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Calculus of Variations

Integral Harnack estimates and the rate of extinction of singular fractional diffusion

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Abstract

We prove several integral Harnack-type inequalities for local weak solutions of parabolic equations with measurable and bounded coefficients, describing singular s-fractional p-Laplacian diffusion. Then we apply such estimates to evaluate the decay rate of the local mass and supremum of the solutions as they approach a possible extinction time. Yet we show consistency of our general decay estimates by studying the extinction phenomenon for weak solutions of the Cauchy-Dirichlet problem, by means of an approximation procedure that carefully avoids the use of an integrable time derivative.

Mathematics Subject Classification 35K67 · 35B65 · 35K92 · 35Q35

1 Introduction and main results

1.1 Heuristics

When describing the flow of a non-Newtonian fluid in a simple situation (as in a pipe), the momentum balance law written for a power-like stress tensor can prescribe a dissipation of energy that distinguishes between *dilatant* fluids, which starting still, stay immobile until a time T^* and *pseudoplastic* fluids, that become immobile after a finite time T^* has passed (see [4] Ch.IV Section 7.6). We are here interested in the latter phenomenon (as opposite to the former one), which we rephrase under the more general principle of extinction of a diffusion process after a finite time. As anticipated, this principle is a consequence of the dissipation of energy involved in the evolutive process, that in general can be supplied by a particular

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source or, as in our case, by the unbalance between the energy of the propagation (parabolic energy terms with power-growth ≈ 2) and the one of the diffusion (elliptic energy terms with a slower power-growth $\approx p < 2$). We refer to the classic books [4, 5] for a presentation of energy methods for the study of the localization of the solutions to parabolic equations.

When the aim is the description of materials with memory or media that exhibit long-range elastic or plastic deformation, these models account for the fact that stresses or strains at one point in a material can be influenced by other regions over a nonlocal range, rather than just the local neighborhood (see for instance [6, 10, 12, 27, 38, 42] and references therein) and the diffusion is termed *nonlocal*. It is the precise scope of this work to address the study of the interplay between local and nonlocal effects caused by the phenomenon of extinction and the regularity properties of weak solutions of diffusion processes described, in particular, by the fractional p-Laplacian equation

$$u_t(x,t) = \int_{\mathbb{R}^N} \frac{|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))}{|x-y|^{N+sp}} \mu(x,y,t)\,dy\,, \qquad x \in \Omega,\ t \in [0,T], \tag{1.1}$$

where $p \in (1, 2)$, $s \in (01)$, and μ is a measurable bounded function, prescribed by the anisotropy of the medium and reflecting the impossibility to measure the properties described by the model without interfering with themselves (see [23] discussion at 3.1 Ch.I).

1.2 Framing of the Topic and Main Results

We consider the following parabolic nonlinear, nonlocal equation

$$u_t + \mathcal{L}_K u = 0, \tag{1.2}$$

in the cylindrical set $\Omega_T = \Omega \times [0, T]$, with $\Omega \subset \mathbb{R}^N$ open ($N \geqslant 2$). The diffusion operator $\mathcal{L}_K$ is formally defined by

$$\mathcal{L}_K u(x,t) = 2 \lim_{\varepsilon \to 0^+} \int_{B^c_\varepsilon(x)} |u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))K(x,y,t)\,dy,$$

where $p \in (1, 2)$, $s \in (0, 1)$, and $K : \mathbb{R}^N \times \mathbb{R}^N \times (0, T) \to \mathbb{R}$ is a measurable function satisfying, for constants $0 < C_1 \leqslant C_2$, the following structural properties almost everywhere:

(K_1) $K(x, y, t) = K(y, x, t)$;

(K_2) $C_1 \leqslant K(x, y, t)|x - y|^{N+ps} \leqslant C_2$.

We remark that when $C_1 = C_2 = 1$, the operator $\mathcal{L}_K$ reduces to the prototype s-fractional p-Laplacian

$$(-\Delta)^s_p u(x,t) = 2 \lim_{\varepsilon \to 0^+} \int_{B^c_\varepsilon(x)} \frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{N+ps}}\,dy.$$

For the precise definition of solution adopted we refer to Section 2. For some related results in the elliptic framework, see [13, 26] and the survey paper [28]. Our choice of $p \in (1, 2)$ qualifies the diffusion operator as *singular*, since when $u(x, t) = u(y, t)$ the elliptic term of the diffusion dominates the process. The former quality of the operator has significant consequences on the properties of the solutions. Indeed, consider the associated Cauchy-Dirichlet problem:

$$\begin{cases} u_t + \mathcal{L}_K u = 0 & \text{in } \Omega_T \\ u = 0 & \text{in } \Omega^c \times (0,T) \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.3}$$

with Ω bounded and initial datum $u_0 \in W_0^{s,p}(\Omega)$, i.e., $u_0 \in W^{s,p}(\mathbb{R}^N)$ and $u_0 = 0$ in Ω^c (see Section 2 for details). We will prove that (global) weak solutions to (1.3) extinguish within a time T_* that depends on some L^q norm of u_0 (see Theorem 1.5 for the precise statement), while local[1] weak solutions satisfy an integral Harnack-type inequality such as, for fixed $\rho, t > 0$,

$$\gamma^{-1} \sup_{0<\tau<t} \int_{B_\rho} u(x,\tau)\,dx \le \inf_{0<\tau<t} \int_{B_{2\rho}} u(x,\tau)\,dx + \left(\frac{t}{\rho^{\lambda_1}}\right)^{\frac{1}{2-p}} + \mathcal{T}, \qquad \gamma > 1,$$

where the second term on the right-hand side takes into account the possible global effects in time, while the term $\mathcal{T}$ involves the long-range values of the solution in space, through the quantity

$$\mathrm{Tail}\big(u, x_0, \rho, t_1, t_2\big) = \sup_{t_1<\tau<t_2} \left[\rho^{ps} \int_{B^c_\rho(x_0)} \frac{|u(x,\tau)|^{p-1}}{|x-x_0|^{N+ps}}\,dx\right]^{\frac{1}{p-1}}, \tag{1.4}$$

for fixed $0 < t_1 \le t_2 \le T$, $x_0 \in \Omega$, that we refer to as the *nonlocal tail* of $u : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ (see for instance [24, 25] for the origin of the term, and [35] for an alternative definition of tail and its consequences).

Both the property of extinction and this integral Harnack-type inequality are a consequence of the fact that the operator is *singular*, paralleling the description of the pseudoplastic fluids of Section 1.1. We address the previous integral Harnack inequality, following the terminology of [19], as *an L^1-L^1 Harnack inequality*, since it is an Harnack-type estimate for the function $t \to \|u(\cdot,t)\|_{L^1(B_\rho)}$. Another peculiarity of the range $1 < p < 2$ is the fact that local weak solutions are not necessarily locally bounded (see [19] Ch. V for a comparison with the p-Laplacian case). A byproduct of our analysis shows with a quantitative estimate that, if p, s satisfy the following relation

$$\frac{2N}{N+2s} =: p_c < p < 2, \tag{1.5}$$

then local weak solutions to (1.2) are locally bounded, provided that the tail terms are adequately controlled (see Remark 3.2 below for the details). Therefore we address the exponent p_c as *critical*, and we will say that the equation (1.2) lies in the *subcritical regime* if $1 < p < p_c$, and in the *supercritical regime* if $p_c < p < 2$, respectively. Connecting the L^1-L^1 Harnack inequality with the estimates for the local boundedness (Theorem 1.1), we show some very useful L^1-L^∞ Harnack-type estimates (Theorem 1.3) that can be used directly to check the rate of decay toward extinction, and show that the behaviour of local weak solutions is, roughly speaking, dictated by the rule

$$\gamma^{-1}\|u(\cdot,\tau)\|_{L^\infty(B_\rho)} \le \left(\inf_{[0,\tau]} \|u(\cdot,\tau)\|_{L^1(B_{2\rho})}\right)^{\frac{ps}{\lambda_1}} \tau^{-\frac{N}{\lambda_1}} + \left(\frac{\tau}{\rho^{ps}}\right)^{\frac{1}{2-p}} + \hat{\mathcal{T}},$$

[1] Meaning with this, weak solutions that are irrespective of any boundary or initial datum, see Section 2 for the precise definition.

where again $\hat{\mathcal{T}}$ is a perturbation term involving the long-range values of the solution in space and $\lambda_1 = N(p-2) + ps$ is the fractional Barenblatt number (see [40] Thm 1.1).

We now present our main results. For all $r \geqslant 1$ we define

$$\lambda_r = N(p-2) + rps,$$

and we note that $p_c = 2N/(N+2s)$ implies $\lambda_2 > 0$ for all $p > p_c$. In order to ease notation, say that a constant γ depends only on the data if it depends only on $\{N, p, s, C_1, C_2\}$, where C_1, C_2 are the constants appearing in (K_2).

Our first result states that, if u is a locally bounded, local weak solution of (1.2), then the supremum of u in a cylinder is controlled by the average of u^r in a larger cylinder plus a perturbative term that depends on the tail and on the ratio t/ρ^{ps}, which accounts for long-range behavior with respect to time.

Theorem 1.1 (L^r-L^∞ estimate) *Let $r \geqslant 1$ be such that $\lambda_r > 0$, with $1 \leq r < 2$ when $p > p_c$. If u is a locally bounded solution of (1.2), non-negative in $B_{4\rho}(x_0) \times (0, t) \subset \Omega_T$, then there exists $\gamma > 0$ depending on the data and r, s.t.*

$$\begin{aligned}\sup_{B_{\rho/2}(x_0)\times(t/2,t)} u \leqslant & \gamma\Big[⨌_{B_\rho(x_0)\times(0,t)} u^r(x,\tau)\,dx\,d\tau\Big]^{\frac{ps}{\lambda_r}}\Big(\frac{t}{\rho^{ps}}\Big)^{-\frac{N}{\lambda_r}} + \\ & + \gamma\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}} \max\Big\{1, \Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1-p}{2-p}} \mathrm{Tail}\Big(u_+, x_0, \frac{\rho}{2}, 0, t\Big)^{p-1}\Big\}.\end{aligned}$$

Moreover, γ blows up when λ_r vanishes.

For ease of presentation we prefer to state Theorem 1.1 for locally bounded solutions that are locally non-negative. Nevertheless, these two requirements are redundant, see Remarks 3.2, 3.5 for a more detailed analysis.

Our second result compares the supremum of the integral of u in a ball, over a time interval, to the infimum of the integral of u in a larger ball.

Theorem 1.2 (L^1-L^1 estimate) *Let u be a solution of (1.2), non-negative in $B_{4\rho}(x_0) \times (0, t) \subset \Omega_T$. Then, there exists $\gamma > 0$ depending on the data s.t.*

$$\begin{aligned}\sup_{0<\tau<t} \int_{B_\rho(x_0)} u(x,\tau)\,dx \leqslant & \gamma\Big[\inf_{0<\tau<t} \int_{B_{2\rho}(x_0)} u(x,\tau)\,dx\Big] \\ & + \gamma\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}} \max\Big\{1, \Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1-p}{2-p}} \mathrm{Tail}\Big(u_-, x_0, \frac{\rho}{2}, 0, t\Big)^{p-1}\Big\} \\ & + \gamma\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}} \max\Big\{1, \Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1-p}{2-p}} \mathrm{Tail}\Big(u_-, x_0, \frac{\rho}{2}, 0, t\Big)^{p-1}\Big\}^{\frac{p-1}{2-p}}.\end{aligned}$$

Note that Theorem 1.2 does not require that the solution u is locally bounded, as it only involves integrals of u; moreover, it is valid for every $1 < p < 2$. From Theorems 1.1 and 1.2 above, for the restricted supercritical regime

$$\frac{2N}{N+s} < p < 2$$

(which ensures $\lambda_1 > 0$), we derive the following further result that relates the supremum of u in a cylinder to the infimum of the integral of u over a larger time interval.

Theorem 1.3 (L^1-L^∞ estimate) *Let $\lambda_1 > 0$, let u be a locally bounded solution of (1.2) non-negative in $B_{4\rho}(x_0) \times (0, t) \subset \Omega_T$. Then, there exists $\gamma > 0$ depending on the data s.t.*

$$\begin{aligned}\sup_{B_{\rho/2}(x_0)\times(t/2,t)} u \leqslant \gamma\Big[\inf_{0<\tau<t}\int_{B_{2\rho}(x_0)} u(x,\tau)\,dx\Big]^{\frac{ps}{\lambda_1}} t^{-\frac{N}{\lambda_1}} \\ + \gamma\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}} \max\Big\{1, \Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1-p}{2-p}} \mathrm{Tail}\Big(u_+, x_0, \frac{\rho}{2}, 0, t\Big)^{p-1}\Big\} \\ + \gamma\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}} \max\Big\{1, \Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1-p}{2-p}} \mathrm{Tail}\Big(u_-, x_0, \frac{\rho}{2}, 0, t\Big)^{p-1}\Big\}^{\frac{ps}{\lambda_1}} \\ + \gamma\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}} \max\Big\{1, \Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1-p}{2-p}} \mathrm{Tail}\Big(u_-, x_0, \frac{\rho}{2}, 0, t\Big)^{p-1}\Big\}^{\frac{(p-1)ps}{(2-p)\lambda_1}}.\end{aligned}$$

The following estimate relates the L^r-norm of a solution u, spanning a time interval $(0, t)$, to its initial value at time 0, provided the L^r-norm of the positive part u_+ concentrates in a ball.

Theorem 1.4 (Backward L^r-L^r estimate) *Let u be a locally bounded solution of (1.2), non-negative in $B_{4\rho}(x_0) \times (0, t) \subset \Omega_T$. Then, there exists $\gamma > 0$ depending on the data, s.t.*

$$\sup_{0<\tau<t}\int_{B_\rho(x_0)} u^r(x,\tau)\,dx \leqslant \gamma \max\Big\{\sup_{0<\tau<t}\int_{B^c_\rho(x_0)} u^r_+(x,\tau)\,dx, \int_{B_{2\rho}(x_0)} u^r(x,0)\,dx + \Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}}\Big\}.$$

The quantitative estimate of Theorem 1.4 has to be read as void when the right hand side is unbounded.

Besides free solutions of (1.2), we also consider the associated Cauchy-Dirichlet problem (1.3), under the assumptions of Section 1.2. Our first result on problem (1.3) is a qualitative one, ensuring a finite extinction time for globally non-negative solutions.

Theorem 1.5 (Extinction time) *Let Ω be a bounded open set, $u_0 \in W^{s,p}_0(\Omega) \cap L^\infty(\mathbb{R}^N)$ s.t. $u_0 \geqslant 0$ in Ω, and $u \geqslant 0$ be a bounded weak solution of (1.3). Then, there exists $T_* \in (0, \infty)$ s.t. $u(\cdot, t) = 0$ in $\mathbb{R}^N$, for all $t \geqslant T_*$. In addition, there exists a constant $\gamma_* > 0$ depending only on $\{N, p, s\}$ such that*

(i) if $1 < p < p_c$, then

$$T_* = \frac{\gamma_*}{C_1}\|u_0\|^{2-p}_{L^q(\Omega)}, \qquad q = \frac{N(2-p)}{ps};$$

(ii) if $p_c \leqslant p < 2$, then

$$T_* = \frac{\gamma_*}{C_1}\|u_0\|^{2-p}_{L^2(\Omega)}|\Omega|^{\frac{\lambda_2}{2N}}.$$

The quantitative counterpart of Theorem 1.5, following from Theorems 1.2 and 1.3 above, provides us with an estimate of the decay rate (see Figure 1) as t approaches the extinction time T_*, as follows.

Theorem 1.6 (Decay Rate) *Let $u \geqslant 0$ be a locally bounded local weak solution of (1.2) in $\Omega_T = \Omega \times [0, T]$ and $T_* \in (0, T)$ an extinction time for u. Let us assume $B_{4\rho}(x_0) \times (0, t) \subset \Omega \times (0, T_*)$. Then, there exists a constant $\gamma > 0$ depending only on the data s.t.*

(i) for all $1 < p < 2$ the local mass decays as

$$\int_{B_\rho(x_0)} u(x,t)\,dx \leqslant \gamma\Big(\frac{T_* - t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}};$$

(*ii*) *if* $\lambda_1 > 0$, *then*

$$\sup_{B_{\rho/2}(x_0)} u\Big(\cdot, \frac{t+T_*}{2}\Big) \leqslant \gamma \Big(\frac{T_*-t}{\rho^{ps}}\Big)^{\frac{1}{2-p}} \max\Big\{1, \Big(\frac{T_*-t}{\rho^{ps}}\Big)^{\frac{1-p}{2-p}} \mathrm{Tail}\Big(u, x_0, \frac{\rho}{2}, t, T_*\Big)^{p-1}\Big\}.$$

Note that the L^∞-bound in (ii) is a stronger bound than the L^1-bound of (i) but the latter also holds for $p < 2N/(N+s)$ and it does not suffer from the behavior of u_+ in $B_\rho(x_0) \setminus B_{\rho/2}(x_0)$.

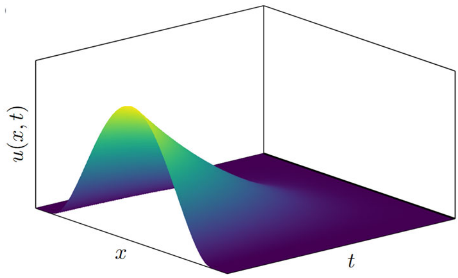

Fig. 1 Illustration of the extinction decay for a local weak solution to (1.3)

1.3 Novelty and Significance

To the best of our knowledge, L^1-L^1 Harnack-type estimates are first found in [20], with the aim of studying the existence of initial traces for very weak solutions to the singular p-Laplacian prototype. A couple of years later they were used in [21] for both the cases of the p-Laplacian equation and the porous medium one, with the aim of evaluating the rate of extinction of non-negative solutions. The method of [20] for the p-Laplacian has been reported in [19], Chap. VII for solutions to the prototype singular equation ($1 < p < 2$). A proof for p-Laplacian type equations with full quasilinear structure can be found first in the paper [22] and then in the monograph [23]. See also [8] and [23] for a later treatment of the sub-critical case.

In the case of fractional p-Laplacian diffusion, such Harnack-type estimates are new. For a different notion of solution and in the whole space, the author in [40] has proved some global Harnack bounds. Existence theory for different definitions of solutions than ours can be found in [36] and [41], see also the book [3]. There the notion of solution involves the existence of a Sobolev time-derivative, that we have carefully avoided here through the approximation procedures of Appendix 1. The main difference lies in considering a general measurable and bounded kernel as in (K_1), (K_2). The discontinuity of the field into the divergence term of (1.2) results in a lack of regularity of the time derivative u_t, see [33] Sec. 13 of Ch. III (pages 224-233). The same lack of regularity in time pertains the possible weak solutions to the Cauchy-Dirichlet problem, whose analysis of the extinction phenomenon has been performed in [1], Theorem 5.1. There the authors point out a particular feature of the nature of fractional nonlinear diffusion, that is, it does not necessarily meet finite speed of propagation in $\mathbb{R}^N$ when $p > 2$. On the other hand, similarly to the classical p-Laplacian, when $p > 2$ local boundedness is inherent in the definition of local weak solution, see [39] (and later [9] for the prototype equation, see also [2]).

Here, *en passant*, we prove that the same is valid for local weak solutions in the super-critical range (see Remark 3.2), see also Theorem 1.1 of [31] for a study on the best tail condition and [37] in the case with no phase ($a = 0$) for a whole satisfying picture in the special case of parabolic minima with a boundary datum. An analysis of the L^1-L^∞ estimates with the finer tail condition of [29] and [35] can be carried from ours with few effort. This is indeed the aim of a future project, where these estimate will be necessary for a different aim. Here we directly employ the definition (1.4), since we aim to the evaluation of the extinction decay of u as power of time, i.e. $(T^* - t)^{1/(2-p)}$, disregarding of the optimality of the long-range effects. Concerning the study of the extinction of solutions, the works [40, 43] (see Section 8) investigate the decay in the whole space through an appropriate comparison principle. For a similar definition of solution, the authors in [18] study the decay rate towards extinction for an equation with double-nonlinearity in time. What diversifies the extinction rates of Theorem 1.6 is the concept of solution (see Definition 2.1), and the circumvention of any comparison principle. Indeed, the formulation of the decay rate of Theorem 1.6 doesn't actually require u to satisfy any Cauchy-Dirichlet conditions.

1.4 The Method

We rely on the energy estimates of [34] to apply the iterative De Giorgi scheme to prove Theorem 1.1. Then the L^1-L^1 Harnack-type estimates are found here, following the approach of [23] within the refined technique of [14, 15], but with a precise nonlocal method appropriately devised, see the proof of Lemma 4.1 for details. On the other hand, the L^r-L^∞ bounds of Theorem 1.1 are more similar to the classic [19], but again the tail terms impose an either/or alternative that is more in the spirit of Lemma 3.1 of [34] (see also Theorem 1.8 of [29] in the linear case with the finest tail condition). Finally, the backwards L^r- estimates cannot run the argument of [23] and had to adapt the original idea of [20] with a proper condition on the long-range effects of the solution. The extinction phenomenon is reminiscent of the classic energy method (see for instance [19] Ch. VII section 2, or [4]), while the lack of time-derivative called for a precise time-mollification in the spirit of [32].

1.5 Future Applications

The L^1-L^1 estimate is very useful to transport uniformly measure information in time. It can be used, among other possibilities, to give a simpler proof of the Hölder continuity of solutions (see [17]) using the main tools developed in [34]. At the same time L^1-L^1 Harnack-type inequalities are fundamental in the study of the initial traces, as in [20] (see [19] for a more comprehensive overview), while backward L^r estimates as the ones of Theorem 1.4 can be used to show that a precise local integrability of the initial datum u_0 (i.e., $u_0 \in L^r_{\mathrm{loc}}(\mathbb{R}^N)$) yields directly locally bounded and locally Hölder-continuous solutions by Ascoli-Arzelà's principle, see Remark 3.6 or [21]. When an oscillation estimate is at stake (as in [34]), it is again possible to study the decay rate to extinction, in a different local geometry than ours, see Corollary 1.4 of [7] or [30] for an approach with the extension trick of [11].

1.6 Structure of the paper

In Section 2 we recall the basic definitions and some preliminary results, including the parabolic fractional embeddings, energy estimates and various iteration Lemmas; in Section 3 we prove Theorem 1.1; in Section 4 we prove Theorems 1.2 and 1.3; in Section 5 we prove

Theorem 1.4; in Section 6 we prove Theorems 1.5 and 1.6; finally, in Appendix 1 we deal with the technical issue of time mollifications.

