

# M-integral for finite anti-plane shear of a nonlinear elastic matrix with rigid inclusions

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## ABSTRACT

The path-independent M-integral plays an important role in analysis of solids with inhomogeneities. However, the available applications are almost limited to linear-elastic or physically non-linear power law type materials under the assumption of infinitesimal strains. In this paper we formulate the M-integral for a class of hyperelastic solids undergoing finite anti-plane shear deformation. As an application we consider the problem of rigid inclusions embedded in a Mooney–Rivlin matrix material. With the derived M-integral we compute weighted averages of the shear stress acting on the inclusion surface. Furthermore, we prove that a system of rigid inclusions can be replaced by one effective inclusion.

## 1. Introduction

The analysis of stress field in an elastic matrix around a rigid or elastic inclusion is a key step in design and modeling of composite materials. Starting from the pioneering work of Eshelby (1957) on an ellipsoidal inclusion embedded in an elastic medium, this problem is widely used in the analysis of effective properties of composites, see e.g. Kachanov and Sevostianov (2018), Lakes (2020), Nemat-Nasser and Hori (1993) and Yee and Ghayesh (2023). However the most studies in this field deal with the linear elasticity under the assumption of infinitesimal strains. Only few closed form analytical solutions are available for non-linear elastic materials undergoing finite deformations. Examples include the analysis of linear-elastic spherical inclusions in a non-linear elastic matrix under hydrostatic stress state (Hashin, 1985), cylindrical inclusions embedded in a neo-Hookean matrix under anti-plane shear and extensional loadings (deBotton et al., 2006), and elliptical inclusions in a neo-Hookean matrix under anti-plane shear deformation (Avazmohammadi et al., 2009). In this paper we address the analysis of stress concentration around a rigid inclusion embedded in an incompressible nonlinear elastic material under the anti-plane shear deformation. The class of considered materials includes Mooney–Rivlin material and some others. The assumption of this loading type leads to a simplified structure of the governing equations since only one scalar displacement field has to be found. The assumption regarding the constitutive behavior allows the reduction of the equilibrium conditions to a single partial differential equation for the displacement. For example, in the case of Mooney–Rivlin material the latter takes the form of the Laplace equation. The same equation is known from the linear theory of elasticity. However, the boundary value problem of finite anti-plane shear remains non-linear due to static boundary conditions. One feature of the Mooney–Rivlin material in comparison to the neo-Hookean one is the appearance of normal stresses on the contact surface under shear, see also recent discussion on the  $I_2$  invariant by Anssari-Benam et al. (2021). This second order

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effect in sense of Reiner and Abir (1964) and Truesdell and Noll (2004) may contribute to the debonding of the inclusion from the surrounding matrix. Let us note nowadays that Mooney–Rivlin model and its particular case such as neo-Hookean material are widely used for modeling of solids undergoing large applications, see e.g. recent papers by Carpenter et al. (2020), Eremeyev and Naumenko (2015), Haldar (2021), Islam et al. (2021), Khaniki and Ghayesh (2023), Khaniki et al. (2023) and Wang et al. (2021).

Despite the particular kind of loading and governing equations, the anti-plane shear plays an important role in the non-linear solid mechanics since various aspects of the general non-linear problem may be examined in a simple setting (Horgan, 1995). The conditions for the existence of non-trivial anti-plane shear deformation modes are given by Knowles (1976) for incompressible and by Knowles (1977b) for compressible non-linear elastic materials. Abeyaratne (1981), Knowles (1977a), Knowles and Sternberg (1980) and Rajagopal and Walton (2011) analyzed stress fields near cracks under finite anti-plane shear. Mathematical studies to the finite anti-plane shear boundary value problem with non-convex strain energy functions are presented by Gurtin and Temam (1981) and Raymond (1993). Many examples of finite anti-plane shear can be found in textbooks on the non-linear elasticity (Antman, 2005; Green & Adkins, 1960; Gurtin, 1983; Lurie, 1990).

One way to estimate stresses and deformations around inclusions, notches, cavities, cracks and other concentration sources is based on conservation laws and invariant J- and M-integrals (Gurtin, 2000; Kienzler & Herrmann, 2000; Maugin, 1993, 2011; Olver, 1984a, 1984b), see also more recent paper by Agiasofitou and Lazar (2017). By this approach basic features of the stress (deformation) field in the concentration zone can be analyzed without the solution of the general boundary value problem, see, for example, Altenbach et al. (2011), Kienzler and Kordisch (1990), Kienzler, Rohde and Schröder (2010) and Kienzler, Rohde, Schröder and Kutz (2010). Note that the more conservation laws and invariant integrals are known, the more features of the solution can be obtained. The conservation laws are related to the variational symmetries of the corresponding Lagrangian. In particular, J- and M-integrals can be derived applying the variational symmetries of the infinitesimal translation and the self-similar expansion (scaling), respectively. In other words, they are related to the homogeneity of the material and the linearity of the constitutive equation, see Kienzler and Herrmann (2000). Invariant integrals are well established and widely used within the linear theory. However, only the J-integral can be applied to non-linear elastic materials while the M-integral can only be used either for linear or for power law type constitutive equations (Lubarda & Markenscoff, 2007), that is the case when the strain energy density is a homogeneous function of infinitesimal strains. Conservation laws and path-independent integrals are closely related to the energy release rate in vicinity of singularities, see Ait-Bachir et al. (2012), Banks-Sills et al. (2013), Budiansky and Rice (1973), Chen (2002), Eischen and Herrmann (1987), Lubarda and Markenscoff (2007) and Piccolroaz et al. (2021) and the references therein. For applications of conservation laws in statics and dynamics we refer to Atkinson and Aparicio (1999), Cahill et al. (2014), Chen and Shield (1977), Chudnovsky (2014), Deng, Yan and Okabe (2022), Deng, Yan and Zhu (2022), El Kabir et al. (2018, 2016), Fletcher (1976), Freund (1978), Guo and Li (2015), Judt and Ricoeur (2015, 2016), Kim and Paulino (2002), Ma and Hills (2022), Markenscoff (2020), Morini et al. (2013), Park and Earmme (1986), Peng et al. (2021), Shi (2011), Shield (1977), Song and Paulino (2006), Yu et al. (2018, 2017), Zhang et al. (2017) and Zuo and Feng (2013). Analysis of the stress intensity factor (and thus J-integral) for crack moving along the stiff (even rigid interface) is discussed by Mishuris et al. (2006). Influence of the linear but essential change in geometry and material properties of the stiff and (in limiting case) rigid interface has been discussed by Mishuris (2004), where it was shown that the number of singular terms maybe more than one.

