leq \sum_{\alpha\in\kappa^+} \kappa^+ \cdot |X| = \kappa^+ \cdot (\kappa^+ \cdot |X|) = \kappa^+ \cdot |X|.$$

Since $G_A(X) \subseteq A$ then, $|G_A(X)| \leq |A|$ and $|G_A(X)| \leq \min\{\kappa^+ \cdot |X|, |A|\}$. ■

By the proposition above and by Proposition 2.5-3 the following result is immediate.

Corollary 2.9. *Let $A \in \mathcal{H}_\infty$ and $X \subseteq A$. Then $G_A(X) = \bigcup_{\alpha\in|A|} \mathbb{E}_\alpha(X)$ defines the subalgebra of A generated by X which also belongs to $\mathcal{H}_\infty$.* ■

3. Filters and Congruences

The aim of this section is to study the filter theory and congruences in $\mathcal{H}_\kappa$ and $\mathcal{H}_\infty$. For this we first recall some basic results about filters on Heyting algebras.

Let A be a Heyting algebra. A non empty subset $F \subseteq A$ is a *filter* or *Heyting filter* to avoid confusion, iff it is satisfies the following two conditions:

$$1 \in F, \ \text{ if } x \in F \text{ and } x \to y \in F \text{ then } y \in F. \tag{10}$$

It is easy to verify that the non-empty subset F is a Heyting filter iff it is an increasing set (i.e., if $a \in F$ and $a \leq b$ then $b \in F$) and if $a, b \in F$ then $a \wedge b \in F$. The Heyting filter F is said to be *proper* iff 0 does not belong to F. We shall denote by $Filt_H(A)$ the set of all Heyting filters in A. Since the intersection of any family of Heyting filters of A is a Heyting filter of A, $\langle Filt_H(A), \subseteq\rangle$ is a complete lattice. We denote by $\langle X\rangle_H$ the Heyting filter generated by $X \subseteq A$, i.e., the intersection of all Heyting filters of A containing X. It is well known that:

$$\langle X\rangle_H = \{x \in A : \exists \ x_1 \ldots x_n \in X \ \text{ such that } \ x \geq x_1 \wedge \ldots \wedge x_n\}. \tag{11}$$

We abbreviate $\langle a\rangle_H$ when $X = \{a\}$ and, in this case, $\langle a\rangle_H$ is called *principal filter associated to* a. Note that $\langle a\rangle_H = [a) = \{x \in A : a \leq x\}$. To avoid confusion, a congruence on the Heyting algebra A is referred as *Heyting congruence* and its congruences lattice is denoted by $Con_H(A)$. For any Heyting filter F of A, $\theta_F = \{(x, y) \in A^2 : x \to y \in F, y \to x \in F\}$ is a Heyting congruence on A and $F = \{x \in A : (x, 1) \in \theta_F\}$. Conversely, if $\theta \in Con_H(A)$ then, $F_\theta = \{x \in A : (x, 1) \in \theta\}$ is a Heyting filter and, $(x, y) \in \theta$ iff $(x \to y, 1) \in \theta$ and $(y \to x, 1) \in \theta$. Thus, the correspondence $F \to \theta_F$ is an order isomorphism from $Filt_H(A)$ onto $Con_H(A)$.

Let A be a Heyting κ-frame. A *κ-congruence* on A is a Heyting congruence in the underlying Heyting structure of A satisfying the following compatibility condition:

$$\left(\bigvee_{i\in\kappa} x_i, \bigvee_{i\in\kappa} y_i\right) \in \theta \quad \text{whenever for each } i \in \kappa, (x_i, y_i) \in \theta. \tag{12}$$

We denote by $Con_\kappa(A)$ the set of all κ-congruences. Let us notice that $Con_\kappa(A)$ is ordered by inclusion and $Con_\kappa(A) \subseteq Con_H(A)$. Let $\theta \in Con_\kappa(A)$. By Theorem 1.5-2, the quotient algebra $A/_\theta$ is a Heyting κ-frame and the natural application $p_\theta : A \to A/_\theta$ is a $\mathcal{H}_\kappa$-homomorphism. Consequently, for each family $(x_i)_{i\in\kappa}$ in A we have that $\bigvee_{i\in\kappa}(x_i/_\theta) = (\bigvee_{i\in\kappa} x_i)/_\theta$.

Remark 3.1. In [22, § 5] a notion of congruence on a κ-frame A is introduced. It is an equivalence relation $\theta \subseteq A \times A$ which is also a sub κ-frame of $A \times A$. In this way, for a Heyting κ-frame A, θ is a κ-congruence on A iff it is a congruence in the κ-frame reduct $\langle A, \wedge, \bigsqcup_{i\in\kappa}\rangle$ compatible with the operation $\to$.

Definition 3.2. Let A be a Heyting κ-frame. A non-empty subset $F \subseteq A$ is a *κ-filter* iff

1. F is a Heyting filter of the underlying Heyting structure of A.
2. If $(x_i \to y)_{i\in\kappa}$ is a subset of F then $(\bigvee_{i\in\kappa} x_i) \to y \in F$.

We denote by $Filt_\kappa(A)$ the set of all κ-filters of A. Let us notice that $Filt_\kappa(A)$ is ordered by inclusion and $Filt_\kappa(A) \subseteq Filt_H(A)$.

Proposition 3.3. *Let A be a Heyting κ-frame and F be a κ-filter. Then,*

1. *The second condition in Definition 3.2 is equivalent to $\bigwedge_{i\in\kappa}(x_i \to y) \in F$ whenever $(x_i \to y)_{i\in\kappa} \subseteq F$.*
2. *If $(x_i)_{i\in\kappa} \subseteq F$ then $\bigwedge_{i\in\kappa} \neg\neg x_i \in F$.*

PROOF. (1) Follows by Proposition 2.1-11. (2) Since F is an increasing set and $x_i \leq \neg\neg x_i$ we have that $\neg\neg x_i = \neg x_i \to 0 \in F$. Thus, by item 1, $\bigwedge_{i\in\kappa} \neg\neg x_i = \bigwedge_{i\in\kappa}(\neg x_i \to 0) = (\bigvee_{i\in\kappa} \neg x_i) \to 0 \in F$. ■

By Proposition 3.3-2 immediately follows that if $A \in \mathcal{B}_\kappa$ and F is a Heyting filter of the underlying Heyting structure of A then, F is a κ-filter iff it is closed under $\bigwedge_\kappa$ infima.

EXAMPLE 3.4. Let A be a Heyting κ-frame.

a. If $a \in A$ then the principal Heyting filter $[a)$ is a κ-filter.

b. Since $Ds(A)$ is a Heyting filter, by Proposition 2.2-4, it immediately follows that $Ds(A)$ is a κ-filter.

c. If A is a totally ordered Heyting algebra then, each Heyting filter of A is a κ-filter. Indeed: We first note that a filter in a totally ordered Heyting algebra can be either principal (and then it is a κ-filter) or it has the form $F = \{x \in A : a < x\}$ for some $a \in A$. In this second case we shall prove that F is a κ-filter. Let $(x_i \to y)_{i\in\kappa}$ be a subset of F. Let us consider two possible cases: *Case:* $y \leq a$) By Eq. (5), it is clear that $x_i \leq y$ otherwise $x_i \to y = y \notin F$. Thus $x_i \to y = 1$ and $(\bigvee_{i\in\kappa} x_i) \to y = \bigwedge_{i\in\kappa}(x_i \to y) = 1 \in F$. *Case:* $y > a$) By Eq. (5), $x_i \to y \in \{y, 1\}$ and then $a < y \leq \bigwedge_{i\in\kappa}(x_i \to y) = (\bigvee_{i\in\kappa} x_i) \to y$. It proves that $(\bigvee_{i\in\kappa} x_i) \to y \in F$ and F is a κ-filter.

d. If $f : A \to B$ is a $\mathcal{H}_\kappa$-homomorphism, $Ker(f) = \{x \in A : f(x) = 1\}$ is a κ-filter. Indeed: We first note that $Ker(f)$ is a Heyting filter. Suppose that $x_i \to y \in Ker(f)$ for each $i \in \kappa$. Thus, $1 = f(x_i \to y) = f(x_i) \to f(y)$ and, by Proposition 2.1-1, $f(x_i) \leq f(y)$. Since f preserves κ-joins, $f(\bigvee_{i\in\kappa} x_i) = \bigvee_{i\in\kappa} f(x_i) \leq f(y)$. Then, $f(\bigvee_{i\in\kappa} x_i \to y) = f(\bigvee_{i\in\kappa} x_i) \to f(y) = 1$ and $\bigvee_{i\in\kappa} x_i \to y \in Ker(f)$. Hence, $Ker(f)$ is a κ-filter.

THEOREM 3.5. *Let A be a Heyting κ-frame. Then, the maps $F \mapsto \theta_F$ and $\theta \mapsto F_\theta$ are mutually inverse order isomorphisms between $Con_\kappa(A)$ and $Filt_\kappa(A)$.*

PROOF. We first prove that if $F \in Filt_\kappa(A)$ then $\theta_F \in Con_\kappa(A)$. Since $\theta_F \in Con_H(A)$, we have to prove that if $(x_i, y_i)_{i\in\kappa}$ is an indexed subset of θ_F then $(\bigvee_{i\in\kappa} x_i, \bigvee_{i\in\kappa} y_i) \in \theta_F$. Indeed: Let us notice that for each $i \in \kappa$, $x_i \to y_i \in F$ and $y_i \to x_i \in F$. Since $x_i \leq \bigvee_{i\in\kappa} x_i$, by Proposition 2.1-2, $y_i \to x_i \leq y_i \to \bigvee_{i\in\kappa} x_i$ and then, $y_i \to \bigvee_{i\in\kappa} x_i \in F$ because F is an increasing set. Thus $(\bigvee_{i\in\kappa} y_i) \to (\bigvee_{i\in\kappa} x_i) \in F$ since F is a κ-filter. With the same argument we can prove that $(\bigvee_{i\in\kappa} x_i) \to (\bigvee_{i\in\kappa} y_i) \in F$. Hence $\theta_F \in Con_\kappa(A)$.

Now let us suppose that $\theta \in Con_\kappa(A)$. Since $F_\theta \in Filt_H(A)$, we shall prove that if $(x_i \to y)_{i\in\kappa}$ is a family in F_θ then $(\bigvee_{i\in\kappa} x_i) \to y \in F_\theta$. Indeed: Let us notice that for each $i \in \kappa$, $(x_i \to y, 1) \in \theta$ and $(x_i, x_i) \in \theta$. Then, by H2, we have that $(x_i \wedge y, x_i) = (x_i \wedge (x_i \to y), x_i \wedge 1) \in \theta$. Thus $(y \wedge \bigvee_{i\in\kappa} x_i, \bigvee_{i\in\kappa} x_i) = (\bigvee_{i\in\kappa}(x_i \wedge y), \bigvee_{i\in\kappa} x_i) \in \theta$. Since $(y, y) \in \theta$ then we have that

$$(1, (\bigvee_{i\in\kappa} x_i) \to y) = ((y \wedge \bigvee_{i\in\kappa} x_i) \to y, (\bigvee_{i\in\kappa} x_i) \to y) \in \theta$$

and, consequently, $(\bigvee_{i\in\kappa} x_i) \to y \in F_\theta$. Hence $F_\theta \in Filt_\kappa(A)$.

Finally, taking into account that $F \mapsto \theta_F$ and $\theta \mapsto F_\theta$ are mutually inverse order isomorphisms between $Con_H(A)$ and $Filt_H(A)$, the theorem follows immediately. ■

Since there is a one-to-one correspondence between the set $Con_\kappa(A)$ and the set $Filt_\kappa(A)$, for each $F \in Filt_\kappa(A)$ we denote by $A/_F$ the quotient algebra $A/_{\theta_F}$ and by $[x]_F$ the equivalence class of x modulo θ_F for $x \in A$. In order to study quotient algebras in $\mathcal{H}_\infty$ we can take advantage of the above results. Indeed: Let $A \in \mathcal{H}_\infty$. We say that $\theta \subseteq A^2$ is a ∞-*congruence* on A iff θ is a κ-congruence for each cardinal number κ. A cardinality argument show that θ is a ∞-congruence iff θ is a $|A|$-congruence. Since $A/_\theta$ is a $|A|$-frame and $|A/_\theta| \leq |A|$ then $A/_\theta \in \mathcal{H}_\infty$. Thus, if we denote by $Con_\infty(A)$ the set of ∞-congruences of A we have that $Con_\infty(A) = Con_{|A|}(A)$.

