Weakly singular linear Volterra integral equations: a Nyström method in weighted spaces of continuous functions *

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Abstract

This paper provides a Nyström method for the numerical solution of Volterra integral equations whose kernels contain singularities of algebraic type. It is proved that the method is stable and convergent in suitable weighted spaces. An error estimate is also given as well as several numerical tests are presented.

Keyword: Volterra integral equations, Nyström method, Lagrange interpolation, Orthogonal Polynomials

MSC: 65R20, 41A05, 45D05

1 Introduction

In this paper we introduce a new Nyström method for solving weakly singular Volterra integral equations of the second kind of the form

$$f(y) + \int_{-1}^{y} k(x,y) f(x) (y-x)^{\alpha} (1+x)^{\beta} dx = g(y), \quad y \in (-1,1), \qquad (1)$$

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where $\alpha, \beta > -1$, f is the unknown solution and k and g are given functions.

In the case $-1 < \alpha < 0$, $\beta = 0$, i.e. the simplest form of second kind Abel-type integral equation, (1) is a model for many applications, arising in mathematical physics, electrochemistry, crystal growth, biophysics, viscoelasticity, heat transfer model, analysis of thin sections in transmission electron microscopy and in the solution of BVP. For the large variety of applications, a great attention has been posed in the literature and several numerical methods for solving these equations have been proposed. For an extensive bibliography the interested reader can consult the survey paper [1, 2, 22] (see also [6]). In the case $\alpha = \beta = 0$ we recall the iterated collocation methods [5, 26, 32, 36] and the spectral collocation methods [21, 33, 35]. For a complete bibliography, we refer to [6, Chapter 2] and the reference therein.

As it is known, for negative values of α, the weak singularity along the boundary $y = x$ and along the side $y = -1$ is inherited by the solution f, even if the right-hand side function g is smooth (see, for instance, [6, Chapter 6], [31], [10]).

In the present paper, besides to consider $\alpha > -1$, we approach the cases of functions k and g smooth inside the open sets $\dot{D} := (-1, 1) \times (-1, 1)$ and $(-1, 1)$, respectively, but possibly presenting algebraic singularities on the border ∂D and/or at the endpoints ± 1. Hence, these equations have usually solutions whose derivatives are unbounded at the left endpoint of the integration interval. By virtue of this peculiarity, they could be better handled in weighted spaces of functions to "absorb" the endpoints singularities of the solution. Hence, denoted by $u(x) = (1 - x)^{\gamma}(1 + x)^{\delta}$ a Jacobi weight with $\gamma, \delta \geq 0$, in this paper we study the equation in the space C_u of functions f that are locally continuous in $(-1, 1)$ and such that $fu \in C^0([-1, 1])$. The weighted Nyström method we develop employs a product integration rule that exactly integrates the singular factor $(y - x)^{\alpha}$, and only the kernel k is approximated in C_u. The final linear system we obtain allows to construct the weighted Nyström interpolant. We study stability, convergence and well conditioning of the system, providing error estimates in suitable subspaces of C_u. The accuracy of the method is confirmed by some numerical tests too. Comparisons with other recent methods [18, 22] are also presented. We point out that a relevant advantage of a Nyström scheme w.r.t. to a collocation approach, is a faster convergence even if the right–hand side g in (1) is not so smooth. Roughly speaking, the effective speed of convergence of Nyström methods is "more" dependent on the smoothness of the kernel k, than on the one of g.

The approach we propose represents a not negligible contribution in the framework of numerical methods devoted to solving this kind of equation. Indeed, numerical methods usually developed in $C^0([-1, 1])$ require additional smoothing transformations to improve their performance. This is for instance the approach proposed in [24], where a piecewise polynomial collocation method is introduced. The same happens in the more recent paper [18], where a projection method based on Lagrange interpolation in $C^0([-1,1])$ is proposed. According to our knowledge, a such kind of study has never been proposed before. Indeed, a Nyström method has been developed in [4, 20] but for other classes of Volterra integral equations. In [4] weakly singular equations with non-compact operators have been considered, and in [20] Volterra equations with highly oscillatory kernels have been studied. In both cases, the methods were analyzed in un-weighted spaces of functions.

We remark that the numerical treatment of several other kinds of integral equations in weighted spaces has been widely proposed in many recent papers. Among them we recall: second kind Fredholm integral equations [11, 19, 30], integro-differential equations [7, 13, 14], Cauchy singular equations [9, 8, 23, 28, 12], Mellin-type integral equations [16, 17, 25].

The paper is structured as follows. In Section 2 we collect some preliminaries, useful in the sequel. In Section 3 we gather new basic results necessary to introduce the Nyström method developed in Section 4. In Section 5 we show our numerical examples and in Section 6 we give the proofs of the main results.

2 Notations and preliminary results

Throughout the paper we use $\mathcal{C}$ in order to denote a positive constant, which may have different values at different occurrences, and we write $\mathcal{C} \neq \mathcal{C}(n, f, \ldots)$ to mean that $\mathcal{C} > 0$ is independent of $n, f, \ldots$.

For a given bivariate function $k(x,y)$, we will write k_y (or k_x) to regard it as function of the only variable x (or y).

2.1 Function spaces

Let u be the Jacobi weight defined as

$$u(x) = v^{\gamma,\delta}(x) := (1-x)^\gamma (1+x)^\delta, \qquad x \in (-1,1), \quad \gamma, \delta \geq 0.$$

We denote by C_u the Banach space of the functions $f \in C^0((-1,1))$ s. t. $fu \in C^0([-1,1])$ and $\lim_{x\to 1}(fu)(x) = 0$ if $\gamma > 0$ as well as $\lim_{x\to -1}(fu)(x) = 0$ if $\delta > 0$, equipped with the norm

$$\|f\|_{C_u} := \|fu\|_\infty = \max_{|x|\leq 1} |(fu)(x)|\,.$$

The limit conditions are necessary to assure that (see for instance [27])

$$\lim_{m\to\infty} E_m(f)_u = 0, \quad \forall f \in C_u$$

where, denoted by $\mathbb{P}_m$ the space of all algebraic polynomials of degree at most m,

$$E_m(f)_u := \inf_{P \in \mathbb{P}_m} \|f - P\|_{C_u}$$

is the error of best polynomial approximation of $f \in C_u$.