1.7 Notation

Throughout the paper, for all $U \subset \mathbb{R}^N$ we will denote by $|U|$ the N-dimensional Lebesgue measure of U, $U^c = \mathbb{R}^N \setminus U$. By $B_\rho(x)$ we will denote the open ball of radius ρ centered at x. Writings like $u \leqslant v$ in U will mean that $u(x) \leqslant v(x)$ for a.e. $x \in U$. By u_+ (resp., u_-) we will denote the positive (resp., negative) part of u. By $\inf_U u$ (resp., $\sup_U u$) we will denote the essential infimum (resp., supremum) of u in U. Most important, γ will denote several positive constants, only depending on the data $N, p, s, \Omega, C_1, C_2$ of the problem, except when explicitly noted.

2 Preliminaries

This section includes some preliminary definitions which provide a rigorous framework for our problems, and some technical results that will be used in the proofs of our main results. We begin by recalling that, for any open set $\Omega \subseteq \mathbb{R}^N$ $(N \geqslant 2)$, a function $u : \Omega \to \mathbb{R}$ is said to belong to the fractional Sobolev space $W^{s,p}(\Omega)$ $0 < s < 1, 1 < p < 2)$ if $u \in L^p(\Omega)$ and

$$\iint_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dx\, dy < \infty.$$

Clearly, if Ω is unbounded, we say that $u \in W^{s,p}_{\text{loc}}(\Omega)$ if $u \in W^{s,p}(\Omega')$ for all $\Omega' \Subset \Omega$. Also, if Ω is bounded, we say that $u \in W^{s,p}_0(\Omega)$ if $u \in W^{s,p}(\mathbb{R}^N)$ and $u = 0$ a.e. in Ω^c. We recall Sobolev's embedding for fractional order spaces: if Ω is bounded, then there exists $\gamma > 0$ depending on N, p, s s.t. for all $u \in W^{s,p}_0(\Omega)$

$$\Big[\int_\Omega |u(x)|^{p_s^*}\, dx \Big]^{\frac{1}{p_s^*}} \leqslant \gamma \Big[\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}}\, dx\, dy \Big]^{\frac{1}{p}}, \quad p_s^* = \frac{Np}{N - ps} \tag{2.1}$$

(recall that $N > ps$). For evolutive equations, we will always identify $u(\cdot, t)$ with $u(t)$.

Now let $\Omega \subset \mathbb{R}^N$ be an open set, $T \in (0, \infty]$. We define solutions of our equation as in [34, Subsection 1.2.2]:

Definition 2.1 We say that $u \in C(0, T, L^2_{\text{loc}}(\Omega)) \cap L^p_{\text{loc}}(0, T, W^{s,p}_{\text{loc}}(\Omega))$ is a (local, weak) solution of (1.2) if for all $\Omega' \Subset \Omega$ and all $0 < t_1 < t_2 < T$ the following conditions are satisfied:

(i) for all $\varphi \in W^{1,2}_{\text{loc}}(0, T, L^2(\Omega')) \cap L^p_{\text{loc}}(0, T, W^{s,p}_0(\Omega'))$

$$\begin{aligned} 0 = &\int_{\Omega'} u(x,\tau)\varphi(x,\tau)\, dx \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_{\Omega'} u(x,\tau)\varphi_\tau(x,\tau)\, dx\, d\tau \\ &+ \int_{t_1}^{t_2} \iint_{\mathbb{R}^N\times\mathbb{R}^N} |u(x,\tau) - u(y,\tau)|^{p-2}(u(x,\tau) \\ &- u(y,\tau))(\varphi(x,\tau) - \varphi(y,\tau))K(x,y,\tau)\, dx\, dy\, d\tau; \end{aligned}$$

(ii) $$\sup_{t_1<\tau<t_2} \int_{\mathbb{R}^N} \frac{|u(x,\tau)|^{p-1}}{(1 + |x|)^{N+ps}}\, dx < \infty.$$

Solutions of the Cauchy-Dirichlet problem (1.3) are defined as follows:

Definition 2.2 Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $u_0 \in W_0^{s,p}(\Omega)$. We say that $u \in C(0,T,L^2(\Omega)) \cap L^p(0,T,W_0^{s,p}(\Omega))$ is a (weak) solution of (1.3) if

(i) for all $\varphi \in W^{1,2}(0,T,L^2(\Omega)) \cap L^p(0,T,W_0^{s,p}(\Omega))$

$$
\begin{aligned}
0 = & \int_\Omega u(x,\tau)\varphi(x,\tau)\,dx\Big|_0^T - \int_0^T \int_\Omega u(x,\tau)\varphi_\tau(x,\tau)\,dx\,d\tau \\
& + \int_0^T \iint_{\mathbb{R}^N \times \mathbb{R}^N} |u(x,\tau) - u(y,\tau)|^{p-2}(u(x,\tau) \\
& - u(y,\tau))(\varphi(x,\tau) - \varphi(y,\tau))K(x,y,\tau)\,dx\,dy\,d\tau;
\end{aligned}
$$

(ii) $u(\cdot,0) = u_0$ in $\mathbb{R}^N$.

Note that Definition 2.2 implies Definition 2.1, as condition (ii) of the latter follows from $u(\cdot,\tau) \in W_0^{s,p}(\Omega)$. We keep the definition of nonlocal tails as in (1.4). We will next recall some basic properties of solutions, beginning from an embedding inequality from [34, Proposition A.3]:

Proposition 2.3 (Embedding) *Let $x_0 \in \mathbb{R}^N$, $0 < \rho_1 < \rho_2 < \rho$, $0 < t_1 < t_2$ be s.t. $B_\rho(x_0) \times (t_1,t_2) \subset \Omega_T$ and $u \in L^p(t_1,t_2,W^{s,p}(B_\rho(x_0))) \cap L^\infty(t_1,t_2,L^2(B_\rho(x_0)))$ be s.t. $\mathrm{supp}(u(\cdot,\tau)) \subset B_{\rho_1}(x_0)$ for a.e. $\tau \in (t_1,t_2)$. Then, there exists $\gamma > 0$ depending on N, p, s s.t.*

$$
\begin{aligned}
& \int_{t_1}^{t_2} \int_{B_{\rho_2}(x_0)} |u(x,\tau)|^{\frac{(N+2s)p}{N}}\,dx\,d\tau \leqslant \gamma \Big[\sup_{t_1<\tau<t_2} ⨍_{B_{\rho_2}(x_0)} u^2(x,\tau)\,dx\Big]^{\frac{ps}{N}} \\
& \cdot \Big[\rho_2^{ps} \int_{t_1}^{t_2} \iint_{B_{\rho_2}(x_0)\times B_{\rho_2}(x_0)} \frac{|u(x,\tau)-u(y,\tau)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau \\
& + \Big(\frac{\rho_2}{\rho_2-\rho_1}\Big)^{N+ps} \int_{t_1}^{t_2} \int_{B_{\rho_2}(x_0)} |u(x,\tau)|^p\,dx\,d\tau\Big].
\end{aligned}
$$

The following energy estimate from [34, Corollary 2.1] will be one of our primary tools:

Proposition 2.4 (Energy estimate) *Let u be a solution of (1.2), $x_0 \in \mathbb{R}^N$, $0 < \rho_1 < \rho_2$, $0 < t_1 < t_2 < t$ be s.t. $B_{\rho_2}(x_0) \times (0,t) \subset \Omega_T$, $k > 0$, $w = u - k$. Then, there exists $\gamma > 0$ depending on the data s.t.*

$$
\begin{aligned}
& \sup_{t-t_1<\tau<t} \int_{B_{\rho_1}(x_0)} w_\pm^2(x,\tau)\,dx + \int_{t-t_1}^t \iint_{B_{\rho_1}(x_0)\times B_{\rho_1}(x_0)} \frac{|w_\pm(x,\tau)-w_\pm(y,\tau)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau \\
& \leqslant \frac{\gamma}{t_2-t_1} \int_{t-t_2}^t \int_{B_{\rho_2}(x_0)} w_\pm^2(x,\tau)\,dx\,d\tau + \frac{\gamma\rho_2^{p-ps}}{(\rho_2-\rho_1)^p} \int_{t-t_2}^t \int_{B_{\rho_2}(x_0)} w_\pm^p(x,\tau)\,dx\,d\tau \\
& + \frac{\gamma\rho_2^N}{(\rho_2-\rho_1)^{N+ps}} \Big[\int_{t-t_2}^t \int_{B_{\rho_2}(x_0)} w_\pm(x,\tau)\,dx\,d\tau\Big] \mathrm{Tail}(w_\pm,x_0,\rho_2,t-t_2,t)^{p-1}.
\end{aligned}
$$

We also recall two useful technical results involving recursive relations. The first is [23, Lemma 5.1]:

Lemma 2.5 (Fast convergence) *Let (Y_n) be a sequence of non-negative real numbers, b, $C > 1$, $\eta > 0$ s.t.*

(i) $Y_{n+1} \leqslant C b^n Y_n^{1+\eta}$ *for all* $n \in \mathbb{N}$;

(ii) $Y_0 \leqslant C^{-\frac{1}{\eta}} b^{-\frac{1}{\eta^2}}$.

Then, $Y_n \to 0$ *as* $n \to \infty$.

A somewhat dual property is the following [23, Lemma 5.2]:

Lemma 2.6 (Interpolation) *Let* (Y_n) *be a bounded sequence of non-negative real numbers,* $b > 1$, $C > 0$, $\eta \in (0, 1)$ *s.t.* $Y_n \leqslant C b^n Y_{n+1}^{1-\eta}$ *for all* $n \in \mathbb{N}$. *Then,*

$$Y_0 \leqslant \left(2 C b^{\frac{1-\eta}{\eta}}\right)^{\frac{1}{\eta}}.$$

Finally, we recall some elementary inequalities from [1, Lemma 2.7]:

Lemma 2.7 *Let* $p \in (1, 2)$, $a, b \geqslant 0$. *Then:*

(i) *for all* $q > 1$ *there exists* $\gamma > 0$ *s.t.*

$$\left|a^{\frac{p+q-2}{p}} - b^{\frac{p+q-2}{p}}\right|^p \leqslant \gamma |a-b|^{p-2}(a-b)(a^{q-1} - b^{q-1});$$

(ii) *for all* $q \geqslant 2$ *there exists* $\gamma > 0$ *s.t.*

$$(a+b)^{q-2}|a-b|^p \leqslant \gamma \left|a^{\frac{p+q-2}{p}} - b^{\frac{p+q-2}{p}}\right|^p.$$

We will also use the following weighted Young's inequality: for all $q > 1$, there exists $\gamma > 0$ s.t. for all $a, b \geqslant 0$, $\varepsilon \in (0, 1)$

$$ab \leqslant \varepsilon a^q + \frac{\gamma}{\varepsilon^{q-1}} b^{q'}. \tag{2.2}$$

We note the following consequence of (2.2) (see [34, eq. (2.4)]): there exists $\gamma > 0$ s.t. for all $a \geqslant b \geqslant 0$, $\varepsilon \in (0, 1)$

$$a^p - b^p \leqslant \varepsilon a^p + \frac{\gamma}{\varepsilon^{p-1}} (a-b)^p. \tag{2.3}$$

3 L^r-L^∞ estimate

In this section we display the proof of Theorem 1.1, which is divided into several steps. To ease notation we set, for $\sigma \in (0, 1)$, the perturbative term

$$P_\sigma = \left(\frac{t}{\rho^{ps}}\right)^{\frac{1}{2-p}} \max\left\{1, \left(\frac{t}{\rho^{ps}}\right)^{\frac{1-p}{2-p}} \mathrm{Tail}\Big(u_+, x_0, \sigma\rho, 0, t\Big)^{p-1}\right\}.$$

We first establish two preliminary lemmas, one for the supercritical regime $p > p_c$ and the other for the subcritical regime $p \leqslant p_c$. As it will appear clear, in the first case the local boundedness is a consequence of the definition of local weak solution. We state our result for nonnegative solutions, in order to ease the reading, though minor standard modifications clarify that the result is still valid for signed super or sub-solutions, see Remark 3.5.

Lemma 3.1 *Let* u *be a local weak solution of (1.2), non-negative in* $B_{4\rho}(x_0) \times (0, t) \subset \Omega_T$. *Let* $p_c < p < 2$, $r \in [1, 2]$ *s.t.* $\lambda_r > 0$, $\sigma \in (0, 1)$, $t_0 \in (0, t]$. *Then, there exist* $\alpha, \gamma > 0$ *depending on the data s.t.*

$$\sup_{B_{\sigma\rho}(x_0)\times(t-\sigma t_0, t)} u \leqslant \frac{\gamma}{[\sigma(1-\sigma)]^\alpha} \left[\sup_{B_\rho(x_0)\times(t-t_0,t)} u^{\frac{ps(2-r)}{\lambda_2}}\right] \left[\iint_{B_\rho(x_0)\times(t-t_0,t)} u^r(x,\tau)\, dx\, d\tau\right]^{\frac{ps}{\lambda_2}} \left(\frac{t_0}{\rho^{ps}}\right)^{-\frac{N}{\lambda_2}} + P_\sigma.$$

Proof Let $k > 0$ be a number to be determined later and perform a decreasing iteration by setting for all $n \in \mathbb{N}$

$$\rho_n = \rho\Big(\sigma + \frac{1-\sigma}{2^n}\Big),\ t_n = t_0\Big(\sigma + \frac{1-\sigma}{2^n}\Big),\ k_n = k\Big(1 - \frac{1}{2^n}\Big).$$

In addition we set

$$\hat{\rho}_n = \frac{3\rho_n + \rho_{n+1}}{4},\ \tilde{\rho}_n = \frac{\rho_n + \rho_{n+1}}{2},\ \check{\rho}_n = \frac{\rho_n + 3\rho_{n+1}}{4},$$

so we have a shrinking geometric configuration and increasing levels. Indeed, if we define the balls

$$B_n = B_{\rho_n}(x_0),\ \hat{B}_n = B_{\hat{\rho}_n}(x_0),\ \tilde{B}_n = B_{\tilde{\rho}_n}(x_0),\ \check{B}_n = B_{\check{\rho}_n}(x_0),$$

and the cylinders $Q_n = B_n \times (t - t_n, t)$, then

$$B_n \supset \hat{B}_n \supset \tilde{B}_n \supset \check{B}_n \supset B_{n+1}, \quad \text{and} \quad Q_n \supset Q_{n+1},$$

with initial cylinder $Q_0 = B_\rho(x_0) \times (t - t_0, t)$ and limit cylinder $Q_\infty = B_{\sigma\rho}(x_0) \times (t - \sigma t_0, t)$. Next, we set $w_n = (u - k_n)_+$ and define the related energy

$$E_n = \sup_{t-t_{n+1}<\tau<t} \int_{\tilde{B}_n} w_{n+1}^2(x,\tau)\,dx + \int_{t-t_{n+1}}^t \iint_{\tilde{B}_n \times \tilde{B}_n} \frac{|w_{n+1}(x,\tau) - w_{n+1}(y,\tau)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau.$$

For any pair of exponents $1 \leqslant q_1 \leqslant q_2$ and each $n \in \mathbb{N}$ the following Chebychev-type inequality holds true:

$$\begin{aligned} \iint_{Q_n} w_{n+1}^{q_1}(x,\tau)\,dx\,d\tau &\leqslant \int_{t-t_n}^t \int_{B_n \cap \{u > k_{n+1}\}} \frac{(u(x,\tau) - k_n)^{q_2}}{(k_{n+1} - k_n)^{q_2 - q_1}}\,dx\,d\tau \\ &\leqslant \frac{\gamma 2^{(q_2-q_1)n}}{k^{q_2-q_1}} \iint_{Q_n} w_n^{q_2}(x,\tau)\,dx\,d\tau. \end{aligned} \tag{3.1}$$

We estimate E_n by applying Proposition 2.4 with radii $0 < \tilde{\rho}_n < \rho_n < \rho$, times $0 < t_{n+1} < t_n < t$, and level k_{n+1}:

$$\begin{aligned} E_n &\leqslant \frac{\gamma}{t_n - t_{n+1}} \iint_{Q_n} w_{n+1}^2(x,\tau)\,dx\,d\tau + \frac{\gamma \rho_n^{p-ps}}{(\rho_n - \tilde{\rho}_n)^p} \iint_{Q_n} w_{n+1}^p(x,\tau)\,dx\,d\tau \\ &\quad + \frac{\gamma \rho_n^N}{(\rho_n - \tilde{\rho}_n)^{N+ps}} \Big[\iint_{Q_n} w_{n+1}(x,\tau)\,dx\,d\tau\Big] \mathrm{Tail}(w_{n+1}, x_0, \rho_n, t - t_n, t)^{p-1} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

We estimate separately each term above. For I_1 we use the definition of (t_n) and (3.1) with $q_1 = q_2 = 2$:

$$\begin{aligned} I_1 &\leqslant \frac{\gamma 2^n}{(1-\sigma)t_0} \iint_{Q_n} w_{n+1}^2(x,\tau)\,dx\,d\tau \\ &\leqslant \frac{\gamma 2^n}{(1-\sigma)t_0} \iint_{Q_n} w_n^2(x,\tau)\,dx\,d\tau. \end{aligned}$$

For I_2 we use the definition of (ρ_n), $(\tilde{\rho}_n)$, and (3.1) with $q_1 = p$, $q_2 = 2$:

$$\begin{aligned} I_2 &\leqslant \frac{\gamma 2^{pn}}{(1-\sigma)^p \rho^{ps}} \iint_{Q_n} w_{n+1}^p(x,\tau)\,dx\,d\tau \\ &\leqslant \frac{\gamma 2^{2n}}{(1-\sigma)^p \rho^{ps} k^{2-p}} \iint_{Q_n} w_n^2(x,\tau)\,dx\,d\tau. \end{aligned}$$

Finally, for I_3 we use (3.1) with $q_1 = 1$, $q_2 = 2$, and we estimate the tail term recalling that $w_{n+1} \leqslant u_+$:

$$\begin{aligned} I_3 &\leqslant \frac{\gamma 2^{(N+ps)n}}{(1-\sigma)^{N+ps}\rho^{ps}} \Big[\iint_{Q_n} w_{n+1}(x,\tau)\,dx\,d\tau \Big] \Big[\sup_{t-t_{n+1}<\tau<t} \rho_n^{ps} \int_{B_n^c} \frac{w_{n+1}^{p-1}(x,\tau)}{|x-x_0|^{N+ps}}\,dx \Big] \\ &\leqslant \frac{\gamma 2^{(N+ps+1)n}}{(1-\sigma)^{N+ps}\rho^{ps}k} \Big[\iint_{Q_n} w_n^2(x,\tau)\,dx\,d\tau \Big] \Big(\frac{\sigma\rho}{\rho_n}\Big)^{-ps} \Big[\sup_{0<\tau<t} (\sigma\rho)^{ps} \int_{B_{\sigma\rho}^c(x_0)} \frac{u_+^{p-1}(x,\tau)}{|x-x_0|^{N+ps}}\,dx \Big] \\ &\leqslant \frac{\gamma 2^{(N+ps+1)n}}{(1-\sigma)^{N+ps}(\sigma\rho)^{ps}k} \Big[\iint_{Q_n} w_n^2(x,\tau)\,dx\,d\tau \Big] \mathrm{Tail}\Big(u_+, x_0, \sigma\rho, 0, t\Big)^{p-1}. \end{aligned}$$

Plugging such estimates into the previous inequality, we find $\alpha, \beta, \gamma > 1$ depending on the data s.t.

$$E_n \leqslant \frac{\gamma 2^{\beta n}}{[\sigma(1-\sigma)]^\alpha} \Big[\frac{1}{t_0} + \frac{1}{\rho^{ps}k^{2-p}} + \frac{1}{\rho^{ps}k} \mathrm{Tail}\Big(u_+, x_0, \sigma\rho, 0, t\Big)^{p-1} \Big] \iint_{Q_n} w_n^2(x,\tau)\,dx\,d\tau.$$

Now we introduce our first condition on k:

$$k > P_\sigma. \tag{3.2}$$

So, from the previous inequality and (3.2) we have for all $n \in \mathbb{N}$

$$E_n \leqslant \frac{\gamma 2^{\beta n}}{[\sigma(1-\sigma)]^\alpha t_0} \iint_{Q_n} w_n^2(x,\tau)\,dx\,d\tau. \tag{3.3}$$

We aim at an iterative estimate for the sequence

$$Y_n = \fint\!\!\fint_{Q_n} w_n^2(x,\tau)\,dx\,d\tau.$$

Set for all $n \in \mathbb{N}$

$$M_n = \Big| Q_n \cap \{u > k_n\} \Big|^{\frac{\lambda_2}{p(N+2s)}},$$

which has a positive exponent due to the choice of p. To proceed, we fix a cutoff function $\xi_n \in C_c^\infty(\check{B}_n)$ s.t. $\xi_n = 1$ in B_{n+1} and satisfies in $\mathbb{R}^N$

$$0 \leqslant \xi_n \leqslant 1, \ |\nabla \xi_n| \leqslant \frac{\gamma 2^n}{(1-\sigma)\rho}.$$

We apply Proposition 2.3 to the function $w_{n+1}\xi_n$, with radii $0 < \check{\rho}_n < \tilde{\rho}_n$ and times $0 < t - t_{n+1} < t$,

$$\int_{t-t_{n+1}}^{t}\int_{\tilde{B}_n} |w_{n+1}(x,\tau)\xi_n(x)|^{\frac{p(N+2s)}{N}}\,dx\,d\tau \leqslant \gamma\Big[\sup_{t-t_{n+1}<\tau<t} \fint_{\tilde{B}_n} |w_{n+1}(x,\tau)\xi_n(x)|^2\,dx\Big]^{\frac{ps}{N}}$$
$$\cdot\Big[\tilde{\rho}_n^{ps}\int_{t-t_{n+1}}^{t}\iint_{\tilde{B}_n\times\tilde{B}_n} \frac{|w_{n+1}(x,\tau)\xi_n(x)-w_{n+1}(y,\tau)\xi_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau$$
$$+\Big(\frac{\tilde{\rho}_n}{\tilde{\rho}_n-\check{\rho}_n}\Big)^{N+ps}\int_{t-t_{n+1}}^{t}\int_{\tilde{B}_n} |w_{n+1}(x,\tau)\xi_n(x)|^p\,dx\,d\tau\Big].$$