A set $F \subseteq A$ is a ∞-*filter* iff F is a κ-filter for each cardinal number κ. Similarly we can prove that F is a ∞-filter iff F is a $|A|$-filter. If we denote by $Filt_\infty(A)$ the set of ∞-filters of A we have that $Filt_\infty(A) = Filt_{|A|}(A)$. Thus, by Proposition 3.5, we can establish the following identification

$$Con_{|A|}(A) = Con_\infty(A) \approx Filt_\infty(A) = Filt_{|A|}(A). \tag{13}$$

Proposition 3.6. *The algebra $\mathbf{2}_\kappa$ is the only simple algebra in $\mathcal{H}_\kappa$ for $\omega \leq \kappa \leq \infty$.*

Proof. It follows from Proposition 3.5 and Example 3.4-c because each non-zero element in a Heyting κ-frame determines a non trivial κ-filter. ■

Let A be a Heyting κ-frame. Observe that the intersection of any family of κ-filters of A is a κ-filter of A. Thus $\langle Filt_\kappa(A), \subseteq\rangle$ is a complete lattice. Let X be a non-empty subset of A. We denote by $\langle X\rangle_\kappa$ the κ-filter generated by X, i.e., the intersection of all κ-filters of A containing X. Note that if $X = \{a\}$ then, $\langle a\rangle_\kappa = \langle a\rangle_H = [a)$. As we will see in Proposition 3.7

the generated κ-filter $\langle X\rangle_\kappa$ can be realized through the following family of subsets of A indexed by ordinals:

$$\mathbb{F}_0(X) = \langle X\rangle_H,$$
$$\mathbb{F}_{\alpha+1}(X) = \Big\langle \{(\bigvee_{i\in\kappa}(x_i \to y)) \to y : y \in A \ \ and \ \ x_i \in \mathbb{F}_\alpha(X)\} \Big\rangle_H,$$
$$\mathbb{F}_\gamma(X) = \bigcup_{\alpha<\gamma} \mathbb{F}_\alpha(X) \ \text{ if } \gamma \text{ is a limit ordinal.} \tag{14}$$

Proposition 3.7. *Let A be a Heyting κ-frame, X be a non-empty subset of A and let us consider the ordinal sequence of Heyting filters defined in Eq. (14). Then,*

1. *If $\alpha \le \beta$ then $\mathbb{F}_\alpha(X) \subseteq \mathbb{F}_\beta(X)$.*
2. *For each ordinal α, $\mathbb{F}_\alpha(X)$ is a Heyting filter.*
3. *There exists an ordinal number $\alpha \le |A|$ such that $\mathbb{F}_\alpha(X) = \mathbb{F}_\beta(X)$ for each $\beta > \alpha$, i.e. $(\mathbb{F}_\alpha(X))_\alpha$ is an ascending stationary chain of Heyting filters in A.*

Proof. (1) Clearly we only need to prove that $\mathbb{F}_\alpha(X) \subseteq \mathbb{F}_{\alpha+1}(X)$. Let $x \in \mathbb{F}_\alpha(X)$. Since $x = (x \to x) \to x$ then it immediately follows that $x \in \mathbb{F}_{\alpha+1}(X)$.

(2) We use transfinite induction. The cases $\mathbb{F}_0(X)$ and $\mathbb{F}_{\alpha+1}(X)$ follow immediate from Eq. (14). If γ is a limit ordinal, by item 1, $(\mathbb{F}_\alpha(X))_{\alpha<\gamma}$ is an up-directed family of Heyting filters under set-inclusion. Then, it is immediate to see that $\mathbb{F}_\gamma(X) = \bigcup_{\alpha<\gamma}\mathbb{F}_\alpha(X)$ is a Heyting filter. Hence our claim.

(3) Let us suppose that $(\mathbb{F}_\alpha(X))_\alpha$ is not stationary for any ordinal $\le |A|$. Then, by an argument of cardinality, there exists a sub family $(\mathbb{F}_{\alpha_i}(X))_{i\in I}$ such that I is a totally ordered set, $|I| > |A|$ and $\mathbb{F}_{\alpha_i}(X) \subsetneqq \mathbb{F}_{\alpha_j}(X)$ whenever $i < j$ in I. Consequently, there exists a family $(x_i)_{i\in I} \subseteq A$ such that $x_i \in \mathbb{F}_{\alpha_i}(X)$ and $x_i \neq x_j$ whenever $i \neq j$ which is a contradiction since $|A| < |I| = |(x_i)_{i\in I}| \le |A|$. Hence, there exists an ordinal number $\alpha \le |A|$ such that $\mathbb{F}_\alpha(X) = \mathbb{F}_\beta(X)$ for each $\beta > \alpha$. ■

Let A be a Heyting κ-frame, X be a non-empty subset of A and let us consider the ordinal sequence of Heyting filters introduced in Eq. (14). Then, by Proposition 3.7-3, we define:

$$\alpha_X = \min\{\alpha \in \mathbf{ON} : \text{if } \beta > \alpha \text{ then } \mathbb{F}_\alpha(X) = \mathbb{F}_\beta(X)\ \}. \tag{15}$$

Proposition 3.8. *Let A be a Heyting κ-frame X be a non-empty subset of A and a family $(\mathbb{F}_\alpha(X))_\alpha$ of Heyting filters as defined in Eq. (14). Let us* *consider the ordinal* α_X *introduced in Eq.* (15)*. Then:*

$$\langle X\rangle_\kappa = \mathbb{F}_{\alpha_X}(X).$$

Proof. We first prove that $\mathbb{F}_{\alpha_X}(X)$ is a κ-filter. Indeed: By Proposition 3.7 we first note that $\mathbb{F}_{\alpha_X}(X)$ is a Heyting filter of A. Let $(a_i \to y)_{i\in\kappa}$ be an indexed subset of elements of $\mathbb{F}_{\alpha_X}(X)$. By definition of $\mathbb{F}_{\alpha_X}(X)$ for each $i \in \kappa$ there exists $x_i \in \mathbb{F}_{\alpha_i}(X)$ such that $x_i \leq a_i \to y$ and $\alpha_i < \alpha_X$. By Proposition 2.1-6 we also have that $(x_i \to y) \to y \leq a_i \to y$. Thus, $\big(\bigvee_{i\in\kappa}(x_i \to y)\big) \to y = \bigwedge_{i\in\kappa}\big((x_i \to y) \to y\big) \leq \bigwedge_{i\in\kappa}(a_i \to y) = (\bigvee_{i\in\kappa} a_i) \to y$ and $(\bigvee_{i\in\kappa} a_i) \to y \in \mathbb{F}_{\alpha_X+1}(X)$ because of Eq. (14). By Proposition 3.7-3, $\mathbb{F}_{\alpha_X}(X) = \mathbb{F}_{\alpha_X+1}(X)$ and consequently $(\bigvee_{i\in\kappa} a_i) \to y \in \mathbb{F}_{\alpha_X}(X)$. Hence $\mathbb{F}_{\alpha_X}(X)$ is a κ-filter of A.

Now we prove that $\mathbb{F}_{\alpha_X}(X) = \langle X\rangle_\kappa$. Since $X \subseteq \mathbb{F}_{\alpha_X}(X)$ then $\langle X\rangle_\kappa \subseteq \mathbb{F}_{\alpha_X}(X)$. Conversely, let F be a κ-filter containing X. By transfinite induction, we shall prove that for each ordinal α, $\mathbb{F}_\alpha(X) \subseteq F$.

Since F is a Heyting filter we have that $\mathbb{F}_0(X) \subseteq F$. Let us assume that $\mathbb{F}_\alpha(X) \subseteq F$. If $x \in \mathbb{F}_{\alpha+1}(X)$ then there exist n indexed sets contained in $\mathbb{F}_\alpha(X)$ of the form $(x_i^1)_{i\in\kappa}, \ldots, (x_i^n)_{i\in\kappa}$ such that $w = \bigwedge_{j=1}^n \big(\bigvee_{i\in\kappa}(x_i^j \to y_j) \to y_j\big) \leq x$. Since $x_i^j \leq (x_i^j \to y_j) \to y_j$, $(x_i^j \to y_j) \to y_j \in F$ and, taking into account that F is a κ-filter, we have that $\big(\bigvee_{i\in\kappa}(x_i^j \to y_j)\big) \to y_j \in F$ for each $1 \leq j \leq n$. Therefore, $w \in F$ and, consequently, $x \in F$. The case in which γ is a limit ordinal is immediate. Thus, for each ordinal number α, $\mathbb{F}_\alpha(X) \subseteq F$ and then $\mathbb{F}_{\alpha_X}(X) \subseteq F$. It proves that $\mathbb{F}_{\alpha_X}(X) \subseteq \langle X\rangle_\kappa$. ■

Proposition 3.9. *Let A be a Heyting κ-frame and X be a non-empty subset of A closed under κ-meets. Then,*

$$\langle X\rangle_\kappa = \{x \in A : \exists (x_i)_{i\leq\kappa} \subseteq X \text{ such that } \bigwedge_{i\in\kappa} x_i \leq x\}.$$

Proof. It is straightforward to see that the set $G = \{x \in A : \exists (x_i)_{i\leq\kappa} \subseteq X$ such that $\bigwedge_{i\in\kappa} x_i \leq x\}$ is an increasing set closed under finite meets i.e., it is a Heyting filter. Since X is closed under κ-meets, by Proposition 3.3, G is a κ-filter too. Let F be a κ-filter such that $X \subseteq F$. It is also not difficult to show that $G \subseteq F$ because X is closed under κ-meets. It proves that G is the minimum filter containing the X. Hence our claim. ■

Proposition 3.10. *Let A be a Heyting κ-frame and X be a non-empty subset of A. If $a \in Reg(A) \cap \langle X\rangle_\kappa$ then there exists a family $(x_i)_{i\leq\kappa} \subseteq X$ such that $\bigwedge_{i\in\kappa} \neg\neg x_i \leq a$.*

Proof. We use transfinite induction in the family $(F_\alpha(X))_\alpha$ introduced in Eq. (14). If $a \in F_0(X)$ it is immediate from the definition of generated

Heyting filter (see Eq. 11). Let us assume that the result holds for $a \in F_\alpha(X)$. If $a \in F_{\alpha+1}(X)$ then there exist n indexed sets contained in $\mathbb{F}_\alpha(X)$ of the form $(x_i^1)_{i\in\kappa}, \ldots, (x_i^n)_{i\in\kappa}$ such that $\bigwedge_{j=1}^n \bigwedge_{i\in\kappa} \big((x_i^j \to y_j) \to y_j\big) \leq a$. Since $a \in Reg(A)$, by Proposition 2.1-(7 and 9) we have that

$$\begin{aligned} a &\geq \neg\neg \bigwedge_{j=1}^{n} \bigwedge_{i\in\kappa} \big((x_i^j \to y_j) \to y_j\big) = \bigwedge_{j=1}^{n} \neg\neg \bigwedge_{i\in\kappa} \big((x_i^j \to y_j) \to y_j\big) \\ &= \bigwedge_{j=1}^{n} \neg\neg\big((\bigvee_{i\in\kappa} x_i^j \to y_j) \to y_j\big) = \bigwedge_{j=1}^{n} \big(\neg\neg(\bigvee_{i\in\kappa} x_i^j \to y_j) \to \neg\neg y_j\big) \\ &= \bigwedge_{j=1}^{n} \big(\neg\neg(\bigvee_{i\in\kappa} \neg\neg x_i^j \to \neg\neg y_j) \to \neg\neg y_j\big). \end{aligned}$$

Since $\neg\neg y_j \in Reg(A)$, by Proposition 2.2-2,

$$\neg\neg(\bigvee_{i\in\kappa} \neg\neg x_i^j \to \neg\neg y_j) \to \neg\neg y_j = (\bigvee_{i\in\kappa} \neg\neg x_i^j \to \neg\neg y_j) \to \neg\neg y_j.$$

Therefore, by Proposition 2.1-5,

$$\begin{aligned} a &\geq \bigwedge_{j=1}^{n} (\bigvee_{i\in\kappa} \neg\neg x_i^j \to \neg\neg y_j) \to \neg\neg y_j = \bigwedge_{j=1}^{n} \bigwedge_{i\in\kappa} (\neg\neg x_i^j \to \neg\neg y_j) \to \neg\neg y_j \\ &\geq \bigwedge_{j=1}^{n} \bigwedge_{i\in\kappa} \neg\neg x_i^j. \end{aligned}$$

Note that $\neg\neg x_i^j$ is a regular element belonging to $F_\alpha(X)$. Then, by inductive hypothesis, for each pair i, j there exists an indexed set $(x_{i,j,s})_{s\leq\kappa}$ in X such that $\bigwedge_{s\in\kappa} \neg\neg x_{i,j,s} \leq \neg\neg x_i^j$. Taking into account that $|(x_{i,j,s})_{i,j,s}| \leq \kappa$, $\bigwedge_{i,j,s} \neg\neg x_{i,j,s}$ exists and it is less than a. Suppose that the result holds for each $F_\alpha(X)$ such that $\alpha \leq \gamma$ where γ is a limit ordinal. By definition of $F_\gamma(X)$, if $a \in Reg(A) \cap F_\gamma(X)$ then there exists $\alpha < \gamma$ such that $a \in F_\alpha(X)$. Thus, by inductive hypothesis $\bigwedge_{i\in\kappa} \neg\neg x_i \leq a$ for some family $(x_i)_{i\leq\kappa}$ in X. ■

In what follows we establish a necessary and sufficient condition so that a subset of a Heyting κ-frame is able to generate a proper κ-filter. For this we first introduce some terminology.

Let X be a subset of a Heyting κ-frame A. We say that X has the κ-*meet property* iff each subset $Y \subseteq X$ such that $|Y| \leq \kappa$ admits infimum and $\bigwedge Y > 0$. We also define the set $\neg\neg X$ as $\neg\neg X = \{\neg\neg x : x \in X\}$.

Proposition 3.11. *Let A be a Heyting κ-frame and X be a non-empty subset of A. Then the following sentences are equivalent:*

1. *$\neg\neg X$ has the κ-meet property,*
2. *$\langle X\rangle_\kappa$ is a proper κ-filter.*

Proof. Let us assume that $\neg\neg X$ has the κ-meet property. Suppose that $0 \in \langle X\rangle_\kappa$. Since $0 \in Reg(A) \cap \langle X\rangle_\kappa$, by Proposition 3.10, there exists an indexed set $(x_i)_{i\in\kappa} \subseteq X$ such that $\bigwedge_{i\in\kappa} \neg\neg x_i \leq 0$ which is a contradiction because $\neg\neg X$ has the κ-meet property. Hence $\langle X\rangle_\kappa$ is a proper κ-filter. By Proposition 3.3-2 the other direction is immediate. ■

The rest of the section is devoted to establish some results about the extension of κ-filters.