The paper is organized as follows. In Section 2 we present the basic equations of the finite elasticity and derive two types of integral identities related to conservation laws. We consider both three- and two-dimensional problems. We formulate the path-independent J- and M-integrals for non-linear elastic incompressible solids. For three-dimensional problems the M-integral cannot be reduced to the surface one, in general. Consequently a path independent line integral does not exist for the general two-dimensional case. Considering the finite anti-plane shear of the Mooney–Rivlin material in Section 3, we derive the path-independent M-integral as a line integral. We show that for the anti-plane shear the strain energy density is a null-Lagrangian. In addition we consider essentially nonlinear material. Finally, in Section 4 we apply the derived M-integral to the analysis of elastic solids with rigid inclusions.

Throughout the paper we use the direct tensor calculus in the sense of Gibbs' notation. Basic rules are given in Eremeyev et al. (2018), Lurie (1990) and Simmonds (1994) among others.

## 2. Basic equations of the non-linear elasticity and some integral identities

Let us consider a body made of isotropic homogeneous incompressible non-linear elastic material which occupies in its undeformed state (reference placement) a region  $B \subset \mathbb{R}^3$ . A deformation of the body is described by a smooth one-to-one mapping

$$\mathbf{x} = \mathbf{x}(\mathbf{X}), \quad (1)$$

where  $\mathbf{x}$  and  $\mathbf{X} \in B$  are the position vectors of a given particle in the actual (deformed) and reference (undeformed) placements, respectively. We define the deformation gradient tensor  $\mathbf{F}$  as follows

$$\mathbf{F} = \nabla \mathbf{x}, \quad (2)$$

where  $\nabla$  is the nabla-operator in the initial placement. The left Cauchy–Green strain tensor and its principal invariants are given by

$$\mathbf{C} = \mathbf{F} \cdot \mathbf{F}^T, \quad (3)$$

$$I_1 = \text{tr } \mathbf{C}, \quad I_2 = \frac{1}{2}(\text{tr}^2 \mathbf{C} - \text{tr } \mathbf{C}^2), \quad I_3 = \det \mathbf{C}.$$

The incompressibility of the material requires that

$$\det \mathbf{F} = 1, \quad I_3 = 1.$$

The mechanical response of a hyperelastic material is determined by the strain energy density  $W$ . For only isotropic incompressible materials  $W$  depends on two invariants  $I_1$  and  $I_2$

$$W = W(I_1, I_2). \tag{4}$$

The first Piola–Kirchhoff stress tensor  $\mathbf{P}$  is related to  $W$  by the formula (Lurie, 1990)

$$\mathbf{P} = -p\mathbf{F}^{-T} + 2\frac{\partial W}{\partial I_1}\mathbf{F} + 2\frac{\partial W}{\partial I_2}(I_1\mathbf{I} - \mathbf{C}) \cdot \mathbf{F}, \tag{5}$$

where  $p$  is a hydrostatic pressure,  $\mathbf{I}$  is the unit tensor, and  $\mathbf{F}^{-T}$  denotes the inverse of the transpose of  $\mathbf{F}$ .

In the absence of body forces, the Lagrangian equilibrium equation takes the form

$$\nabla \cdot \mathbf{P} = \mathbf{0}. \tag{6}$$

For a homogeneous non-linear elastic material we introduce the Eshelby tensor (Maugin, 1993)

$$\mathbf{B} = W\mathbf{I} - \mathbf{P} \cdot \mathbf{F}^T, \tag{7}$$

which satisfies the relation

$$\nabla \cdot \mathbf{B} = \mathbf{0}. \tag{8}$$

Eq. (8) can be proved by straightforward calculations, see Gurtin (2000), Kienzler and Herrmann (2000) and Maugin (1993, 2011) for details. Integrating (8) over an arbitrary region  $V \subset B$  and using the divergence theorem we obtain the identity

$$\int_A \mathbf{n} \cdot \mathbf{B} \, dA = \mathbf{0}, \tag{9}$$

where  $A = \partial V$  and  $\mathbf{n}$  is the unit vector normal to  $A$ . Identity (9) is the origin of so-called path-independent integrals like as the J-integral used in the fracture mechanics (Kienzler & Herrmann, 2000; Maugin, 1993). Obviously, from (6) it also follows

$$\int_A \mathbf{n} \cdot \mathbf{P} \, dA = \mathbf{0}, \tag{10}$$

which expresses the balance of momentum.

Using (6)–(8) another integral identity related to the M-integral in the linear elasticity can be derived. Applying the identities

$$\nabla \cdot (\mathbf{B} \cdot \mathbf{X}) = \text{tr } \mathbf{B} \quad \text{and} \quad \text{tr } \mathbf{B} = 3W - \mathbf{P} : \mathbf{F},$$

where  $:$  stands for the double-dot product of second-order tensors, we obtain the relation

$$\int_A \mathbf{n} \cdot \mathbf{B} \cdot \mathbf{X} \, dA = \int_V \nabla \cdot (\mathbf{B} \cdot \mathbf{X}) \, dV = 3 \int_V W \, dV - \int_V \mathbf{P} : \mathbf{F} \, dV. \tag{11}$$

With the divergence theorem and the equilibrium condition (6) the last term in (11) can be transformed as it follows

$$\int_V \mathbf{P} : \mathbf{F} \, dV = - \int_V (\nabla \cdot \mathbf{P}) \cdot \mathbf{x} \, dV + \int_A \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{x} \, dA = \int_A \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{x} \, dA.$$