By Hölder's inequality and the previous embedding estimate we have

$$\begin{aligned}
\iint_{Q_{n+1}} w_{n+1}^2(x,\tau)\,dx\,d\tau &\leqslant \Big[\iint_{Q_{n+1}} w_{n+1}(x,\tau)^{\frac{p(N+2s)}{N}}\,dx\,d\tau\Big]^{\frac{2N}{p(N+2s)}} M_{n+1} \qquad (3.4)\\
&\leqslant \Big[\int_{t-t_{n+1}}^{t}\int_{\tilde{B}_n} |w_{n+1}(x,\tau)\xi_n(x)|^{\frac{p(N+2s)}{N}}\,dx\,d\tau\Big]^{\frac{2N}{p(N+2s)}} M_{n+1}\\
&\leqslant \gamma\Big[\sup_{t-t_{n+1}<\tau<t}\fint_{\tilde{B}_n} |w_{n+1}(x,\tau)\xi_n(x)|^2\,dx\Big]^{\frac{2s}{N+2s}}\\
&\cdot\Big[\rho^{ps}\int_{t-t_{n+1}}^{t}\iint_{\tilde{B}_n\times\tilde{B}_n} \frac{|w_{n+1}(x,\tau)\xi_n(x)-w_{n+1}(y,\tau)\xi_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau\\
&+\frac{2^{(N+ps)n}}{(1-\sigma)^{N+ps}}\int_{t-t_{n+1}}^{t}\int_{\tilde{B}_n} |w_{n+1}(x,\tau)\xi_n(x)|^p\,dx\,d\tau\Big]^{\frac{2N}{p(N+2s)}} M_{n+1}.
\end{aligned}$$

To estimate the 'gradient' term above, we split the integrand and use the properties of ξ_n:

$$\begin{aligned}
&\int_{t-t_{n+1}}^{t}\iint_{\tilde{B}_n\times\tilde{B}_n} \frac{|w_{n+1}(x,\tau)\xi_n(x)-w_{n+1}(y,\tau)\xi_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau\\
&\leqslant \gamma\int_{t-t_{n+1}}^{t}\iint_{\tilde{B}_n\times\tilde{B}_n} \frac{|w_{n+1}(x,\tau)-w_{n+1}(y,\tau)|^p}{|x-y|^{N+ps}}\xi_n^p(y)\,dx\,dy\,d\tau\\
&\quad+\gamma\int_{t-t_{n+1}}^{t}\iint_{\tilde{B}_n\times\tilde{B}_n} w_{n+1}^p(x,\tau)\frac{|\xi_n(x)-\xi_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau\\
&\leqslant \gamma\int_{t-t_{n+1}}^{t}\iint_{\tilde{B}_n\times\tilde{B}_n} \frac{|w_{n+1}(x,\tau)-w_{n+1}(y,\tau)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau\\
&\quad+\frac{\gamma 2^{pn}}{(1-\sigma)^p\rho^p}\int_{t-t_{n+1}}^{t}\iint_{\tilde{B}_n\times\tilde{B}_n} \frac{w_{n+1}^p(x,\tau)}{|x-y|^{N+ps-p}}\,dx\,dy\,d\tau\\
&\leqslant \gamma E_n+\frac{\gamma 2^{pn}}{(1-\sigma)^p\rho^p}\Big[\int_{t-t_{n+1}}^{t}\int_{\tilde{B}_n} w_{n+1}^p(x,\tau)\,dx\,d\tau\Big]\Big[\sup_{x\in\tilde{B}_n}\int_{\tilde{B}_n}\frac{dy}{|x-y|^{N+ps-p}}\Big]\\
&\leqslant \gamma E_n+\frac{\gamma 2^{pn}}{(1-\sigma)^p\rho^{ps}}\int_{t-t_{n+1}}^{t}\int_{\tilde{B}_n} w_{n+1}^p(x,\tau)\,dx\,d\tau.
\end{aligned}$$

Plug the last inequality into (3.4), taking into account the mean value and the definition of E_n:

$$\begin{aligned}
&\iint_{Q_{n+1}} w_{n+1}^2(x,\tau)\,dx\,d\tau \\
&\leqslant \gamma\rho^{-\frac{2Ns}{N+2s}}\Big[\sup_{t-t_{n+1}<\tau<t}\int_{\tilde{B}_n} w_{n+1}^2(x,\tau)\,dx\Big]^{\frac{2s}{N+2s}} \\
&\quad\cdot\Big[\rho^{ps}E_n + \frac{2^{\beta n}}{(1-\sigma)^\alpha}\int_{t-t_{n+1}}^{t}\int_{\tilde{B}_n} w_{n+1}^p(x,\tau)\,dx\,d\tau\Big]^{\frac{2N}{p(N+2s)}} M_{n+1} \\
&\leqslant \frac{\gamma 2^{\beta n}}{(1-\sigma)^\alpha}\rho^{-\frac{2Ns}{N+2s}} E_n^{\frac{2s}{N+2s}}\Big[\rho^{ps}E_n + \int_{t-t_{n+1}}^{t}\int_{\tilde{B}_n} w_{n+1}^p(x,\tau)\,dx\,d\tau\Big]^{\frac{2N}{p(N+2s)}} M_{n+1},
\end{aligned}$$

with different $\alpha, \beta, \gamma > 1$ depending on the data. We use (3.3) on E_n, (3.1) with $q_1 = p$ and $q_2 = 2$, and again (3.2) to estimate k:

$$\begin{aligned}
\iint_{Q_{n+1}} w_{n+1}^2(x,\tau)\,dx\,d\tau &\leqslant \frac{\gamma 2^{\beta n}}{[\sigma(1-\sigma)]^\alpha}\rho^{-\frac{2Ns}{N+2s}} t_0^{-\frac{2s}{N+2s}}\Big[\iint_{Q_n} w_n^2(x,\tau)\,dx\,d\tau\Big]^{\frac{2s}{N+2s}} \\
&\quad\cdot\Big[\frac{\rho^{ps}}{t_0}\iint_{Q_n} w_n^2(x,\tau)\,dx\,d\tau + \iint_{Q_n} w_{n+1}^p(x,\tau)\,dx\,d\tau\Big]^{\frac{2N}{p(N+2s)}} M_{n+1} \\
&\leqslant \frac{\gamma 2^{\beta n}}{[\sigma(1-\sigma)]^\alpha}\rho^{-\frac{2Ns}{N+2s}} t_0^{-\frac{2s}{N+2s}}\Big(\frac{\rho^{ps}}{t_0} + \frac{1}{k^{2-p}}\Big)^{\frac{2N}{p(N+2s)}} \\
&\quad\Big[\iint_{Q_n} w_n^2(x,\tau)\,dx\,d\tau\Big]^{\frac{2(N+ps)}{p(N+2s)}} M_{n+1} \\
&\leqslant \frac{\gamma 2^{\beta n}}{[\sigma(1-\sigma)]^\alpha} t_0^{-\frac{2(N+ps)}{p(N+2s)}}\Big[\iint_{Q_n} w_n^2(x,\tau)\,dx\,d\tau\Big]^{\frac{2(N+ps)}{p(N+2s)}} M_{n+1}.
\end{aligned}$$

The inequality above is 'almost' the desired iterative estimate, due to the presence of the measure term M_{n+1}. The latter can be estimated as follows, via a version of (3.1) with exponents $q_1 = 0$, $q_2 = 2$:

$$\begin{aligned}
\Big|Q_{n+1}\cap\{u > k_{n+1}\}\Big| &\leqslant \int_{t-t_{n+1}}^{t}\int_{B_n\cap\{u>k_{n+1}\}} \frac{(u(x,\tau)-k_n)^2}{(k_{n+1}-k_n)^2}\,dx\,d\tau \\
&\leqslant \frac{2^{2n+2}}{k^2}\iint_{Q_n} w_n^2(x,\tau)\,dx\,d\tau.
\end{aligned}$$

Combining the last two inequalities, and recalling that $|Q_n|$, $|Q_{n+1}|$ are comparable, we get the iterative estimate

$$\begin{aligned}
&⨌_{Q_{n+1}} w_{n+1}^2(x,\tau)\,dx\,d\tau \\
&\leqslant \frac{1}{|Q_{n+1}|}\frac{\gamma 2^{\beta n}}{[\sigma(1-\sigma)]^\alpha} t_0^{-\frac{2(N+ps)}{p(N+2s)}} k^{-\frac{2\lambda_2}{p(N+2s)}}\Big[\iint_{Q_n} w_n^2(x,\tau)\,dx\,d\tau\Big]^{1+\frac{2s}{N+2s}} \\
&\leqslant \frac{\gamma 2^{\beta n}}{[\sigma(1-\sigma)]^\alpha}\Big(\frac{t_0}{\rho^{ps}}\Big)^{-\frac{2N}{p(N+2s)}} k^{-\frac{2\lambda_2}{p(N+2s)}}\Big[⨌_{Q_n} w_n^2(x,\tau)\,dx\,d\tau\Big]^{1+\frac{2s}{N+2s}}.
\end{aligned}$$

Our next step consists in applying Lemma 2.5 to the sequence (Y_n). Set

$$C = \frac{\gamma}{[\sigma(1-\sigma)]^\alpha}\left(\frac{t_0}{\rho^{ps}}\right)^{-\frac{2N}{p(N+2s)}} k^{-\frac{2\lambda_2}{p(N+2s)}},\ b = 2^\beta, \eta = \frac{2s}{N+2s}.$$

Clearly $C > 0$, $b > 1$, and $\eta \in (0, 1)$ are independent of n. Our iterative estimate then rephrases as

$$Y_{n+1} \leqslant C b^n Y_n^{1+\eta}. \tag{3.5}$$

Non we give a precise definition of k by setting

$$k = \left[\frac{\gamma 2^{\frac{\beta(N+2s)}{2s}}}{[\sigma(1-\sigma)]^\alpha}\right]^{\frac{p(N+2s)}{2\lambda_2}} \left[\iint_{Q_0} u^2(x,\tau)\,dx\,d\tau\right]^{\frac{ps}{\lambda_2}} \left(\frac{t_0}{\rho^{ps}}\right)^{-\frac{N}{\lambda_2}} + P_\sigma.$$

Such choice clearly complies with (3.2). In addition, recalling that $u \geqslant 0$ in Q_0, we have

$$\begin{aligned} Y_0 &= \iint_{Q_0} u^2(x,\tau)\,dx\,d\tau \\ &\leqslant \left[\frac{\gamma}{[\sigma(1-\sigma)]^\alpha}\left(\frac{t_0}{\rho^{ps}}\right)^{-\frac{2N}{p(N+2s)}} k^{-\frac{2\lambda_2}{p(N+2s)}}\right]^{-\frac{N+2s}{2s}} 2^{-\beta(\frac{N+2s}{2s})^2} \\ &= C^{-\frac{1}{\eta}} b^{-\frac{1}{\eta^2}}. \end{aligned} \tag{3.6}$$

Due to (3.5) and (3.6) we may apply Lemma 2.5, yielding $Y_n \to 0$ as $n \to \infty$. Passing to the limit we have

$$\iint_{Q_\infty} (u(x,\tau) - k)_+^2\,dx\,d\tau = 0,$$

implying $u \leqslant k$ almost everywhere in Q_∞. Thus, recalling our choice of k, we have for convenient $\alpha, \gamma > 1$ depending on the data

$$\sup_{Q_\infty} u \leqslant \frac{\gamma}{[\sigma(1-\sigma)]^\alpha}\left[\iint_{Q_0} u^2(x,\tau)\,dx\,d\tau\right]^{\frac{ps}{\lambda_2}} \left(\frac{t_0}{\rho^{ps}}\right)^{-\frac{N}{\lambda_2}} + P_\sigma\,. \tag{3.7}$$

Now, we use the local boundedness implied by (3.7) plus our assumption on r, that is equivalent to

$$\frac{N(2-p)}{ps} < r < 2,$$

which is an admissible range since $p > p_c$, to obtain

$$\sup_{Q_\infty} u \leqslant \frac{\gamma}{[\sigma(1-\sigma)]^\alpha}\left[\sup_{Q_0} u^{\frac{ps(2-r)}{\lambda_2}}\right]\left[\iint_{Q_0} u^r(x,\tau)\,dx\,d\tau\right]^{\frac{ps}{\lambda_2}} \left(\frac{t_0}{\rho^{ps}}\right)^{-\frac{N}{\lambda_2}} + P_\sigma\,.$$

Recalling the definitions of Q_∞ and Q_0, respectively, we conclude. □

Remark 3.2 Formula (3.7) alone shows that local weak solutions are locally bounded in the range $p_c < p \leq 2$. We observe that if $r > 2$, then it is possible to bound the supremum of u with its L^r norm by a simple use of Hölder inequality.

We show a similar Lemma for the sub-critical regime, this time assuming u to be locally bounded.

Lemma 3.3 *Let u be a locally bounded local weak solution of (1.2), non-negative in $B_{4\rho}(x_0)\times(0,t)\subset\Omega_T$. Let $1<p\leqslant p_c$ and $\lambda_r>0$. Let us fix $\sigma\in(0,1)$ and $t_0\in(0,t]$. Then, there exist $\alpha,\gamma>0$ depending on the data and r, s.t.*

$$\sup_{B_{\sigma\rho}(x_0)\times(t-\sigma t_0,t)} u \leqslant \frac{\gamma}{[\sigma(1-\sigma)]^{\alpha}}\Bigg[\sup_{B_\rho(x_0)\times(t-t_0,t)} u^{\frac{Nr-p(N+2s)}{(r-2)(N+ps)}}\Bigg]$$
$$\Bigg[\fint\!\!\fint_{B_\rho(x_0)\times(t-t_0,t)} u^r(x,\tau)\,dx\,d\tau\Bigg]^{\frac{ps}{(r-2)(N+ps)}}$$
$$\cdot\left(\frac{t_0}{\rho^{ps}}\right)^{-\frac{N}{(r-2)(N+ps)}}+P_\sigma.$$

Moreover, we have that γ, $\alpha\to+\infty$ as $r\to 2$, while both constants remain stable when $r\to+\infty$.

Proof First note that the assumption $\lambda_r>0$, along with $p\leqslant p_c$, implies

$$r>\frac{N(2-p)}{ps}\geqslant 2.$$

Let $k>0$ be a number to be determined later and define sequences of cylinders and levels, functions w_n and energies E_n, as in Lemma 3.1. Arguing as in Lemma 3.1, applying Proposition 2.4 and (3.1) with several pairs of exponents (here we use $r>2$), we find

$$E_n\leqslant\frac{\gamma 2^{\beta n}}{[\sigma(1-\sigma)]^{\alpha}}\Bigg[\frac{1}{t_0k^{r-2}}+\frac{1}{\rho^{ps}k^{r-p}}+\frac{1}{\rho^{ps}k^{r-1}}\mathrm{Tail}\Big(u_+,x_0,\sigma\rho,0,t\Big)^{p-1}\Bigg]\iint_{Q_n} w_n^r(x,\tau)\,dx\,d\tau,$$

with $\alpha,\beta,\gamma>0$ depending on the data and r. Again if we assume

$$k>P_\sigma, \tag{3.8}$$

then the n-th energy is homogeneously controlled by

$$E_n\leqslant\frac{\gamma 2^{\beta n}}{[\sigma(1-\sigma)]^{\alpha}t_0k^{r-2}}\iint_{Q_n} w_n^r(x,\tau)\,dx\,d\tau. \tag{3.9}$$

This time we will derive an iterative estimate for the following sequence (with no mean value):

$$Y_n=\iint_{Q_n} w_n^r(x,\tau)\,dx\,d\tau.$$

Define the cutoff function ξ_n as in Lemma 3.1. Then, apply Proposition 2.3 to the function $w_{n+1}\xi_n$, the properties of ξ_n, (3.8), and (3.9):

$$\begin{aligned}
&\int_{t-t_{n+1}}^{t}\int_{\tilde B_n}|w_{n+1}(x,\tau)\\
&\xi_n(x)|^{\frac{p(N+2s)}{N}}\,dx\,d\tau \leqslant \gamma\Big[\sup_{t-t_{n+1}<\tau<t}\fint_{\tilde B_n}|w_{n+1}(x,\tau)\xi_n(x)|^2\,dx\Big]^{\frac{ps}{N}}\\
&\quad\cdot\Big[\tilde\rho_n^{ps}\int_{t-t_{n+1}}^{t}\iint_{\tilde B_n\times\tilde B_n}\frac{|w_{n+1}(x,\tau)\xi_n(x)-w_{n+1}(x,\tau)\xi_n(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau\\
&\quad+\Big(\frac{\tilde\rho_n}{\tilde\rho_n-\check\rho_n}\Big)^{N+ps}\int_{t-t_{n+1}}^{t}\int_{\tilde B_n}|w_{n+1}(x,\tau)\xi_n(x)|^p\,dx\,d\tau\Big]\\
&\leqslant \frac{\gamma}{\tilde\rho_n^{ps}}E_n^{\frac{ps}{N}}\Big[\tilde\rho_n^{ps}E_n+\Big(\frac{\tilde\rho_n}{\tilde\rho_n-\check\rho_n}\Big)^{N+ps}\iint_{Q_n}w_{n+1}^p(x,\tau)\,dx\,d\tau\Big]\\
&\leqslant \frac{\gamma 2^{psn}}{[\sigma(1-\sigma)]^{ps}\rho^{ps}}E_n^{\frac{ps}{N}}\Big[\rho^{ps}E_n+\frac{2^{(N+ps+r-p)n}}{[\sigma(1-\sigma)]^{N+ps}k^{r-p}}\iint_{Q_n}w_n^r(x,\tau)\,dx\,d\tau\Big]\\
&\leqslant \frac{\gamma 2^{\beta n}}{[\sigma(1-\sigma)]^{\alpha}}\Big[\frac{1}{t_0k^{r-2}}\iint_{Q_n}w_n^r(x,\tau)\,dx\,d\tau\Big]^{\frac{N+ps}{N}},
\end{aligned}$$

again with $\alpha,\beta,\gamma>0$ depending on the data and r. Further, noting that $p(N+2s)/N\leqslant 2\leqslant r$, recalling that u is assumed locally bounded, and that $\xi_n=1$ in B_{n+1}, we get

$$\begin{aligned}
\iint_{Q_{n+1}}w_{n+1}^r(x,\tau)\,dx\,d\tau &\leqslant \Big[\int_{t-t_{n+1}}^{t}\int_{\tilde B_n}|w_{n+1}(x,\tau)\xi_n(x)|^{\frac{p(N+2s)}{N}}\,dx\,d\tau\Big]\Big[\sup_{Q_0}u\Big]^{\frac{Nr-p(N+2s)}{N}}\\
&\leqslant \frac{\gamma 2^{\beta n}}{[\sigma(1-\sigma)]^{\alpha}}\Big[\frac{1}{t_0k^{r-2}}\iint_{Q_n}w_n^r(x,\tau)\,dx\,d\tau\Big]^{\frac{N+ps}{N}}\Big[\sup_{Q_0}u\Big]^{\frac{Nr-p(N+2s)}{N}},
\end{aligned}$$

with the supremum of u in Q_0 term replacing the measure multiplier of the corresponding estimate of Lemma 3.1 (here the assumption that u is locally bounded is essential). We will now apply Lemma 2.5 to (Y_n), this time with

$$C=\frac{\gamma}{[\sigma(1-\sigma)]^{\alpha}}\Big(\frac{1}{t_0k^{r-2}}\Big)^{\frac{N+ps}{N}}\Big[\sup_{Q_0}u\Big]^{\frac{Nr-p(N+2s)}{N}},\ b=2^{\beta},\ \eta=\frac{ps}{N}.$$

Note that $C,b>1$ and $\eta\in(0,1)$ are independent of n. The previous estimate then rephrases as

$$Y_{n+1}\leqslant Cb^nY_n^{1+\eta}. \tag{3.10}$$

Define k by setting

$$k=\Big[\frac{\gamma 2^{\frac{\beta N}{ps}}}{[\sigma(1-\sigma)]^{\alpha}}\Big]^{\frac{N}{(r-2)(N+ps)}}\Big[\iint_{Q_0}u^r(x,\tau)\,dx\,d\tau\Big]^{\frac{ps}{(r-2)(N+ps)}}\Big[\sup_{Q_0}u\Big]^{\frac{Nr-p(N+2s)}{(r-2)(N+ps)}}t_0^{-\frac{1}{r-2}}+P_\sigma,$$

complying in particular with (3.8). Then we have

$$
\begin{aligned}
Y_0 &= \iint_{Q_0} u^r(x,\tau)\,dx\,d\tau \\
&\leqslant \Big[\frac{\gamma}{[\sigma(1-\sigma)]^\alpha}\Big(\frac{1}{t_0 k^{r-2}}\Big)^{\frac{N+ps}{N}} \sup_{Q_0} u^{\frac{Nr-p(N+2s)}{N}}\Big]^{-\frac{N}{ps}} 2^{-\beta(\frac{N}{ps})^2} \\
&= C^{-\frac{1}{\eta}} b^{-\frac{1}{\eta^2}}.
\end{aligned}
\tag{3.11}
$$

Due to (3.10) and (3.11) we may apply Lemma 2.5 and find that $Y_n \to 0$ as $n \to \infty$. Passing to the limit, we get

$$
\iint_{Q_\infty} (u(x,\tau)-k)_+^r\,dx\,d\tau = 0,
$$

hence $u \leqslant k$ a.e. in Q_∞. Therefore, by the definition of k we have

$$
\begin{aligned}
\sup_{Q_\infty} u &\leqslant \frac{\gamma}{[\sigma(1-\sigma)]^\alpha}\Big[\sup_{Q_0} u^{\frac{Nr-p(N+2s)}{(r-2)(N+ps)}}\Big]\Big[\iint_{Q_0} u^r(x,\tau)\,dx\,d\tau\Big]^{\frac{ps}{(r-2)(N+ps)}} t_0^{-\frac{1}{r-2}} + P_\sigma \\
&\leqslant \frac{\gamma}{[\sigma(1-\sigma)]^\alpha}\Big[\sup_{Q_0} u^{\frac{Nr-p(N+2s)}{(r-2)(N+ps)}}\Big]\Big[⨌_{Q_0} u^r(x,\tau)\,dx\,d\tau\Big]^{\frac{ps}{(r-2)(N+ps)}} \Big(\frac{t_0}{\rho^{ps}}\Big)^{-\frac{N}{(r-2)(N+ps)}} + P_\sigma,
\end{aligned}
$$

for some $\alpha, \gamma > 0$ depending on the data and r. This concludes the proof. □