Let A be a sub Heyting κ-frame of B where $\omega \leq \kappa \leq \infty$. We say that $F_A \in Filt_\kappa(A)$ can be *extended*, or has an *extension* on B, iff there exists a $F_B \in Filt_\kappa(B)$ such that $F_A = F_B \cap A$. We then refer to F_B as an *extension of F_A on B*. We say that B has the *congruence extension property* (CEP) iff, for every sub Heyting κ-frame A of B, each $F_A \in Filt_\kappa(A)$ has an extension to B. A subclass of $\mathcal{H}_\kappa$ has the CEP iff every algebra in this subclass satisfies CEP. In what follows $\langle F_A\rangle^B_\kappa$ denotes the κ-filter generated by F_A in B.

Proposition 3.12. *Let A be a sub Heyting κ-frame of B where $\omega \leq \kappa \leq \infty$ and $F_A \in Filt_\kappa(A)$. Then,*

$$\langle F_A\rangle^B_\kappa \cap Reg(A) = F_A \cap Reg(A).$$

Proof. Let us notice that if $\kappa = \infty$ then $\langle F_A\rangle^B_\infty = \langle F_A\rangle^B_{|B|}$. Thus, the case $\kappa = \infty$ becomes a particular case in $\mathcal{H}_{|B|}$. In this way we can confine our proof to $\kappa < \infty$. Suppose that A is a sub Heyting κ-frame of B. Let $x \in \langle F_A\rangle^B_\kappa \cap Reg(A)$. By Proposition 3.10 there exists an indexed set $(x_i)_{i\in\kappa} \subseteq F_A$ such that $\bigwedge_{i\in\kappa} \neg\neg x_i \leq x$. Since F_A is a κ-filter, by Proposition 3.3, $\bigwedge_{i\in\kappa} \neg\neg x_i \in F_A$ and then $x \in F_A$. Thus, $\langle F_A\rangle^B_\kappa \cap Reg(A) \subseteq F_A \cap Reg(A)$. The other inclusion is immediate. ■

Theorem 3.13. *Let A be a sub Heyting κ-frame of B and $F_A \in Filt_\kappa(A)$ where $\omega \leq \kappa \leq \infty$. Then the following assertions are equivalent:*

1. *F_A has an extension on B.*
2. *$\langle F_A\rangle^B_\kappa$ is an extension of F_A on B.*
3. *$\langle F_A\rangle^B_\kappa \cap Ds(A) = F_A \cap Ds(A)$.*

Proof. Like the proof of Proposition 3.12, the case $\kappa = \infty$ becomes a particular case in $\mathcal{H}_{|B|}$. Thus we also confine our proof to $\kappa < \infty$.

(1) $\Longrightarrow$ (2) Let F_B be an extension of F_A on B i.e., $F_A = F_B \cap A$. Since F_B is a κ-filter of B containing F_A, then we have that $F_A \subseteq \langle F_A \rangle_\kappa^B \subseteq F_B$. Thus, $F_A = F_A \cap A \subseteq \langle F_A \rangle_\kappa^B \cap A \subseteq F_B \cap A = F_A$. Hence, $\langle F_A \rangle_\kappa^B$ is an extension of F_A on B.

(2) $\Longrightarrow$ (3) By hypothesis we have that $\langle F_A \rangle_\kappa^B \cap A = F_A$. Thus, $\langle F_A \rangle_\kappa^B \cap Ds(A) = \langle F_A \rangle_\kappa^B \cap A \cap Ds(A) = F_A \cap Ds(A)$.

(3) $\Longrightarrow$ (1) Let us assume that $\langle F_A \rangle_\kappa^B \cap Ds(A) = F_A \cap Ds(A)$. Let $x \in \langle F_A \rangle_\kappa^B \cap A$. By Eq. (6), $x = \neg\neg x \wedge x_d$ where $\neg\neg x \in Reg(A)$ and $x_d \in Ds(A)$. Since $x \leq \neg\neg x$, $\neg\neg x \in \langle F_A \rangle_\kappa^B \cap Reg(A)$ and, by Proposition 3.12, $\neg\neg x \in F_A$. Since $x \leq x_d$, $x_d \in \langle F_A \rangle_\kappa^B \cap Ds(A)$ and, by hypothesis, $x_d \in F_A$. Thus, $x = \neg\neg x \wedge x_d \in F_A$ and $\langle F_A \rangle_\kappa^B \cap A \subseteq F_A$. Since the other inclusion is immediate we have that $\langle F_A \rangle_\kappa^B$ is an extension of F_A on B. ■

The above theorem shows that the extension of a κ-filters depends only on the dense elements contained in the κ-filter.

Example 3.14. Let A be a sub Heyting κ-frame of B and $F_A \in Filt_\kappa(A)$ where F_A is proper.

a. Let $a \in A$ and the principal κ-filter $[a)^A$. Then, it is immediate to see that $[a)^B = \{x \in B : a \leq x\}$ is an extension of $[a)^A$ on B.

b. If $Ds(A) \subseteq F_A$ then $\langle F_A \rangle_\kappa^B$ is an extension of F_A on B. Indeed: It is immediate to see that $\langle F_A \rangle_\kappa^B \cap Ds(A) = Ds(A) = F_A \cap Ds(A)$. Then, by Theorem 3.13, this claim follows. In particular, since $Ds(A)$ is a κ-filter of A (see Example 3.4-b), it admits an extension on B.

c. Let us notice that in a Heyting κ-frame, in general, the existence of maximal κ-filters is not guaranteed. However we can prove that if F_A is a maximal κ-filter in A then $\langle F_A \rangle_\kappa^B$ is an extension of F_A on B. In order to do this we prove that $Ds(A) \subseteq F_A$. Indeed: Let us suppose that there exists $x \in Ds(A)$ such that $x \notin F(A)$. By the maximality of $F(A)$ we have that $\langle F(A) \cup \{x\} \rangle_\kappa = A$. Thus, by Proposition 3.11, $\neg\neg(F(A) \cup \{x\})$ does not have the κ-meet property. It implies that there exists an indexed subset $(x_i)_{i\in\kappa}$ of F_A such that $0 = \neg\neg x \wedge \bigwedge_{i\in\kappa} \neg\neg x_i = 1 \wedge \bigwedge_{i\in\kappa} \neg\neg x_i = \bigwedge_{i\in\kappa} \neg\neg x_i$ which is a contradiction since F_A is a proper κ-filter. Thus, $Ds(A) \subseteq F_A$ and, by the item above, $\langle F_A \rangle_\kappa^B$ is an extension of F_A on B.

d. Let us suppose that $F_A = \langle X \rangle_\kappa^A$ where X is closed under κ-meets. Then $\langle F_A \rangle_\kappa^B$ is an extension of F_A to B. Indeed: By Proposition 3.9 F_A has the form $F_A = \{x \in A : \exists (x_i)_{i\in\kappa} \subseteq X \text{ such that } \bigwedge_{i\in\kappa} x_i \leq x\}$ and $\langle F_A \rangle_\kappa^B = \{x \in B : \exists (x_i)_{i\in\kappa} \subseteq X \text{ such that } \bigwedge_{i\in\kappa} x_i \leq x\}$. Clearly $F_A \subseteq \langle F_A \rangle_\kappa^B \cap A$. To see the converse, let $a \in \langle F_A \rangle_\kappa^B \cap A$. Then there exists $(x_i)_{i\in\kappa} \subseteq X$

such that $\bigwedge_{i\in\kappa} x_i \leq a$. Since $\bigwedge_{i\in\kappa} x_i \in X \subset F_A$ and F_A is upward closed, it follows that $a \in F_A$. Hence our claim.

e. If F_A is closed under κ-meets then $\langle F_A\rangle_\kappa^B$ is an extension of F_A to B. Indeed: Since $F_A = \langle F_A\rangle_\kappa^A$ then F_A is generate by a set closed under κ-meets. Hence, by item d, our claim.

f. If F_A is a κ-filter generated by a set of regular elements of A then $\langle F_A\rangle_\kappa^B$ is an extension of F_A to B. Indeed: Let us suppose that $\langle X\rangle_\kappa^A = F_A$ where $X \subseteq Reg(A)$. By Proposition 3.3-2 we have that for each set $(x_i)_{i\in\kappa} \subseteq X$, $\bigwedge_{i\in\kappa} x_i = \bigwedge_{i\in\kappa} \neg\neg x_i \in F_A$. If we consider the set $X_1 = \{\bigwedge_{i\in\alpha} x_i : (x_i)_{i\in\alpha\leq\kappa} \subseteq X\}$ then $X \subseteq X_1 \subseteq F_A$ and, consequently, $\langle X_1\rangle_\kappa^A = F_A$. Hence, by item d, $\langle F_A\rangle_\kappa^B$ is an extension of F_A to B because X_1 is closed under κ-meets.

Note that for each $A \in \mathcal{B}_\kappa$, $Ds(A) = \{1\}$. Then, Theorem 3.13 allows us to establish the following well known result:

Corollary 3.15. *$\mathcal{B}_\kappa$ satisfies CEP for $\omega \leq \kappa \leq \infty$.* ■

4. Direct Indecomposabilty

In this section we study the direct decomposition of Heyting κ-frames. We first recall some basic results about direct decomposition of Heyting algebras. Let A be a Heyting algebra. An element $z \in A$ is called a *Boolean* or *central* element of A iff $z \vee \neg z = 1$. We denote by $B(A)$ the set of all boolean elements of A which is also called *the center of* A. It is well known that $\langle B(A), \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean subalgebra of A. Let $z \in B(A)$ and let us consider the segment $(z]$. If we define the binary operation $\rightarrow_z$ in $(z]$ by the formula $x \rightarrow_z y = z \wedge (x \rightarrow y)$ then $[0, z]_H = \langle (z], \wedge, \vee, \rightarrow_z, 0, z\rangle$ is a Heyting algebra. The map

$$B(A) \ni z \mapsto \theta_z = \{(x, y) \in A^2 : x \wedge z = y \wedge z\} \tag{16}$$

is a Boolean isomorphism between $B(A)$ and the Boolean sub lattice of $Con_H(A)$ of factor congruences. The correspondence $x/_{\theta_z} \mapsto x \wedge z$ defines a Heyting isomorphism from $A/_{\theta_z}$ onto $[0, z]_H$ and $x \mapsto (x \wedge z, x \wedge \neg z)$ defines a Heyting isomorphism from A onto $[0, z]_H \times [0, \neg z]_H$. Conversely, if $f : A \rightarrow A_1 \times A_2$ is a Heyting-isomorphism, the element $z \in A$ such that $f(z) = (1, 0)$ is the unique element in $B(A)$ such that A_1 is Heyting isomorphic to $[0, z]_H$ and A_2 is Heyting isomorphic to $[0, \neg z]_H$. In this way, a Heyting algebra A is directly idecomposable iff $B(A) = \{0, 1\}$. We denote by $\mathcal{DI}(\mathcal{H})$ the class of directly idecomposable Heyting algebras and by $\mathcal{DI}(\mathcal{H}_\kappa)$

the class of directly idecomposable Heyting κ-frames. In what follows we shall establish analogous results for Heyting κ-frames.

PROPOSITION 4.1. *Let A be a Heyting κ-frame and $z \in B(A)$. Then:*

1. *The structure $[0, z]_\kappa = \langle (z], \bigvee_{i\in\kappa}, \wedge, \to_z, 0, z\rangle$ is a Heyting κ-frame.*
2. *$\theta_z = \{(x, y) \in A^2 : x \wedge z = y \wedge z\} \in Con_\kappa(A)$ and the correspondence $x/\theta_z \mapsto x \wedge z$ defines an $\mathcal{H}_\kappa$-isomorphism from $A/_{\theta_z}$ onto $[0, z]_\kappa$.*
3. *$x \mapsto (x \wedge z, x \wedge \neg z)$ defines an $\mathcal{H}_\kappa$-isomorphism from A onto the direct product $[0, z]_\kappa \times [0, \neg z]_\kappa$.*

PROOF. (1) Immediate. (2) Taking into account that θ_z is a Heyting-congruence, we only need to prove that if $(x_i, y_i) \in \theta_z$, for each $i \in \kappa$, then $(\bigvee_{i\in\kappa} x_i, \bigvee_{i\in\kappa} y_i) \in \theta_z$. Since $x_i \wedge z = y_i \wedge z$ then we have that $(\bigvee_{i\in\kappa} x_i) \wedge z = \bigvee_{i\in\kappa}(x_i \wedge z) = \bigvee_{i\in\kappa}(y_i \wedge z) = (\bigvee_{i\in\kappa} y_i) \wedge z$. Hence $(\bigvee_{i\in\kappa} x_i, \bigvee_{i\in\kappa} y_i) \in \theta_z$ and $\theta_z \in Con_\kappa(A)$. Thus, $A/_{\theta_z}$ is a Heyting κ-frame. Since $x/_{\theta_z} \mapsto x \wedge z$ defines a Heyting-isomorphism from $A/_{\theta_z}$ onto $[0, z]_\kappa$, it preserves κ-joins. Hence, it is a $\mathcal{H}_\kappa$-isomorphism. (3) Follows from items 2 and 3. ■

An immediate consequence of Proposition 4.1 and Eq. (13) is the following characterization of direct indecomposability on $\mathcal{H}_\kappa$.