To deal with smoother functions, with $\lambda \in \mathbb{R}^+$ we consider the Zygmund type space

$$Z_\lambda(u) = \left\{ f \in C_u \ : \ \sup_{t>0} \frac{\Omega_\varphi^k(f,t)_u}{t^\lambda} < \infty, \quad k \geq 1,\, k > \lambda \right\},$$

where, setting $\varphi(x) = \sqrt{1-x^2}$, the main part of the *$\varphi$-modulus of smoothness* $\Omega_\varphi^k(f,t)_u$ is defined as [15, p. 90]

$$\Omega_\varphi^k(f,t)_u = \sup_{0<h\leq t} \|(\Delta_{h\varphi}^k f)u\|_{I_{kh}}, \quad I_{kh} = [-1+(2kh)^2, 1-(2kh)^2],$$

with

$$\Delta_{h\varphi}^k f(x) = \sum_{i=0}^k (-1)^i \binom{k}{i} f\left(x + \frac{h\varphi(x)}{2}(k-2i)\right).$$

The space $Z_\lambda(u)$ is endowed with the norm

$$\|f\|_{Z_\lambda(u)} \quad = \quad \|fu\|_\infty + \sup_{t>0} \frac{\Omega_\varphi^k(f,t)_u}{t^\lambda}.$$

Zygmund spaces are strictly connected with the Sobolev space of order $r \in \mathbb{N}, r \geq 1$ defined as

$$W_r(u) = \Big\{ f \in C_u : f^{(r-1)} \in AC(-1,1), \\ \|f\|_{W_r(u)} = \|fu\|_\infty + \|f^{(r)}\varphi^r u\|_\infty < \infty \Big\},$$

being $AC(-1,1)$ the set of the functions which are absolutely continuous on every closed subinterval of $(-1,1)$. Indeed,

$$W_{\lfloor\lambda\rceil+1} \subset Z_\lambda \subset W_{\lfloor\lambda\rceil}$$

being $\lfloor\lambda\rceil$ the smallest integer greater than or equal to $\lambda > 0$.

Finally, to estimate $E_m(f)_u$, we recall the following weak-Jackson inequality

$$E_m(f)_u \leq \mathcal{C} \int_0^{\frac{1}{m}} \frac{\Omega_\varphi^k(f,t)_u}{t} dt, \quad \forall f \in C_u, \tag{2}$$

and the Favard inequality (see, e.g., [27, p. 172])

$$E_m(f)_u \leq \frac{\mathcal{C}}{m^r} \|f\|_{W_r(u)}, \quad \forall f \in W_r(u),$$

where, in both cases, $\mathcal{C} \neq \mathcal{C}(m,f)$.

2.2 Lagrange interpolating polynomials

Let

$$w(x) = v^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta, \quad \alpha,\beta > -1,$$

and denote by $\{p_m(w)\}_{m=0}^\infty$ the sequence of the corresponding orthonormal polynomials with positive leading coefficients.

For a given $f \in C_u$, let $\mathcal{L}_m^w(f) \in \mathbb{P}_{m-1}$ be the Lagrange polynomial interpolating f at the zeros $\{x_k\}_{k=1}^m$ of $p_m(w)$, i.e.

$$\mathcal{L}_m^w(f,x) = \sum_{i=1}^m \ell_{m,i}(x) f(x_i), \quad \ell_{m,i}(x) = \lambda_{m,i}(w) \sum_{j=0}^{m-1} p_j(w,x) p_j(w,x_i), \tag{3}$$

where $\lambda_{m,k}(w)$ are the Christoffel numbers w.r.t. w. Let us now recall the conditions under which the sequence of Lebesgue constants $\{\|\mathcal{L}_m^w\|_u\}_m$ associated to the previous interpolating process in C_u, i.e.

$$\|\mathcal{L}_m^w\|_u := \|\mathcal{L}_m^w\|_{C_u \to C_u} = \max_{|x|\le 1} u(x) \sum_{k=1}^{m+1} \frac{|\ell_{m,k}(x)|}{u(x_k)},$$

has a logarithmic growth.

Theorem 2.1. *[27, p.272] Given two Jacobi weights $u = v^{\gamma,\delta}$ with $\gamma,\delta \ge 0$ and $w = v^{\alpha,\beta}$ with $\alpha,\beta > -1$, then*

$$\|\mathcal{L}_m^w\|_u \sim \log m,$$

if and only if

$$\max\left\{0, \frac{\alpha}{2} + \frac{1}{4}\right\} \le \gamma \le \frac{\alpha}{2} + \frac{5}{4}, \quad \max\left\{0, \frac{\beta}{2} + \frac{1}{4}\right\} \le \delta \le \frac{\beta}{2} + \frac{5}{4},$$

are satisfied.

We conclude by recalling a theorem useful to prove convergence and stability of the product quadrature rule that we will introduce in Section 3.