Eq. (11) takes the form

$$\int_A \mathbf{n} \cdot \mathbf{B} \cdot \mathbf{X} \, dA = 3 \int_V W \, dV - \int_A \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{x} \, dA. \tag{12}$$

Substituting (7) into (12) we obtain

$$\int_A [W\mathbf{n} \cdot \mathbf{X} - \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{F}^T \cdot \mathbf{X} + \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{x}] \, dA = 3 \int_V W \, dV. \tag{13}$$

The volume integral in the right hand side of (13) cannot be reduced to a surface one, in general. Such transformation is always possible in the case of linear elastic solids, or for special constitutive relations (power law) of physically non-linear materials, see, for example, Lubarda and Markenscoff (2007). In what follows we show that such transformation is also possible in the case of finite anti-plane shear of specific non-linear elastic materials such as for example the Mooney–Rivlin material.

Let us note that for derivation of (9), (10) and (13) we assumed that all considered fields are continuously differentiable functions as it should be for the application of the divergence theorem. In particular, this means that there are no discontinuities that lead to the loss of ellipticity of the equilibrium conditions. In such a case the surface integrals over the discontinuity surfaces must be added to (9), (10) and (13). For the related analysis we refer to Abeyaratne (1981), Chen (1996) and Knowles and Sternberg (1980).

The integral identities (9), (10) and (13) are derived for the three-dimensional (3D) case. For applications it is also useful to consider two-dimensional (2D) problems. Let  $X_1, X_2, X_3$  be the Cartesian coordinates in the reference placement with the base

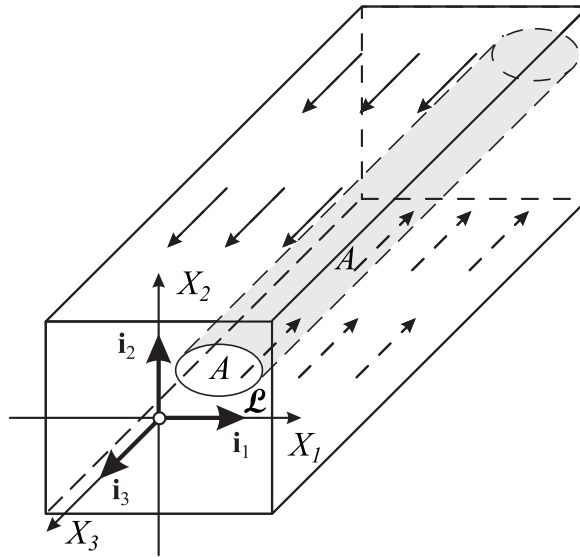


Fig. 1. Cylindrical body  $B$  subjected to anti-plane shear in the reference placement.

vectors  $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ , respectively. Let us assume that  $\mathbf{F}$  depends only on two coordinates, that is  $\mathbf{F} = \mathbf{F}(X_1, X_2)$ . With this assumption several deformation modes including the plain deformation, the anti-plane shear and the axial-symmetric deformation can be analyzed. Hence,  $\mathbf{P}$  and  $\mathbf{B}$  depend also on  $X_1$  and  $X_2$ . Without loss of generality one can introduce the Eshelby tensor in the 2D case by the formula

$$\mathbf{B} = W\mathbf{I}_2 - \mathbf{P} \cdot \mathbf{F}^T, \tag{14}$$

where  $\mathbf{I}_2 = \mathbf{i}_1 \otimes \mathbf{i}_1 + \mathbf{i}_2 \otimes \mathbf{i}_2$  is the 2D unit tensor. Both tensors  $\mathbf{B}$  and  $\mathbf{P}$  satisfy (6) and (8) with the 2D nabla-operator:  $\nabla = \mathbf{i}_1 \frac{\partial}{\partial X_1} + \mathbf{i}_2 \frac{\partial}{\partial X_2}$ . Thus, we have two integral identities

$$\int_{\mathcal{L}} \mathbf{n} \cdot \mathbf{B} \, ds = \mathbf{0}, \tag{15}$$

$$\int_{\mathcal{L}} \mathbf{n} \cdot \mathbf{P} \, ds = \mathbf{0}. \tag{16}$$

Here  $\mathcal{L}$  is the contour that bounds the plane region  $A$  and  $\mathbf{n}$  is the unit normal to  $\mathcal{L}$ ,  $\mathbf{n} \cdot \mathbf{i}_3 = 0$ . The derivation of the 2D analogue of (13) mimics the 3D case and leads to the identity

$$\int_{\mathcal{L}} [W\mathbf{n} \cdot \mathbf{X} - \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{F}^T \cdot \mathbf{X} + \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{x}] \, ds = 2 \int_A W \, dA. \tag{17}$$

Note that (15) and (16) coincide up to notations with (9) and (10), respectively, while (17) differs from (13) by the factor 2 in the right hand side.

### 3. Anti-plane shear and M-integral

For the study of the anti-plane shear we consider a cylindrical body  $B$  deformed in such a way that the displacement vector of each particle remains parallel to the cylinder axis and is independent of the axial coordinate. We introduce the Cartesian coordinates  $x_1, x_2, x_3$  and  $X_1, X_2, X_3$  in the actual and initial placements, respectively, with the third axis parallel to  $B$ , see Fig. 1. Thus, the anti-plane shear is described by

$$\mathbf{x} = \mathbf{X} + u(X_1, X_2)\mathbf{i}_3, \tag{18}$$

or by

$$x_1 = X_1, \quad x_2 = X_2, \quad x_3 = X_3 + u(X_1, X_2),$$

where  $u$  is an unknown function. Hereinafter  $\mathbf{i}_1, \mathbf{i}_2$  and  $\mathbf{i}_3$  are the base unit orthogonal vectors related to  $X_1, X_2, X_3$ .