THEOREM 4.2. *$\mathcal{DI}(\mathcal{H}_\kappa) = \mathcal{H}_\kappa \cap \mathcal{DI}(\mathcal{H})$ for $\omega \leq \kappa \leq \infty$.* ■

Hence, direct indecomposable algebras in $\mathcal{H}_\kappa$ where $\omega \leq \kappa \leq \infty$ are exactly the algebras in which the underlying Heyting structure is direct indecomposable in $\mathcal{H}$ i.e., its center is $\{0, 1\}$. An immediate consequence of this is that $\mathbf{2}_\kappa$ is the unique direct indecomposable algebra in $\mathcal{B}_\kappa$.

5. Glivenko Type Theorem and Regular Amalgamation in $\mathcal{H}_\kappa$

The purpose of this section is to formulate a version of Glivenko theorem extended to Heyting κ-frames and, at the same time, to study some consequences thereof. The Glivenko theorem reads that if A is a Heyting algebra then $\langle Reg(A), \vee^R, \wedge, \to, 0\rangle$ where $x \vee^R y = \neg\neg(x \vee y)$ is a Boolean algebra isomorphic to $A/_{Ds(A)}$. Furthermore, the double negation $\neg\neg$ defines a surjective Heyting homomorphism of the form

$$\neg\neg : A \to Reg(A) \tag{17}$$

where $Reg(A) \approx A/_{Ds(A)}$.

If we consider the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{H}$ then it has a left adjoint i.e, a *reflector* $\mathcal{H} \overset{\mathcal{R}}{\to} \mathcal{B}$, which assigns to every Heyting algebra A the boolean algebra $\mathcal{R}(A) = Reg(A)$ and to every Heyting homomorphism $f : A \to B$ the corresponding restriction $\mathcal{R}(f) = f/_{Reg(A)}$. Moreover $\mathcal{R}$ preserves injective Heyting homomorphisms. In this way $\mathcal{B}$ is a reflective subcategory of $\mathcal{H}$. By Proposition 2.1-12 we can easily establish the following Glivenko type theorem for $\mathcal{H}_\kappa$ where $\kappa \neq \infty$:

Theorem 5.1. *Let $A \in \mathcal{H}_\kappa$. Then, the structure $\langle Reg(A), \bigvee^R_{i\in\kappa}, \wedge, \to, 0\rangle$, where $\bigvee^R_{i\in\kappa} x_i = \neg\neg \bigvee_{i\in\kappa} x_i$, belongs to $\mathcal{B}_\kappa$.*

Proof. Let $(x_i)_{i\in\kappa}$ be an indexed set in $Reg(A)$. We prove that $\bigvee^R_{i\in\kappa} x_i$ is the supremum of the family in $Reg(A)$. We first note that, for each $i \in \kappa$, $x_i \leq \bigvee_{i\in\kappa} x_i \leq \neg\neg \bigvee_{i\in\kappa} x_i = \bigvee^R_{i\in\kappa} x_i$. Let $a \in Reg(A)$ be an upper bound of $(x_i)_{i\in\kappa}$. Then $\bigvee^R_{i\in\kappa} x_i = \neg\neg \bigvee_{i\in\kappa} x_i \leq \neg\neg a = a$. Thus $\bigvee^R_{i\in\kappa}$ defines the κ-supremum in $Reg(A)$. Since $\bigvee^R_{i\in\kappa}$ coincides with $\vee^R$ for finite subsets of $Reg(A)$ then, by the Glivenko theorem for Heyting algebras, we have $Reg(A) \in \mathcal{B}_\kappa$. ■

Let us notice that the restriction $\mathcal{R}/_{\mathcal{H}_\kappa}$ defines a functor $\mathcal{H}_\kappa \overset{\mathcal{R}}{\to} \mathcal{B}_\kappa$ which is the reflector of the inclusion functor $\mathcal{B}_\kappa \hookrightarrow \mathcal{H}_\kappa$. Moreover $\mathcal{R}/_{\mathcal{H}_\kappa}$ also preserves injective $\mathcal{H}_\kappa$-homomorphisms. By Example 3.4-b we also have that the double negation $\neg\neg$ defines a $\mathcal{H}_\kappa$-homomorphism of the form $\neg\neg : A \to \mathcal{R}(A) = Reg(A)$ for each $A \in \mathcal{H}_\kappa$ where $Reg(A) \approx A/_{Ds(A)}$.

Proposition 5.2. *Let $A \in \mathcal{H}_\kappa$ and $t(\overline{x}) \in Term_\kappa(X)$ where $\overline{x} = (x_i)_{i\in\alpha\leq\kappa}$. Then for each indexed set $\overline{a} = (a_i)_{i\in\alpha\leq\kappa}$ in A we have that*

$$\neg\neg t^A(\overline{a}) = t^{Reg(A)}(\overline{\neg\neg a})$$

where $\overline{\neg\neg a} = (\neg\neg a_i)_{i\in\alpha\leq\kappa}$.

Proof. We use induction on the notion of complexity of terms introduced in Eq. (3). If $Comp(t) = 0$ then $t(\overline{x}) \in X$ i.e., t is a variable. Thus the result is immediate. By inductive hypothesis let us assume that the proposition hold for $Comp(t) < \gamma$. If $Comp(t) = \gamma$ then we have to consider the cases $t(\overline{x}) = t_1(\overline{x}) \wedge t_2(\overline{x})$, $t(\overline{x}) = t_1(\overline{x}) \to t_2(\overline{x})$ and $t(\overline{x}) = \bigvee_{i\in\kappa} t_i(\overline{x})$.

By Proposition 2.1-7 and inductive hypothesis we have that

$$\begin{aligned}\neg\neg t^A(\overline{a}) &= \neg\neg\big(t_1^A(\overline{a}) \wedge t_2^A(\overline{a})\big) = \neg\neg t_1^A(\overline{a}) \wedge \neg\neg t_2^A(\overline{a}) \\ &= t_1^{Reg(A)}(\overline{\neg\neg a}) \wedge t_2^{Reg(A)}(\overline{\neg\neg a}) = t^{Reg(A)}(\overline{\neg\neg a}).\end{aligned}$$

By an analogous argument, considering Proposition 2.1-9 and the inductive hypothesis, we can also prove the case $t(\overline{x}) = t_1(\overline{x}) \to t_2(\overline{x})$.

By Proposition 2.1-12, Theorem 5.1 and inductive hypothesis we have that

$$\begin{aligned}\neg\neg t^A(\overline{a}) &= \neg\neg \bigvee_{i\in\kappa} t_i^A(\overline{a}) = \neg\neg \bigvee_{i\in\kappa} \neg\neg t_i^A(\overline{a}) \\ &= \neg\neg \bigvee_{i\in\kappa} t_i^{Reg(A)}(\neg\neg\overline{a}) = \bigvee_{i\in\kappa}^{R} t_i^{Reg(A)}(\neg\neg\overline{a}) \\ &= t^{Reg(A)}(\overline{\neg\neg a}).\end{aligned}$$

Hence our claim. ∎

The result above allows us to establish the following version of the negative Gödel translation for Heyting κ-frames:

Proposition 5.3. *Let $t \in Term_\kappa(X)$ where $\kappa \neq \infty$. Then*

$$\mathcal{B}_\kappa \models t = 1 \quad iff \quad \mathcal{H}_\kappa \models \neg\neg t = 1.$$

Proof. Let us assume thar $\mathcal{B}_\kappa \models t = 1$ but suppose that $\mathcal{H}_\kappa \not\models \neg\neg t = 1$. Clearly we can suppose that t has the form $t(\overline{x})$ where $\overline{x} = (x_i)_{i\in\alpha\leq\kappa}$ are the variables occurring in t. Thus, there exists an indexed set $\overline{a} = (a_i)_{i\in\alpha\leq\kappa}$ in A such that $\neg\neg t^A(\overline{a}) \neq 1$. Then, by Proposition 5.2, we also have $t^{Reg(A)}(\overline{\neg\neg a}) \neq 1$ which is a contradiction since $\overline{\neg\neg a} = (\neg\neg a_i)_{i\in\alpha\leq\kappa}$ is an idexed set in $Reg(A) \in \mathcal{B}_\kappa$. Now let us assume that $\mathcal{H}_\kappa \models \neg\neg t = 1$. Since $\mathcal{B}_\kappa \subseteq \mathcal{H}_\kappa$ we have that $\mathcal{B}_\kappa \models \neg\neg t = 1$. Consequently $\mathcal{B}_\kappa \models t = 1$ because $\neg\neg t = t$ in $\mathcal{B}_\kappa$. ∎

An interesting instance of the negative Gödel translation related to σ-frames [4,6,22] endowed with the residuum of $\wedge$ i.e. the class $\mathcal{H}_\omega$ can be established. In order to do this we use the famous Loomis-Sikorski Theorem, proved independently by Loomis [21] and Sikorski [30]. We first recall the following notion. A *σ-field of sets* related to a nonempty set X is a set of the form $T \subseteq \mathbf{2}^X$ endowed the σ-complete Boolean structure inhered from $\mathbf{2}^X$.

Theorem 5.4. [Loomis-Sikorski] *Let A be a σ-complete Boolean algebra. Then there exists a σ-field of sets T and a surjective $\mathcal{B}_\omega$-homomorphism $f : T \to A$.* ∎

Proposition 5.5. *Let $t \in Term_\omega(X)$. Then:*

$$\mathbf{2}_\omega \models t = 1 \quad iff \quad \mathcal{H}_\omega \models \neg\neg t = 1.$$

Proof. Let $t(\overline{x}) \in Term_\omega(X)$. With regard to the non-trivial direction let us assume that $\mathbf{2}_\omega \models t(\overline{x}) = 1$. Let us notice that for each σ-field of sets T we have that $T \models t(\overline{x}) = 1$ because T is a σ-complete Boolean algebra that can be $\bigvee_\kappa$-embedded into a direct product of the form $\prod_I \mathbf{2}_\omega$. Let $A \in \mathcal{B}_\omega$ and $v : Term_\omega(X) \to A$ be a valuation such that $v(\overline{x}) = \overline{a}$. By Theorem 5.4 there exists a σ-field of sets T and a surjective $\mathcal{B}_\omega$-homomorphism $f : T \to A$. Thus there exists a sequence $\overline{m}$ in T such that $f(\overline{m}) = \overline{a}$ because f is surjective. Since $t^T(\overline{m}) = 1$ then $v(t(\overline{x})) = t^A(\overline{a}) = f(t^T(\overline{m})) = f(1) = 1$. It proves that $A \models t(\overline{x}) = 1$ and the equation holds in $\mathcal{B}_\omega$. Then, by Proposition 5.3, $\mathcal{H}_\omega \models \neg\neg t(\overline{x}) = 1$. ■

Theorem 5.1 also enables us to establish a kind of strong amalgamation property related to regular elements.

Proposition 5.6. *Let $C \overset{g}{\hookleftarrow} A \overset{f}{\hookrightarrow} B$ be a V-formation in $\mathcal{H}_\kappa$ for $\omega \leq \kappa \leq \infty$. Then there exist $\mathcal{H}_\kappa$-homomorphisms $C \overset{g_r}{\to} E \overset{f_r}{\leftarrow} B$ such that:*

1. *The restictions $f_r/_{Reg(B)}$ and $g_r/_{Reg(C)}$ are injectives,*
2. *$f_r f/_{Reg(A)} = g_r g/_{Reg(A)}$,*
3. *$Imag(f_r/_{Reg(B)}) \cap Imag(g_r/_{Reg(C)}) = Imag(f_r f/_{Reg(A)})$.*

Proof. Suppose that $\kappa \neq \infty$. By Eq. (17) let us consider the $\mathcal{B}_\kappa$-homomorphisms

$$Reg(C) \overset{\mathcal{R}(g)}{\longleftarrow} Reg(A) \overset{\mathcal{R}(f)}{\longrightarrow} Reg(B). \tag{18}$$

Since f, g are injectives and $\mathcal{R}$ preserves injective $\mathcal{H}_\kappa$-homomorphisms, $\mathcal{R}(f)$ and $\mathcal{R}(g)$ are injective $\mathcal{B}_\kappa$-homomorphisms defining a V-formation in $\mathcal{B}_\kappa$. In [20] it is proved that $\mathcal{B}_\kappa$ satisfies the strong amalgamation property. Then we can consider a strong amalgam $Reg(C) \overset{i_c}{\to} E \overset{i_B}{\leftarrow} Reg(B)$ in $\mathcal{B}_\kappa$ for the V-formation given in Eq. (18). Let us consider the following diagram

$$
\begin{array}{ccccccc}
A & & \xrightarrow{\quad f \quad} & & B & & \\
\Big\downarrow g & \searrow^{\neg\neg_A} & & & & \searrow^{\neg\neg_B} & \\
 & & Reg(A) & \xrightarrow{\mathcal{R}(f)} & & & Reg(B) \\
C & & \Big\downarrow \mathcal{R}(g) & & & & \Big\downarrow i_B \\
 & \searrow_{\neg\neg_C} & & & & & \\
 & & Reg(C) & \xrightarrow[i_C]{} & & & E
\end{array} \tag{19}
$$

Since the restriction $\neg\neg_X/_{Reg(X)}$ is injective for $X \in \{A, B, C\}$, the $\mathcal{H}_\kappa$-homomorphisms $f_r = i_B \circ \neg\neg_B$ and $g_r = i_C \circ \neg\neg_C$ satisfy the properties enunciated in the proposition.