Theorem 2.2. *[29], [27, p.348] Let $u = v^{\gamma,\delta}$, $w = v^{\alpha,\beta}$ and $\varphi = v^{\frac{1}{2},\frac{1}{2}}$. Assume*

$$\sup_{y\in[-1,\ 1]} \int_{-1}^1 \frac{|d(x,y)|}{u(x)} \log\left(2 + \frac{|d(x,y)|}{u(x)}\right) dx < \infty.$$

Then, for any $f \in C_u$, we have

$$\sup_{-1\le y\le 1} \int_{-1}^1 |\mathcal{L}_m^w(f,x) d(x,y)| dx \le \mathcal{C}\, \|fu\|_\infty, \quad \mathcal{C} \ne \mathcal{C}(m,f)$$

if and only if

$$\sup_{y\in[-1,\ 1]} \int_{-1}^1 \frac{|d(x,y)|}{\sqrt{w(x)\varphi(x)}} dx < \infty, \quad \int_{-1}^1 \frac{\sqrt{w(x)\varphi(x)}}{u(x)} dx < \infty.$$

3 Auxiliary results

Setting

$$(\mathcal{V}f)(y) = \int_{-1}^{y} f(x)k(x,y)(y-x)^{\alpha}(1+x)^{\beta}dx, \tag{4}$$

equation (23) can be rewritten in the form

$$(I+\mathcal{V})f = g.$$

3.1 Solvability of equation (1) in C_u

The first problem we approach is to assure equation (1) is uniquely solvable in C_u.

Theorem 3.1. *Assume that $\gamma < 1+\alpha,\ \delta < 1+\beta$ and*

$$\tilde{k}(y) := v^{-\gamma,1+\alpha+\beta}(y) \sup_{x\in[-1,\ 1]} |k(x,y)| \in Z_{\lambda}(u). \tag{5}$$

For any $g \in C_u$ equation (1) admits a unique solution in C_u.

Remark 3.2. *Note that even if $\tilde{k}$ is singular at $y=1$, $\tilde{k}u$ it is not.*

Now we propose some examples of kernels k satisfying (5) for suitable weights u.

- $k(x,y) \equiv c$, $c \in \mathbb{R}$ (see e.g.[24, Example 1]). Assuming $1+\alpha+\beta \geq 0$, (5) is satisfied with $0 < \lambda \leq 2(1+\alpha+\beta+\delta)$.
- $k(x,y) = e^{xy}(1+y)^2$, and $\alpha = \beta = -\frac{3}{4}$. Setting $\gamma = 0$, one has $\tilde{k}(y) = e^{1+y}v^{0,\frac{3}{2}}(y)$, and (5) is fulfilled with $\lambda = 3+2\delta$.
- $k(x,y) = (x+y^2+1)\sin(x+y)$ and $\alpha = 0$, $\beta = 1/4$. In this case

  $$\tilde{k}(y) = v^{-\gamma,\frac{5}{4}}(y) \sup_{x\in[-1,\ 1]} \big((x+y^2+1)\sin(x+y)\big) \sim v^{-\gamma,\frac{5}{4}}(y),$$

  and $\tilde{k} \in Z_{\frac{5}{2}+2\delta}(u)$, for any $u = v^{\gamma,\delta}$.

3.2 A product integration rule

Now we introduce a product rule on which the weighted Nyström method is based. The rule has been obtained by approximating the function $k_y f$ in (4) by the Lagrange polynomial $\mathcal{L}_m^w(k_y f)$, i.e.

$$(\mathcal{V}f)(y) = (\mathcal{V}_m f)(y) + \mathcal{E}_m(f,y),$$

where $\mathcal{E}_m$ denotes the remainder term and

$$(\mathcal{V}_m f)(y) = \sum_{k=1}^{m} c_k(y) k(x_k, y) f(x_k), \tag{6}$$

with

$$\begin{aligned} c_k(y) &= \int_{-1}^{y} \ell_{m,k}(x)(y-x)^{\alpha}(1+x)^{\beta} dx \\ &= \lambda_{m,k}(w) \sum_{j=0}^{m-1} p_j(w, x_k) \int_{-1}^{y} p_j(w, x)(y-x)^{\alpha}(1+x)^{\beta} dx, \end{aligned} \tag{7}$$

being $\ell_{m,k}$ defined in (3).

Next theorem states conditions under which the integration rule in (6) is stable and convergent.

Theorem 3.3. *Assume that the function k satisfies* (18)*, $k_y \in C_u$ and the weights $u = v^{\gamma,\delta}$ and $w = v^{\alpha,\beta}$ fulfill the following conditions*

$$\begin{aligned} 0 \leq &\gamma < \min\left\{\alpha + 1, \frac{\alpha}{2} + \frac{5}{4}\right\} \\ \max\{0, -(1+\alpha+\beta)\} \leq &\delta < \min\left\{\beta + 1, \frac{\beta}{2} + \frac{5}{4}\right\}. \end{aligned} \tag{8}$$

Then formula (6) *is stable, i.e.*

$$\sup_{y \in [-1,1]} \sup_{m} u(y) \sum_{k=1}^{m} \frac{|c_k(y)|}{u(x_k)} < \infty, \tag{9}$$

and for each $f \in C_u$

$$\lim_{m \to \infty} \|[(\mathcal{V} - \mathcal{V}_m) f u\| = 0.$$

4 The Nyström method

In order to approximate the solution of (1), we introduce here a weighted Nyström method. Considering the sequence $\left\{(\mathcal{V}_m f)(y)\right\}_m$, with $(\mathcal{V}_m f)(y)$ defined in (6), we introduce the following finite-dimensional equation

$$(\mathcal{I} + \mathcal{V}_m) f_m = g, \tag{10}$$

in the unknown f_m. Then, multiplying both sides of the previous equation by the weight u and collocating at the zeros $\{x_i\}_{i=1}^m$ of $p_m(w)$, we get the following linear system

$$\sum_{j=1}^{m} \left[\delta_{ij} + c_j(x_i) \frac{u(x_i)}{u(x_j)} k(x_j, x_i)\right] a_j = (gu)(x_i), \qquad i = 1, ..., m \tag{11}$$

where $a_j = (f_m u)(x_j)$ are the unknowns and

$$\begin{aligned} c_j(x_i) &= \lambda_{m,j} \sum_{k=0}^{m-1} p_k(w, x_j) \int_{-1}^{x_i} p_k(w,x)(x_i - x)^\alpha (1+x)^\beta dx \\ &= \left(\frac{1+x_i}{2}\right)^{1+\alpha+\beta} \lambda_{m,j} \sum_{k=0}^{m-1} p_k(w, x_j) \int_{-1}^{1} p_k(w, \gamma_{x_i}(x)) v^{\alpha,\beta}(x) dx \\ &= \left(\frac{1+x_i}{2}\right)^{1+\alpha+\beta} \lambda_{m,j} \sum_{k=0}^{m-1} p_k(w, x_j) \sum_{s=1}^{m} \lambda_{m,s} p_k(w, \gamma_{x_i}(x_s)), \end{aligned}$$

with $\gamma_x(t) = \left(\frac{1+x}{2}\right) t + \frac{x-1}{2}$.