The deformation gradient tensor  $\mathbf{F}$  corresponding to (18), the left Cauchy–Green strain tensor and its invariants take the form

$$\begin{aligned} \mathbf{F} &= \mathbf{I} + \nabla u \otimes \mathbf{i}_3, & \mathbf{C} &= \mathbf{I} + \nabla u \otimes \mathbf{i}_3 + \mathbf{i}_3 \otimes \nabla u + \nabla u \otimes \nabla u, \\ I_1 = I_2 &= 3 + |\nabla u|^2, & I_3 &= 1, \quad |\nabla u|^2 = \nabla u \cdot \nabla u. \end{aligned} \tag{19}$$

Equilibrium condition (6) includes three scalar equations with respect to two unknown functions  $p$  and  $u$ . This means that the system of Eqs. (6) is overdetermined and the anti-plane shear deformation mode is only possible for specific geometry and/or constitutive equation. The requirements for the existence of anti-plane shear are formulated and proved by Knowles (1976, 1977a). In the following we consider some of these materials.

### 3.1. Mooney–Rivlin material

The Mooney–Rivlin model satisfies the conditions given by Knowles (1976). The corresponded strain energy density takes the form

$$W = C_1(I_1 - 3) + C_2(I_2 - 3), \tag{20}$$

where  $C_1$  and  $C_2$  are the elastic moduli. Further examples satisfying the condition given in Knowles (1976) include materials with the strain energy density in more simple form like this one  $W = W(I_1)$  and some others.

For the Mooney–Rivlin material and the anti-plane shear, Eq. (6) reduces to the Laplace equation

$$\nabla^2 u = 0, \tag{21}$$

and  $W$  takes the form

$$W = K|\nabla u|^2, \quad K = C_1 + C_2. \tag{22}$$

With Eq. (22) the volume integral in (13) and the surface one in (17) are

$$\int_V W \, dV = K \int_A u \mathbf{n} \cdot \nabla u \, dA, \tag{23}$$

$$\int_A W \, dA = K \int_{\mathcal{L}} u \mathbf{n} \cdot \nabla u \, ds, \tag{24}$$

Indeed, the strain energy density  $W$  takes the divergence form

$$W = \nabla \cdot (Ku \nabla u).$$

Therefore, as any divergence,  $W$  is a null Lagrangian for the anti-plane shear of the Mooney–Rivlin material, see Olver (1993). The technique of null-Lagrangians for generating new exact solutions of the non-linear elasticity problems is used by Haughton (2008) where other null strain energy functions for the anti-plane shear are presented.

With Eq. (5) the Piola–Kirchhoff stress tensor is specified as follows

$$\mathbf{P} = -q\mathbf{I} - 2C_2 \nabla u \otimes \nabla u + 2K \nabla u \otimes \mathbf{i}_3 + (p - 2C_2)\mathbf{i}_3 \otimes \nabla u + (K + 2C_2 - p)\mathbf{i}_3 \otimes \mathbf{i}_3, \tag{25}$$

$$q = p - 2C_1 - 2(2 + |\nabla u|^2)C_2,$$

see Knowles (1976) and Lurie (1990) for details. Let us note that although the equilibrium equations (6) are reduced to the linear Laplace equation (21), the boundary value problem remains still non-linear, in general, due to the non-linear static boundary conditions. Indeed, in the case of a free boundary we have the non-linear condition

$$\mathbf{n} \cdot \mathbf{P} = \mathbf{0}.$$

Substituting (23) into (13), we derive the surface integral identity

$$\int_A [W \mathbf{n} \cdot \mathbf{X} - \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{F}^T \cdot \mathbf{X} + \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{x} - 3Ku \mathbf{n} \cdot \nabla u] \, dA = 0. \tag{26}$$

For the anti-plane shear of a cylindrical body it is more convenient to use 2D identity (17). Substituting (24) into (17), we derive the path-independent M-integral in the form of line integral over the contour  $\mathcal{L}$

$$M \equiv \int_{\mathcal{L}} [W \mathbf{n} \cdot \mathbf{X} - \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{F}^T \cdot \mathbf{X} + \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{x} - 2Ku \mathbf{n} \cdot \nabla u] \, ds = 0. \tag{27}$$

Using (19) and (22) we transform  $M$  to

$$M = \int_{\mathcal{L}} [K|\nabla u|^2 \mathbf{n} \cdot \mathbf{X} - \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{i}_3 \nabla u \cdot \mathbf{X} + \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{i}_3 u - 2Ku \mathbf{n} \cdot \nabla u] \, ds. \tag{28}$$

Taking into account (10) and (19), Eq. (9) reduces to the form

$$\int_A [W \mathbf{n} - \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{F}^T] \, dA = \int_A [W \mathbf{n} - \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{i}_3 \nabla u] \, dA = \mathbf{0}. \tag{29}$$

Using (29) in a similar way we introduce the path-independent J-integral for the anti-plane shear as it follows

$$\mathbf{J} \equiv \int_{\mathcal{L}} [K|\nabla u|^2 \mathbf{n} - \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{i}_3 \nabla u] \, ds = \mathbf{0}. \tag{30}$$

In both integral identities (28) and (30) the mixed component of the Piola–Kirchhoff stress tensor  $P_{n3} \equiv \mathbf{n} \cdot \mathbf{P} \cdot \mathbf{i}_3$  is calculated from (25) as follows

$$P_{n3} = 2K\mathbf{n} \cdot \nabla u. \tag{31}$$

Thus, for the finite anti-plane shear of the Mooney–Rivlin material we formulate two integral identities as follows

$$M \equiv \int_{\mathcal{L}} [K|\nabla u|^2 \mathbf{n} \cdot \mathbf{X} - 2K(\mathbf{n} \cdot \nabla u)\nabla u \cdot \mathbf{X}] ds = 0, \tag{32}$$

$$\mathbf{J} \equiv \int_{\mathcal{L}} [K|\nabla u|^2 \mathbf{n} - 2K(\mathbf{n} \cdot \nabla u)\nabla u] ds = \mathbf{0}. \tag{33}$$

Here  $\mathbf{n}$  is a unit vector lying in the  $\mathbf{i}_1$ – $\mathbf{i}_2$  plane, that is  $\mathbf{n} \cdot \mathbf{i}_3 = 0$ , and  $\nabla$  is the two-dimensional operator since  $u$  does not depend on  $X_3$ . Introduced in (32) and (33) the quantities  $M$  and  $\mathbf{J}$  represent the path-independent M- and J-integrals for the Mooney–Rivlin material, respectively. Eqs. (32) and (33) can also be derived considering the Lagrangian

$$L = \int_A K|\nabla u|^2 dA$$

and applying the technique of Olver (1993). In what follows we illustrate the application of (32) and (33) for estimation of stress state around a rigid inclusion.