Now let us consider the case $\kappa = \infty$. By Proposition 2.5-4 we have that the V-formation $C \overset{g}{\hookleftarrow} A \overset{f}{\hookrightarrow} B$ in $\mathcal{H}_\infty$ is **a** V-formation in $\mathcal{H}_\kappa$ where $\kappa = \max\{|A|, |B|, |C|\}$. Since each Boolean algebra can be embedded into a complete Boolean algebra preserving all existing meets and joins, we can assume that E is a complete Boolean algebra in the diagram of Figure (19). Thus, the above diagram is a diagram in $\mathcal{H}_\infty$ and the proposition holds in $\mathcal{H}_\infty$. ■

6. Injective Objects in $\mathcal{H}_\kappa$ and $\mathcal{H}_\infty$

In this section we characterize the injective objects in the categories $\mathcal{H}_\kappa$ for $\omega \leq \kappa \leq \infty$. For this we need following results:

THEOREM 6.1.

1. *$\mathcal{B}_\kappa$ has only trivial injective objects for $\omega \leq \kappa \neq \infty$* [27].
2. *$\mathcal{B}_\infty$ has only trivial injective objects* [5]. ■

Let $A \in \mathcal{H}_\kappa$ where $\omega \leq \kappa \leq \infty$. It is immediate to see that $Ds(A) \cup \{0\}$ is a sub $\mathcal{H}_\kappa$-algebra of A. Let us consider an order extension of $Ds(A) \cup \{0\}$ by adding a new bottom $\bot$. In this way $\langle Ds(A) \cup \{0, \bot\}, \leq \rangle$ is an ordered set in which $\bot < 0$ and 0 is the unique atom. $Ds(A) \cup \{0, \bot\}$ has a $\bigvee_\kappa$-complete lattice ordered structure inherited from $Ds(A) \cup \{0\}$. Moreover, if we define the operation $\to$ in $Ds(A) \cup \{0, \bot\}$ as

$$x \to y = \begin{cases} x \to y, & \text{if } x, y \in Ds(A) \cup \{0\}, \\ \bot, & \text{if } x \neq \bot, y = \bot, \\ 1, & \text{if } x = \bot. \end{cases}$$

then the following algebra

$$D_\bot(A) = \langle Ds(a) \cup \{0\}, \bigvee_\kappa, \wedge, \to, \bot, 1 \rangle \tag{20}$$

belongs to $\mathcal{H}_\kappa$ and 0 is the minimum dense element in $D_\bot(A)$.

THEOREM 6.2. *$\mathcal{H}_\kappa$ has only trivial injective objects for $\omega \leq \kappa \leq \infty$.*

PROOF. Let us suppose that A is an injective object in $\mathcal{H}_\kappa$ for $\omega \leq \kappa \leq \infty$. Note that A can not be a Boolean algebra, otherwise A would be injective object in $\mathcal{B}_\kappa$ which is a contradiction by Theorem 6.1. Thus, $Ds(A) \neq \{1\}$. We first prove that there exists $d = \min Ds(A)$. Let us consider the algebra $Ds(A) \cup \{0\}$ in $\mathcal{H}_\kappa$ and the $\mathcal{H}_\kappa$-embeddings $\iota : Ds(A) \cup \{0\} \to A$ and $\iota_1 : Ds(A) \cup \{0\} \to D_\perp(A)$. Since A is an injective object in $\mathcal{H}_\kappa$ then there exists an $\mathcal{H}_\kappa$-homomorphism $f : D_\perp(A) \to A$ such that the following diagram is commutative:

$$\begin{array}{ccc} Ds(A) \cup \{0\} & \xrightarrow{\ \iota\ } & A \\ {\scriptstyle \iota_1}\big\downarrow & \equiv & \nearrow {\scriptstyle f} \\ D_\perp(A) & & \end{array}$$

Note that 0 is the minimum dense element in $D_\perp(A)$. Then $f(0) \in Ds(A)$ and, by the commutativity of the above diagram, $f(0) \leq x$ for each $x \in Ds(A)$. It proves that $d = f(0) = \min Ds(A)$. Thus, $\perp < 0 < d$ in $D_\perp(A)$. By the commutativity of the above diagram $d = f(0) = f(d)$. Therefore $d = f(0) = f(d \to 0) = f(d) \to f(0) = d \to d = 1$ which is a contradiction. Hence A is a trivial algebra. ■

7. Hilbert Style Calculus for Heyting κ-Frames

The development of infinitary languages was encouraged by Tarski and Henkin who organized a seminar on this topic at Berkeley in the fall of 1956. The interest of Tarski in this area led to a series of new developments in set theory that grew out of William Hanf's work on models of infinitary languages at the beginning of the 1960s [14]. In the same decade Carol Karp published his book where an infinitary Hilbert style calculus for a classical propositional system was introduced [19, § 5]. In this section we develop an infinitary Hilbert style calculus for $\mathcal{H}_\kappa$ for $\kappa \neq \infty$.

Taking into account the operation symbols $\langle \bigsqcup_\kappa, \wedge, \to, 0\rangle$ defining $Term_\kappa(X)$ (see Definition 1.4) we formally introduce the following syntactic abbreviations:

$$p \vee q \text{ is a syntactic abbreviation of } \bigsqcup_\kappa (p, q, 0, 0, \ldots),$$

$$\neg p \text{ is a syntactic abbreviation of } p \to 0,$$

$$1 \text{ is a syntactic abbreviation of } 0 \to 0.$$

A term $t \in Term_\kappa(X)$ is a *tautology* iff $\mathcal{H}_\kappa \models t = 1$. Each subset $T \subseteq Term_\kappa(X)$ is referred to as a *theory*. If v is a valuation, $v(T) = 1$ means that for each $t \in T$, $v(t) = 1$. We use the notation $T \models_{\mathcal{H}_\kappa} t$, read as t *is a* *semantic consequence* of T, to indicate that for each valuation v, $v(T) = 1$ implies $v(t) = 1$.

Definition 7.1. The calculus $\langle Term_\kappa(X), \vdash\rangle$ is given by the following set of axioms:

HK1. $0 \to p$,

HK2. $p \to (q \to p)$,

HK3. $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$,

HK4. $(p \wedge q) \to p$,

HK5. $(p \wedge q) \to q$,

HK6. $(p \to q) \to ((r \to q) \to ((p \vee r) \to q))$,

HK7. $p \to (q \to r) \to ((p \wedge q) \to r)$,

HK8. $((p \wedge q) \to r) \to (p \to (q \to r))$,

HK9. $p_i \to \bigsqcup_{i\in\kappa} p_i$,

HK10. $(p \wedge \bigsqcup_{i\in\kappa} q_i) \to \bigsqcup_{i\in\kappa}(p \wedge q_i)$,

and the following inference rules:

$$\frac{p, p\to q}{q} \quad \textit{modus ponens } (MP)$$

$$\frac{(p_i \to q)_{i\in\kappa}}{(\bigsqcup_{i\in\kappa} p_i)\to q} \quad \bigsqcup-\textit{rule}$$

We introduce the concept of κ-proof that generalizes the usual notion of proof in the standard Hilbert style calculus for intuitionistic logic. Let T be a theory in $Term_\kappa(X)$. A *κ-proof* from T is a subset of $Term_\kappa(X)$ ordered by ordinal numbers $t_0 \dots t_\alpha \dots t_\beta$ such that for each $\alpha \leq \beta$, t_α is either, an axiom or is inferred from earlier formulas t_γ, $\gamma < \alpha$ by modus ponens or $\bigsqcup$-rule. $T \vdash t$ means that t is *κ-provable* from T, that is, t is the last member of a proof from T. If $T = \emptyset$ we use the notation $\vdash t$ and, in this case, we will say that t is a *κ-theorem* of the calculus $\langle Term_\kappa(X), \vdash\rangle$. A theory T is *inconsistent* iff $T \vdash 0$; otherwise it is *consistent*. By Axiom HK1 we can show that T is inconsistent iff $T \vdash t$ for each $t \in Term_\kappa(X)$.

Theorem 7.2. [Soundness] *Let T be theory in $Term_\kappa(X)$. Then:*

$$T \vdash t \Longrightarrow T \models_{\mathcal{H}_\kappa} t.$$

Proof. Let $q \in Term_\kappa(X)$. Note that if q is an axiom HK1 ... HK10 then for each possible valuation v, $v(q) = 1$. It is clear that modus ponens preserves valuations equal to 1. We have to prove that the $\bigsqcup$-rule preserves valuations equal to 1. Suppose that $1 = v(p_i \to q) = v(p_i) \to v(q)$ for each $i \in \kappa$.

Then $v((\bigsqcup_{i\in\kappa} p_i) \to q) = (\bigsqcup_{i\in\kappa} v(p_i)) \to v(q) = \bigwedge_{i\in\kappa}(v(p_i) \to v(q)) = 1$. Thus the $\bigsqcup$-rule preserves valuations equal to 1. Hence the theorem is easily proved using transfinite induction. ■

With the usual argument used in intuitionistic logic we can establish the following technical result.

LEMMA 7.3. *Let T be theory in $Term_\kappa(X)$. Then:*

1. $\vdash p \to p$,
2. $\vdash p \to (q \to (p \wedge q))$,
3. $\vdash p \to (p \vee q)$ *and* $\vdash q \to (p \vee q)$,
4. *if* $T \vdash p$ *and* $T \vdash q$ *then* $T \vdash p \wedge q$,
5. *if* $T \vdash p$ *then* $T \vdash q \to p$ *for each* $q \in Term_\kappa(X)$. ■

The proof of the above lemma is a repetition of the proofs given in [12] with the obvious corrections due to the differences between both axiomatic systems.

The following theorem provides a version of the Deduction Theorem for the calculus $\langle Term_\kappa(X), \vdash\rangle$ taking into account the presence of an infinitary connective $\bigsqcup_{i\in\kappa}$.

THEOREM 7.4. [Deduction Theorem] *Let T be a theory in $Term_\kappa(X)$ and p, q be terms in $Term_\kappa(X)$. Then:*

$$T \cup \{p\} \vdash q \ \ iff \ \ T \vdash p \to q.$$

PROOF. Let $t_1 \dots t_\alpha \dots q$ be a κ-proof of q from $T \cup \{p\}$. Let us prove, by transfinite induction, that $T \vdash p \to t_\alpha$ for each t_α in the κ-proof. Note that t_1 must be either in T or an axiom or p itself. The first two cases follow by HK2. The third case follows by Lemma 7.3-1. Thus $T \vdash p \to t_1$. Let us assume that $T \vdash p \to t_\alpha$ for $\alpha < \beta$. If t_β is an axiom or $t_\beta \in T \cup \{p\}$, then $T \vdash p \to t_\beta$ follows as in the case t_1. If t_β follows by modus ponens from some t_α and t_γ where $\alpha < \beta$ and $\gamma < \beta$, then $T \vdash p \to t_\beta$ follows by inductive hypothesis and HK2 as in the standard intuitionistic calculus. We have to discuss the case of the $\bigsqcup$-rule. Suppose t_β has the form $(\bigsqcup_{i\in\kappa} p_i) \to t$ and $T \cup \{p\} \vdash p_i \to t$. Then:

(1) $T \vdash p \to (p_i \to t)$	by inductive hypothesis
(2) $T \vdash (p \wedge p_i) \to t$	by MP 1, HK7
(3) $T \vdash (\bigsqcup_{i\in\kappa}(p \wedge p_i)) \to t$	by $\bigsqcup$-rule, 2
(4) $T \vdash (p \wedge \bigsqcup_{i\in\kappa} p_i) \to (\bigsqcup_{i\in\kappa}(p \wedge p_i)) \to t)$	by Lemma 7.3-5

(5) $\vdash [(p \wedge \bigsqcup_{i\in\kappa} p_i) \to (\bigsqcup_{i\in\kappa}(p \wedge p_i)) \to t)] \to$
$[((p \wedge \bigsqcup_{i\in\kappa} p_i) \to \bigsqcup_{i\in\kappa}(p \wedge p_i)) \to ((p \wedge \bigsqcup_{i\in\kappa} p_i) \to t)]$ HK3

(6) $T \vdash ((p \wedge \bigsqcup_{i\in\kappa} p_i) \to \bigsqcup_{i\in\kappa}(p \wedge p_i)) \to ((p \wedge \bigsqcup_{i\in\kappa} p_i) \to t)$ by MP 4,5

(7) $T \vdash (p \wedge \bigsqcup_{i\in\kappa} p_i) \to t$ by MP 6, HK 10

(8) $T \vdash p \to ((\bigsqcup_{i\in\kappa} p_i) \to t)$ by MP 7, HK 8

The converse is immediate. It proves that $T \cup \{p\} \vdash q$ iff $T \vdash p \to q$. ■

LEMMA 7.5. *Let T be a theory in $Term_\kappa(X)$. Then:*

1. if $T \vdash p \to q$ and $T \vdash q \to r$ then $T \vdash p \to r$,

2. if $T \vdash p \to q$ and $T \vdash r \to s$ then $T \vdash (p \wedge r) \to (q \wedge s)$,

3. $\vdash \bigsqcup_{i\in\kappa}(p \wedge q_i) \to (p \wedge \bigsqcup_{i\in\kappa} q_i)$.