A matrix representation of system (11) is given by

$$(I_m + D_m K_m D_m^{-1})\mathbf{a} = \mathbf{b}, \tag{12}$$

where $\mathbf{a} = (a_1, \ldots, a_m)^T$, $\mathbf{b} = (b_1, \ldots, b_m)^T$ with $b_i = u(x_i)g(x_i)$, I_m is the identity matrix of order m and

$$D_m = \operatorname{diag}(u(x_1), \ldots, u(x_m)), \quad (K_m)_{ij} = c_j(x_i) k(x_j, x_i).$$

System (12) is equivalent to the finite-dimensional equation (10). In fact, the solution $\mathbf{a}^*$ of (12), if it exists, allow one to costruct the weighted Nyström interpolant

$$(f_m u)(y) = (gu)(y) - u(y) \sum_{j=1}^{m} \frac{c_j(y)}{u(x_j)} k(x_j, y)\, a_j^* \tag{13}$$

which is the solution of (10). Vice-versa, the latter furnishes a solution to (12). Merely evaluate (13) at the nodes x_i, $i = 1, \ldots, m$.

Next theorem states the convergence of the above described Nyström method.

Theorem 4.1. *Assume that*

$$\max\left\{0, \frac{\alpha}{2} + \frac{1}{4}\right\} \le \gamma < \min\left\{\alpha + 1, \frac{\alpha}{2} + \frac{5}{4}\right\}, \tag{14}$$

$$\max\left\{0, -(1+\alpha+\beta), \frac{\beta}{2} + \frac{1}{4}\right\} \le \delta < \min\left\{\beta + 1, \frac{\beta}{2} + \frac{5}{4}\right\}, \tag{15}$$

and

$$\tilde{k}(y) := v^{-\gamma, 1+\alpha+\beta}(y) \sup_{x \in [-1,\, 1]} |k(x,y)| \in Z_\lambda(u).$$

Then, for m sufficiently large, the operators $(I + \mathcal{V}_m)^{-1}$ exist and are uniformly bounded, and system (12) is well conditioned.

Moreover, denoted by f^* *and* f_m^* *the unique solution of equation* (1) *and* (10) *respectively, if* $g \in Z_\lambda(u)$ *the following convergence estimate holds true*

$$\|(f^* - f_m^*)u\|_\infty = \mathcal{O}\left(\frac{1}{m^\lambda}\right). \tag{16}$$

About the error estimate, we remark once again that the Nyström interpolant converges to the exact solution with the same order of convergence of the quadrature formula (see, for instance, Theorem 6.4) that is estimated by the error of best polynomial approximation of $k_y f$ (see, estimate (20)).

Moreover, system (12) is well-conditioned that is its condition number is bounded by a constant which does not depend on the size of the system m and the magnitude of such a constant depends on the condition nummber of the operator $\mathcal{I} + \mathcal{V}_m$.

Remark 4.2. *Let us note that for* $\alpha > -1$*, we can always choose a parameter* γ *according to* (14)*, while there are cases for which the parameter* δ *does not satisfy the condition* $\delta > -(1+\alpha+\beta)$*. Certainly, this does not happen if* $1+\alpha+\beta \geq 0$ *or* $\beta > -\frac{5}{6} - \frac{2}{3}\alpha$*. Indeed, in these two cases* $\max\left\{0, \frac{\beta}{2} + \frac{1}{4}\right\}$*, coincides with* 0 *and* $\frac{\beta}{2} + \frac{1}{4}$*, respectively.*

5 Numerical Tests

In this section, we apply the proposed Nyström method to solve numerically equations of the type (1). For increasing m, we will report in the tables the weighted maximum errors

$$\epsilon_m = \max_{i=1:1000} |(f - f_m)(y_i)u(y_i)|$$

where f_m is the Nyström interpolant defined in (13), f is the exact solution and $(y_i)_{i=1,\dots,1000}$ are equally spaced points of $(-1,1)$. In the case f is unknown, we consider exact the approximated solution f_M, for M sufficiently large, declaring M in each test. In these cases, we will denote the errors by ϵ_m^M.

Moreover, for each example we report the condition number of system (12), i.e. $\text{cond}(I_m + D_m K_m D_m^{-1})$, computed in infinity norm, and the estimated order of convergence

$$EOC_m = \frac{\log(\epsilon_m/\epsilon_{2m})}{\log 2}.$$

All the numerical experiments were performed using Matlab R2021a in double precision on an Intel Core i7-2600 system (8 cores), under the Debian GNU/Linux operating system.

Example 5.1. Let us consider the following equation

$$f(y) + \int_{-1}^{y} e^{x+y} f(x)(y-x)^{-\frac{1}{3}} dx = g(y)$$

with

$$g(y) = e^y + 2^{-\frac{2}{3}} e^{3y} \left(\Gamma\left(\frac{2}{3}\right) - \Gamma\left(\frac{2}{3}, 2(1+y)\right) \right).$$

Here $\Gamma(z)$ is the Euler Gamma function and $\Gamma(a,z)$ is the incomplete Gamma function. The unique solution is the smooth function $f(y) = e^y$. Since $\alpha = -1/3$, $\beta = 0$, according to (14)-(15), we can fix $\gamma = \delta = 1/2$. Either g and k are analytical functions and also $\tilde{k}$ in (5) is it. The results are reported in Table 1. As we could expect, the convergence is very fast. Indeed, almost the machine precision is attained by solving a square system of order 32.