We can now complete the proof of the main result:

Proof of Theorem 1.1 (conclusion). The second part of the proof is based on a further iterative argument, with increasing cylinders. Fix $\theta \in (0,1)$ (to be determined later) and set $\rho'_0 = \theta\rho$, $t'_0 = \theta t$. Also, set for all $n \in \mathbb{N}$

$$
\rho'_n = \rho\Big[\theta + (1-\theta)\sum_{i=1}^n \frac{1}{2^i}\Big],\ t'_n = t\Big[\theta + (1-\theta)\sum_{i=1}^n \frac{1}{2^i}\Big],
$$

so that both (ρ'_n) and (t'_n) are increasing and $\rho'_n \to \rho$, $t'_n \to t$ as $n \to \infty$. Also, for all $n \in \mathbb{N}$ we set $Q'_n = B_{\rho'_n}(x_0) \times (t - t'_n, t)$ so $Q'_n \subset Q'_{n+1}$ and the limit cylinder is $Q'_\infty = B_\rho(x_0) \times (0,t)$. Now recall that $u \geqslant 0$ in Q_∞ and u is locally bounded, so we may define a bounded sequence of positive numbers by setting for all $n \in \mathbb{N}$

$$
S_n = \sup_{Q'_n} u,
$$

We are going to prove an iterative estimate on the sequence (S_n), by considering two cases:

(a) If $p_c < p < 2$, then recall that $r < 2$. We apply Lemma 3.1 with radius ρ'_{n+1}, time $t_0 = t'_{n+1}$, and

$$
\sigma = \frac{\theta + (1-\theta)\sum_{i=1}^n 2^{-i}}{\theta + (1-\theta)\sum_{i=1}^{n+1} 2^{-i}} \in (0,1),
$$

so that $\sigma\rho'_{n+1} = \rho'_n$, $\sigma t'_{n+1} = t'_n$. We get for some $\gamma, \alpha > 0$ depending on the data and θ

$$\sup_{Q'_n} u \leqslant \frac{\gamma}{[\sigma(1-\sigma)]^\alpha}\Big[\sup_{Q'_{n+1}} u^{\frac{ps(2-r)}{\lambda_2}}\Big]\Big[\iint\!\!\!\!\!\!-_{Q'_{n+1}} u^r(x,\tau)\,dx\,d\tau\Big]^{\frac{ps}{\lambda_2}}\Big(\frac{t'_{n+1}}{(\rho'_{n+1})^{ps}}\Big)^{-\frac{N}{\lambda_2}} \tag{3.12}$$
$$+\Big(\frac{t}{(\rho'_{n+1})^{ps}}\Big)^{\frac{1}{2-p}} \max\Big\{1, \Big(\frac{t}{(\rho'_{n+1})^{ps}}\Big)^{\frac{1-p}{2-p}}\mathrm{Tail}\Big(u_+, x_0, \rho'_n, 0, t\Big)^{p-1}\Big\}$$
$$= J_1 + J_2$$

Now we estimate J_1 and J_2, separately. First note that

$$\sigma^\alpha(1-\sigma)^\alpha \geqslant \frac{(1-\theta)^\alpha}{2^{\alpha(n+2)}}.$$

Also, we have for some $\gamma > 0$ independent of n the inequality

$$\iint\!\!\!\!\!\!-_{Q'_{n+1}} u^r(x,\tau)\,dx\,d\tau = \frac{1}{|Q'_{n+1}|}\iint_{Q'_{n+1}} u^r(x,\tau)\,dx\,d\tau \le \gamma \iint\!\!\!\!\!\!-_{Q'_\infty} u^r(x,\tau)\,dx\,d\tau.$$

Therefore, since ρ'_{n+1} and t'_{n+1} are comparable to ρ and t, respectively, we find

$$J_1 \leqslant \frac{\gamma 2^{\alpha n}}{(1-\theta)^\alpha}\Big[\sup_{Q'_{n+1}} u^{\frac{ps(2-r)}{\lambda_2}}\Big]\Big[\iint\!\!\!\!\!\!-_{Q'_\infty} u^r(x,\tau)\,dx\,d\tau\Big]^{\frac{ps}{\lambda_2}}\Big(\frac{t}{\rho^{ps}}\Big)^{-\frac{N}{\lambda_2}}.$$

To estimate J_2 we note that $\rho'_n \geqslant \theta\rho$, hence the tail can be treated as

$$\mathrm{Tail}\Big(u_+, x_0, \rho'_n, 0, t\Big)^{p-1} \leqslant \theta^{-ps}\mathrm{Tail}\Big(u_+, x_0, \theta\rho, 0, t\Big)^{p-1}.$$

Therefore, again by comparability, we have

$$J_2 \leqslant \gamma\theta^{-ps}\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}} \max\Big\{1, \Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1-p}{2-p}}\mathrm{Tail}\Big(u_+, x_0, \theta\rho, 0, t\Big)^{p-1}\Big\} =: \gamma\theta^{-ps}P_\theta.$$

Plugging such estimates into (3.12) yields, for some $\alpha, \gamma > 0$ depending on the data and θ,

$$S_n \leqslant \gamma 2^{\alpha n} S_{n+1}^{\frac{ps(2-r)}{\lambda_2}}\Big[\iint\!\!\!\!\!\!-_{Q'_\infty} u^r(x,\tau)\,dx\,d\tau\Big]^{\frac{ps}{\lambda_2}}\Big(\frac{t}{\rho^{ps}}\Big)^{-\frac{N}{\lambda_2}} + \gamma P_\theta.$$

Recall the bounds on r seen in Lemma 3.1. We can then apply Young's inequality (2.2) to the first and second terms, with exponents

$$q = \frac{\lambda_2}{ps(2-r)}, \qquad q' = \frac{\lambda_2}{\lambda_r}$$

and $\varepsilon \in (0,1)$ (to be determined later), to find

$$S_n \leqslant \varepsilon S_{n+1} + \frac{\gamma 2^{\alpha n}}{\varepsilon^\alpha}\Big[\iint\!\!\!\!\!\!-_{Q'_\infty} u^r(x,,\tau)\,dx\,d\tau\Big]^{\frac{ps}{\lambda_r}}\Big(\frac{t}{\rho^{ps}}\Big)^{-\frac{N}{\lambda_r}} + P_\theta,$$

with $\alpha, \gamma > 0$ depending on the data and θ.

(b) If $1 < p \leqslant p_c$, then recall that $r > 2$. We apply Lemma 3.3 with the same choice of ρ, t_0, σ as in case (a), and with analogous estimates we get

$$S_n \leqslant \gamma 2^{\alpha n} S_{n+1}^{\frac{Nr-p(N+2s)}{(r-2)(N+ps)}} \Big[\iint_{Q'_\infty} u^r(x,\tau)\,dx\,d\tau \Big]^{\frac{ps}{(r-2)(N+ps)}} \Big(\frac{t}{\rho^{ps}}\Big)^{-\frac{N}{(r-2)(N+ps)}} + \gamma P_\theta.$$

This time we apply (2.2) with exponents

$$q = \frac{(r-2)(N+ps)}{Nr-p(N+2s)},\ q' = \frac{(r-2)(N+ps)}{\lambda_r},$$

and $\varepsilon \in (0,1)$ (to be determined later), to find again

$$S_n \leqslant \varepsilon S_{n+1} + \frac{\gamma 2^{\alpha n}}{\varepsilon^\alpha} \Big[\iint_{Q'_\infty} u^r(x,,\tau)\,dx\,d\tau \Big]^{\frac{ps}{\lambda_r}} \Big(\frac{t}{\rho^{ps}}\Big)^{-\frac{N}{\lambda_r}} + P_\theta,$$

with $\alpha, \gamma > 0$ depending on the data, r, and θ.

In both cases above, we have obtained the same iterative inequality for the sequence (S_n). Iterating such estimate from 0 to n, we have

$$\begin{aligned} S_0 &\leqslant \varepsilon S_1 + \frac{\gamma}{\varepsilon^\alpha} \Big[\iint_{Q'_\infty} u^r(x,\tau)\,dx\,d\tau \Big]^{\frac{ps}{\lambda_r}} \Big(\frac{t}{\rho^{ps}}\Big)^{-\frac{N}{\lambda_r}} + \gamma P_\theta \\ &\leqslant \varepsilon^2 S_2 + \frac{\gamma}{\varepsilon^\alpha}(1+\varepsilon 2^\alpha) \Big[\iint_{Q'_\infty} u^r(x,\tau)\,dx\,d\tau \Big]^{\frac{ps}{\lambda_r}} \Big(\frac{t}{\rho^{ps}}\Big)^{-\frac{N}{\lambda_r}} + \gamma(1+\varepsilon) P_\theta \ldots \\ &\leqslant \varepsilon^n S_n + \frac{\gamma}{\varepsilon^\alpha} \sum_{i=0}^{n-1} (\varepsilon 2^\alpha)^i \Big[\iint_{Q'_\infty} u^r(x,\tau)\,dx\,d\tau \Big]^{\frac{ps}{\lambda_r}} \Big(\frac{t}{\rho^{ps}}\Big)^{-\frac{N}{\lambda_r}} + \gamma \sum_{i=0}^{n-1} \varepsilon^i P_\theta. \end{aligned}$$

As soon as we take $0 < \varepsilon < 2^{-\alpha}$, the series above are convergent. Thus, recalling that (S_n) is bounded and passing to the limit as $n \to \infty$, we have for some $\gamma > 0$ depending on the data, r, and θ

$$\sup_{B_{\theta\rho}\times(t-\theta t,t)} u = S_0 \leqslant \gamma \Big[\iint_{B_\rho(x_0)\times(0,t)} u^r(x,\tau)\,dx\,d\tau \Big]^{\frac{ps}{\lambda_r}} \Big(\frac{t}{\rho^{ps}}\Big)^{-\frac{N}{\lambda_r}} + \gamma P_\theta.$$

Finally, choosing $\theta = 1/2$ and recalling the definition of P_θ, we find

$$\begin{aligned} \sup_{B_{\rho/2}(x_0)\times(t/2,t)} u &\leqslant \gamma \Big[\iint_{B_\rho(x_0)\times(0,t)} u^r(x,\tau)\,dx\,d\tau \Big]^{\frac{ps}{\lambda_r}} \Big(\frac{t}{\rho^{ps}}\Big)^{-\frac{N}{\lambda_r}} \\ &\quad + \gamma \Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}} \max\Big\{1, \Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1-p}{2-p}} \mathrm{Tail}\Big(u_+, x_0, \frac{\rho}{2}, 0, t\Big)^{p-1}\Big\}, \end{aligned}$$

with $\gamma > 0$ depending on the data and r, which concludes the proof. □

Remark 3.4 We note that, in Theorem 1.1, the functional dependence of the constant γ on r degenerates when λ_r tends to 0. However, we observe that in the supercritical regime

$$\lambda_r = N(p-2) + rps > 0 \qquad \forall\, r \geq 1\,.$$

Therefore, we have the following situation:

- if $1 < p < \frac{2N}{N+s}$, the constant γ depends on r and becomes unbounded when λ_r vanishes;

- if $\frac{2N}{N+s} < p < 2$ the constant γ does not degenerate with any choice of r.

The functional independence of γ on the integrability exponent r in the supercritical range indeed reflects the fact that the solutions are automatically bounded, and therefore r-integrable. Since this property must be assumed in the subcritical regime, the estimate of Theorem 1.1 trivializes as soon as λ_r vanishes. The critical regime $p = 2N/(N+s)$ is the threshold and needs a different inspection, in the spirit of [16].

Remark 3.5 The whole double iterative procedure can be performed, as usual, for any signed sub-solution, thereby showing the L^r-L^∞ estimate for u_+. In case u is a solution, then $-u$ is a sub-solution to a similar equation and a similar bound is available for the negative part of u. Summarizing, if u is a locally bounded local weak solution to (1.2) and $\lambda_r > 0$ (with $1 < r \le 2$ if $p > p_c$), then the L^r-L^∞ estimate is valid for $|u|$,

$$\sup_{B_{\rho/2}(x_0)\times(t/2,t)} |u| \leqslant \gamma\Big[\iint_{B_\rho(x_0)\times(0,t)} |u|^r(x,\tau)\,dx\,d\tau\Big]^{\frac{ps}{\lambda_r}}\Big(\frac{t}{\rho^{ps}}\Big)^{-\frac{N}{\lambda_r}} + \gamma\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}}\max\Big\{1,\ \Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1-p}{2-p}}\mathrm{Tail}\Big(u, x_0, \frac{\rho}{2}, 0, t\Big)^{p-1}\Big\}.$$

4 L^1-L^1 and L^1-L^∞ estimates

In this section we focus on estimates involving the L^1-norm of the solution, proving Theorems 1.2 and 1.3. First we assume u to be a (possibly unbounded) solution of (1.2), s.t. $u \geqslant 0$ in $B_{4\rho}(x_0)\times(0,t) \subset \Omega_T$. Set

$$P_\pm = \max\Big\{1,\ \Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1-p}{2-p}}\mathrm{Tail}\Big(u_\pm, x_0, \frac{\rho}{2}, 0, t\Big)^{p-1}\Big\}. \tag{4.1}$$

We need a technical lemma, yielding a special energy estimate for u. We shall use it at one step only (see (4.18) below), but it is a crucial one:

Lemma 4.1 *Let* $0 < \sigma < \sigma' < 1$, $\gamma' > 0$, $\xi \in C^1_c(B_{\sigma'\rho}(x_0))$ *s.t.*

$$\xi = 1 \text{ in } B_{\sigma\rho}(x_0),\ 0 \leqslant \xi \leqslant 1,\ |\nabla\xi| \leqslant \frac{\gamma'}{(\sigma'-\sigma)\rho} \text{ in } \mathbb{R}^N.$$

Then, there exists $\gamma > 0$ *depending on the data and* γ', *s.t.*

$$\begin{aligned}&\int_0^t \tau^{\frac{1}{p}}\iint_{A_\tau}\frac{(u(x,\tau)-u(y,\tau))^p}{|x-y|^{N+ps}}\Big[u(x,\tau)+\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}}\Big]^{-\frac{2}{p}}\xi^p(x)\,dx\,dy\,d\tau\\ &\leqslant \gamma\rho^s\max\Big\{\frac{1}{(\sigma'-\sigma)^p},\ \frac{1}{(1-\sigma')^{N+ps}}\Big\}\\ &\Big[\sup_{0<\tau<t}\int_{B_\rho(x_0)}u(x,\tau)\,dx+\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}}\Big]^{\frac{2(p-1)}{p}}\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{p}}P_-,\end{aligned}$$

where for all $\tau \in (0,t)$

$$A_\tau = \big\{(x,y)\in B_{\sigma'\rho}(x_0)\times B_{\sigma'\rho}(x_0):\ u(x,\tau) > u(x,\tau),\ \xi(x) > \xi(y)\big\}.$$

Proof Set for brevity

$$B = B_\rho(x_0),\ \check{B} = B_{\sigma\rho}(x_0),\ \hat{B} = B_{\sigma'\rho}(x_0),$$

so that $\check{B} \subset \hat{B} \subset B$, and

$$\nu = \Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}} > 0.$$

Define for all $(x, \tau) \in \mathbb{R}^N \times (0, T)$

$$\varphi(x,\tau) = -\tau^{\frac{1}{p}}(u(x,\tau)+\nu)^{\frac{p-2}{p}}\xi^p(x).$$

The function φ is well defined since $\xi = 0$ in $\hat{B}^c$, yet it is not an admissible test function in (1.2) as it is not differentiable in τ, in general. Nevertheless, using a convenient mollification procedure (the proof is postponed to Lemma A.2), we get the following inequality, with $\gamma > 0$ depending on the data:

$$\begin{aligned} 0 &\geqslant -\frac{pt^{\frac{1}{p}}}{2(p-1)}\int_B (u(x,t)+\nu)^{\frac{2(p-1)}{p}}\xi^p(x)\,dx \\ &\quad + \frac{1}{2(p-1)}\int_0^t \tau^{\frac{1-p}{p}}\int_B (u(x,\tau)+\nu)^{\frac{2(p-1)}{p}}\xi^p(x)\,dx\,d\tau \\ &\quad + \gamma\int_0^t \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x,\tau)-u(y,\tau)|^{p-2}(u(x,\tau)-u(y,\tau))}{|x-y|^{N+ps}}(\varphi(x,\tau)-\varphi(y,\tau))\,dx\,dy\,d\tau \\ &= I_1 + I_2. \end{aligned} \tag{4.2}$$

We first deal with the evolutive term I_1. Since the second integral of I_1 is positive, we dismiss it. Besides, on the first term we apply Hölder's inequality and $\xi \leqslant 1$:

$$\begin{aligned} I_1 &\geqslant -\frac{pt^{\frac{1}{p}}}{2(p-1)}\int_B (u(x,t)+\nu)^{\frac{2(p-1)}{p}}\xi^p(x)\,dx \\ &\geqslant -\gamma t^{\frac{1}{p}}\Big[\int_B (u(x,t)+\nu)\,dx\Big]^{\frac{2(p-1)}{p}}|B|^{\frac{2-p}{p}} \\ &\geqslant -\gamma\rho^{\frac{N(2-p)}{p}}t^{\frac{1}{p}}\Big[\sup_{0<\tau<t}\int_B u(x,\tau)\,dx + \nu\rho^N\Big]^{\frac{2(p-1)}{p}}. \end{aligned}$$

Recalling the definitions of λ_1 and ν, we find

$$I_1 \geqslant -\gamma\rho^s\Big[\sup_{0<\tau<t}\int_B u(x,\tau)\,dx + \Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}}\Big]^{\frac{2(p-1)}{p}}\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{p}}. \tag{4.3}$$

Now we turn to the diffusive term I_2. For simplicity, we fix $\tau \in (0, t)$ and focus on the space integral only (via a sign change):

$$\begin{aligned} I_3 = \iint_{\mathbb{R}^N\times\mathbb{R}^N} &\frac{|u(x,\tau)-u(y,\tau)|^{p-2}(u(x,\tau)-u(y,\tau))}{|x-y|^{N+ps}} \\ &\Big[(u(x,\tau)+\nu)^{\frac{p-2}{p}}\xi^p(x) - (u(y,\tau)+\nu)^{\frac{p-2}{p}}\xi^p(y)\Big]\,dx\,dy. \end{aligned}$$

By symmetry, we may rephrase such integral by setting

$$\begin{aligned}I_3 &= 2\iint_{A_\tau^+} \frac{(u(x,\tau)-u(y,\tau))^{p-1}}{|x-y|^{N+ps}}\Big[(u(x,\tau)+\nu)^{\frac{p-2}{p}}\xi^p(x)-(u(y,\tau)+\nu)^{\frac{p-2}{p}}\xi^p(y)\Big]\,dx\,dy\\ &\quad+2\iint_{A_\tau^-} \frac{(u(x,\tau)-u(y,\tau))^{p-1}}{|x-y|^{N+ps}}\Big[(u(x,\tau)+\nu)^{\frac{p-2}{p}}\xi^p(x)-(u(y,\tau)+\nu)^{\frac{p-2}{p}}\xi^p(y)\Big]\,dx\,dy\\ &= I_4+I_5,\end{aligned}$$

where we have set

$$A_\tau^+ = \Big\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N : u(x,\tau)>u(y,\tau),\ u(y,\tau)\geqslant 0\Big\},$$

$$A_\tau^- = \Big\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N : u(x,\tau)>u(y,\tau),\ u(y,\tau)< 0\Big\}.$$

We first consider I_4. Note that for all $(x,y)\in A_\tau^+$ s.t. $\xi(x)\leqslant\xi(y)$ the integrand is negative, so we may restrict ourselves to the subdomain where $\xi(x)>\xi(y)$ and split the integrand as follows:

$$\begin{aligned}I_4 &\leqslant \gamma\iint_{A_\tau^+\cap\{\xi(x)>\xi(y)\}} \frac{(u(x,\tau)-u(y,\tau))^{p-1}}{|x-y|^{N+ps}}\Big[(u(x,\tau)+\nu)^{\frac{p-2}{p}}-(u(y,\tau)+\nu)^{\frac{p-2}{p}}\Big]\xi^p(x)\,dx\,dy\\ &\quad+\gamma\iint_{A_\tau^+\cap\{\xi(x)>\xi(y)\}} \frac{(u(x,\tau)-u(y,\tau))^{p-1}}{|x-y|^{N+ps}}(u(y,\tau)+\nu)^{\frac{p-2}{p}}(\xi^p(x)-\xi^p(y))\,dx\,dy\\ &= I_6+I_7.\end{aligned}$$

In order to deal with I_6, I_7, respectively, we need two pointwise inequalities holding for all $(x,y)\in A_\tau^+$ s.t. $\xi(x)>\xi(y)$. First, by concavity we have

$$(u(x,\tau)+\nu)^{\frac{p-2}{p}}-(u(y,\tau)+\nu)^{\frac{p-2}{p}}\leqslant\frac{p-2}{p}(u(x,\tau)+\nu)^{-\frac{2}{p}}(u(x,\tau)-u(y,\tau)).$$

So, I_6 is estimated by a negative quantity as follows:

$$I_6\leqslant-\gamma\iint_{A_\tau^+\cap\{\xi(x)>\xi(y)\}}\frac{(u(x,\tau)-u(y,\tau))^p}{|x-y|^{N+ps}}(u(x,\tau)+\nu)^{-\frac{2}{p}}\xi^p(x)\,dx\,dy. \tag{4.4}$$

Besides, for all $(x,y)\in A_\tau^+\cap\{\xi(x)>\xi(y)\}$ we apply inequality (2.3) with $a=\xi(x)>\xi(y)=b$, $\delta\in(0,1)$ to be determined later, and $\varepsilon\in(0,1)$ defined by

$$\begin{aligned}\varepsilon &= \delta\frac{u(x,\tau)-u(y,\tau)}{u(y,\tau)+\nu}\Big[\frac{u(y,\tau)+\nu}{u(x,\tau)+\nu}\Big]^{\frac{2}{p}}\\ &\leqslant\Big[\frac{u(y,\tau)+\nu}{u(x,\tau)+\nu}\Big]^{\frac{2-p}{p}}<1,\end{aligned}$$

so that the following pointwise inequality holds with $\gamma>0$ independent of δ:

$$\begin{aligned}\xi^p(x)-\xi^p(y) &\leqslant \delta\frac{u(x,\tau)-u(y,\tau)}{u(y,\tau)+\nu}\Big[\frac{u(y,\tau)+\nu}{u(x,\tau)+\nu}\Big]^{\frac{2}{p}}\xi^p(x)\\ &\quad+\frac{\gamma}{\delta^{p-1}}\Big[\frac{u(y,\tau)+\nu}{u(x,\tau)-u(y,\tau)}\Big]^{p-1}\Big[\frac{u(x,\tau)+\nu}{u(y,\tau)+\nu}\Big]^{\frac{2(p-1)}{p}}(\xi(x)-\xi(y))^p.\end{aligned}$$