PROOF. (1) Immediate from Theorem 7.4. (2) By HK4 and Theorem 7.4 it is immediate that $T \cup \{p \wedge r\} \vdash q$ and $T \cup \{p \wedge r\} \vdash s$. Then, by Lemma 7.3-4 and Theorem 7.4, $T \vdash (p \wedge r) \to (q \wedge s)$. (3)

(1) $\vdash q_i \to \bigsqcup_{i\in\kappa} q_i$ HK9

(2) $\vdash p \to p$ by Lemma 7.3-1

(3) $\vdash (p \wedge q_i) \to (p \wedge \bigsqcup_{i\in\kappa} q_i)$ by 1,2 and item 2

(4) $\vdash \bigsqcup_{i\in\kappa}(p \wedge q_i) \to (p \wedge \bigsqcup_{i\in\kappa} q_i)$ by $\bigsqcup$-rule in 3 ■

THEOREM 7.6. *Let T be a theory in $Term_\kappa(X)$. Consider the following relation in $Term_\kappa(X)$:*

$$p \equiv_T q \qquad iff \qquad T \vdash (p \to q) \wedge (q \to p).$$

Then $\equiv_T$ is an equivalence relation in $Term_\kappa(X)$. Moreover if we define the following operations in the quotient $L_T(X) = Term_\kappa(X)/_{\equiv_T}$:

$[\alpha]_T \wedge [\beta]_T = [\alpha \wedge \beta]_T$, $\qquad [p]_T \to [q]_T = [p \to q]_T$,

$\bigsqcup_{i\in\kappa}[p_i]_T = [p_i]_T$, $\qquad 0 = [0]_T$,

then:

1. $T \vdash t$ if and only if $[t]_T = 1$,

2. $\langle L_T(X), \bigsqcup_{i\in\kappa}, \wedge, \to, 0\rangle$ is a Heyting κ-frame,

3. $\langle L_\emptyset(X), \bigsqcup_{i\in\kappa}, \wedge, \to, 0\rangle$ is a free object in $\mathcal{H}_\kappa$ on X generators.

PROOF. It is immediate that $\equiv_T$ is symmetric. By Lemma 7.3-1 and Lemma 7.5-1, it is reflexive and transitive. Thus $\equiv_T$ is an equivalence relation in $Term_\kappa(X)$.

(1) Immediate from Lemma 7.3-(1 and 5).

(2) With the same argument used in intuitionistic logic we can see that $\wedge$ and $\rightarrow$ are well defined in $L_T(X)$. We have to prove that $\bigsqcup$ is well defined in $L_T(X)$. Suppose that $[p_i]_T = [q_i]_T$ for each $i \in \kappa$.

(1) $T \vdash p_i \rightarrow q_i$	by hypothesis
(2) $\vdash q_i \rightarrow \bigsqcup_{i \in \kappa} q_i$	HK9
(3) $T \vdash p_i \rightarrow \bigsqcup_{i \in \kappa} q_i$	by 1,2 and Lemma7.5-1
(4) $T \vdash (\bigsqcup_{i \in \kappa} p_i) \rightarrow \bigsqcup_{i \in \kappa} q_i$	$\bigsqcup$-rule in 3

Analogously we can prove that $T \vdash (\bigsqcup_{i \in \kappa} q_i) \rightarrow \bigsqcup_{i \in \kappa} p_i$. Therefore $\bigsqcup_{i \in \kappa}$ is well defined in $L_T(X)$. Now we prove that $L_T(X)$ is a Heyting κ-frame. Note that Axioms HK1 ... HK6, HK9 plus Lemma 7.3-3 define the usual intuitinistic calculus. Then $\langle L_T(X), \vee, \wedge, \rightarrow, 0 \rangle$ is a Heyting algebra. By HK9 and item 1 we have that $1 = [p_i]_{T,X} \rightarrow \bigsqcup_{i \in \kappa} [p_i]_{T,X}$ thus $L_T(X)$ satisfies the axiom $\kappa 2$ of Heyting κ-frames. By item 1, Lemma 7.5-4 and HK10, $\bigsqcup_{i \in \kappa}([p]_T \wedge [q_i]_T) = [p]_T \wedge \bigsqcup_{i \in \kappa} [q_i]_T$ i.e. $L_T(X)$ satisfies the axiom $\kappa 3$ of Heyting κ-frames. It proves that $\langle L_T(X), \bigsqcup_{i \in \kappa}, \wedge, \rightarrow, 0 \rangle$ is a Heyting κ-frame.

(3) It is clear that we can identify the set X with $\{[x]_\emptyset \in L_\emptyset(X) : x \in X\}$. Let A be a Heyting κ-frame and $f : X \rightarrow A$ be a function. Then we can extend f to a unique valuation $v_f : Term_\kappa(X) \rightarrow A$. By Theorem 7.2 we can see that if $\vdash p$ then $v_f(p) = 1$. Thus, if $[p]_\emptyset = [q]_\emptyset$ i.e. $\vdash p \rightarrow q$ and $\vdash q \rightarrow p$ then $v_f(p) = v_f(q)$. This implies that the assignment $L_\emptyset(X) \ni [p]_\emptyset \mapsto v([p]_\emptyset) = v_f(p)$ is a well defined map of the form $v : L_\emptyset(X) \rightarrow A$. Note that v is a $\mathcal{H}_\kappa$-homomorphism because v_f is a valuation. Since v_f uniquely extends f to a $\mathcal{H}_\kappa$-homomorphism then v is the unique $\mathcal{H}_\kappa$-homomorphism such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{f} & A \\ 1_X \downarrow & \equiv & \nearrow v \\ L_\emptyset(X) & & \end{array}$$

Hence, $L_\emptyset(X)$ is a free object in $\mathcal{H}_\kappa$ on X generators. ■

Remark 7.7. Let us notice that in an equational class of infinitary algebras, the free algebra on X generators always exists (see [31, Proposition 8.3]). In our case, Proposition 7.6-3 provides a representation of the free Heyting κ-frame on X generators built up from an infinitary Hilbert style calculus.

Theorem 7.8. [Completeness] *Let X be a nonempty set, $t \in Term_\kappa(X)$ and $T \subseteq Term_\kappa(X)$. Then:*

1. $L_\emptyset(X) \models t = 1$ *iff* $\mathcal{H}_\kappa \models t = 1$.

2. $T \models_{\mathcal{H}_\kappa} t$ *iff* $T \vdash t$.

Proof. (1) Suppose that $L_\emptyset(X) \models t = 1$. Assume that there exist a Heyting κ-frame A and a valuation $v : X \to A$ such that $v(t) \neq 1$. Since $L_\emptyset(X)$ is a free object then there exists an $\mathcal{H}_\kappa$-homomorphisms $v^* : L_\emptyset(X) \to A$ such that the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{v} & A \\ {\scriptstyle 1_X}\downarrow & \equiv & \nearrow {\scriptstyle v^*} \\ L_\emptyset(X) & & \end{array}$$

Thus $v^*([t]_\emptyset) \neq 1$ and $[t]_\emptyset \neq 1$ which is a contradiction. It proves that $\mathcal{H}_\kappa \models t = 1$. The other direction is trivial.

(2) Suppose that $T \models_{\mathcal{H}_\kappa} t$ but $T \not\vdash t$. Then $v : X \to L_T(X)$ such that $v(x) = [x]_T$ defines a valuation satisfying $v(T) = 1$ and $v(t) = [t]_T \neq 1$ which is a contradiction. The converse is immediate. ■

Remark 7.9. In the standard algebraic logic in which only connectives with finite arity are allowed, completeness type theorems like the one above are formulated taking into account the Lindenbaum algebra generated by a denumerable set. The reason to privilege this algebra is due to the fact that the quantity of variables in a term is finite. In the infinitary case, the validity of an equation can be studied in a Lindenbaum algebra whose generators contain all the variables that appear in the equation.

Theorem 7.10. *Let A be a Heyting κ-frame. Then there exist a set X and a theory $T \subseteq Term_\kappa(X)$ such that*

$$A \cong L_T(X).$$

Proof. Let X be a subset of A generating A. Then the inclusion map $i : X \to A$ defines a valuation $i^* : Term_\kappa(X) \to A$. Let us consider the following theory in $Term_\kappa(X)$:

$$T = \{t \in Term_\kappa(X) : i^*(t) = 1\}.$$

We shall prove that $f : L_T(X) \to A$ such that $f([t]_T) = i^*(t)$ is a $\mathcal{H}_\kappa$-isomorphism. Note that if $[t]_T = [q]_T$ then, $T \vdash (t \to q) \wedge (q \to t)$ and $i^*((t \to q) \wedge (q \to t)) = 1$. Thus $i^*(t) = i^*(q)$ and f is well defined. If $i^*(t) = 1$ then $t \in T$ and $[t]_T = 1$ in $L_T(X)$. This argument proves that f is injective. Note that f is a $\mathcal{H}_\kappa$-homomorphisms since i^* is a valuation. Now we prove that f is a surjective map. For this, by Proposition 2.8, we use induction in the family $\mathbb{E}_i(X)$ introduced in Eq. (9). Let $a \in A$.

If $a \in X = \mathbb{E}_0(X)$ then $f([a]_T) = i^*(a) = a$. Let γ be an ordinal number such that $\gamma < \kappa$. By inductive hypothesis let us assume that for each $a \in \mathbb{E}_i(X)$ where $i < \gamma$, there exists $[t]_T \in L_T(X)$ such that $f([t]_T) = a$. Suppose that $a \in \mathbb{E}_\gamma(X)$. If $a = \bigsqcup_{i\in\kappa} b_i$ such that $(b_i)_{i\in\kappa} \subseteq \bigcup_{i\in\gamma} \mathbb{E}_i(X)$ then there exists a family $([t_i]_T)_{i\in\kappa}$ in $L_T(X)$ such that $f([t_i]_T) = b_i$. Therefore $[\bigsqcup_{i\in\kappa} t_i]_T \in L_T(X)$ and $f([\bigsqcup_{i\in\kappa} t_i]_T) = \bigsqcup_{i\in\kappa} f([t_i]_T) = \bigsqcup_{i\in\kappa} b_i = a$. With the same argument we can see that there exists $[t]_T \in L_T(X)$ such that $f([t]_T) = a$ when, $a = b \star c$, $\{b, c\} \subseteq \bigcup_{i\in\gamma} \mathbb{E}_i(X)$ and $\star \in \{\wedge \rightarrow\}$. Thus f is surjective. Hence f is a $\mathcal{H}_\kappa$-isomorphism. ■

8. Equational Completeness for Heyting κ-Frames

Subdirectly irreducible algebras play a crucial role in many aspects of the standard universal algebra. One of these is a natural completeness theorem for equational classes of algebras arising from the Birkhoff's Subdirect Representation Theorem [9, Theorem 8.6]. In general, this argument could be not valid for infinitary algebras because the Birkhoff theorem may not be valid and the notion of subdirectly irreducible algebra becomes meaningless in this extended algebraic framework. However, in this section, an equational completeness theorem with respect to a subclass of subdirecly irreducible Heyting algebras is obtained without resorting to the Birkhoff's theorem.

Theorem 8.1. [2, § IX, Theorem 5] *A Heyting algebra is subdirectly irreducible iff it has exactly one coatom.* ■

We denote by $\mathbf{SI}(\mathcal{H})$ the class of subdirectly irreducibles Heyting algebras. For the class $\mathcal{H}_\kappa$ where $\kappa \neq \infty$ we define the following class

$$\mathbf{SI}(\mathcal{H}_\kappa) = \mathcal{H}_\kappa \cap \mathbf{SI}(\mathcal{H}).$$

Proposition 8.2. *Let A, B be two Heyting κ-frames and $f : A \rightarrow B$ be a function such that $\bigsqcup_{i\in\kappa} f(x_i) \leq f(\bigsqcup_{i\in\kappa} x_i)$, $f(x) \wedge f(y) \leq f(x \wedge y)$, $f(0) = 0$ and $f(1) = 1$. Let us consider the set*

$$A \times^f B = \{(a, b) \in A \times B : b \leq f(a)\}$$

endowed with the following operations

$(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2, b_1 \wedge b_2)$,

$(a_1, b_1) \rightarrow (a_2, b_2) = (a_1 \rightarrow a_2, (b_1 \rightarrow b_2) \wedge f(a_1 \rightarrow a_2))$,

$\bigsqcup_{i\in\kappa}(a_i, b_i) = (\bigsqcup_{i\in\kappa} a_i, \bigsqcup_{i\in\kappa} b_i)$,

$1 = (1, 1), \quad 0 = (0, 0).$

Then:

1. *f is order preserving.*
2. *$\langle A \times^f B, \wedge, \rightarrow, \bigsqcup_{i\in\kappa}, 0, 1\rangle$ is a Heyting κ-frame.*
3. *$\pi_A : A \times^f B \rightarrow A$ such that $\pi_A(a, b) = a$ is a $\mathcal{H}_\kappa$-homomorphism.*

PROOF. (1) Let $a, b \in A$ such that $a \leq b$. Then $f(a) \leq f(a) \vee f(b) \leq f(a \vee b) = f(b)$.

(2) We first prove that $\langle A \times^f B, \bigsqcup_{i\in\kappa}, \wedge, 0, 1\rangle$ is a bounded distributive $\bigvee_\kappa$-complete lattice. For this we show that the operations $\langle \wedge, \bigsqcup_{i\in\kappa}\rangle$ are closed in $A \times^f B$. Indeed:

Let (a_1, b_1), (a_2, b_2) be two elements in $A \times^f B$, i.e. $b_i \leq f(a_i)$ for $i = 1, 2$. Then $b_1 \wedge b_2 \leq f(a_1) \wedge f(a_2) \leq f(a_1 \wedge a_2)$ and $(a_1 \wedge a_2, b_1 \wedge b_2) \in A \times^f B$. Thus $\wedge$ is closed in $A \times^f B$.