Table 1: Numerical results for Example 5.1

m	ϵ_m	$\text{cond}(I_m + D_m K_m D_m^{-1})$	EOC_m
4	5.79e-02	5.75e+00	11.57
8	1.90e-05	7.50e+00	29.59
16	2.35e-14	8.70e+00	2.41
32	4.44e-15	9.82e+00	

Example 5.2. The aim of this example is to give a comparison of performance between the proposed Nyström method and the projection method developed by the same authors in [18]. Moreover we want to asses the main advantages obtained by treating the equation in weighted spaces of continuous functions by our Nyström method instead of the collocation method proposed there. Consider the equation [18, Example 3.1 and Example 4.1]

$$f(y) + \int_{-1}^{y} e^{xy} \sin(\sqrt{1+x}) f(x) \sqrt{1+x}\, dx = e^{(1+y)^{\frac{1}{3}}}.$$

According to the assumptions of Theorem 4.1, we look for the solution in C_u with $u(x) = v^{\frac{1}{2},\frac{4}{5}}(x)$. The expected theoretical order of convergence is $\mathcal{O}(m^{-\frac{34}{15}})$, since $g \in Z_{34/15}(u)$ and k satisfies (5) with $\lambda = 23/5 \sim 4.6$. By Table 2, easily we can see that Nyström method turns out to be accurate. The numerical errors as well as the EOC_m are slightly better than the theoretical estimate, suggesting that the constant in the error term is significantly small. Moreover, the linear systems (12) are always well conditioned.

Let us now compare the results presented here with those achieved by the collocation method developed in [18]. We just recall that the method proposed there improves its performance after introducing the smoothing transformation

$$\phi_q(z) = 2^{1-q}(1+z)^q - 1, \qquad q \in \mathbb{N} \tag{17}$$

and as q increases more relevant benefits are produced if the involved functions or their derivatives present singularities at the endpoint -1 of the interval $[-1, 1]$. For the convenience of the reader, we report in Table 3 the errors $\widetilde{e}_m^{700} = \max_{i=1,2,\dots 10^3} |(\widetilde{f}_{700} - \widetilde{f}_m)(y_i)|$, being $\widetilde{f}_m$ the approximate solution of the collocation method, computed for different choices of q.

By Table 3, we can assert that the proposed Nyström method furnishes very satisfactory results especially if compared with the projection method in the case when no smoothing techniques are applied ($q = 1$) or if the transformation (17) is employed with $q = 2$. If $q = 3$, by solving a system of order $m = 512$ the Nyström method gives one more correct digit.

Table 2: Numerical results for Example 5.2

m	ϵ_m^{700}	$\mathrm{cond}(I_m + D_m K_m D_m^{-1})$	EOC_m
4	3.38e-03	2.95e+00	
8	1.75e-04	3.70e+00	4.03
16	1.07e-05	4.06e+00	3.78
32	7.83e-07	4.21e+00	3.78
64	5.70e-08	4.29e+00	3.74
128	4.26e-09	4.31e+00	3.74
256	3.19e-10	4.32e+00	4.19
512	1.75e-11	4.33e+00	

Table 3: Numerical results for Example 5.2 achieved in [18]

m	q	$\tilde{\epsilon}_m^{700}$	q	$\tilde{\epsilon}_m^{700}$	q	$\tilde{\epsilon}_m^{700}$
4	1	1.14e-02	2	1.29e-02	3	4.94e-02
8		8.29e-03		1.30e-03		5.59e-03
16		4.39e-03		9.44e-04		3.14e-05
32		1.46e-03		2.89e-04		7.96e-10
64		4.58e-04		8.41e-05		8.52e-10
128		1.68e-04		2.41e-05		8.53e-10
256		3.77e-05		7.75e-06		7.67e-10
512		2.58e-05		3.09e-06		9.47e-10

Example 5.3. Consider the following equation

$$f(y) - \frac{\sqrt{2}}{2} \int_{-1}^{y} f(x)\,(y-x)^{-1/2}\,dx = \sqrt{\frac{1+y}{2}} - \frac{\pi}{4}(1+y).$$

The solution is $f(y) = \sqrt{\frac{1+y}{2}}$. By fixing $u(x) = v^{\frac{2}{5},1}(x)$, $g \in W_3(u)$. The expected convergence order $\mathcal{O}\left(m^{-3}\right)$ is confirmed by the weighted errors and the EOC_m values displayed in Table 4 . The conditioning of the linear system grows a little bit, remaining acceptable. We point out that the same equation has been considered in [22, Example 4.1]. In that paper the authors apply two different approaches for Volterra equations in $[0,1]$, in the case $\beta = 0$. In details, they propose an interpolation postprocessing technique to the collocation solution under graded mesh, and an hybrid collocation solution under "looser"graded mesh. So, the methods presented in [22] belong to a different family of methods with respect to our global approximation method. Therefore, the comparison can be performed only by observing the values stated in the tables given there. It seems that our approach provides satisfactory results.