Let us note that the volume integral in (13) or the surface one in (17) can be reduced to the surface or line integrals not only for the Mooney–Rivlin material.

### 3.2. Essentially nonlinear material

Another admissible model is a so-called essentially nonlinear material studied by Zubov and Rudev (1993, 1996). Here  $W$  is given by

$$W = \frac{C}{\alpha} (I_1 - 3)^\alpha,$$

where  $C > 0$  is an elastic modulus and  $\alpha \geq 1/2$  is an additional parameter. For  $\alpha \neq 1$  this model demonstrates essential nonlinearity even in the case of small deformations. Note this model satisfies to the Hadamard condition for all deformations (Zubov & Rudev, 1993). One can find some similarities between this model and equations of implicit elasticity developed by Rajagopal (2007, 2010, 2011). For antiplane deformations  $W$  is a homogeneous function of  $\nabla u$  such as

$$W = \frac{C}{\alpha} |\nabla u|^{2\alpha}.$$

The corresponding equilibrium equation has the form

$$\nabla \cdot [(\nabla u \cdot \nabla u)^{\alpha-1} \nabla u] = 0.$$

Now the technique by Lubarda and Markenscoff (2007) can be applied here with minor changes in order to derive M-integral. Indeed, here we have the formulae

$$\int_V W dV = \frac{C}{\alpha} \int_A u(\nabla u \cdot \nabla u)^{\alpha-1} \mathbf{n} \cdot \nabla u dA, \tag{34}$$

$$\int_A W dA = \frac{C}{\alpha} \int_{\mathcal{L}} u(\nabla u \cdot \nabla u)^{\alpha-1} \mathbf{n} \cdot \nabla u ds, \tag{35}$$

which demonstrate that the energy integral can be transformed into a surface or to a contour integral.

## 4. Anti-plane shear of elastic solids with rigid inclusions

As an example of application of the M-integral we consider the finite anti-plane shear of an elastic body with rigid inclusions. The main idea of application of (32) and (33) is to select the contour  $\mathcal{L}$  as the union of two contours,  $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ , such that the integrands in (32) and (33) are known or can be easily calculated for one of them. Hence, without the solution of the problem we obtain additional information on the displacement and stresses for the second contour.

### 4.1. Axially symmetric anti-plane shear of an elastic tube with a circular rigid inclusion

To illustrate the idea let us start from the simplest case of the axial anti-plane shear of an elastic circular tube. The full closed-form analytical solution to this problem is available, see, for example, Lurie (1990). Here we have  $u = u(R)$  where  $R = \sqrt{X_1^2 + X_2^2}$  is the radius of a point in the reference placement, see Fig. 2. We assume the following boundary conditions

$$u(a) = 0, \quad u(b) = U,$$

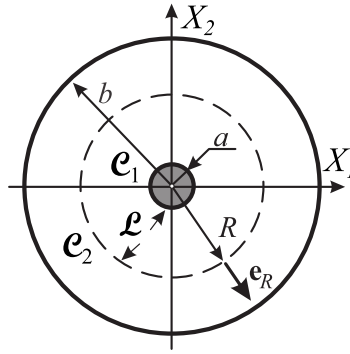


Fig. 2. Cross-section of a circular cylinder with a rigid inclusion.

where  $a$  and  $b$  are the internal and external radii of the tube, respectively, and  $U$  is the given displacement. The displacement has the form (Lurie, 1990)

$$u(R) = U \frac{\ln R/a}{\ln b/a}. \tag{36}$$

For the axial anti-plane shear we have  $\nabla u = u' \mathbf{e}_R$ , where prime stands for the derivative with respect to  $R$ ,  $(\dots)' = d(\dots)/dR$ , and  $\mathbf{e}_R$  is the base vector corresponding to the polar coordinate  $R$ .

It would seem at first sight that the problem is linear as in linear elasticity, but it is not the case. Indeed, unlike the linear elasticity here the normal stresses are non-zero. It is an example of the so-called second-order effects in the finite elasticity (Lurie, 1990; Reiner & Abir, 1964; Truesdell & Noll, 2004). For example, the normal stress  $P_{RR}$  is given by

$$P_{RR} = C_2 U^2 \left[ \frac{2}{(b^2 - a^2) \ln b/a} + \frac{1}{R^2 (\ln b/a)^2} \right],$$

see Lurie (1990) for details. Non-zero tensile stress  $P_{RR}$  may contribute to debonding of the inclusion from the surrounding matrix.

Let us choose  $\mathcal{L}$  as the union of two circles  $C_1$  and  $C_2$  with radii  $a$  and  $R$ , respectively. With this choice Eq. (33) reduces to a trivial identity. Indeed, for any circle  $C$  of radius  $R$  and any function  $f(R)$  we have the identity

$$\int_C f(R) \mathbf{e}_R ds = R f(R) \int_0^{2\pi} \mathbf{e}_R d\Phi = 0,$$

to which reduces (33).