Plugging such inequality into the integrand of I_7, and recalling that $u(y,\tau)+\nu \geqslant \nu$, we have

$$\begin{aligned} I_7 &\leqslant \gamma\delta \iint_{A_\tau^+\cap\{\xi(x)>\xi(y)\}} \frac{(u(x,\tau)-u(y,\tau))^p}{|x-y|^{N+ps}} (u(x,\tau)+\nu)^{-\frac{2}{p}} \xi^p(x)\,dx\,dy \\ &\quad + \frac{\gamma}{\delta^{p-1}\nu^{2-p}} \iint_{A_\tau^+\cap\{\xi(x)>\xi(y)\}} (u(x,\tau)+\nu)^{\frac{2(p-1)}{p}} \frac{(\xi(x)-\xi(y))^p}{|x-y|^{N+ps}}\,dx\,dy. \end{aligned} \tag{4.5}$$

Choosing $\delta \in (0,1)$ small enough, we can reabsorb the first term of (4.5) into (4.4), so that for some $\gamma > 0$ depending on the data

$$\begin{aligned} I_4 &\leqslant -\gamma \iint_{A_\tau^+\cap\{\xi(x)>\xi(y)\}} \frac{(u(x,\tau)-u(y,\tau))^p}{|x-y|^{N+ps}} (u(x,\tau)+\nu)^{-\frac{2}{p}} \xi^p(x)\,dx\,dy \\ &\quad + \frac{\gamma}{\nu^{2-p}} \iint_{A_\tau^+\cap\{\xi(x)>\xi(y)\}} (u(x,\tau)+\nu)^{\frac{2(p-1)}{p}} \frac{(\xi(x)-\xi(y))^p}{|x-y|^{N+ps}}\,dx\,dy \\ &= I_8 + I_9. \end{aligned} \tag{4.6}$$

In the following lines, we leave I_8 alone and estimate I_9. First note that, for all $(x,y) \in A_\tau^+ \cap \{\xi(x) > \xi(y)\}$, we have in particular $\xi(x) > 0$ and hence $x \in \hat{B}$. Therefore, we may split the space integral as follows:

$$\begin{aligned} I_9 &\leqslant \frac{\gamma}{\nu^{2-p}} \iint_{\hat{B}\times B} (u(x,\tau)+\nu)^{\frac{2(p-1)}{p}} \frac{(\xi(x)-\xi(y))^p}{|x-y|^{N+ps}}\,dx\,dy \\ &\quad + \frac{\gamma}{\nu^{2-p}} \iint_{\hat{B}\times B^c} (u(x,\tau)+\nu)^{\frac{2(p-1)}{p}} \frac{(\xi(x)-\xi(y))^p}{|x-y|^{N+ps}}\,dx\,dy \\ &= I_{10} + I_{11}. \end{aligned}$$

On I_{10} we act using the gradient bound on ξ and Hölder's inequality:

$$\begin{aligned} I_{10} &\leqslant \frac{\gamma}{(\sigma'-\sigma)^p \rho^p \nu^{2-p}} \iint_{\hat{B}\times B} \frac{(u(x,\tau)+\nu)^{\frac{2(p-1)}{p}}}{|x-y|^{N+ps-p}}\,dx\,dy \\ &\leqslant \frac{\gamma}{(\sigma'-\sigma)^p \rho^p \nu^{2-p}} \Big[\int_{\hat{B}} (u(x,\tau)+\nu)^{\frac{2(p-1)}{p}}\,dx \Big] \Big[\sup_{x\in\hat{B}} \int_B \frac{dy}{|x-y|^{N+ps-p}}\,dy \Big] \\ &\leqslant \frac{\gamma}{(\sigma'-\sigma)^p \rho^p \nu^{2-p}} \Big[\int_B (u(x,\tau)+\nu)\,dx \Big]^{\frac{2(p-1)}{p}} |B|^{\frac{2-p}{p}} \Big[\int_{B_{2\rho}(0)} \frac{dz}{|z|^{N+ps-p}} \Big] \\ &\leqslant \frac{\gamma \rho^{s-ps-\frac{\lambda_1}{p}}}{(\sigma'-\sigma)^p \nu^{2-p}} \Big[\int_B u(x,\tau)\,dx + \nu\rho^N \Big]^{\frac{2(p-1)}{p}}. \end{aligned} \tag{4.7}$$

To estimate I_{11}, we note that for all $x \in \hat{B}$, $y \in B^c$

$$|x-y| \geqslant |y-x_0| - |x-x_0| \geqslant (1-\sigma')|y-x_0|,$$

so by $0 \leqslant \xi \leqslant 1$ and Hölder's inequality again we get

$$\begin{aligned} I_{11} &\leqslant \frac{\gamma}{(1-\sigma')^{N+ps}\nu^{2-p}} \iint_{\hat{B}\times B^c} \frac{(u(x,\tau)+\nu)^{\frac{2(p-1)}{p}}}{|y-x_0|^{N+ps}}\,dx\,dy \\ &\leqslant \frac{\gamma}{(1-\sigma')^{N+ps}\nu^{2-p}} \Big[\int_B (u(x,\tau)+\nu)^{\frac{2(p-1)}{p}}\,dx\Big]\Big[\int_{B^c} \frac{dy}{|y-x_0|^{N+ps}}\Big] \\ &\leqslant \frac{\gamma}{(1-\sigma')^{N+ps}\rho^{ps}\nu^{2-p}} \Big[\int_B (u(x,\tau)+\nu)\,dx\Big]^{\frac{2(p-1)}{p}} |B|^{\frac{2-p}{p}} \\ &\leqslant \frac{\gamma\rho^{s-ps-\frac{\lambda_1}{p}}}{(1-\sigma')^{N+ps}\nu^{2-p}} \Big[\int_B u(x,\tau)\,dx + \nu\rho^N\Big]^{\frac{2(p-1)}{p}}. \end{aligned} \tag{4.8}$$

By (4.7) and (4.8) we have

$$I_9 \leqslant \frac{\gamma\rho^{s-ps-\frac{\lambda_1}{p}}}{\nu^{2-p}} \max\Big\{\frac{1}{(\sigma'-\sigma)^p}, \frac{1}{(1-\sigma')^{N+ps}}\Big\}\Big[\int_B u(x,\tau)\,dx + \nu\rho^N\Big]^{\frac{2(p-1)}{p}}.$$

Going back to (4.6) and recalling the definition of ν we have

$$\begin{aligned} I_4 &\leqslant -\gamma \iint_{A_\tau^+\cap\{\xi(x)>\xi(y)\}} \frac{(u(x,\tau)-u(y,\tau))^p}{|x-y|^{N+ps}} \Big[u(x,\tau) + \Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}}\Big]^{-\frac{2}{p}} \xi^p(x)\,dx\,dy \\ &\quad + \frac{\gamma\rho^{s-\frac{\lambda_1}{p}}}{t} \max\Big\{\frac{1}{(\sigma'-\sigma)^p}, \frac{1}{(1-\sigma')^{N+ps}}\Big\}\Big[\int_B u(x,\tau)\,dx + \Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}}\Big]^{\frac{2(p-1)}{p}}. \end{aligned} \tag{4.9}$$

There remains to estimate I_5. First we observe that, for all $(x, y) \in A_\tau^-$ we have $u(y, \tau) < 0$ and hence by assumption $y \in B^c$. This in turn implies $\xi(y) = 0$, so we may reduce the integrand. Also, by subadditivity we have

$$(u(x,\tau)-u(y,\tau))^{p-1} \leqslant u^{p-1}(x,\tau) + u_-^{p-1}(y,\tau).$$

Therefore, we can estimate I_5 as follows, further reducing the integration domain by $\xi = 0$ in $\hat{B}^c$:

$$I_5 \leqslant \gamma \iint_{A_\tau^-} \frac{u^{p-1}(x,\tau)+u_-^{p-1}(y,\tau)}{|x-y|^{N+ps}} (u(x,\tau)+\nu)^{\frac{p-2}{p}} \xi^p(x)\,dx\,dy \tag{4.10}$$

$$\begin{aligned} &\leqslant \frac{\gamma}{\nu^{2-p}} \iint_{\hat{B}\times B^c} \frac{(u(x,\tau)+\nu)^{\frac{2(p-1)}{p}}}{|x-y|^{N+ps}}\,dx\,dy \\ &\quad + \frac{\gamma}{\nu} \iint_{\hat{B}\times B^c} \frac{(u(x,\tau)+\nu)^{\frac{2(p-1)}{p}}}{|x-y|^{N+ps}} u_-^{p-1}(y,\tau)\,dx\,dy \\ &= I_{12} + I_{13}. \end{aligned} \tag{4.11}$$

Now I_{12} is estimated similarly to (4.8):

$$I_{12} \leqslant \frac{\gamma\rho^{s-ps-\frac{\lambda_1}{p}}}{(1-\sigma')^{N+ps}\nu^{2-p}} \Big[\int_B u(x,\tau)\,dx + \nu\rho^N\Big]^{\frac{2(p-1)}{p}}. \tag{4.12}$$

The estimate of I_{13} is more delicate, as it involves a tail-type integral of u_- (see (1.4)). As usual, for all $(x, y) \in \hat{B} \times B^c$ we have

$$|x-y| \geqslant (1-\sigma')|y-x_0|,$$

hence we can separate the integrals as follows:

$$
\begin{aligned}
I_{13} &\leqslant \frac{\gamma}{(1-\sigma')^{N+ps}\nu}\Big[\int_{\hat{B}}(u(x,\tau)+\nu)^{\frac{2(p-1)}{p}}\,dx\Big]\Big[\int_{B^c}\frac{u_-^{p-1}(y,\tau)}{|y-x_0|^{N+ps}}\,dy\Big] \\
&\leqslant \frac{\gamma}{(1-\sigma')^{N+ps}\rho^{ps}\nu}\Big[\int_B(u(x,\tau)+\nu)\,dx\Big]^{\frac{2(p-1)}{p}}|B|^{\frac{2-p}{p}}\Big[\Big(\frac{\rho}{2}\Big)^{ps}\int_{B^c}\frac{u_-^{p-1}(y,\tau)}{|y-x_0|^{N+ps}}\,dy\Big] \\
&\leqslant \frac{\gamma\rho^{s-ps-\frac{\lambda_1}{p}}}{(1-\sigma')^{N+ps}\nu}\Big[\int_B u(x,\tau)\,dx+\nu\rho^N\Big]^{\frac{2(p-1)}{p}}\mathrm{Tail}\Big(u_-,x_0,\frac{\rho}{2},0,t\Big)^{p-1}.
\end{aligned}
\tag{4.13}
$$

Plugging (4.12) and (4.13) back into (4.10), and recalling the definition of ν, we get

$$
I_5 \leqslant \frac{\gamma\rho^{s-\frac{\lambda_1}{p}}}{(1-\sigma')^{N+ps}t}\Big[\int_B u(x,\tau)\,dx+\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}}\Big]^{\frac{2(p-1)}{p}}P_-, \tag{4.14}
$$

where $P_-\geqslant 1$ is defined by (4.1). Note that (4.9) and (4.14) provide us with homogeneous estimates of I_4 and I_5, respectively (except for the constants and the perturbation P_-)), so we have for all $\tau\in(0,t)$

$$
\begin{aligned}
I_3 &= I_4+I_5 \\
&\leqslant -\gamma\iint_{A_\tau^+\cap\{\xi(x)>\xi(y)\}}\frac{(u(x,\tau)-u(y,\tau))^p}{|x-y|^{N+ps}}\Big[u(x,\tau)+\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}}\Big]^{-\frac{2}{p}}\xi^p(x)\,dx\,dy \\
&+\frac{\gamma\rho^{s-\frac{\lambda_1}{p}}}{t}\max\Big\{\frac{1}{(\sigma'-\sigma)^p},\frac{1}{(1-\sigma')^{N+ps}}\Big\}\Big[\int_B u(x,\tau)\,dx+\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}}\Big]^{\frac{2(p-1)}{p}}P_-.
\end{aligned}
$$

A time integration yields

$$
\begin{aligned}
I_2 &\geqslant \gamma\int_0^t\tau^{\frac{1}{p}}\iint_{A_\tau^+\cap\{\xi(x)>\xi(y)\}}\frac{(u(x,\tau)-u(y,\tau))^p}{|x-y|^{N+ps}}\Big[u(x,\tau)+\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}}\Big]^{-\frac{2}{p}}\xi^p(x)\,dx\,dy\,d\tau \\
&-\gamma\rho^s\max\Big\{\frac{1}{(\sigma'-\sigma)^p},\frac{1}{(1-\sigma')^{N+ps}}\Big\}\Big[\sup_{0<\tau<t}\int_B u(x,\tau)\,dx+\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}}\Big]^{\frac{2(p-1)}{p}}\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{p}}P_-.
\end{aligned}
\tag{4.15}
$$

Finally, we combine (4.2), (4.3), and (4.15) and reduce the space integration domain in the first term, to find

$$
\begin{aligned}
0 &\geqslant I_1+I_2 \\
&\geqslant \gamma\int_0^t\tau^{\frac{1}{p}}\iint_{A_\tau}\frac{(u(x,\tau)-u(y,\tau))^p}{|x-y|^{N+ps}}\Big[u(x,\tau)+\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}}\Big]^{-\frac{2}{p}}\xi^p(x)\,dx\,dy\,d\tau \\
&-\gamma\rho^s\max\Big\{\frac{1}{(\sigma'-\sigma)^p},\frac{1}{(1-\sigma')^{N+ps}}\Big\}\Big[\sup_{0<\tau<t}\int_B u(x,\tau)\,dx+\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}}\Big]^{\frac{2(p-1)}{p}}\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{p}}P_-,
\end{aligned}
$$

which yields the conclusion, with a different $\gamma>0$ depending on the data and on γ'. □

We can now prove the L^1-L^1 estimate:

Proof of Theorem 1.2 (conclusion). We perform an increasing iteration by setting for all $n\in\mathbb{N}$

$$
\rho_n=\rho\sum_{i=0}^n\frac{1}{2^i},\ \check{\rho}_n=\frac{3\rho_n+\rho_{n+1}}{4},\ \hat{\rho}_n=\frac{\rho_n+3\rho_{n+1}}{4},
$$

so that $\rho_n < \check{\rho}_n < \hat{\rho}_n < \rho_{n+1}$, with $\rho_0 = \rho$ and limit $\rho_\infty = 2\rho$ as $n \to \infty$. Also set

$$B_n = B_{\rho_n}(x_0),\ \ \check{B}_n = B_{\check{\rho}_n}(x_0),\ \ \hat{B}_n = B_{\hat{\rho}_n}(x_0),$$

so $B_n \subset \check{B}_n \subset \hat{B}_n \subset B_{n+1}$, and $B_\infty = B_{2\rho}(x_0)$. For all $n \in \mathbb{N}$ let $t_1 \in [0, t]$ be s.t.

$$\int_{B_n} u(x, t_1)\, dx = \sup_{0<\tau<t} \int_{B_n} u(x, \tau)\, dx = S_n,$$

noting as well that (S_n) is a bounded sequence in $[0, +\infty)$ as $u \in C(0, T, L^1(B_{2\rho}(x_0))$ (see Definition 2.1). Also, we can find $t_2 \in [0, t]$, independent of n, s.t.

$$\int_{B_\infty} u(x, t_2)\, dx = \inf_{0<\tau<t} \int_{B_\infty} u(x, \tau)\, dx = J_1.$$

Let us fix $n \in \mathbb{N}$. Without loss of generality, we shall assume henceforth that $0 \leqslant t_1 \leqslant t_2 \leqslant t$. In addition, we pick $\xi_n \in C_c^\infty(\check{B}_n)$ s.t.

$$\xi_n = 1 \text{ in } B_n,\ 0 \leqslant \xi_n \leqslant 1,\ \ |\nabla \xi_n| \leqslant \frac{\gamma 2^n}{\rho} \text{ in } \mathbb{R}^N.$$

We use $\xi_n^{p+1} \in C_c^1(\check{B}_n)$ as a (stationary) test function for (1.2) in the cylinder $\check{B}_n \times (t_1, t_2)$, to get

$$\begin{aligned} 0 = &\int_{\check{B}_n} u(x, \tau)\xi_n^{p+1}(x)\, dx \Big|_{t_1}^{t_2} \\ &+ \int_{t_1}^{t_2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} |u(x, \tau) - u(y, \tau)|^{p-2}(u(x, \tau) \\ &- u(y, \tau))(\xi_n^{p+1}(x) - \xi_n^{p+1}(y))K(x, y, t)\, dx\, dy\, d\tau. \end{aligned}$$

This in turn, by the properties of ξ_n, implies

$$\begin{aligned} S_n &\leqslant \int_{\check{B}_n} u(x, t_1)\xi_n^{p+1}(x)\, dx \\ &\leqslant \int_{B_\infty} u(x, t_2)\, dx + \gamma \int_{t_1}^{t_2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x, \tau) - u(y, \tau)|^{p-2}(u(x, \tau) - u(y, \tau))}{|x - y|^{N+ps}} \\ &\quad (\xi_n^{p+1}(x) - \xi_n^{p+1}(y))\, dx\, dy\, d\tau \\ &= J_1 + J_2. \end{aligned} \tag{4.16}$$

We focus on the diffusive term J_2. First, using symmetry and the properties of ξ_n, we split such term into three integrals:

$$\begin{aligned}
J_2 &\leqslant \gamma \int_{t_1}^{t_2} \iint_{\hat{B}_n \times \hat{B}_n} \frac{|u(x,\tau) - u(y,\tau)|^{p-2}(u(x,\tau) - u(y,\tau))}{|x-y|^{N+ps}} (\xi_n^{p+1}(x) - \xi_n^{p+1}(y))\, dx\, dy\, d\tau \\
&\quad + 2\gamma \int_{t_1}^{t_2} \iint_{\hat{B}_n \times \hat{B}_n^c} \frac{|u(x,\tau) - u(y,\tau)|^{p-2}(u(x,\tau) - u(y,\tau))}{|x-y|^{N+ps}} \xi_n^{p+1}(x)\, dx\, dy\, d\tau \\
&\leqslant \gamma \int_{t_1}^{t_2} \iint_{\hat{B}_n \times \hat{B}_n^c} \frac{|u(x,\tau) - u(y,\tau)|^{p-2}(u(x,\tau) - u(y,\tau))}{|x-y|^{N+ps}} (\xi_n^{p+1}(x) - \xi_n^{p+1}(y))\, dx\, dy\, d\tau \\
&\quad + \gamma \int_{t_1}^{t_2} \iint_{\hat{B}_n \times \hat{B}_n^c} \frac{u^{p-1}(x,\tau)}{|x-y|^{N+ps}} \xi_n^{p+1}(x)\, dx\, dy\, d\tau \\
&\quad + \gamma \int_{t_1}^{t_2} \iint_{\hat{B}_n \times \hat{B}_n^c} \frac{u_-^{p-1}(y,\tau)}{|x-y|^{N+ps}} \xi_n^{p+1}(x)\, dx\, dy\, d\tau \\
&= J_3 + J_4 + J_5.
\end{aligned}$$

We are going to separately estimate J_3, J_4, and J_5. First let us deal with J_3. Note that, for all $\tau \in (0,t)$ and $(x,y) \in \hat{B}_n \times \hat{B}_n$, we have $u(y,\tau) \geqslant 0$, in addition the integrand in J_3 is positive iff the differences $u(x,\tau) - u(y,\tau)$, $\xi_n(x) - \xi_n(y)$ have the same sign. Therefore, exploiting also symmetry and expanding the time integration interval, we have

$$J_3 \leqslant \gamma \int_0^t \iint_{A_{n,\tau}} \frac{(u(x,\tau) - u(y,\tau))^{p-1}}{|x-y|^{N+ps}} (\xi_n^{p+1}(x) - \xi_n^{p+1}(y))\, dx\, dy\, d\tau,$$

where for all $\tau \in (0,t)$ we have set

$$A_{n,\tau} = \big\{(x,y) \in \hat{B}_n \times \hat{B}_n : u(x,\tau) > u(y,\tau),\ \xi_n(x) > \xi_n(y)\big\}.$$

Next, note that for all $(x,y) \in A_{n,\tau}$

$$0 \leqslant \xi_n^{p+1}(x) - \xi_n^{p+1}(y) \leqslant \frac{\gamma 2^n \xi_n^p(x)}{\rho} |x-y|,$$

with $\gamma > 0$ independent of n. We then set

$$\nu_n = \Big(\frac{t}{\rho_{n+1}^{ps}}\Big)^{\frac{1}{2-p}} > 0,$$

so by the previous estimate on the cut-off function and weighted Hölder's inequality we have

$$\begin{aligned} J_3 &\leqslant \frac{\gamma 2^n}{\rho}\int_0^t \iint_{A_{n,\tau}} \frac{(u(x,\tau)-u(y,\tau))^{p-1}}{|x-y|^{N+ps-1}}\xi_n^p(x)\,dx\,dy\,d\tau \\ &\leqslant \frac{\gamma 2^n}{\rho}\int_0^t \iint_{A_{n,\tau}} \Big[(u(x,\tau)-u(y,\tau))^{p-1}(u(x,\tau)+\nu_n)^{\frac{2(1-p)}{p^2}}\tau^{\frac{p-1}{p^2}}\Big] \\ &\quad\cdot\Big[(u(x,\tau)+\nu_n)^{\frac{2(p-1)}{p^2}}\tau^{\frac{1-p}{p^2}}|x-y|\Big]\frac{\xi_n^p(x)}{|x-y|^{N+ps}}\,dx\,dy\,d\tau \\ &\leqslant \frac{\gamma 2^n}{\rho}\Big[\int_0^t \tau^{\frac{1}{p}}\iint_{A_{n,\tau}} \frac{(u(x,\tau)-u(y,\tau))^{p}}{|x-y|^{N+ps}}(u(x,\tau)+\nu_n)^{-\frac{2}{p}}\xi_n^p(x)\,dx\,dy\,d\tau\Big]^{\frac{p-1}{p}} \\ &\quad\cdot\Big[\int_0^t \tau^{\frac{1-p}{p}}\iint_{A_{n,\tau}} \frac{(u(x,\tau)+\nu_n)^{\frac{2(p-1)}{p}}}{|x-y|^{N+ps-p}}\xi_n^p(x)\,dx\,dy\,d\tau\Big]^{\frac{1}{p}} = \frac{\gamma 2^n}{\rho}J_6^{\frac{p-1}{p}}J_7^{\frac{1}{p}}. \end{aligned} \tag{4.17}$$