Table 4: Numerical results for Example 5.3

m	ϵ_m	$\text{cond}(I_m + D_m K_m D_m^{-1})$	EOC_m
4	1.96e-01	2.09e+02	
8	2.17e-02	5.51e+02	3.07
16	2.58e-03	9.58e+02	3.02
32	3.18e-04	1.45e+03	3.01
64	3.96e-05	2.02e+03	3.00
128	4.94e-06	2.67e+03	3.00
256	6.18e-07	3.39e+03	3.00
512	7.72e-08	4.20e+03	3.00
1024	9.65e-09	5.07e+03	

Example 5.4. Consider

$$f(y) + \int_{-1}^{y} \log(x+y+2) f(x)(y-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}\,dx = (y^2+1)|\sin y|^{\frac{7}{2}}.$$

According to (14)-(15), the equation can be considered in C_u with $u = v^{\frac{3}{4},\frac{3}{4}}$. The function k satisfies (5) with $\lambda = 11/2$ and $g \in Z_{7/2}(u)$. Consequently, we expect a convergence of the order $\mathcal{O}(m^{-7/2})$. However, the numerical results given in Table 5 are better than the theoretical expectations. Finally, for m increasing all the systems (12) are well conditioned.

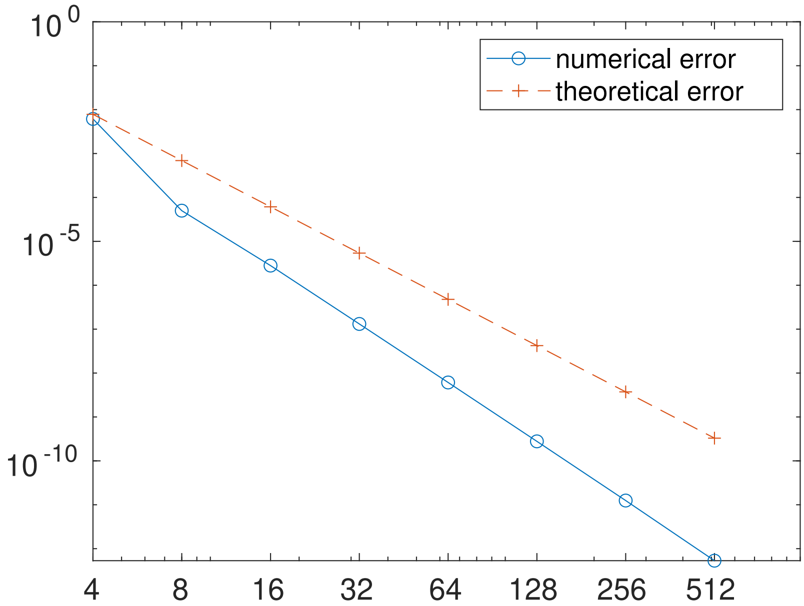

Figure 1: Numerical and theoretical errors of Example 5.4

Table 5: Numerical results for Example 5.4

m	ϵ_m^{1024}	$\text{cond}(I_m + D_m K_m D_m^{-1})$	EOC_m
4	5.31e-03	2.42e+00	
8	5.37e-05	2.77e+00	4.51
16	2.35e-06	2.96e+00	4.58
32	9.84e-08	3.01e+00	4.47
64	4.45e-09	3.02e+00	4.46
128	2.02e-10	3.03e+00	4.48
256	9.05e-12	3.03e+00	4.55
512	3.86e-13	3.03e+00	

Example 5.5. Let us consider the equation

$$f(y) + \int_{-1}^{y} |x|^{\frac{9}{2}} (xy + 5) f(x)(1+x)^{\frac{1}{3}} dx = \frac{e^{y-2}}{y^2+1}$$

in the space C_u with $u = v^{\frac{1}{2},1}$. In Table 6 we report our numerical errors which are better than those we expect from our theoretical estimate that is in this case $\mathcal{O}(m^{-9/2})$.

Table 6: Numerical results for Example 5.5

m	ϵ_m^{1024}	$\text{cond}(I_m + D_m K_m D_m^{-1})$	EOC_m
4	7.07e-03	6.99e+00	
8	3.71e-04	1.34e+01	7.02
16	2.86e-06	1.94e+01	6.34
32	3.53e-08	2.35e+01	5.58
64	7.36e-10	2.65e+01	5.53
128	1.59e-11	2.85e+01	5.50
256	3.51e-13	2.98e+01	5.60
512	7.22e-15	3.06e+01	

6 Proofs

Lemma 6.1. *Assume that $\gamma < 1+\alpha,\ \delta < 1+\beta$ and*

$$N_k = \sup_{y\in[-1,\ 1]} \left(v^{0,1+\alpha+\beta+\delta}(y) \sup_{x\in[-1,\ 1]} |k(x,y)| \right) < \infty. \tag{18}$$

Then, $\mathcal{V} : C_u \to C_u$ is bounded.

Proof. For any $y \in [-1, 1]$, if $\delta < \beta + 1$ and $\gamma < \alpha + 1$, then

$$\begin{aligned}
|(\mathcal{V}f)(y)u(y)| &\le u(y) \int_{-1}^{y} |k(x,y)|\,|(fu)(x)| \frac{(y-x)^\alpha}{(1-x)^\gamma}(1+x)^{\beta-\delta}\,dx \\
&\le \|fu\|_\infty u(y) \sup_{x\in[-1,\ 1]} |k(x,y)| \int_{-1}^{y} \frac{(y-x)^\alpha}{(1-x)^\gamma}(1+x)^{\beta-\delta}\,dx \\
&\le \mathcal{C}\,\|fu\|_\infty \sup_{x\in[-1,\ 1]} |k(x,y)|u(y)v^{-\gamma,1+\alpha+\beta}(y) \int_{-1}^{1} v^{\alpha,\beta-\delta}(t)dt \\
&\le \mathcal{C}\,\|fu\|_\infty \sup_{y\in[-1,\ 1]} \left(v^{0,1+\alpha+\beta+\delta}(y) \sup_{|x|\le 1} |k(x,y)| \right),
\end{aligned}$$

that is the thesis. □

Lemma 6.2. *Assume that $\gamma < 1+\alpha,\ \delta < 1+\beta$. Then, under the assumption (5), $\mathcal{V} : C_u \to C_u$ is compact.*