Unlike (33) the M-integral leads to less trivial results. Applying (32) to  $\mathcal{L} = C_1 \cup C_2$  we obtain the identity

$$K u'^2(R) R^2 = K u'^2(a) a^2, \tag{37}$$

which represents the first integral of (21) in the case of axial symmetry. Indeed, for the axially symmetric deformation Eq. (21) takes the form

$$u''(R) + \frac{u'(R)}{R} = \frac{1}{R} (R u'(R))' = 0$$

and has the following first integral

$$R u'(R) = \text{const}$$

coinciding with (37). Obviously, the displacement (36) satisfies (37). Moreover, because the closed-form solution is available, there is no need for the use of path-independent integrals. Indeed, the M-integral leads to another, more complicated way of solution of the problem. On the other hand this simple example demonstrates that unlike the J-integral, the M-integral gives an additional information on solution of the problem and can be used in more complicated cases where the analytical solution is unknown.

#### 4.2. Anti-plane shear of a tube with an arbitrary inclusion or a system of inclusions

Let us extend the previous results to the case of an arbitrary rigid inclusion, see Fig. 3 a. For such a geometry  $u$  depends on both polar coordinates,  $R$  and  $\Phi$ . The boundary conditions are

$$u|_{\Gamma} = 0, \quad u(b) = U,$$

where  $\Gamma$  is the contour of the inclusion. Choosing  $\mathcal{L}$  as the union of the circle  $C$  of radius  $R$  and  $\Gamma$  from (32) we obtain

$$\int_C \left[ K |\nabla u|^2 - 2K \left( \frac{\partial u}{\partial R} \right)^2 \right] R ds - \int_{\Gamma} K \left( \frac{\partial u}{\partial n} \right)^2 \mathbf{n} \cdot \mathbf{X} ds$$

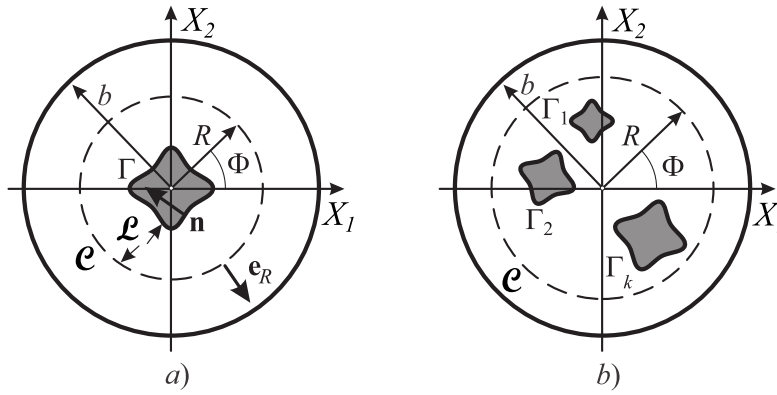


Fig. 3. Cross-section of a circular cylinder with an arbitrary rigid inclusion (a) and with a system of rigid inclusions (b).

$$= K \int_0^{2\pi} \left[ \left( \frac{\partial u}{\partial \Phi} \right)^2 - R^2 \left( \frac{\partial u}{\partial R} \right)^2 \right] d\Phi - K \int_{\Gamma} \left( \frac{\partial u}{\partial n} \right)^2 \mathbf{n} \cdot \mathbf{X} ds = 0. \tag{38}$$

Substituting  $R = b$  into (38) and taking into account the boundary condition at  $R = b$  we reduce (38) to

$$- b^2 \int_0^{2\pi} \left( \frac{\partial u}{\partial R} \right)^2 \Big|_{R=b} d\Phi = \int_{\Gamma} \left( \frac{\partial u}{\partial n} \right)^2 \mathbf{n} \cdot \mathbf{X} ds. \tag{39}$$

With (31) we transform (39) to

$$- b^2 \int_0^{2\pi} P_{R3}^2 \Big|_{R=b} d\Phi = \int_{\Gamma} P_{n3}^2 \mathbf{n} \cdot \mathbf{X} ds. \tag{40}$$

Eq. (40) gives the averaged value of the shear stress  $P_{n3}$ . It shows that, we prove the averaged value of  $P_{n3}$  with the weight function  $\mathbf{n} \cdot \mathbf{X}$  has the same value for any shape of the inclusion.

In addition to (40) from (16) it follows another integral identity

$$- b \int_0^{2\pi} P_{R3} \Big|_{R=b} d\Phi = \int_{\Gamma} P_{n3} ds. \tag{41}$$

The relations (38), (40), and (41) can be useful for the verification of numerical analysis as well as for the estimation of mean values of shear stresses on the inclusion surface.

Obviously, one can consider a system of rigid inclusions, see Fig. 3b. Considering  $\mathcal{L} = C \cup \Gamma_1 \cup \Gamma_2 \dots \Gamma_k$  where  $\Gamma_i, i = 1, 2, \dots, k$ , are the contours of the inclusions we obtain the formulae

$$\int_0^{2\pi} \left[ \left( \frac{\partial u}{\partial \Phi} \right)^2 - R^2 \left( \frac{\partial u}{\partial R} \right)^2 \right] d\Phi = \sum_{i=1}^k \int_{\Gamma_i} \left( \frac{\partial u}{\partial n} \right)^2 \mathbf{n} \cdot \mathbf{X} ds, \tag{42}$$

$$- b^2 \int_0^{2\pi} P_{R3}^2 \Big|_{R=b} d\Phi = \sum_{i=1}^k \int_{\Gamma_i} P_{n3}^2 \mathbf{n} \cdot \mathbf{X} ds, \tag{43}$$

$$- b \int_0^{2\pi} P_{R3} \Big|_{R=b} d\Phi = \sum_{i=1}^k \int_{\Gamma_i} P_{n3} ds. \tag{44}$$

Comparing (40), (41) and (43), (44) we conclude that far from inclusions the system of inclusions can be replaced by one effective inclusion.