To estimate J_6, we apply Lemma 4.1 with the following choices:

$$\rho=\rho_{n+1},\ \sigma=\frac{\rho_n}{\rho_{n+1}},\ \sigma'=\frac{\check{\rho}_n}{\rho_{n+1}},\ \xi=\xi_n.$$

Note that $0<\sigma<\sigma'<1$ with differences estimated respectively by

$$\sigma'-\sigma=\frac{\rho_{n+1}-\rho_n}{4\rho_{n+1}}\geqslant\frac{1}{2^{n+4}},\quad 1-\sigma'=\frac{3\rho_{n+1}-3\rho_n}{4\rho_{n+1}}\geqslant\frac{3}{2^{n+4}}.$$

Also we have $\xi_n\in C_c^1(\check{B}_n)$, $\xi_n=1$ in B_n, and in all of $\mathbb{R}^N$

$$0\leqslant\xi_n\leqslant 1,\ |\nabla\xi_n|\leqslant\frac{\gamma'}{(\sigma'-\sigma)\rho},$$

for some $\gamma'>0$ depending on the data. In addition, $A_{n,\tau}$ coincides with the domain A_τ of Lemma 4.1 for all $\tau\in(0,t)$. Therefore, recalling that ρ_{n+1} and ρ are comparable via numerical constants, we have

$$\begin{aligned} J_6 &\leqslant \gamma\rho_{n+1}^s \max\big\{2^{pn},\,2^{(N+ps)n}\big\}\Big[\sup_{0<\tau<t}\int_{B_{n+1}} u(x,\tau)\,dx+\Big(\frac{t}{\rho_{n+1}^{\lambda_1}}\Big)^{\frac{1}{2-p}}\Big]^{\frac{2(p-1)}{p}}\Big(\frac{t}{\rho_{n+1}^{\lambda_1}}\Big)^{\frac{1}{p}} \\ &\quad\cdot\max\Big\{1,\,\Big(\frac{t}{\rho_{n+1}^{ps}}\Big)^{\frac{1-p}{2-p}}\mathrm{Tail}\Big(u_-,x_0,\frac{\rho_{n+1}}{2},0,t\Big)^{p-1}\Big\} \\ &\leqslant \gamma b^n\rho^s\Big[S_{n+1}+\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}}\Big]^{\frac{2(p-1)}{p}}\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{p}}P_-, \end{aligned} \tag{4.18}$$

where P_- is defined by (4.1) and $b > 1$ depends on the data. For J_7 we have the following estimate:

$$J_7 \leqslant \Big[\int_0^t \tau^{\frac{1-p}{p}}\,d\tau\Big]\Big[\sup_{0<\tau<t}\int_{B_{n+1}}(u(x,\tau)+\nu_n)^{\frac{2(p-1)}{p}}\,dx\Big]\Big[\sup_{x\in B_{n+1}}\int_{B_{n+1}}\frac{dy}{|x-y|^{N+ps-p}}\Big] \tag{4.19}$$

$$\leqslant \gamma t^{\frac{1}{p}}\Big[\sup_{0<\tau<t}\int_{B_{n+1}}(u(x,\tau)+\nu_n)\,dx\Big]^{\frac{2(p-1)}{p}}|B_{n+1}|^{\frac{2-p}{p}}\Big[\int_{B_{4\rho(0)}}\frac{dz}{|z|^{N+ps-p}}\Big]$$

$$\leqslant \gamma t^{\frac{1}{p}}\rho^{\frac{N(2-p)}{p}+p-ps}\Big[\sup_{0<\tau<t}\int_{B_{n+1}}u(x,\tau)\,dx+\nu_n\rho_{n+1}^N\Big]^{\frac{2(p-1)}{p}}$$

$$\leqslant \gamma\rho^{p+s-ps}\Big[S_{n+1}+\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}}\Big]^{\frac{2(p-1)}{p}}\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{p}}.$$

Plugging (4.18) and (4.19) back into (4.17), we find $\gamma, b > 1$ depending on the data s.t.

$$J_3 \leqslant \gamma b^n\Big[S_{n+1}+\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}}\Big]^{\frac{2(p-1)}{p}}\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{p}}P_-^{\frac{p-1}{p}}$$

Next we apply Young's inequality (2.2) with $q = p/(2(p-1)) > 1$ and $\varepsilon \in (0,1)$ to be determined later:

$$J_3 \leqslant \varepsilon\Big[S_{n+1}+\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}}\Big]+\gamma_\varepsilon b^n\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}}P_-^{\frac{p-1}{2-p}}, \tag{4.20}$$

with $\gamma_\varepsilon \to \infty$ as $\varepsilon \to 0$ and $b > 1$ depending on the data. The integrals J_4, J_5 are easier to deal with. To estimate J_4, we first recall that $\xi_n = 0$ in $\check{B}_n^c$ and $0 \leqslant \xi_n \leqslant 1$ in all of $\mathbb{R}^N$, hence

$$J_4 \leqslant \gamma\int_0^t\iint_{\check{B}_n\times\hat{B}_n^c}\frac{u^{p-1}(x,\tau)}{|x-y|^{N+ps}}\,dx\,dy\,d\tau.$$

Besides, we note that for all $x \in \check{B}_n$, $y \in \hat{B}_n^c$ we have

$$|x-y| \geqslant |y-x_0|-|x-x_0| \geqslant \frac{|y-x_0|}{\gamma 2^n}.$$

So, using both Hölder's and Young's inequalities (2.2) with $q = 1/(p-1) > 1$ and $\varepsilon \in (0,1)$ to be determined, we get

$$J_4 \leqslant \gamma t\Big[\sup_{0<\tau<t}\int_{\check{B}_n}u^{p-1}(x,\tau)\,dx\Big]\Big[\sup_{x\in\check{B}_n}\int_{\hat{B}_n^c}\frac{dy}{|x-y|^{N+ps}}\Big] \tag{4.21}$$

$$\leqslant \gamma 2^{(N+ps)n}t\Big[\sup_{0<\tau<t}\int_{B_{n+1}}u^{p-1}(x,\tau)\,dx\Big]\Big[\int_{B_\rho^c(0)}\frac{dz}{|z|^{N+ps}}\Big]$$

$$\leqslant \frac{\gamma b^n t}{\rho^{ps}}\Big[\sup_{0<\tau<t}\int_{B_{n+1}}u(x,\tau)\,dx\Big]^{p-1}|B_{n+1}|^{2-p}$$

$$\leqslant \frac{\gamma b^n t}{\rho^{\lambda_1}}S_{n+1}^{p-1} \leqslant \varepsilon S_{n+1}+\gamma_\varepsilon b^n\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}},$$

again with $\gamma_\varepsilon \to \infty$ as $\varepsilon \to 0$ and $b > 1$ depending on the data. The estimate of J_5 begins as above, but involves a further tail term:

$$\begin{aligned} J_5 &\leqslant \gamma \int_0^t \iint_{\check{B}_n \times \hat{B}_n^c} \frac{u_-^{p-1}(y,\tau)}{|x-y|^{N+ps}}\, dx\, dy\, d\tau \\ &\leqslant \gamma 2^{(N+ps)n} t |\check{B}_n| \Big[\sup_{0<\tau<t} \int_{\hat{B}_n^c} \frac{u_-^{p-1}(y,\tau)}{|y-x_0|^{N+ps}}\, dy \Big] \\ &\leqslant \gamma b^n t \rho^{N-ps} \mathrm{Tail}\Big(u_-, x_0, \frac{\hat{\rho}_n}{2}, 0, t\Big)^{p-1} \\ &\leqslant \gamma b^n \Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}} P_-, \end{aligned} \tag{4.22}$$

where we used comparability of ρ and $\hat{\rho}_n$, and as usual $b > 1$ only depends on the data. Next we gather all estimates from (4.20), (4.21), and (4.22) to find the following estimate for J_2, holding for $\varepsilon \in (0, 1)$ to be determined, $\gamma_\varepsilon \to \infty$ as $\varepsilon \to 0$, and $b > 1$ depending on the data:

$$J_2 \leqslant \varepsilon S_{n+1} + \gamma_\varepsilon b^n \Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}} \big(P_- + P_-^{\frac{p-1}{2-p}}\big),$$

where we have used $P_- \geqslant 1$, but we could not choose a unique exponent for P_- since $(p-1)/(2-p)$ spans the whole interval $(0, \infty)$ for $p \in (1, 2)$. Now (4.16) yields an iterative bound for the sequence (S_n):

$$\begin{aligned} S_n &\leqslant J_1 + \varepsilon S_{n+1} + \gamma_\varepsilon b^n \Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}} \big(P_- + P_-^{\frac{p-1}{2-p}}\big) \\ &\leqslant \varepsilon S_{n+1} + \gamma_\varepsilon b^n \Big[J_1 + \Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}} \big(P_- + P_-^{\frac{p-1}{2-p}}\big) \Big]. \end{aligned} \tag{4.23}$$

Iterating on (4.23) from 0 to n, we find

$$\begin{aligned} S_0 &\leqslant \varepsilon S_1 + \gamma \Big[J_1 + \Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}} \big(P_- + P_-^{\frac{p-1}{2-p}}\big) \Big] \\ &\leqslant \varepsilon^2 S_2 + \gamma(1+\varepsilon b) \Big[J_1 + \Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}} \big(P_- + P_-^{\frac{p-1}{2-p}}\big) \Big] \dots \\ &\leqslant \varepsilon^n S_n + \gamma \sum_{i=0}^{n-1} (\varepsilon b)^i \Big[J_1 + \Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}} \big(P_- + P_-^{\frac{p-1}{2-p}}\big) \Big]. \end{aligned}$$

Choose now $\varepsilon \in (0, 1/b)$ so that the series on the right hand side converges, and recall that (S_n) is a bounded sequence. So, letting $n \to \infty$, we get

$$S_0 \leqslant \gamma \Big[J_1 + \Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}} \big(P_- + P_-^{\frac{p-1}{2-p}}\big) \Big],$$

which rephrases as

$$\sup_{0<\tau<t} \int_{B_\rho(x_0)} u(x,\tau)\, dx \leqslant \gamma \inf_{0<\tau<t} \int_{B_{2\rho}(x_0)} u(x,\tau)\, dx + \gamma \Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}} \big(P_- + P_-^{\frac{p-1}{2-p}}\big),$$

yielding the conclusion as soon as we recall the definition of P_-. □

Combining Theorems 1.1 and 1.2, it is not difficult to obtain a $L^1 - L^\infty$ estimate, provided the solution u is locally bounded in addition to the previous assumptions:

Proof of Theorem 1.3. First, from Theorem 1.2 we derive the following inequality involving the mean values of u in different balls:

$$\sup_{0<\tau<t} \fint_{B_\rho(x_0)} u(x,\tau)\,dx \leqslant \gamma\Big[\inf_{0<\tau<t} \fint_{B_{2\rho}(x_0)} u(x,\tau)\,dx\Big] + \frac{\gamma}{\rho^N}\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}}\big(P_- + P_-^{\frac{p-1}{2-p}}\big) \tag{4.24}$$

$$\leqslant \gamma\Big[\inf_{0<\tau<t} \fint_{B_{2\rho}(x_0)} u(x,\tau)\,dx\Big] + \gamma\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}}\big(P_- + P_-^{\frac{p-1}{2-p}}\big),$$

where γ depends on the data. Besides, since $\lambda_1 > 0$ and u is locally bounded, we may apply Theorem 1.1 with $r = 1$:

$$\sup_{B_{\rho/2}(x_0)\times(t/2,t)} u \leqslant \gamma\Big[\fint\!\!\fint_{B_\rho(x_0)\times(0,t)} u(x,\tau)\,dx\,d\tau\Big]^{\frac{ps}{\lambda_1}}\Big(\frac{t}{\rho^{ps}}\Big)^{-\frac{N}{\lambda_1}} + \gamma\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}} P_+, \tag{4.25}$$

with a possibly bigger $\gamma > 0$ depending on the data. Concatenating (4.24) and (4.25) and changing the constant γ conveniently, we find

$$\begin{aligned}
\sup_{B_{\rho/2}(x_0)\times(t/2,t)} u &\leqslant \gamma\Big[\sup_{0<\tau<t} \fint_{B_\rho(x_0)} u(x,\tau)\,dx\Big]^{\frac{ps}{\lambda_1}}\Big(\frac{t}{\rho^{ps}}\Big)^{-\frac{N}{\lambda_1}} + \gamma\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}} P_+ \\
&\leqslant \gamma\Big[\inf_{0<\tau<t} \fint_{B_{2\rho}(x_0)} u(x,\tau)\,dx \\
&\quad + \Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}}\big(P_- + P_-^{\frac{p-1}{2-p}}\big)\Big]^{\frac{ps}{\lambda_1}}\Big(\frac{t}{\rho^{ps}}\Big)^{-\frac{N}{\lambda_1}} + \gamma\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}} P_+ \\
&\leqslant \gamma\Big[\inf_{0<\tau<t} \fint_{B_{2\rho}(x_0)} u(x,\tau)\,dx\Big]^{\frac{ps}{\lambda_1}}\Big(\frac{t}{\rho^{ps}}\Big)^{-\frac{N}{\lambda_1}} \\
&\quad + \gamma\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{ps}{(2-p)\lambda_1} - \frac{N}{\lambda_1}}\big(P_- + P_-^{\frac{p-1}{2-p}}\big)^{\frac{ps}{\lambda_1}} + \gamma\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}} P_+ \\
&\leqslant \gamma\Big[\inf_{0<\tau<t} \int_{B_{2\rho}(x_0)} u(x,\tau)\,dx\Big]^{\frac{ps}{\lambda_1}} t^{-\frac{N}{\lambda_1}} \\
&\quad + \gamma\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}}\big(P_+ + P_-^{\frac{ps}{\lambda_1}} + P_-^{\frac{(p-1)ps}{(2-p)\lambda_1}}\big),
\end{aligned}$$

which concludes the proof. □

It is worth pointing out the following special cases of Theorems 1.2 and 1.3 for *globally non-negative* solutions (for which $P_- = 1$):

Corollary 4.2 (Globally non-negative solutions) *Let u be a solution of (1.2), s.t. $u \geqslant 0$ in $\mathbb{R}^N \times (0,T)$, and $B_{4\rho}(x_0) \times (0,t) \subset \Omega_T$. There exists a constant $\gamma > 0$ depending only on the data s.t.*

(i) for all $1 < p < 2$ and λ_1 of any sign,

$$\sup_{0<\tau<t} \int_{B_\rho(x_0)} u(x,\tau)\,dx \leqslant \gamma\Big[\inf_{0<\tau<t} \int_{B_{2\rho}(x_0)} u(x,\tau)\,dx\Big] + \gamma\Big(\frac{t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}};$$

(ii) *if* $\lambda_1 > 0$ *and* u *is locally bounded, then*

$$\sup_{B_{\rho/2}(x_0)\times(t/2,t)} u \leqslant \gamma\Big[\inf_{0<\tau<t}\int_{B_{2\rho}(x_0)} u(x,\tau)\,dx\Big]^{\frac{ps}{\lambda_1}} t^{-\frac{N}{\lambda_1}} + \gamma\Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1}{2-p}} \max\Big\{1, \Big(\frac{t}{\rho^{ps}}\Big)^{\frac{1-p}{2-p}} \mathrm{Tail}\Big(u, x_0, \frac{\rho}{2}, 0, t\Big)^{p-1}\Big\}.$$

5 Backward $L^r - L^r$ estimate

In this section we prove Theorem 1.4. With this aim in mind, we assume u to be a locally bounded solution of (1.2) s.t. $u \geqslant 0$ in $B_{4\rho}(x_0)\times(0,t) \subset \Omega_T$, and we assume that the right-hand side is finite, i.e

$$\sup_{0<\tau<t}\int_{B^c_\rho(x_0)} u^r_+(x,\tau)\,dx + \int_{B_{2\rho}(x_0)} u^r(x,0)\,dx + \Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}} < \infty,$$

otherwise there is nothing to prove. We argue in a dichotomic form, assuming

$$\sup_{0<\tau<t}\int_{B_\rho(x_0)} u^r(x,\tau)\,dx > \sup_{0<\tau<t}\int_{B^c_\rho(x_0)} u^r_+(x,\tau)\,dx. \tag{5.1}$$

The crucial step is the following technical lemma:

Lemma 5.1 *Let (5.1) hold,* $\sigma \in (0,1)$*. Then, there exists* $\gamma > 0$ *depending on the data and* r*, s.t.*

$$\sup_{0<\tau<t}\int_{B_\rho(x_0)} u^r(x,\tau)\,dx \leqslant \int_{B_{(1+\sigma)\rho}(x_0)} u^r(x,0)\,dx + \frac{\gamma}{\sigma^{N+ps}}\Big[\sup_{0<\tau<t}\int_{B_{(1+\sigma)\rho}(x_0)} u^r(x,\tau)\,dx\Big]^{\frac{p+r-2}{r}} \Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}}.$$

Proof Fix $t_0 \in (0,t)$ and set for brevity

$$B = B_\rho(x_0),\ \tilde{B} = B_{(1+\sigma/2)\rho}(x_0),\ \hat{B} = B_{(1+\sigma)\rho}(x_0),$$

so that $B \subset \tilde{B} \subset \hat{B}$. Also, let $\xi \in C^\infty_c(\tilde{B})$ be a cutoff function s.t. $\xi = 1$ in B, $0 \leqslant \xi \leqslant 1$, $|\nabla\xi| \leqslant \gamma/(\sigma\rho)$ in $\mathbb{R}^N$. As in previous cases, the idea is to use as a test function in (1.2)

$$\varphi(x,\tau) = u^{r-1}(x,\tau)\xi^p(x),$$

but this is prevented by several reasons (non-differentiability in time, possible singularity if $r < 2$). Nevertheless, by applying a convenient mollification procedure (see Lemma A.3 below), we can find $\gamma > 1$ depending on the data and r, s.t.

$$\begin{aligned} 0 &\geqslant \frac{1}{r}\int_{\tilde{B}} u^r(x,\tau)\xi^p(x)\,dx\Big|_0^{t_0} \\ &\quad + \frac{1}{\gamma}\int_0^{t_0}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x,\tau)-u(y,\tau)|^{p-2}(u(x,\tau)-u(y,\tau))}{|x-y|^{N+ps}} \\ &\quad \Big[u^{r-1}(x,\tau)\xi^p(x) - u^{r-1}(y,\tau)\xi^p(y)\Big]\,dx\,dy\,d\tau \\ &= I_1 + I_2 \end{aligned} \tag{5.2}$$

(note that multiplication by ξ^p selects non-negative values of u in all terms above). We first deal with the evolutive term I_1, by exploiting the properties of ξ:

$$I_1 \geqslant \frac{1}{r}\int_B u^r(x,t_0)\,dx - \frac{1}{r}\int_{\tilde{B}} u^r(x,0)\,dx. \tag{5.3}$$

We now turn to the diffusive term I_2. Set for all $\tau \in (0, t_0)$

$$A_\tau = \big\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : u(x,\tau) > u(y,\tau)\big\},$$
$$A_\tau^+ = \big\{(x,y) \in A_\tau : \xi(x) < \xi(y)\big\}.$$

Note that for all $(x, y) \in A_\tau$ two cases may occur:

(a) if $\xi(x) \geqslant \xi(y)$, then

$$u^{r-1}(x,\tau)\xi^p(x) \geqslant u^{r-1}(y,\tau)\xi^p(y),$$

in particular the integrand of I_2 becomes non-negative;

(b) if $\xi(x) < \xi(y)$ (i.e., $(x, y) \in A_\tau^+$), then we must have $y \in \tilde{B}$ and

$$u(x,\tau) > u(y,\tau) \geqslant 0.$$

We fix $\tau \in (0, t_0)$ and use symmetry to reduce the space integration domain to A_τ, then we recall (a) above to dismiss positive contributions, and (b) to separate u^{r-1} from the corresponding multiplier ξ^p:

$$\begin{aligned}
&\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x,\tau)-u(y,\tau)|^{p-2}(u(x,\tau)-u(y,\tau))}{|x-y|^{N+ps}} \\
&\quad\Big[u^{r-1}(x,\tau)\xi^p(x) - u^{r-1}(y,\tau)\xi^p(y)\Big]\,dx\,dy \\
&\geqslant 2\iint_{A_\tau^+} \frac{(u(x,\tau)-u(y,\tau))^{p-1}}{|x-y|^{N+ps}}\Big[u^{r-1}(x,\tau)\xi^p(x) - u^{r-1}(y,\tau)\xi^p(y)\Big]\,dx\,dy \\
&= 2\iint_{A_\tau^+} \frac{(u(x,\tau)-u(y,\tau))^{p-1}}{|x-y|^{N+ps}} u^{r-1}(x,\tau)(\xi^p(x)-\xi^p(y))\,dx\,dy \\
&\quad + 2\iint_{A_\tau^+} \frac{(u(x,\tau)-u(y,\tau))^{p-1}}{|x-y|^{N+ps}} (u^{r-1}(x,\tau)-u^{r-1}(y,\tau))\xi^p(y)\,dx\,dy \\
&= I_3 + I_4.
\end{aligned}$$

To estimate I_3 we use inequality (2.3) with $a = \xi(y) > \xi(x) = b, \delta \in (0, 1)$ to be determined, and

$$\varepsilon = \delta\,\frac{u(x,\tau)-u(y,\tau)}{u(x,\tau)} \in (0,1),$$

so we get the following pointwise estimate for all $(x, y) \in A_\tau^+$:

$$\xi^p(y) - \xi^p(x) \leqslant \delta\,\frac{u(x,\tau)-u(y,\tau)}{u(x,\tau)}\xi^p(y) + \frac{\gamma}{\delta^{p-1}}\,\frac{u^{p-1}(x,\tau)}{(u(x,\tau)-u(y,\tau))^{p-1}}(\xi(y)-\xi(x))^p.$$