Let $(a_i, b_i)_{i\in\kappa}$ be a subset of $A \times^f B$, i.e. $b_i \leq f(a_i)$ for each $i \in \kappa$. Then $\bigsqcup_{i\in\kappa} b_i \leq \bigsqcup_{i\in\kappa} f(a_i) \leq f(\bigsqcup_{i\in\kappa} a_i)$ and $(\bigsqcup_{i\in\kappa} a_i, \bigsqcup_{i\in\kappa} b_i) \in A \times^f B$. Thus $\bigsqcup_{i\in\kappa}$ is closed in $A \times^f B$.

In this way $A\times^f B$ inherits the lattice order structure of the product $A\times B$ proving that $\langle A \times^f B, \bigsqcup_{i\in\kappa}, \wedge, 0, 1\rangle$ is a bounded distributive $\bigvee_\kappa$-complete lattice.

Now we prove that $\rightarrow$ is the residuum of $\wedge$ in $A \times^f B$. It is immediate to see that $\rightarrow$ is a closed operation in $A \times^f B$.

On the one hand let us suppose that $(x, y) \wedge (a_1, b_1) \leq (a_2, b_2)$. Since the lattice order structure of $A \times^f B$ is componentwise, taking into account the residuum of $\wedge$ in A and in B respectively, we have that

$$x \leq a_1 \rightarrow a_2 \quad and \quad y \leq b_1 \rightarrow b_2. \tag{21}$$

By item 1, f is order preserving and then, by Eq. (21), we have that $f(x) \leq f(a_1 \rightarrow a_2)$. Thus $y \leq f(a_1 \rightarrow a_2)$ because $y \leq f(x)$. Therefore, by Eq. (21), $y \leq (b_1 \rightarrow b_2) \wedge f(a_1 \rightarrow a_2)$ proving that

$$(x, y) \leq (a_1 \rightarrow a_2, (b_1 \rightarrow b_2) \wedge f(a_1 \rightarrow a_2)) = (a_1, b_1) \rightarrow (a_2, b_2).$$

On the other hand let us assume that $(x, y) \leq (a_1, a_2) \rightarrow (b_1, b_2)$. Since $x \leq a_1 \rightarrow b_1$, by residuation in A, it is immediate that $x \wedge a_1 \leq b_1$. Since $y \leq (b_1 \rightarrow b_2) \wedge f(a_1 \rightarrow a_2) \leq b_1 \rightarrow b_2$, by residuation in B, we have that $y \wedge b_1 \leq b_2$. It proves that

$$(x, y) \wedge (a_1, b_1) \leq (a_2, b_2).$$

Hence $\to$ is the residuum of $\wedge$ in $A \times^f B$ and $\langle A \times^f B, \wedge, \to, \bigsqcup_{i\in\kappa}, 0, 1\rangle$ is a Heyting κ-frame.

(3) Immediate. ■

Proposition 8.3. *Let κ be an infinite cardinal and let us consider the free algebra $L_\emptyset(X)$ in $\mathcal{H}_\kappa$ introduced in Theorem 7.6-3. Then there exists an $\mathcal{H}_\kappa$-embedding $i_S : L_\emptyset(X) \to A$ such that $A \in \mathbf{SI}(\mathcal{H}_\kappa)$.*

Proof. Let us consider the function $f : L_\emptyset(X) \to \mathbf{2}$ such that

$$f(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{otherwise.} \end{cases}$$

We prove that $\bigsqcup_{i\in\kappa} f(x_i) \leq f(\bigsqcup_{i\in\kappa} x_i)$. Suppose that $\bigsqcup_{i\in\kappa} f(x_i) = 1$. Then there exists $i_0 \in \kappa$ such that $f(x_{i_0}) = 1$ and then $x_{i_0} = 1$. Thus $\bigsqcup_{i\in\kappa} x_i = 1$ and $f(\bigsqcup_{i\in\kappa} x_i) = 1$. It proves that $\bigsqcup_{i\in\kappa} f(x_i) \leq f(\bigsqcup_{i\in\kappa} x_i)$.

Now we prove that $f(x) \wedge f(y) \leq f(x \wedge y)$. Suppose that $f(x \wedge y) = 0$. Then $x \wedge y \neq 1$ and consequently, $x \neq 1$ or $y \neq 1$. Thus $f(x) = 0$ or $f(y) = 0$ and, $f(x) \wedge f(y) = 0 \leq f(x \wedge y)$. It proves that $f(x) \wedge f(y) \leq f(x \wedge y)$.

Since $f(0) = 0$ and $f(1) = 1$, by Proposition 8.2-1, $A = L_\emptyset(X) \times^f \mathbf{2}$ is a Heyting κ-frame. By definition of $L_\emptyset(X) \times^f \mathbf{2}$ we have that $(t, 0) \in A$ for each $t \in L_\emptyset(X)$ and $(t, 1) \in A$ iff $1 \leq f(t)$ iff $t = 1$. Thus A has the form

$$A = \{(t, 0) : t \in L_\emptyset(X)\} \cup \{(1, 1)\}$$

where $(1, 0)$ is the unique coatom in A. Consequently, $A \in \mathbf{SI}(\mathcal{H}_\kappa)$.

Let us consider the injective function $i : X \to A$ such that $i(x) = (x, 0)$. Since $L_\emptyset(X)$ is a free object in $\mathcal{H}_\kappa$, there exists an unique $\mathcal{H}_\kappa$-homomorphism $i_S : L_\emptyset(X) \to A$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\;i\;} & A \\ {\scriptstyle 1_X}\downarrow & \equiv & \nearrow {\scriptstyle i_S} \\ L_\emptyset(X) & & \end{array}$$

By Proposition 8.2-3, if we consider the $\mathcal{H}_\kappa$-homomorphism $\pi_{L_\emptyset(X)}$ then the external diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\;1_X\;} & L_\emptyset(X) \\ {\scriptstyle 1_X}\downarrow & \equiv \quad \nearrow {\scriptstyle 1_{L_\emptyset(X)}} & \uparrow {\scriptstyle \pi_{L_\emptyset(X)}} \\ L_\emptyset(X) & \xrightarrow[\;i_S\;]{} & A \end{array}$$

Then $\pi_{L_\emptyset(X)} i_S = 1_{L_\emptyset(X)}$ since, by Proposition 7.6-3, $L_\emptyset(X)$ is a free object in $\mathcal{H}_\kappa$. It proves that i_S is injective. Hence our claim. ■

THEOREM 8.4. *Let κ be an infinite cardinal number and $t \in Term_\kappa(X)$. Then,*

$$\mathbf{SI}(\mathcal{H}_\kappa) \models t = 1 \ \ iff \ \ \mathcal{H}_\kappa \models t = 1.$$

PROOF. Let us assume that $\mathbf{SI}(\mathcal{H}_\kappa) \models t = 1$. Suppose that there exists $\mathcal{H}_\kappa \not\models t = 1$. Then, by Theorem 7.8-1, $L_\emptyset(X) \not\models t = 1$. By Proposition 8.3, $L_\emptyset(X)$ can be $\mathcal{H}_\kappa$-embedded into a Heying κ-frame $A \in \mathbf{SI}(\mathcal{H}_\kappa)$. Thus there exists a valuation $v : Term_\kappa(X) \to A$ such that $v(t) \neq 1$ which is a contradiction. Hence $\mathcal{H}_\kappa \models t = 1$. The converse is immediate. ■

Intuitionism requires that the disjunction of two sentences is provable iff we can prove one of the two disjuncts. In algebraic terms it can be expresses by saying that free Heyting algebras are *well-connected* [25] i.e., if $x \vee y = 1$ in a free Heyting algebra then $x = 1$ or $y = 1$. Another interest consequence of Proposition 8.3 is the following generalization of the property mentioned above.

PROPOSITION 8.5. *Let $(t_i)_{i\in\kappa}$ be a set of terms in $Term_\kappa(X)$. Then*

1. *If $L_\emptyset(X) \models \bigsqcup_{i\in\kappa} t_i = 1$ then there exists $i_0 \in \kappa$ such that $L_\emptyset(X) \models t_{i_0} = 1$.*
2. *If $\vdash \bigsqcup_{i\in\kappa} t_i$ then there exists $i_0 \in \kappa$ such that $\vdash t_{i_0}$.*

PROOF. (1) Suppose that $\bigsqcup_{i\in\kappa} t_i = 1$ in $L_\emptyset(X)$. By Proposition 8.3, there exists a $\mathcal{H}_\kappa$-embedding $j : L_\emptyset(X) \to A$ where A has a unique coatom c_A. Then, $1 = j(\bigsqcup_{i\in\kappa} t_i) = \bigsqcup_{i\in\kappa} j(t_i) > c_A$. It implies that there exists $i_0 \in \kappa$ such that $j(t_{i_0}) = 1$ otherwise, c_A would be an upper bound of $(t_i)_{i\in\kappa}$ and $\bigsqcup_{i\in\kappa} j(t_i) < c_A$ which is a contradiction. Hence $t_{i_0} = 1$ in $L_\emptyset(X)$ because j is injective.

(2) Immediate from item 1 and Theorem 7.6-1. ■

9. Equational Type Completeness for $\mathcal{H}_\infty$

In this section we provide an equational type completeness theorem for $\mathcal{H}_\infty$ based on the Heyting κ-frame structure. For this we first introduce a language suitable for the notion of term in $\mathcal{H}_\infty$.

For each infinite cardinal number κ we define the following interval of cardinal numbers

$$[\omega, \kappa] = \{\eta \in \mathbf{Card} : \omega \leq \eta \leq \kappa\}.$$

Let us noticed that the objects in $\mathcal{H}_\infty$ can be seen as algebras of the form

$$\langle A, \wedge, \rightarrow, (\bigsqcup_\kappa)_{\omega \leq \kappa \in \mathbf{Card}}, 0\rangle \tag{22}$$

where, for each cardinal number $\kappa \geq \omega$, the reduct $\langle A, \wedge, \rightarrow, \bigsqcup_\kappa, 0\rangle$ defines a Heyting κ-frame. If $\kappa \geq \omega$ is a cardinal number then we denote by τ_κ the type of algebras related to the reduct $\langle \wedge, \rightarrow, (\bigsqcup_\eta)_{\eta \in [\omega,\kappa]_{Card}}, 0\rangle$ of the frames. For each set of variables X and for each type of algebras τ_κ we consider the set of terms $Term_{\tau_\kappa}(X)$ introduced in Definition 1.4. In this way, the proper class $Term_\infty(X)$ is introduced as follows:

$$t \in Term_\infty(X) \quad iff \quad \exists \kappa \in \mathbf{Card} \ \ s.t., \ \ t \in Term_{\tau_\kappa}(X). \tag{23}$$

Recalling that the set $Term_{\tau_\kappa}(X)$ is built up from the family $(X_\gamma^{\tau_\kappa})_\gamma$, indexed by ordinals (see Eq. (1)), then we can observe that:

$$Term_\kappa(X) \subseteq Term_{\tau_\kappa}(X). \tag{24}$$

Moreover, it is easy to check that $X_0^{\tau_{\kappa_1}} = X_0^{\tau_{\kappa_2}}$ for all infinite cardinals numbers κ_1, κ_2. In this way, and regardless of the cardinal κ, the elements of $X_0^{\tau_\kappa}$ are referred as *atomic* terms in the language of $\mathcal{H}_\infty$.

In order to establish a completeness type theorem for $Term_{\mathcal{H}_\infty}(X)$-equations in $\mathcal{H}_\infty$ we first introduce a kind of translation terms map sending each term of $Term_\infty(X)$ on a term in $Term_\kappa(X)$ for an appropriate cardinal number κ. The following concept will be crucial to establish the mentioned translation terms map.

Definition 9.1. Let $t \in Term_{\mathcal{H}_\infty}(X)$. By considering Eq. (23) then we define the *translation index* $\eta(t)$ of t as follows:

$$\eta(t) = \min\{k \in Card : t \in Term_{\tau_k}\}.$$

From all possible operation symbols $\bigsqcup_{i\in -}$ in $t \in Term_{\mathcal{H}_\infty}(X)$, the translation index $\eta(t)$ highlights the maximum of the arities of such operation symbols.

Definition 9.2. Let $t \in Term_\infty(X)$ and κ be cardinal number such that $\eta(t) \leq \kappa$. Then we define the *κ-translation* $N_\kappa(t)$ of t as follows:

if t is an atomic term then $N_\kappa(t) = t$,

if t is of the form $t_1 \star t_2$, $N_\kappa(t) = N_\kappa(t_1) \star N_\kappa(t_2)$ where $\star \in \{\wedge, \rightarrow\}$,

if t is of the form $\bigsqcup_{i\in\alpha\leq\kappa} t_i$ then $N_\kappa(t) = \bigsqcup_{i\in\kappa} t_i^*$ where

$$t_i^* = \begin{cases} N_\kappa(t_i), & \text{if } i < \alpha \\ 0, & \text{if } \alpha \leq i \leq \kappa. \end{cases}$$

Let us notice that if $t \in Term_\infty(X)$ then $N_\kappa(t) \in Term_\kappa(X)$.