### 4.3. Anti-plane shear of an elastic solid with a symmetric rigid inclusion

Let us consider the anti-plane shear of a material with a rigid inclusion having two perpendicular axes of symmetry, see Fig. 4. Far from the inclusion we have the homogeneous anti-plane shear with  $u$  given by

$$u \approx kX_2 \quad \text{as } R \rightarrow \infty, \tag{45}$$

where  $k$  is the dimensionless parameter. Due to symmetry the displacement on the inclusion surface vanishes

$$u = 0. \tag{46}$$

Let us use in (32) and (33) the closed contour  $\mathcal{L} = ABCDEFA$ , see Fig. 4. Therefore we consider the integrands of (33) and (32) along the path  $ABCDEFA$ . Along  $AB$  and  $DE$  we have

$$u = 0, \quad \frac{\partial u}{\partial X_1} = 0, \quad \mathbf{X} = X_1 \mathbf{i}_1, \quad \mathbf{n} = -\mathbf{i}_2. \tag{47}$$



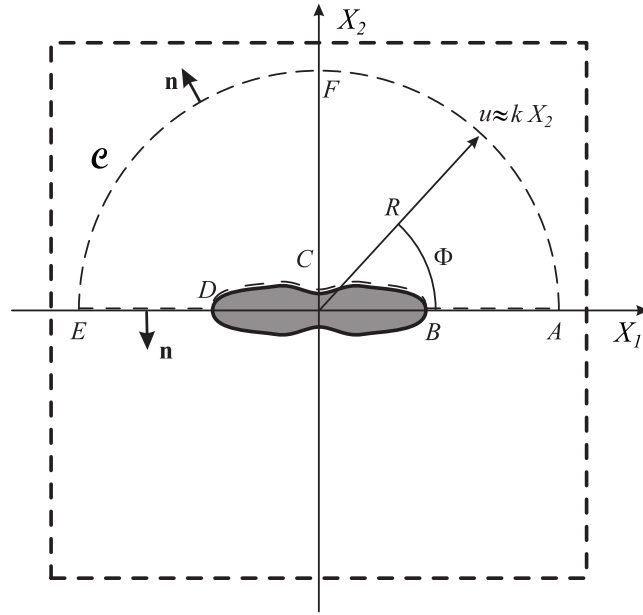


Fig. 4. Cross-section of an arbitrary cylindrical body with a symmetric rigid inclusion.

Along  $BCD$  we obtain

$$u = 0, \quad \nabla u = -\frac{\partial u}{\partial n} \mathbf{n}, \tag{48}$$

while along  $EFA$  we use (45) to compute

$$\nabla u = k \mathbf{i}_2, \quad \mathbf{n} = \mathbf{e}_R, \quad \mathbf{X} = R \mathbf{e}_R. \tag{49}$$

Substituting (47), (48) and (49) into (32) the M-integral is

$$\begin{aligned} & \int_{BCD} [K|\nabla u|^2 \mathbf{n} \cdot \mathbf{X} - 2K(\mathbf{n} \cdot \nabla u) \nabla u \cdot \mathbf{X}] \, ds \\ & + \int_{EFA} [K|\nabla u|^2 \mathbf{n} \cdot \mathbf{X} - 2K(\mathbf{n} \cdot \nabla u) \nabla u \cdot \mathbf{X}] \, ds \\ & = - \int_{BCD} K \left| \frac{\partial u}{\partial n} \right|^2 \mathbf{n} \cdot \mathbf{X} \, ds + \int_0^\pi [k^2 R - 2Kk^2 R(\mathbf{e}_R \cdot \mathbf{i}_2)^2] R \, d\Phi = 0. \end{aligned}$$

With the identity  $\mathbf{e}_R \cdot \mathbf{i}_2 = \sin \phi$  it takes the form

$$\int_{BCD} K \left| \frac{\partial u}{\partial n} \right|^2 \mathbf{n} \cdot \mathbf{X} \, ds = 0. \tag{50}$$

In addition, with the selected contour  $\mathcal{L}$  we calculate the projection of the vector  $\mathbf{J}$  on the vector  $\mathbf{i}_1$  as follows

$$\begin{aligned} \mathbf{J} \cdot \mathbf{i}_1 &= \int_{BCD} [K|\nabla u|^2 \mathbf{n} - 2K(\mathbf{n} \cdot \nabla u) \nabla u] \, ds \cdot \mathbf{i}_1 \\ &+ \int_{EFA} [K|\nabla u|^2 \mathbf{n} - 2K(\mathbf{n} \cdot \nabla u) \nabla u] \, ds \cdot \mathbf{i}_1 \\ &= - \int_{BCD} K \left| \frac{\partial u}{\partial n} \right|^2 \mathbf{n} \cdot \mathbf{i}_1 \, ds. \end{aligned}$$

From the symmetry of the problem under consideration we find that the latter integral vanish identically. The use of the J-integral leads again to the trivial result as in the case of axially symmetric anti-plane shear.

Let us analyze (50). Obviously, if  $\frac{\partial u}{\partial n}|_\Gamma = 0$  then (50) is fulfilled. Vice versa, from (50) it follows that  $\frac{\partial u}{\partial n} = 0$  when  $\mathbf{X} \cdot \mathbf{n} < 0$  along  $\Gamma$ . This requirement is satisfied for an inclusion with convex cross-section, for example for circular or elliptic inclusion.

Hence, assuming  $\mathbf{X} \cdot \mathbf{n} < 0$  for almost every point of  $\Gamma$  we obtain that

$$\frac{\partial u}{\partial n} \Big|_\Gamma = 0 \quad \text{or} \quad P_{n3} \Big|_\Gamma = 0.$$

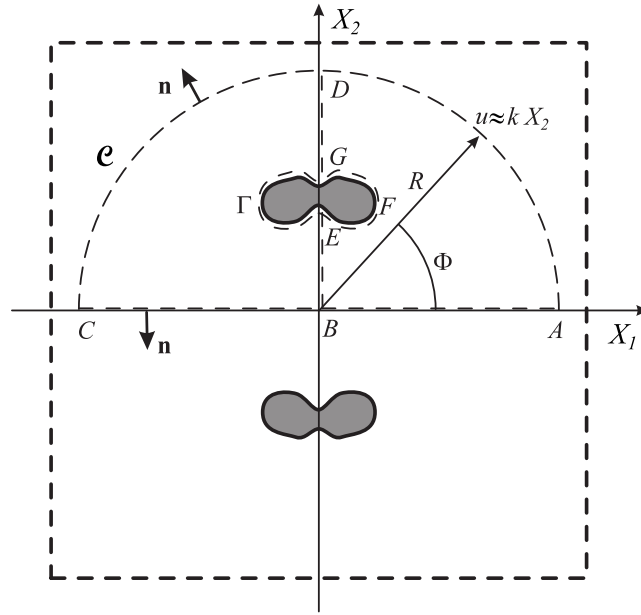


Fig. 5. Cross-section of an arbitrary cylindrical body with a symmetric system of two rigid inclusions.