Proof. Let us prove that for any $f \in C_u$,

$$\Omega_\varphi^r(\mathcal{V}f, t)_u \le \mathcal{C}t^\lambda \|fu\|_\infty. \tag{19}$$

By (4) we have

$$
\begin{aligned}
&|u(y)\Delta^r_{h\varphi}(\mathcal{V}f)(y)| \\
&\leq \|fu\|_\infty u(y) \left| \Delta^r_{h\varphi(y)} \left(\int_{-1}^{y} k(x,y)(y-x)^\alpha \frac{(1+x)^{\beta-\delta}}{(1-x)^\gamma} dx \right) \right| \\
&\leq \mathcal{C}\|fu\|_\infty u(y) \left| \Delta^r_{h\varphi(y)} \left(\sup_{|x|\leq 1} |k(x,y)| v^{-\gamma,1+\alpha+\beta}(y) \right) \right| \int_{-1}^{1} v^{\alpha,\beta-\delta}(t)dt.
\end{aligned}
$$

Then, taking the supremum on $y \in I_{rh} = [-1+(2rh)^2, 1-(2rh)^2]$ and setting

$$\tilde{k}(y) = v^{-\gamma,1+\alpha+\beta}(y) \sup_{|x|\leq 1} |k(x,y)|$$

we have for $0 < h \leq t$,

$$\sup_{t>0} \frac{\Omega^r_\varphi(\mathcal{V}f,t)_u}{t^\lambda} \leq \mathcal{C}\|fu\|_\infty \sup_{t>0} \frac{\Omega^r_\varphi(\tilde{k},t)_u}{t^\lambda},$$

and thus estimate (19), by virtue of assumption (5).

In this way, by (2) we can assert that $E_m(\mathcal{V}f)_u \leq \mathcal{C}m^{-\lambda}\|fu\|_\infty$, from which we deduce

$$\lim_m \left(\sup_{\|f\|_{C_u}=1} E_m(\mathcal{V}f)_u \right) = 0.$$

Hance, by virtue of [34, p. 44] and in view of the boundedness of the operator, we can claim that $\mathcal{V} : C_u \to C_u$ is compact.

□

Proof of Theorem 3.1. The theorem follows, by Lemmas 6.1 and 6.2, taking into account that condition (5) implies (18) since $\|\tilde{k}u\|_\infty = N_k$. □

Proof of Theorem 3.3. First, let us note that setting,

$$\widetilde{v}(x,y) = \begin{cases} (y-x)^\alpha(1+x)^\beta, & y > x \\ 0, & y < x \end{cases}$$

under the assumptions (8), we can apply Theorem 2.2, deducing that

$$\sup_{|y|\leq 1} u(y) \int_{-1}^{1} |\mathcal{L}^w_m(k_y f, x)\, \tilde{v}(x,y)|\, dx \leq \mathcal{C} \sup_{|y|\leq 1} u(y)\|k_y f u\|_\infty \quad \mathcal{C} \neq \mathcal{C}(m,f,k_y).$$

This allow us to state that the formula is stable in C_u.

Now, denoting by $P_{m-1}(x,y)$ the best polynomial approximation of $k_y f \in C_u$ of degree $m-1$ in both variables, we have

$$
\begin{aligned}
|[(\mathcal{V}f)(y)-(\mathcal{V}_m f)(y)]u(y)| \le u(y) \Bigg[\int_{-1}^{1} |(k_y f)(x) - P_{m-1,y}(x)|\, \widetilde{v}(x,y)dx + \\
\int_{-1}^{1} |L_m^w(k_y f - P_{m-1,y}, x)\, \widetilde{v}(x,y)|\, dx \Bigg] \\
\le \mathcal{C} u(y) E_{m-1}(k_y f)_u \int_{-1}^{1} \frac{\tilde{v}(x,y)}{u(x)} dx \\
\le \mathcal{C}\,(1+y)^{1+\alpha+\beta+\delta}\, E_{m-1}(k_y f)_u. \qquad (20)
\end{aligned}
$$

Now, since for any $f_1, f_2 \in C_\omega,\ \omega = v^{\rho,\sigma},\ \rho,\sigma \ge 0$ we have [27, p.384]

$$
E_{2m}(f_1 f_2)_\omega \le \mathcal{C}\left(\|f_1\omega\|_\infty E_m(f_2)_\omega + E_m(f_1)_\omega \|f_2\omega\|_\infty \right),
$$

by the assumptions on k and (8) we can state that

$$
\begin{aligned}
&|[(\mathcal{V}f)(y)-(\mathcal{V}_m f)(y)]u(y)| \\
&\le \mathcal{C}\,(1+y)^{1+\alpha+\beta+\delta} \left[\|k_y u\|_\infty E_{[\frac{m-1}{2}]}(f)_u + \|fu\|_\infty E_{[\frac{m-1}{2}]}(k_y)_u \right] \\
&\le \mathcal{C} \left[N_k E_{[\frac{m-1}{2}]}(f)_u + \|fu\|_\infty \sup_{|y|\le 1} \left((1+y)^{1+\alpha+\beta+\delta}\, E_{[\frac{m-1}{2}]}(k_y)_u \right) \right]. \qquad (21)
\end{aligned}
$$

Then, being $f, k_y \in C_u$, we deduce the convergence of the formula. □

Next result is essential to prove the convergence of the introduced Nyström method.