Reversing the inequality above, and recalling that $u(x, \tau) > 0$, we get

$$\begin{aligned}
I_3 \geqslant &-\delta \iint_{A_\tau^+} \frac{(u(x,\tau)-u(y,\tau))^p}{|x-y|^{N+ps}} u^{r-2}(x,\tau)\xi^p(y)\,dx\,dy \\
&- \frac{\gamma}{\delta^{p-1}} \iint_{A_\tau^+} u^{p+r-2}(x,\tau)\frac{(\xi(y)-\xi(x))^p}{|x-y|^{N+ps}}\,dx\,dy.
\end{aligned} \tag{5.4}$$

To estimate I_4, we distinguish two cases:

(a) if $r \geqslant 2$, then we apply Lemma 2.7, concatenating (i) and (ii), with $q = r$, $a = u(x,\tau) > u(y,\tau) = b$, to get the pointwise estimate

$$
\begin{aligned}
&(u(x,\tau) - u(y,\tau))^{p-1}(u^{r-1}(x,\tau) - u^{r-1}(y,\tau)) \\
&\quad \geqslant \frac{1}{\gamma}(u(x,\tau) - u(y,\tau))^p (u(x,\tau) + u(y,\tau))^{r-2},
\end{aligned}
$$

which in turn produces

$$
\begin{aligned}
I_4 &\geqslant \frac{1}{\gamma} \iint_{A_\tau^+} \frac{(u(x,\tau) - u(y,\tau))^p}{|x-y|^{N+ps}} (u(x,\tau) + u(y,\tau))^{r-2} \xi^p(y)\, dx\, dy \\
&\geqslant \frac{1}{\gamma} \iint_{A_\tau^+} \frac{(u(x,\tau) - u(y,\tau))^p}{|x-y|^{N+ps}} u^{r-2}(x,\tau) \xi^p(y)\, dx\, dy;
\end{aligned}
$$

(b) if $1 < r < 2$, then we apply Lagrange's rule:

$$
\begin{aligned}
I_4 &\geqslant \frac{1}{\gamma} \iint_{A_\tau^+} \frac{(u(x,\tau) - u(y,\tau))^p}{|x-y|^{N+ps}} \min\left\{u^{r-2}(x,\tau),\, u^{r-2}(y,\tau)\right\} \xi^p(y)\, dx\, dy \\
&= \frac{1}{\gamma} \iint_{A_\tau^+} \frac{(u(x,\tau) - u(y,\tau))^p}{|x-y|^{N+ps}} u^{r-2}(x,\tau) \xi^p(y)\, dx\, dy.
\end{aligned}
$$

Not that the estimates in (a), (b) above coincide, up to a different constant $\gamma > 1$ depending on the data and r. Now choose $\delta \in (0,1)$ in (5.4) so small that the first term can be reabsorbed in the estimate of I_4. Subsequently, we dismiss positive contributions to get for all $\tau \in (0, t_0)$

$$
\begin{aligned}
&\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x,\tau) - u(y,\tau)|^{p-2}(u(x,\tau) - u(y,\tau))}{|x-y|^{N+ps}} \\
&\quad \left[u^{r-1}(x,\tau)\xi^p(x) - u^{r-1}(y,\tau)\xi^p(y)\right] dx\, dy \\
&\quad \geqslant \left(\frac{1}{\gamma} - \delta\right) \iint_{A_\tau^+} \frac{(u(x,\tau) - u(y,\tau))^p}{|x-y|^{N+ps}} u^{r-2}(x,\tau)\xi^p(y)\, dx\, dy \\
&\quad\quad - \frac{\gamma}{\delta^{p-1}} \iint_{A_\tau^+} u^{p+r-2}(x,\tau) \frac{(\xi(y) - \xi(x))^p}{|x-y|^{N+ps}}\, dx\, dy \\
&\quad \geqslant -\gamma \iint_{A_\tau^+} u^{p+r-2}(x,\tau) \frac{(\xi(y) - \xi(x))^p}{|x-y|^{N+ps}}\, dx\, dy.
\end{aligned}
$$

Next integrate in time and recall that $y \in \tilde{B}$ for all $(x,y) \in A_\tau^+$:

$$
\begin{aligned}
I_2 &\geqslant -\gamma \int_0^{t_0} \iint_{A_\tau^+} u^{p+r-2}(x,\tau) \frac{(\xi(y) - \xi(x))^p}{|x-y|^{N+ps}}\, dx\, dy\, d\tau \\
&\geqslant -\gamma \int_0^{t_0} \iint_{\hat{B} \times \tilde{B}} u^{p+r-2}(x,\tau) \frac{(\xi(y) - \xi(x))^p}{|x-y|^{N+ps}}\, dx\, dy\, d\tau \\
&\quad - \gamma \int_0^{t_0} \iint_{\hat{B}^c \times \tilde{B}} u_+^{p+r-2}(x,\tau) \frac{(\xi(y) - \xi(x))^p}{|x-y|^{N+ps}}\, dx\, dy\, d\tau \\
&= I_5 + I_6.
\end{aligned} \tag{5.5}
$$

To estimate I_5, we apply the properties of ξ:

$$\begin{aligned}
I_5 &\geqslant -\frac{\gamma}{\sigma^p\rho^p}\int_0^{t_0}\iint_{\hat{B}\times\tilde{B}}\frac{u^{p+r-2}(x,\tau)}{|x-y|^{N+ps}}\,dx\,dy\,d\tau\\
&\geqslant -\frac{\gamma t}{\sigma^p\rho^p}\Big[\sup_{0<\tau<t}\int_{\hat{B}}u^{p+r-2}(x,\tau)\,dx\Big]\Big[\sup_{x\in\hat{B}}\int_{\tilde{B}}\frac{dy}{|x-y|^{N+ps-p}}\Big]\\
&\geqslant -\frac{\gamma t}{\sigma^p\rho^p}\Big[\sup_{0<\tau<t}\int_{\hat{B}}u^r(x,\tau)\,dx\Big]^{\frac{p+r-2}{r}}|\hat{B}|^{\frac{2-p}{r}}\Big[\int_{B_{2\rho}(0)}\frac{dz}{|z|^{N+ps-p}}\Big]\\
&\geqslant -\frac{\gamma}{\sigma^p}\Big[\sup_{0<\tau<t}\int_{\hat{B}}u^r(x,\tau)\,dx\Big]^{\frac{p+r-2}{r}}\Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}}.
\end{aligned}$$

The estimate of I_6 is more delicate. First note that for all $(x,y)\in\hat{B}^c\times\tilde{B}$

$$|x-y|\geqslant|x-x_0|-|y-x_0|\geqslant\frac{\sigma}{2+2\sigma}|x-x_0|,$$

so by boundedness of ξ and weighted Hölder's inequality we have

$$\begin{aligned}
I_6 &\geqslant -\gamma\Big(\frac{2+2\sigma}{\sigma}\Big)^{N+ps}\int_0^t\iint_{\hat{B}^c\times\tilde{B}}\frac{u_+^{p+r-2}(x,\tau)}{|x-x_0|^{N+ps}}\,dx\,dy\,d\tau\\
&\geqslant -\frac{\gamma t}{\sigma^{N+ps}}|\tilde{B}|\Big[\sup_{0<\tau<t}\int_{\hat{B}^c}\frac{u_+^{p+r-2}(x,\tau)}{|x-x_0|^{N+ps}}\,dx\Big]\\
&\geqslant -\frac{\gamma\rho^N t}{\sigma^{N+ps}}\Big[\sup_{0<\tau<t}\int_{\hat{B}^c}u_+^r(x,\tau)\,dx\Big]^{\frac{p+r-2}{r}}\Big[\int_{\hat{B}^c}|x-x_0|^{-\frac{(N+ps)r}{2-p}}\,dx\Big]^{\frac{2-p}{r}}\\
&\geqslant -\frac{\gamma}{\sigma^{N+ps}}\Big[\sup_{0<\tau<t}\int_{B^c}u_+^r(x,\tau)\,dx\Big]^{\frac{p+r-2}{r}}\Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}}.
\end{aligned}$$

Now we recall (5.1) to see that

$$I_6\geqslant-\frac{\gamma}{\sigma^{N+ps}}\Big[\sup_{0<\tau<t}\int_B u^r(x,\tau)\,dx\Big]^{\frac{p+r-2}{r}}\Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}}.$$

Plug these estimates into (5.5), recalling that $\sigma^p>\sigma^{N+ps}$ and $B\subset\hat{B}$, to find the following estimate of the diffusive term:

$$I_2\geqslant-\frac{\gamma}{\sigma^{N+ps}}\Big[\sup_{0<\tau<t}\int_{\hat{B}}u^r(x,\tau)\,dx\Big]^{\frac{p+r-2}{r}}\Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}}. \tag{5.6}$$

Finally, we concatenate (5.2) with (5.3) and (5.6):

$$\frac{1}{r}\int_B u^r(x,t_0)\,dx\leqslant\frac{1}{r}\int_{\hat{B}}u^r(x,0)\,dx-\frac{\gamma}{\sigma^{N+ps}}\Big[\sup_{0<\tau<t}\int_{\hat{B}}u^r(x,\tau)\,dx\Big]^{\frac{p+r-2}{r}}\Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}},$$

which yields the conclusion as soon as we multiply by r and take the supremum over $t_0\in(0,t)$. ◻

We can now complete the proof of the main result:

Proof of Theorem 1.4 (conclusion). We assume that (5.1), otherwise there is nothing to prove. We are going to perform an iteration, setting $\sigma_0 = 0$, $\rho_0 = \rho$, and for all $n \geqslant 1$

$$\sigma_n = \sum_{i=1}^{n} \frac{1}{2^i}, \ \ \rho_n = (1 + \sigma_n)\rho,$$

so that (σ_n), (ρ_n) are increasing with $\sigma_n \to 1$, $\rho_n \to 2\rho$. Besides, we set $B_n = B_{\rho_n}(x_0)$, hence $B_n \subset B_{n+1}$. We fix $n \geqslant 0$ and apply Lemma 5.1 with $\rho = \rho_n$ and

$$\sigma = \frac{\sigma_{n+1} - \sigma_n}{1 + \sigma_n} \in (0, 1),$$

implying $(1 + \sigma)\rho = \rho_{n+1}$. Note that, by (5.1) and the inclusion $B_0 \subset B_n$, we have

$$\begin{aligned}\sup_{0<\tau<t} \int_{B_n} u^r(x,\tau)\,dx &\geqslant \sup_{0<\tau<t} \int_{B_0} u^r(x,\tau)\,dx \\ &\geqslant \sup_{0<\tau<t} \int_{B_0^c} u^r(x,\tau)\,dx \geqslant \sup_{0<\tau<t} \int_{B_n^c} u^r(x,\tau)\,dx.\end{aligned}$$

Starting from the inequality of Lemma 5.1, and exploiting the definitions above, we get

$$\begin{aligned}\sup_{0<\tau<t} \int_{B_n} u^r(x,\tau)\,dx \leqslant& \int_{B_{n+1}} u^r(x,0)\,dx \\ &+ \gamma\Big(\frac{1+\sigma_n}{\sigma_{n+1}-\sigma_n}\Big)^{N+ps}\Big[\sup_{0<\tau<t}\int_{B_{n+1}} u^r(x,\tau)\,dx\Big]^{\frac{p+r-2}{r}}\Big(\frac{t^r}{\rho_n^{\lambda_r}}\Big)^{\frac{1}{r}} \qquad (5.7)\\ \leqslant& \Big[\int_{B_{n+1}} u^r(x,0)\,dx\Big]^{\frac{p+r-2}{r}}\Big[\int_{B_{2\rho}(x_0)} u^r(x,0)\,dx\Big]^{\frac{2-p}{r}} \\ &+ \gamma 2^{(N+ps)n}\Big[\sup_{0<\tau<t}\int_{B_{n+1}} u^r(x,\tau)\,dx\Big]^{\frac{p+r-2}{r}}\Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}} \\ \leqslant& \gamma 2^{(N+ps)n}\Big[\sup_{0<\tau<t}\int_{B_{n+1}} u^r(x,\tau)\,dx\Big]^{\frac{p+r-2}{r}} \\ &\Big\{\Big[\int_{B_{2\rho}(x_0)} u^r(x,0)\,dx\Big]^{\frac{2-p}{r}} + \Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}}\Big\}, \qquad (5.8)\end{aligned}$$

with $\gamma > 0$ independent of n. We are going to apply Lemma 2.6 to the sequence

$$Y_n = \sup_{0<\tau<t} \int_{B_n} u^r(x,\tau)\,dx,$$

which is a bounded sequence of non-negative real numbers by assumption $u \in L^r(\mathbb{R}^N \times (0,T))$. We also set

$$b = 2^{N+ps}, \ \ \eta = \frac{2-p}{r}, \ \ C = \gamma\Big\{\Big[\int_{B_{2\rho}(x_0)} u^r(x,0)\,dx\Big]^{\frac{2-p}{r}} + \Big(\frac{t^r}{\rho^{\lambda_r}}\Big)^{\frac{1}{r}}\Big\},$$

so $b > 1$, $0 < \eta < 1$, $C > 0$, and (5.7) implies for all $n \in \mathbb{N}$

$$Y_n \leqslant C b^n Y_{n+1}^{1-\eta}.$$

Therefore, we have

$$Y_0 \leqslant \left(2Cb^{\frac{1-\eta}{\eta}}\right)^{\frac{1}{\eta}} \leqslant \gamma \int_{B_{2\rho}(x_0)} u^r(x,0)\,dx + \gamma\left(\frac{t^r}{\rho^{\lambda_r}}\right)^{\frac{1}{2-p}},$$

for some $\gamma > 0$ depending on the data and r. In view of (5.1), we conclude. □

6 Extinction time and decay estimates

In this section we deal with solutions of problem (1.3), set in a bounded domain Ω with a positive initial datum $u_0 \in W_0^{s,p}(\Omega)$. First we prove that any such solutions has a finite time of extinction, estimated in terms of the initial datum and possibly the domain's volume:

Proof of Theorem 1.5. First we consider case (i), i.e., $1 < p < p_c$. Then, an elementary calculation shows that

$$q = \frac{N(2-p)}{ps} > 2.$$

The first step consists in testing problem (1.3) with u^{q-1}, which of course is forbidden by the lack of time-differentiability of u. Choosing a convenient test function in Definition 2.2 and applying a mollification procedure (see Lemma A.4 (i)), we can prove that there exists $\gamma > 0$ depending on N, p, s s.t. for all $\psi \in W_0^{1,2}(0,T)$, $\psi \geqslant 0$ in $(0,T)$

$$\int_0^T \left[-\|u(\cdot,\tau)\|^q_{L^q(\Omega)}\psi'(\tau) + \frac{C_1}{\gamma}\|u(\cdot,\tau)\|^{p+q-2}_{L^q(\Omega)}\psi(\tau)\right] d\tau \leqslant 0, \tag{6.1}$$

where C_1 is as in (K_2). We rephrase (6.1) as follows. Set for all $t \in (0,T)$

$$U(t) = \|u(\cdot,t)\|^q_{L^q(\Omega)}.$$

Since the Sobolev weak time-derivative of $t \to \|u(\cdot,t)\|^q_{L^q(\Omega)}$ is bounded by (6.1), the function U is absolutely continuous on $[0,T]$ (up to the choice of a representative). Hence, from (6.1) we see that U satisfies the following ordinary differential inequality:

$$\begin{cases} U'(t) + \dfrac{C_1}{\gamma} U^{\frac{p+q-2}{q}}(t) \leqslant 0 & \text{a.e. in } (0,T) \\ U(0) = \|u_0\|^q_{L^q(\Omega)}. \end{cases} \tag{6.2}$$

The mapping

$$t \mapsto U^{\frac{2-p}{q}}(t) = U^{\frac{ps}{N}}(t)$$

is as well a.e. differentiable. So, integrating on (6.2), we have for all $t \in (0,T)$

$$\begin{aligned} U^{\frac{2-p}{q}}(t) - U^{\frac{2-p}{q}}(0) &= \frac{2-p}{q}\int_0^t U^{\frac{2-p-q}{q}}(\tau)U'(\tau)\,d\tau \\ &\leqslant -\frac{2-p}{q}\frac{C_1}{\gamma}\int_0^1 1\,d\tau = -\frac{C_1 t}{\gamma_*}, \end{aligned}$$

with $\gamma_* > 0$ depending on N, p, s. Equivalently, we have

$$\|u(\cdot,t)\|^{2-p}_{L^q(\Omega)} \leqslant \|u_0\|^{2-p}_{L^q(\Omega)} - \frac{C_1 t}{\gamma_*},$$

which implies that $u(\cdot,t)$ vanishes a.e. in Ω as soon as

$$t \geqslant \frac{\gamma_*}{C_1}\|u_0\|^{2-p}_{L^q(\Omega)}.$$

Recalling that u vanishes in Ω^c at any time, we obtain (i).

We now turn to case (ii). The proof goes essentially as in the previous case, except we replace the exponent q with 2. Note, in this connection, that $p \geqslant p_c$ implies both $\lambda_2 \geqslant 0$ and $p_s^* \geqslant 2$. By Hölder's inequality, then, we have for all $t \in (0,T)$

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leqslant |\Omega|^{\frac{\lambda_2}{2Np}}\|u(\cdot,t)\|_{L^{p_s^*}(\Omega)}.$$

Through a convenient testing procedure (see Lemma A.4 (ii)), we find $\gamma > 0$ depending on N, p, s s.t. for all $\psi \in W_0^{1,2}(0,T)$, $\psi \geqslant 0$ in $(0,T)$

$$\int_0^T \Big[-\|u(\cdot,\tau)\|^2_{L^2(\Omega)}\psi'(\tau) + \frac{C_1}{\gamma}|\Omega|^{-\frac{\lambda_2}{2N}}\|u(\cdot,\tau)\|^p_{L^2(\Omega)}\psi(\tau)\Big]\,d\tau \leqslant 0. \tag{6.3}$$

Reasoning as in the previous case, we set for all $t \in (0,T)$

$$U(t) = \|u(\cdot,t)\|^2_{L^2(\Omega)},$$

so again U is absolutely continuous in $[0,T]$ a.e. differentiable, and satisfies the ordinary differential inequality

$$\begin{cases} U'(t) + \dfrac{C_1}{\gamma}|\Omega|^{-\frac{\lambda_2}{2N}}U^{\frac{p}{2}}(t) \leqslant 0 \quad \text{a.e. in } (0,T) \\ U(0) = \|u_0\|^2_{L^2(\Omega)}. \end{cases} \tag{6.4}$$

In turn, integrating on (6.4), we get for all $t \in (0,T)$

$$\begin{aligned} U^{\frac{2-p}{2}}(t) - U^{\frac{2-p}{2}}(0) &= \frac{2-p}{2}\int_0^t U^{-\frac{p}{2}}(\tau)U'(\tau)\,d\tau \\ &\leqslant -\frac{2-p}{2}\frac{C_1}{\gamma}|\Omega|^{-\frac{\lambda_2}{2N}}\int_0^t 1\,d\tau = -\frac{C_1 t}{\gamma_*}|\Omega|^{-\frac{\lambda_2}{2N}}, \end{aligned}$$

with $\gamma_* > 0$ depending on N, p, s. The latter inequality rephrases as

$$\|u(\cdot,t)\|^{2-p}_{L^2(\Omega)} \leqslant \|u_0\|^{2-p}_{L^2(\Omega)} - \frac{C_1 t}{\gamma_*}|\Omega|^{-\frac{\lambda_2}{2N}}.$$

Finally, we have $u(\cdot,t) = 0$ in Ω as soon as

$$t \geqslant \frac{\gamma_*}{C_1}\|u_0\|^{2-p}_{L^2(\Omega)}|\Omega|^{\frac{\lambda_2}{2N}},$$

which yields the conclusion since u vanishes in Ω^c at any time. □

Finally, we estimate the rate of extinction of u as time approaches T_*.

Proof of Theorem 1.6. We may see u as a non-negative, local weak solution of (1.2) in Ω_{T_*}, which in addition satisfies $u(\cdot,t) = 0$ in $\mathbb{R}^N$ for all $t \geqslant T_*$. Up to a time translation, we apply

Corollary 4.2 (i) to u in the time interval (t, T_*). Therefore, there exists $\gamma > 0$ depending on the data s.t.

$$\begin{aligned}\int_{B_\rho(x_0)} u(x,t)\,dx &\leqslant \sup_{t<\tau<T_*} \int_{B_\rho(x_0)} u(x,\tau)\,dx \\ &\leqslant \gamma\Big[\inf_{t<\tau<T_*} \int_{B_{2\rho}(x_0)} u(x,\tau)\,dx\Big] + \gamma\Big(\frac{T_*-t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}} \\ &= \gamma\Big(\frac{T_*-t}{\rho^{\lambda_1}}\Big)^{\frac{1}{2-p}},\end{aligned}$$

which proves (i).

Now assume $p_* < p < 2$, in particular $\lambda_1 > 0$. Set

$$P = \max\Big\{1, \Big(\frac{T_*-t}{\rho^{ps}}\Big)^{\frac{1-p}{2-p}} \mathrm{Tail}\Big(u, x_0, \frac{\rho}{2}, t, T_*\Big)^{p-1}\Big\} > 0.$$

By Corollary 4.2 (ii), again applied in the time interval (t, T_*), and Theorem 1.5, there exists $\gamma > 0$ depending on the data s.t.

$$\begin{aligned}\sup_{B_{\rho/2}(x_0)} u\Big(\cdot, \frac{t+T_*}{2}\Big) &\leqslant \sup_{B_{\rho/2}(x_0)\times((t+T_*)/2,T_*)} u \\ &\leqslant \gamma\Big[\inf_{t<\tau<T_*} \int_{B_{2\rho}(x_0)} u(x,\tau)\,dx\Big]^{\frac{ps}{\lambda_1}} (T_*-t)^{-\frac{N}{\lambda_1}} + \gamma\Big(\frac{T_*-t}{\rho^{ps}}\Big)^{\frac{1}{2-p}} P \\ &= \gamma\Big(\frac{T_*-t}{\rho^{ps}}\Big)^{\frac{1}{2-p}} P,\end{aligned}$$

which proves (ii). Thus, the proof is concluded. □

Remark 6.1 Clearly, the estimate of Theorem 1.6 (ii) is sharper than that in (i), up to a rescaling of the cylinder $B_\rho(x_0) \times (t, T_*)$ and under appropriate conditions on the tail (recall that u vanishes in Ω^c). Nevertheless, the fist estimate holds even in the subcritical regime $1 < p \leqslant p_*$ and does not require additional assumptions.