At first sight this conclusion may seem somewhat surprising because we already have  $u = 0$  along  $\Gamma$ . However this is a consequence of our assumption regarding the symmetry of the considered problem. From the physical point of view this means that such rigid inclusion does not increase the resistance of the composite if compared with the response of the matrix without the inclusion.

4.4. Anti-plane shear of an elastic solid with a symmetric system of rigid inclusions

Let us now consider a system of inclusions. For the sake of simplicity we analyze two inclusions arranged symmetrically with respect to the  $X_1$ -axis, see Fig. 5. We choose  $\mathcal{L} = ABCDA \cup \Gamma$  and assume that

$$u|_{\Gamma} = 0,$$

where  $\Gamma$  is the contour of the first inclusion.

Similarly to (50) we again substitute (47), (48) and (49) into (32) and find that

$$\int_{\Gamma} K \left| \frac{\partial u}{\partial n} \right|^2 \mathbf{n} \cdot \mathbf{X} \, ds = 0 \quad \text{or} \quad \int_{\Gamma} P_{n3}^2 \mathbf{n} \cdot \mathbf{X} \, ds = 0. \tag{51}$$

Considering the contour  $ABEFGDA$  and taking into account that  $\mathbf{n} \cdot \mathbf{X} = 0$  and  $\frac{\partial u}{\partial X_1} = 0$  at  $X_1 = 0$  from (32) the same integral identities but now for the half the contour  $\Gamma$  can be derived

$$\int_{EFG} K \left| \frac{\partial u}{\partial n} \right|^2 \mathbf{n} \cdot \mathbf{X} \, ds = 0 \quad \text{or} \quad \int_{EFG} P_{n3}^2 \mathbf{n} \cdot \mathbf{X} \, ds = 0. \tag{52}$$

Unlike to results given in Section 4.3 from (51) or (52) it does not follow that  $P_{n3} = 0$  along  $\Gamma$  because  $\mathbf{n} \cdot \mathbf{X}$  can take both positive and negative values.

Eqs. (51) or (52) were derived from the M-integral and provide the averaged weighted values of the shear stress along the inclusion surface. The result can be used for verification of finite element calculations or for the analysis of homogenized properties of a composite. In a similar way one can consider any system of rigid inclusions symmetrically distributed with respect to the  $X_1$  and  $X_2$ -axes.

Conclusions

In this paper we shown that for a certain class of hyperelastic materials and for the finite anti-plane shear deformation mode the path-independent M-integral exists and can be applied for the analysis of composites with rigid inclusions. The introduced class of materials satisfies conditions by Knowles (1976, 1977a) for antiplane deformations. In addition, under antiplane shear the strain energy density should be a null-Lagrangian in sense of Olver (1983) and Olver and Sivaloganathan (1988). With the help of the M-integral relations for the averaged weighted shear stress on the inclusion surface are derived for one inclusion or a system of inclusions embedded in an incompressible Mooney–Rivlin matrix material. Furthermore, it was possible to prove that a system of

rigid inclusions can be replaced by an effective one. The obtained results can be useful in estimation of homogenized properties of non-linear elastic composites. For example, the described in this paper approach could be useful in biomechanics (Carpenter et al., 2020), where Mooney–Rivlin or neo-Hookean models are widely used.

Let us note that the application of the M-integral is not limited to composites with rigid inclusions. The M-integral was extensively used for linear elastic solids with notches, inclusions, systems of voids, cracks and dislocations (Asadpoure & Mohammadi, 2007; Chang & Peng, 2004; Chen & Lee, 2004; Hui & Chen, 2010; Lv et al., 2019; Radi, 2011; Suo, 2000; Yi-Feng & Yi-Heng, 2009; Yu & Li, 2013; Yu et al., 2012), see also reviews by Chen and Lu (2003) and Hou et al. (2022). Notion of M-integral was also extended to dynamics (Gommerstadt, 2001, 2014; Markenscoff, 2006) and to inelastic materials in Banks-Sills and Dolev (2004), Defa et al. (2006), Hou et al. (2020), Jeon and Im (2001), Meyer et al. (2017), Pitti et al. (2008), Tian and Rajapakse (2005), Yu et al. (2013a, 2013b) and Zhang et al. (2021). Conservative laws and related path-independent integrals are discussed also for 2D structures, see Eremeyev (2023) and Kienzler and Herrmann (2000) and the references therein. M-integral was also used for evaluation of surface energy for solids with multiple cracks (Chang & Guo, 2020; Chang & Yang, 2020). So this approach could be useful for a priori estimations of some effective mechanical properties and stress concentration phenomena considering surface stresses or other surface-related phenomena, see recent papers by Dai and Schiavone (2023), Gorbushin et al. (2020), Jiang et al. (2022), Krasnitckii et al. (2023), Mikhasev et al. (2023), Piccolroaz et al. (2021), Pronina et al. (2020), Shugailo et al. (2023) and Yang et al. (2023) and the references therein. As in the case of J-integral, M-integral could be useful for movable defects such as propagating cracks or phase interfaces. Our results indicate, that such problems can also be analyzed within the non-linear finite elasticity at least for certain deformation modes.

### CRedit authorship contribution statement

**Victor A. Eremeyev:** Writing – review & editing, Writing – original draft, Visualization, Investigation, Funding acquisition, Formal analysis, Conceptualization. **Konstantin Naumenko:** Writing – review & editing, Investigation, Formal analysis, Conceptualization.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### Data availability

No data was used for the research described in the article.

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