Theorem 6.3. *Under the assumptions of Theorem 3.3, Theorem 2.1 and assuming that condition* (5) *is fulfilled, we have*

$$
\lim_{m\to\infty} \|(\mathcal{V}-\mathcal{V}_m)\mathcal{V}_m\|_{C_u \to C_u} = 0.
$$

Proof. By (21) with $\mathcal{V}_m f$ in place of f we get

$$
\begin{aligned}
|[(\mathcal{V}-\mathcal{V}_m)\mathcal{V}_m f(y)]u(y)| \le \mathcal{C} \Bigg[N_k E_{[\frac{m-1}{2}]}(\mathcal{V}_m f)_u \\
+ \|(\mathcal{V}_m f)u\|_\infty \sup_{|y|\le 1} \left((1+y)^{1+\alpha+\beta+\delta}\, E_{[\frac{m-1}{2}]}(k_y)_u \right) \Bigg] \\
(22)
\end{aligned}
$$

Let us now estimate $E_{[\frac{m-1}{2}]}(\mathcal{V}_m f)_u$. By definitions (6) and (7), we can write

$$\begin{aligned}
&|u(y)\Delta^r_{h\varphi}(\mathcal{V}_m f)(y)| \\
&\leq u(y)\sum_{k=1}^m f(x_k)\left|\Delta^r_{h\varphi}\left(c_k(y)k(x_k,y)\right)\right| \\
&\leq \|fu\|_\infty \sum_{k=1}^m \frac{u(y)}{u(x_k)}\left|\Delta^r_{h\varphi}\|k_y\|_\infty\left(\int_{-1}^y \ell_{m,k}(x)(y-x)^\alpha(1+x)^\beta dx\right)\right| \\
&\leq \|fu\|_\infty\left(\max_{|x|\leq 1}\sum_{k=1}^m \frac{|\ell_{m,k}(x)u(x)|}{u(x_k)}\right)u(y)\left|\Delta^r_{h\varphi}\left(\|k_y\|_\infty\int_{-1}^y (y-x)^\alpha v^{-\gamma,\beta-\delta}(x)dx\right)\right|.
\end{aligned}$$

Therefore, by Theorem 2.1 and by proceeding as in the proof of Theorem 6.2, under the assumption (5), we get

$$\sup_{t>0}\frac{\Omega^r_\varphi(\mathcal{V}_m f,t)_u}{t^\lambda} \leq \mathcal{C}\log m\,\|fu\|_\infty,$$

and thus by (2) we have

$$E_{\lfloor\frac{m-1}{2}\rfloor}(\mathcal{V}_m f)_u \leq \mathcal{C}\frac{\log m}{m^\lambda}\|fu\|_\infty.$$

Then, by (22) and taking into account the stability of the formula (9), we obtain the thesis. □

In order to prove Theorem 4.1 we recall a well-know result (see, for instance [3, Theorem 4.1.1 p. 106]

Theorem 6.4. *Let $\mathcal{X}$ be a Banach space, $K:\mathcal{X}\to\mathcal{X}$ be a given bounded compact operator and $K_m:\mathcal{X}\to\mathcal{X}$, $m\in\mathbb{N}$ be a given bounded operator with $\lim_{m\to\infty}\|Kf-K_mf\|=0$ for all $f\in\mathcal{X}$. Let us consider the operator equations*

$$(I-K)f=g, \tag{23}$$

$$(I-K_m)f_m=g, \tag{24}$$

where I is the identity operator and $g\in\mathcal{X}$. If $\lim_{m\to\infty}\|(K-K_m)K_m\|=0$, then $(I-K_m)^{-1}$ exist and is uniformly bounded, with

$$\|(I-K_m)^{-1}\| \leq \frac{1+\|(I-K)^{-1}\|\,\|K_m\|}{1-\|(I-K)^{-1}\|\,\|(K-K_m)K_m\|}.$$

Morever, denoted by $f^*\in\mathcal{X}$ and $f^*_m\in\mathcal{X}$ the unique solution of (23) *and* (24)*, respectively, we have*

$$\|f^*-f^*_m\| \leq \mathcal{C}\|(K-K_m)f^*\|, \quad \mathcal{C}\neq\mathcal{C}(m,f^*,g).$$

Proof of Theorem 4.1. By Theorem 6.4, we can immediately deduce that by virtue of Theorem 3.3 and Theorem 6.3 equations (24), admit a unique solution. About the well conditioning, it can be proved by using the same arguments as [3, p. 113] only by replacing the usual infinity norm with the weighted uniform norm of the space C_u.

Let us now prove (16). By applying again Theorem 6.4, by (21) we can assert

$$\|[f^* - f_m^*]u\|_\infty \leq \mathcal{C} \left[E_{[\frac{m-1}{2}]}(f^*)_u + \frac{1}{m^\lambda} \|f^* u\|_\infty \right].$$

Hence, since by the assumptions on g and k, we have that f^* is at least in $Z_\lambda(u)$ we can deduce that

$$\|[f^* - f_m^*]u\|_\infty \leq \frac{\mathcal{C}}{m^\lambda} \|f^*\|_{Z_\lambda(u)}.$$

□

7 Conclusions and future work

In this paper, we have proposed a Nyström method for weakly singular Volterra integral equations, with kernels presenting singularities along the diagonal $y = x$ and/or at the side $y = -1$ of the square $[-1, 1]^2$. These cases are pathological since the solution inherits the singularities.

After the conditions assuring the equation is unisolvent in C_u, we have proved the numerical method is stable and convergent, providing also error estimates in Zygmund weighted spaces. The novelty of these results depends on the analysis of the method, carried out in spaces of functions endowed with weighted uniform norms. The choice of these spaces allows obtaining better performance w.r.t. similar global procedures like projection methods. Indeed, usually the latter require smoothing transformations to regularize the solution, not required in our method. Hence, our procedure is lighter and requires a lower computational cost.

The numerical experiments have confirmed the theoretical results, showing the stability of the method, the well conditioning of the final linear system we have to solve, and the theoretical order of convergence. In addition, a comparison between the proposed approach and the recent methods developed in [18, 22] supports the satisfactory performance of the Nyström method.

Finally, we would like to mention that the proposed approach can be also extended to the nonlinear or multidimensional case [37, 38, 39]. This will be the subject of a future research work.

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