PROPOSITION 9.3. *Let $t \in Term_{\mathcal{H}_\infty}(X)$ such that $\eta(t) \leq \kappa$. Then:*

$$\mathcal{H}_\infty \models t = N_\kappa(t)$$

PROOF. Let $A \in \mathcal{H}_\infty$ and $v : X \to A$ be a valuation. We use transfinite induction on the ordinal numbers γ for the family $(X_\gamma^{\tau_\kappa})_\gamma$ defining $Term_{\tau_\kappa}(X)$. If $t \in X_0^{\tau_\kappa}$ then it is immediate. By inductive hypothesis we assume that the equation holds for $t \in X_\alpha^{\tau_\kappa}$ where $\alpha < \gamma$. If $t \in X_\gamma^{\tau_\kappa}$ we have to consider two possible cases:

t is of the form $t_1 \star t_2$ where $\star \in \{\wedge \to\}$. Thus $t_1 \in X_{\alpha_1}^{\tau_\kappa}$ and $t_2 \in X_{\alpha_2}^{\tau_\kappa}$, where $\alpha_1, \alpha_2 < \gamma$. Then, by inductive hypothesis, $v(N_\kappa(t)) = v(N_\kappa(t_1)) \star v(N_\kappa(t_2)) = v(t_1) \star v(t_2) = v(t)$.

t is of the form $\bigsqcup_{i\in\beta\leq\eta(t)\leq\kappa} t_i$ where $t_i \in X_{\alpha_i}^{\tau_\kappa}$ and $\alpha_i < \gamma$. By definition of N_κ, we have that $N_\kappa(t) = \bigsqcup_{i\in\kappa} t_i^*$, where

$$t_i^* = \begin{cases} N_\kappa(t_i), & \text{if } i \in \beta \\ 0, & \text{if } \beta \leq i \leq \kappa. \end{cases}$$

By inductive hypothesis, for $i \in \beta$, $v(t_i) = v(N_\kappa(t_i))$ because $t_i \in X_{\alpha_i}^{\tau_\kappa}$ and $\alpha_i < \gamma$. Thus,

$$\begin{aligned} v(N_\kappa(\bigsqcup_{i\in\beta} t_i)) &= \bigvee_{i\in\beta} v(N_\kappa(t_i)) \vee \bigvee_{\beta\leq i<\kappa} v(0) \\ &= \bigvee_{i\in\beta} v(t_i) = v(\bigsqcup_{i\in\beta} t_i) = v(t). \end{aligned}$$

Hence our claim. ■

By the above proposition we can see that the κ-translation transforms a term $t \in Term_\infty(X)$, where its infinitary operation symbols are of the form $\bigsqcup_{i\in\alpha\leq\eta(t)\leq\kappa}$, into an equivalent term $N_\kappa(t)$ written in the language of $\mathcal{H}_\kappa$.

THEOREM 9.4. *Let $t \in Term_{\mathcal{H}_\infty}(X)$ such that $\eta(t) \leq \kappa$. Then*

$$\mathcal{H}_\infty \models t = 1 \quad iff \quad \mathcal{H}_\kappa \models N_\kappa(t) = 1.$$

PROOF. $\Longrightarrow$) Let us assume $\mathcal{H}_\infty \models t = 1$. Therefore, by Proposition 9.3, $\mathcal{H}_\infty \models N_\kappa(t) = 1$. Suppose that there exists a Heyting κ-frame A and a valuation $v : X \to A$ such that $v(N_\kappa(t)) \neq 1$. By Theorem 2.3, there exists $\widehat{A} \in \mathcal{H}_\infty$ and a $\mathcal{H}_\kappa$-embedding $i : A \to \widehat{A}$. Thus, the composition $v \circ i : X \to \widehat{A}$ defines a valuation in the algebra $\widehat{A} \in \mathcal{H}_\infty$ such that $v \circ i(N_\kappa(t)) \neq 1$ which is a contradiction. Hence $\mathcal{H}_\kappa \models N_\kappa(t) = 1$.

$\Longleftarrow$) Suppose that $\mathcal{H}_\kappa \models N_\kappa(t) = 1$. Since $\mathcal{H}_\infty$ is a subclass of $\mathcal{H}_\kappa$, we have that $\mathcal{H}_\infty \models N_\kappa(t) = 1$ and, by Proposition 9.3, $\mathcal{H}_\infty \models t = 1$. ■

The above theorem says that the equational theory of $\mathcal{H}_\infty$ where arbitrary supremum are taken into account can be captured in the classes $\{\mathcal{H}_\kappa\}_{\omega \leq \kappa \in \mathbf{Card}}$ where the types of algebras are sets. Furthermore, in this perspective and by Theorem 8.4, the equational theory of $\mathcal{H}_\infty$ is ruled by the classes $\{\mathbf{SI}(\mathcal{H}_\kappa)\}_{\omega \leq \kappa \in \mathbf{Card}}$.

10. Heyting κ-Frames and Heyting Algebras with a Quantifier

In algebraic logic there exists a kind of algebraic structures whose study is motivated by predicate logic. Its origin lies in a seminal work of Halmos dating back to 1955 [13] where the notion of monadic algebra is introduced. Concretely, a monadic algebra is a Boolean algebra with an additional unary operation whose interpretation represents the algebraic counterpart of a quantifier. Various further investigations have been carried out since [13]. For example, Monteiro and Varsavsky, in 1957, introduced the notion of monadic Heyting algebras [28] to serve the same purpose for the monadic intuitionistic predicate calculus. Soon after the notion of quantifiers on distributive lattices were considered for the first time by Servi [29] but it was Cignoli in [10] who studied them as algebras, which he named Q-distributive lattices. In this framework, a natural generalization of these lattices, called Heyting algebras with a quantifier, were introduced by Abad et al. in [1]. In this section we study a relation between these Heyting algebras and Heyting κ-frames.

A *Heyting algebras with a quantifier*, or Q-Heyting algebra for short, is an algebra $\langle H, \vee, \wedge, \rightarrow, \exists, 0\rangle$ of type $\langle 2, 2, 2, 1, 0\rangle$ satisfying the following conditions:

1. $\langle H, \vee, \wedge, \rightarrow, 0\rangle$ is a Heyting algebra,
2. $\exists 0 = 0$,
3. $x \leq \exists x$,
4. $\exists\exists x = \exists x$,
5. $\exists(x \vee y) = \exists x \vee \exists y$,
6. $\exists(x \wedge \exists y) = \exists x \wedge \exists y$.

Monadic Heyting algebras [8,28] are the simplest examples of Q-Heyting algebras. However, from Heyting κ-frames, we can built an interesting family of examples of Q-Heyting algebras. Indeed:

Let X be a set such that $|X| \leq \kappa$ and $H \in \mathcal{H}_\kappa$. Let us consider the set of all functions $H^X = \{f : X \to H\}$ endowed with the following operations:

$$\begin{aligned} f \vee g \text{ such that } (f \vee g)(x) &= f(x) \vee g(x), \\ f \wedge g \text{ such that } (f \wedge g)(x) &= f(x) \wedge g(x), \\ f \to g \text{ such that } (f \to g)(x) &= f(x) \to g(x), \\ \exists f \text{ such that } (\exists f)(x) &= \bigvee_{y \in X} f(y), \\ 0 \text{ such that } 0(x) &= 0. \end{aligned} \tag{25}$$

In an analogous way as in [13] and [28] we can see that the resulting algebra $\langle H^X, \vee, \wedge, \to, \exists, 0\rangle$ is a Q-Heyting algebra.

These kinds of functional algebras were first defined by considering $H \in \mathcal{H}_\infty$ i.e., a complete Heyting algebra, and X an arbitrary set. In this framework another unary operator, whose interpretation represents the algebraic counterpart of the universal quantifier $\forall$, is also consider. Concretely, for each $f \in H^X$ we define $(\forall f)(x) = \bigwedge_{y \in X} f(y)$. In [8], based on these functional algebras with two quantifiers, a representation theorem for monadic Heyting algebras is obtained.

Summarizing, Heyting k-frames are also related to the algebras motivated by the intuitionistic monadic predicate logic.

11. Concluding Remarks

This article is an attempt to contribute to the study of algebraic structures with infinitary operations. Specifically, by adapting several techniques of universal algebra to algebras with infinitary operations, the class of Heyting κ-frames is studied. In this perspective, basic properties of these infinitary algebras such as, congruences, direct indecomposability, subdirect representations, injective objects, etc. were obtained. Furthermore, an infinitary Hilbert style calculus with a corresponding completeness theorem is obtained. We also note that the structure of Heyting κ-frames allowed to establish an equational type completeness theorem for $\mathcal{H}_\infty$ if its language of algebras defined a proper class.

To conclude, we would like to note that there are still interesting topics to be studied about the structure of Heyting κ-frames, such as the characterisation of the projective objects in $\mathcal{H}_\infty$, the amalgamation property in $\mathcal{H}_\kappa$ for $0 \leq \kappa \leq \infty$, the functors between the categories of Heyting κ-frames for different cardinals κ ecc. It would also be interesting to study possible applications of these infinitary algebras to other mathematical structures.

Acknowledgements. This work is supported by: (1) MIUR project PRIN 2017: "Theory and applications of resource sensitive logics", CUP: 20173WKCM5; (2) FdS (Fondazione di Sardegna) project "Ubiquitous Quantum Reality (UQR): understanding the natural process under the light of quantum-like structures" (Code: F73C22001360007).

Funding Open access funding provided by Università degli Studi di Cagliari within the CRUI-CARE Agreement.

References

[1] Abad, M., J. P. Diaz Varela, L. A. Rueda, and A. M. Suardíaz, Varieties of three-valued Heyting algebras with a quantifier, Studia Logica 65(2):181–198, 2000.

[2] Balbes, R., and Ph. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia, 1974.

[3] Banaschewski, B., Frames and compactifications, in *Extensions Theory of Topological Structures and its Applications*, Deutscher Verlag der Wissenschaften, 1969.

[4] Banaschewski, B., and C. Gilmour, Stone–Cech compactification and dimension theory for regular σ-frames, Journal of the London Mathematical Society 2(127, 39, part 1):1–8, 1989.

[5] Banaschewski, B., On the injectivity of Boolean algebras, Commentationes Mathematicae Universitatis Carolinae 34:501–511, 1993.

[6] Banaschewski, B., The frame envelope of a σ-frame, Quaestiones Mathematicae 16:51–60, 1993.

[7] Bezhanishvili, G., Varieties of monadic Heyting algebras. Part I, Studia Logica 61:367–402, 1998.

[8] Bezhanishvili, G., and J. Harding, Functional monadic Heyting algebras, Algebra Universalis 48:1–10, 2002.
[9] Burris, S., and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, New York, Heidelberg, Berlin, 1982.
[10] Cignoli, R., Quantifiers on distributive lattices, Discrete Mathematics 96:183–197, 1991.
[11] Dowker, C., and D. Papert, Quotient frames and subspaces. Proceedings of the London Mathematical Society 16:275–296, 1966.
[12] Hájek, P., *Metamathematics of Fuzzy Logic*, Kluwer, Dordrecht-Boston-London, 1998.
[13] Halmos, J. P., Algebraic logic I. Monadic Boolean algebras, Composito Mathematica 12:217–249, 1955.
[14] Hanf, W., *Some Fundamental Problems Concerning Languages with Infinitely Long Expressions*, Ph.D. thesis, Berkeley, 1962.
[15] Höhle, U., Commutative, residuated l-monoids, in U. Höhle, and E. P. Klement (eds.), *Non-classical Logic and Their Applications to Fuzzy Subset. A Handbook on the Mathematical Foundations of Fuzzy Set Theory*, Kluwer Academic Publishers, Dordrecht, 1995, pp. 53–107.
[16] Isbel, J. R., Atomless parts of spaces, Mathematica Scandinavica 31:5–32, 1972.
[17] Johnstone, P. T., *Stone Spaces, Cambridge Studies in Advanced Mathematics*, vol. 3, Cambridge University Press, Cambridge, 1982.
[18] Johnstone, P. T., The point of pointless topology, Bulletin of the American Mathematical Society (N.S.) 8:41–53, 1983.
[19] Karp, C., Language with expressions of infinite length, in L. E. J. Brower, E. W. Beth, and A. Heyting, (eds.), *Studies in Logic and the Foundations of Mathematics*, North-Holland Publishing Company, Amsterdam, 1964.
[20] Lagrange, R., Amalgamation and epimorphisms in m complete Boolean algebras, Algebra Universalis 8:277–279, 1974.
[21] Loomis, L. H., On the representation of σ-complete Boolean algebras, Bulletin of the American Mathematical Society 53:757–760, 1947.
[22] Madden, J., κ-frames, Journal of Pure and Applied Algebra 70:107–127, 1991.
[23] Madden, J., and A. Molitor, Epimorphisms of frames, Journal of Pure and Applied Algebra 70:129–132, 1991.
[24] McKinsey, J., and A. Tarski, The algebra of topology, Annals of Mathematics 45:141–191, 1944.
[25] Maksimova, L., On maximal intermediate logic with disjunction property, Studia Logic 45:69–75, 1986.
[26] Manes, E., Algebraic theories, in *Graduate Text in Mathematics*, vol. 46, Springer Verlag MR 54/7578, 1976.
[27] Monk, J. D., Non trivial m-injective Boolean algebras do not exist, Bulletin of the American Mathematical Society 73(4):526–527, 1967.
[28] Monteiro, A., and O. Varsavsky, Algebras de Heyting mon'adicas, Actas de las X Jornadas de la Unión Matemática Argentina, Bahíia Blanca, 1957, pp. 52–62.
[29] Servi, M., Un'assiomatizzazione dei reticoli esistenziali, Bollettino dell'Unione Matematica Italiana A 16(5):298–301, 1979.

[30] Sikorski, R., *Boolean Algebras*, Springer-Verlag, Berlin, Heidelberg, New York, 1964.
[31] Słomiński, J., The theory of abstract algebras with infinitary operations, Rozprawy Matematyczne 18:1–67, 1959.

H. Freytes, G. Sergioli
Department of Philosophy
University of Cagliari
Via Is Mirrionis 1
09123 Cagliari
Italy
hfreytes@gmail.com

H. Freytes
Department of Mathematics and Computer Science
University of Cagliari
Via Ospedale, 72
09124 Cagliari
Italy