Appendix A. Time mollifications

At several steps in our proofs, namely in the proofs of Lemma 4.1, Lemma 5.1, and Theorem 1.5, we have introduced test functions that are not admissible according to Definitions 2.1 and 2.2, respectively. Such procedure is widely known in parabolic regularity theory, and in our case it is formally justified by the following mollification argument.

For all $v : \Omega_T \to \mathbb{R}$ and all $h \in (0, 1)$, we define two finite convolutions of v with exponential weight functions by setting for all $(x, t) \in \Omega_T$

$$v_h(x,t) = \int_0^t \frac{v(x,\tau)}{h} e^{\frac{\tau-t}{h}}\,d\tau,\ \ v_{\bar{h}}(x,t) = \int_t^T \frac{v(x,\tau)}{h} e^{\frac{t-\tau}{h}}\,d\tau.$$

We refer to [34, Appendix B] and [32, Lemma 2.2] for the following properties of v_h, $v_{\bar{h}}$:

Proposition A.1 *Let $v \in L^q(\Omega_T)$ for some $q \in [1, +\infty[$. Then, for all $h \in (0, 1)$ we have:*

(*i*) $v_h, v_{\bar h}$ *are differentiable in t with*

$$\frac{\partial v_h}{\partial t}(x,t) = \frac{v(x,t)-v_h(x,t)}{h}, \quad \frac{\partial v_{\bar h}(x,t)}{\partial t} = \frac{v_{\bar h}(x,t)-v(x,t)}{h},$$

in particular for all $\varphi \in W^{1,2}(0,T,L^{q'}(\Omega))$ *s.t.* $\varphi(\cdot,0)=\varphi(\cdot,T)=0$ *we have*

$$\int_0^T v_h(x,\tau)\varphi_\tau(x,\tau)\,d\tau = -\int_0^T \frac{v(x,\tau)-v_h(x,\tau)}{h}\varphi(x,\tau)\,d\tau,$$

$$\int_0^T v_{\bar h}(x,\tau)\varphi_\tau(x,\tau)\,d\tau = -\int_0^t \frac{v_{\bar h}(x,\tau)-v(x,\tau)}{h}\varphi(x,\tau)\,d\tau;$$

(*ii*) $v_h, v_{\bar h} \in L^q(\Omega_T)$ *and* $v_h, v_{\bar h} \to v$ *in* $L^q(\Omega_T)$ *as* $h \to 0^+$, *also if* $v \geqslant 0$ *in* Ω_T *then* $0 \leqslant v_h, v_{\bar h} \leqslant v$ *in* Ω_T*;*

(*iii*) *if* $v \in L^q(0,T,W^{s,q}(\Omega'))$ *for some* $\Omega' \Subset \Omega$, *then* $v_h, v_{\bar h} \in L^q(0,T,W^{s,q}(\Omega'))$ *with*

$$\int_0^T \iint_{\Omega'\times\Omega'} \frac{|v_h(x,\tau)-v_h(y,\tau)|^q}{|x-y|^{N+qs}}\,dx\,dy\,d\tau \leqslant \|v\|^q_{L^q(0,T,W^{s,q}(\Omega'))},$$

and an analogous estimate holds for $v_{\bar h}$.

The following lemma provides a rigorous proof of inequality (4.2) in Lemma 4.1:

Lemma A.2 *Let* u, $\check B \subset \hat B \subset B$, t, ν, ξ *be as in Lemma 4.1. Then, there exists* $\gamma > 0$ *depending on the data s.t.*

$$\begin{aligned}
0 \geqslant & -\frac{pt^{\frac{1}{p}}}{2(p-1)}\int_B (u(x,t)+\nu)^{\frac{2(p-1)}{p}}\xi^p(x)\,dx \\
& + \frac{1}{2(p-1)}\int_0^t \tau^{\frac{1-p}{p}}\int_B (u(x,\tau)+\nu)^{\frac{2(p-1)}{p}}\xi^p(x)\,dx\,d\tau \\
& - \gamma\int_0^t \tau^{\frac{1}{p}}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x,\tau)-u(y,\tau)|^{p-2}(u(x,\tau)-u(y,\tau))}{|x-y|^{N+ps}} \\
& \Big[(u(x,\tau)+\nu)^{\frac{p-2}{p}}\xi^p(x)-(u(y,\tau)+\nu)^{\frac{p-2}{p}}\xi^p(y)\Big]\,dx\,dy\,d\tau.
\end{aligned}$$

Proof Fix $\varepsilon \in (0,t/2)$ and set for all $\tau \in [0,T]$

$$\psi_\varepsilon(\tau) = \begin{cases} \tau/\varepsilon & \text{if } 0 \leqslant \tau < \varepsilon \\ 1 & \text{if } \varepsilon \leqslant \tau < t-\varepsilon \\ (t-\tau)/\varepsilon & \text{if } t-\varepsilon \leqslant \tau < t \\ 0 & \text{if } t \leqslant \tau \leqslant T, \end{cases}$$

so $\psi_\varepsilon : [0,T] \to [0,1]$ is a Lipschitz mapping. Also, fix $h \in (0,1)$ and set for all $(x,\tau) \in \mathbb{R}^N \times (0,T)$

$$\varphi_h^\varepsilon(x,\tau) = -(u_{\bar h}(x,\tau)+\nu)^{\frac{p-2}{p}}\xi^p(x)\tau^{\frac{1}{p}}\psi_\varepsilon(\tau).$$

Note that the negative exponent in the first term does not produce any singularity as $u(\cdot,\tau) \geqslant 0$ in $\hat B$. By Proposition A.1, $\varphi_h^\varepsilon \in W^{1,2}_{\mathrm{loc}}(0,T,L^2(\hat B)) \cap L^p_{\mathrm{loc}}(0,T,W_0^{s,p}(\hat B))$ is a suitable test function for (1.2), in $\hat{B} \times (0, t)$ (see Definition 2.1), so we have

$$
\begin{aligned}
0 = & -\int_0^t \int_{\hat{B}} u(x,\tau) \frac{\partial \varphi_h^\varepsilon}{\partial \tau}(x,\tau)\, dx\, d\tau \\
& + \int_0^t \iint_{\mathbb{R}^N \times \mathbb{R}^N} |u(x,\tau) - u(y,\tau)|^{p-2}(u(x,\tau) \\
& - u(y,\tau))(\varphi_h^\varepsilon(x,\tau) - \varphi_h^\varepsilon(y,\tau)) K(x,y,\tau)\, dx\, dy\, d\tau \\
= & \, H_1 + H_2,
\end{aligned} \tag{A.1}
$$

where we have also used that $\varphi_h^\varepsilon(\cdot, 0) = \varphi_h^\varepsilon(\cdot, t) = 0$ in $\mathbb{R}^N$ and $\xi = 0$ in $\hat{B}^c$. Our aim is now to estimate both H_1, H_2 from below, and then pass to the limit as $\varepsilon, h \to 0$. We first focus on the evolutive term H_1:

$$
\begin{aligned}
H_1 = & -\int_0^T \int_{\hat{B}} u_{\bar{h}}(x,\tau) \frac{\partial \varphi_h^\varepsilon}{\partial \tau}(x,\tau)\, dx\, d\tau + \int_0^T \int_{\hat{B}} (u_{\bar{h}}(x,\tau) - u(x,\tau)) \frac{\partial \varphi_h^\varepsilon}{\partial \tau}(x,\tau)\, dx\, d\tau \\
= & \, H_3 + H_4.
\end{aligned}
$$

On H_3 we first apply Proposition A.1 (i), then the chain rule of differentiation and the definition of ψ_ε :

$$
\begin{aligned}
H_3 = & \int_0^T \int_{\hat{B}} \frac{\partial u_{\bar{h}}}{\partial \tau}(x,\tau) \varphi_h^\varepsilon(x,\tau)\, dx\, d\tau \\
= & -\int_0^T \int_{\hat{B}} \frac{\partial}{\partial \tau}(u_{\bar{h}}(x,\tau) + \nu)(u_{\bar{h}}(x,\tau) + \nu)^{\frac{p-2}{p}} \xi^p(x) \tau^{\frac{1}{p}} \psi_\varepsilon(\tau)\, dx\, d\tau \\
= & -\frac{p}{2(p-1)} \int_0^T \int_{\hat{B}} \frac{\partial}{\partial \tau}(u_{\bar{h}}(x,\tau) + \nu)^{\frac{2(p-1)}{p}} \xi^p(x) \tau^{\frac{1}{p}} \psi_\varepsilon(\tau)\, dx\, d\tau \\
= & \frac{p}{2(p-1)} \int_0^T \int_{\hat{B}} (u_{\bar{h}}(x,\tau) + \nu)^{\frac{2(p-1)}{p}} \xi^p(x) \frac{\partial}{\partial \tau}(\tau^{\frac{1}{p}} \psi_\varepsilon(\tau))\, dx\, d\tau \\
= & \frac{1}{2(p-1)} \int_0^T \int_{\hat{B}} (u_{\bar{h}}(x,\tau) + \nu)^{\frac{2(p-1)}{p}} \xi^p(x) \tau^{\frac{1-p}{p}} \psi_\varepsilon(\tau)\, dx\, d\tau \\
& + \frac{p}{2(p-1)} \int_0^T \int_{\hat{B}} (u_{\bar{h}}(x,\tau) + \nu)^{\frac{2(p-1)}{p}} \xi^p(x) \tau^{\frac{1}{p}} \psi_\varepsilon'(\tau)\, dx\, d\tau \\
= & \frac{1}{2(p-1)} \int_0^t \int_{\hat{B}} (u_{\bar{h}}(x,\tau) + \nu)^{\frac{2(p-1)}{p}} \xi^p(x) \tau^{\frac{1-p}{p}} \psi_\varepsilon(\tau)\, dx\, d\tau \\
& + \frac{p}{2(p-1)} \Big[\frac{1}{\varepsilon} \int_0^\varepsilon \tau^{\frac{1}{p}} \int_{\hat{B}} (u_{\bar{h}}(x,\tau) + \nu)^{\frac{2(p-1)}{p}} \xi^p(x)\, dx\, d\tau \\
& - \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \tau^{\frac{1}{p}} \int_{\hat{B}} (u_{\bar{h}}(x,\tau) + \nu)^{\frac{2(p-1)}{p}} \xi^p(x)\, dx\, d\tau \Big].
\end{aligned}
$$

First let $h \to 0$ and use Proposition A.1 (ii) with $q = 1$ (note that $2(p-1)/p < 1$):

$$
\begin{aligned}
\lim_{h \to 0} H_3 = & \frac{1}{2(p-1)} \int_0^t \int_{\hat{B}} (u(x,\tau) + \nu)^{\frac{2(p-1)}{p}} \xi^p(x) \tau^{\frac{1-p}{p}} \psi_\varepsilon(\tau)\, dx\, d\tau \\
& + \frac{p}{2(p-1)} \Big[\frac{1}{\varepsilon} \int_0^\varepsilon \tau^{\frac{1}{p}} \int_{\hat{B}} (u(x,\tau) + \nu)^{\frac{2(p-1)}{p}} \xi^p(x)\, dx\, d\tau \\
& - \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \tau^{\frac{1}{p}} \int_{\hat{B}} (u(x,\tau) + \nu)^{\frac{2(p-1)}{p}} \xi^p(x)\, dx\, d\tau \Big].
\end{aligned}
$$

Further, let $\varepsilon \to 0$ and apply the fundamental theorem of calculus:

$$\lim_{\varepsilon,h\to 0} H_3 = \frac{1}{2(p-1)} \int_0^t \int_{\hat{B}} (u(x,\tau)+\nu)^{\frac{2(p-1)}{p}} \xi^p(x) \tau^{\frac{1-p}{p}} \, dx \, d\tau - \frac{pt^{\frac{1}{p}}}{2(p-1)} \int_{\hat{B}} (u(x,t)+\nu)^{\frac{2(p-1)}{p}} \xi^p(x) \, dx.$$

Next, on H_4 we apply Proposition A.1 (i) and positivity of $u_{\bar{h}}$:

$$\begin{aligned} H_4 &= \frac{2-p}{p} \int_0^T \int_{\hat{B}} (u_{\bar{h}}(x,\tau) - u(x,\tau))(u_{\bar{h}}(x,\tau)+\nu)^{-\frac{2}{p}} \frac{\partial u_{\bar{h}}}{\partial \tau}(x,\tau) \tau^{\frac{1}{p}} \psi_\varepsilon(\tau) \xi^p(x) \, dx \, d\tau \\ &\quad - \int_0^T \int_{\hat{B}} (u_{\bar{h}}(x,\tau) - u(x,\tau))(u_{\bar{h}}(x,\tau)+\nu)^{\frac{p-2}{p}} \frac{\partial}{\partial \tau} (\tau^{\frac{1}{p}} \psi_\varepsilon(\tau)) \xi^p(x) \, dx \, d\tau \\ &\geqslant \frac{2-p}{p} \int_0^T \int_{\hat{B}} \frac{(u_{\bar{h}}(x,\tau) - u(x,\tau))^2}{h} (u_{\bar{h}}(x,\tau)+\nu)^{-\frac{2}{p}} \tau^{\frac{1}{p}} \psi_\varepsilon(\tau) \xi^p(x) \, dx \, d\tau \\ &\quad - \int_0^T \int_{\hat{B}} |u_{\bar{h}}(x,\tau) - u(x,\tau)| \nu^{\frac{p-2}{p}} \left| \frac{\partial}{\partial \tau} (\tau^{\frac{1}{p}} \psi_\varepsilon(\tau)) \right| \xi^p(x) \, dx \, d\tau. \end{aligned}$$

The first term above is non-negative, while the second tends to 0 as $h \to 0$, uniformly for all $\varepsilon \in (0, t/2)$, by Proposition A.1 (ii) (with $q = 1$), so we have

$$\liminf_{\varepsilon,h\to 0} H_4 \geqslant 0.$$

By the previous estimates, and recalling that $\xi = 0$ in $\hat{B}^c$, we have

$$\liminf_{\varepsilon,h\to 0} H_1 \geqslant \frac{1}{2(p-1)} \int_0^t \tau^{\frac{1-p}{p}} \int_B (u(x,\tau)+\nu)^{\frac{2(p-1)}{p}} \xi^p(x) \, dx \, d\tau - \frac{pt^{\frac{1}{p}}}{2(p-1)} \int_B (u(x,t)+\nu)^{\frac{2(p-1)}{p}} \xi^p(x) \, dx. \tag{A.2}$$

We now turn to the diffusive term H_2. For all $(x,y,\tau) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,T)$ set

$$G(x,y,\tau) = |u(x,\tau) - u(y,\tau)|^{p-2}(u(x,\tau) - u(y,\tau)),$$

$$F_h(x,y,\tau) = \tau^{\frac{1}{p}} \Big[(u_{\bar{h}}(x,\tau)+\nu)^{\frac{p-2}{p}} \xi^p(x) - (u_{\bar{h}}(y,\tau)+\nu)^{\frac{p-2}{p}} \xi^p(y) \Big],$$

$$F(x,y,\tau) = \tau^{\frac{1}{p}} \Big[(u(x,\tau)+\nu)^{\frac{p-2}{p}} \xi^p(x) - (u(y,\tau)+\nu)^{\frac{p-2}{p}} \xi^p(y) \Big].$$

First note that

$$\int_0^t \iint_{B\times B} \frac{|G(x,y,\tau)|^{\frac{p}{p-1}}}{|x-y|^{N+ps}} \, dx \, dy \, d\tau \leqslant \|u\|^p_{L^p(0,t,W^{s,p}(B))}. \tag{A.3}$$

By Proposition A.1 (ii) we have for a.e. $(x,y,\tau) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,T)$

$$\lim_{h\to 0} F_h(x,y,\tau) = F(x,y,\tau).$$

We first note that, letting $\varepsilon \to 0$ and using the properties of the kernel K, we have

$$\begin{aligned}\lim_{\varepsilon\to 0} H_2 &= -\int_0^t \iint_{\mathbb{R}^N\times\mathbb{R}^N} G(x,y,\tau)F_h(x,y,\tau)K(x,y,\tau)\,dx\,dy\,d\tau \\ &\geqslant -\gamma\int_0^t \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(x,y,\tau)F_h(x,y,\tau)}{|x-y|^{N+ps}}\,dx\,dy\,d\tau.\end{aligned}$$

We now claim that

$$\lim_{h\to 0}\int_0^t \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(x,y,\tau)F_h(x,y,\tau)}{|x-y|^{N+ps}}\,dx\,dy\,d\tau = \int_0^t \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(x,y,\tau)F(x,y,\tau)}{|x-y|^{N+ps}}\,dx\,dy\,d\tau. \tag{A.4}$$

Exploiting symmetry and recalling that $\xi = 0$ in $\hat{B}^c$, we can split the difference as follows:

$$\begin{aligned}&\int_0^t \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(x,y,\tau)}{|x-y|^{N+ps}}\big[F_h(x,y,\tau)-F(x,y,\tau)\big]\,dx\,dy\,d\tau \\ &= \int_0^t \iint_{B\times B} \frac{G(x,y,\tau)}{|x-y|^{N+ps}}\big[F_h(x,y,\tau)-F(x,y,\tau)\big]\,dx\,dy\,d\tau \\ &\quad + 2\int_0^t \tau^{\frac{1}{p}} \iint_{\hat{B}\times B^c} \frac{G(x,y,\tau)}{|x-y|^{N+ps}}\big[(u_{\bar{h}}(x,\tau)+\nu)^{\frac{p-2}{p}} - (u(x,\tau)+\nu)^{\frac{p-2}{p}}\big]\xi^p(x)\,dx\,dy\,d\tau \\ &= H_5 + H_6.\end{aligned}$$

We show separately that both H_5, H_6 tend to 0 as $h \to 0$, up to a subsequence. First we deal with H_5. By subadditivity and $0 \leqslant \xi \leqslant 1$ in $\mathbb{R}^N$ we have

$$\begin{aligned}&\int_0^t \iint_{B\times B} \frac{|F_h(x,y,\tau)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau \\ &\leqslant \gamma\int_0^t \tau \iint_{B\times B} \frac{\big|(u_{\bar{h}}(x,\tau)+\nu)^{\frac{p-2}{p}} - (u_{\bar{h}}(y,\tau)+\nu)^{\frac{p-2}{p}}\big|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau \\ &\quad + \gamma\int_0^t \tau \iint_{B\times B} (u_{\bar{h}}(y,\tau)+\nu)^{p-2}\frac{|\xi^p(x)-\xi^p(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau = H_7 + H_8.\end{aligned}$$

To estimate H_7 we apply Lagrange's rule, inequality $\tau \leqslant t$, and positivity of $u_{\bar{h}}$, then Proposition A.1 *(iii)*:

$$\begin{aligned}H_7 &\leqslant \gamma t\int_0^t \iint_{B\times B} \max\big\{(u_{\bar{h}}(x,\tau)+\nu)^{-\frac{2}{p}},\ (u_{\bar{h}}(x,\tau)+\nu)^{-\frac{2}{p}}\big\}^p \\ &\qquad \frac{|u_{\bar{h}}(x,\tau)-u_{\bar{h}}(x,\tau)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau \\ &\leqslant \frac{\gamma t}{\nu^2}\int_0^t \iint_{B\times B} \frac{|u_{\bar{h}}(x,\tau)-u_{\bar{h}}(x,\tau)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau \\ &\leqslant \frac{\gamma t}{\nu^2}\|u\|^p_{L^p(0,t,W^{s,p}(B))}.\end{aligned}$$

For H_8 we argue similarly, using Lagrange's rule on the increment of ξ^p and Proposition A.1 (ii) (taking this time $q = p$):

$$
\begin{aligned}
H_8 &\leqslant \gamma t \int_0^t \iint_{B\times B} (u_{\bar{h}}(y,\tau)+\nu)^{p-2} \max\left\{\xi^{p-1}(x),\ \xi^{p-1}(y)\right\}^p \frac{|\xi(x)-\xi(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau \\
&\leqslant \frac{\gamma t}{\nu^2\rho^p}\Big[\int_0^t\int_B (u_{\bar{h}}(y,\tau)+\nu)^p\,dy\,d\tau\Big]\Big[\sup_{y\in B}\int_B \frac{dx}{|x-y|^{N+ps-p}}\Big] \\
&\leqslant \frac{\gamma t}{\nu^2\rho^p}\Big[\int_0^t\int_B u_{\bar{h}}^p(y,\tau)\,dy\,d\tau + \nu^p\rho^N t\Big]\Big[\int_{B_{2\rho}(0)} \frac{dz}{|z|^{N+ps-p}}\Big] \\
&\leqslant \frac{\gamma t}{\nu^2\rho^{ps}}\Big[\|u\|^p_{L^p(B\times(0,t))} + \nu^p\rho^N t\Big].
\end{aligned}
$$

Thus, we have found a constant $\gamma > 0$ depending on u, ν, ρ, and t, but not on h, s.t. for all $h \in (0,1)$

$$
\int_0^t \iint_{B\times B} \frac{|F_h(x,y,\tau)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau \leqslant \gamma.
$$

A standard argument based on reflexivity, and exploiting (A.3), shows that up to passing to a sequence of h's converging to 0, we have

$$
\lim_{h\to 0} H_5 = 0. \tag{A.5}
$$

Next we focus on H_6. First note that for all $x \in \hat{B}$, $y \in B^c$

$$
|x-y| \geqslant |y-x_0| - |x-x_0| \geqslant (1-\sigma')|y-x_0|.
$$

Recalling that $0 \leqslant \xi \leqslant 1$, we estimate H_6 as follows:

$$
\begin{aligned}
|H_6| &\leqslant \frac{\gamma t^{\frac{1}{p}}}{(1-\sigma')^{N+ps}} \int_0^t \iint_{\hat{B}\times B^c} \frac{u^{p-1}(x,\tau)}{|y-x_0|^{N+ps}} \Big|(u_{\bar{h}}(x,\tau)+\nu)^{\frac{p-2}{p}} - (u(x,\tau)+\nu)^{\frac{p-2}{p}}\Big|\,dx\,dy\,d\tau \\
&\quad + \frac{\gamma t^{\frac{1}{p}}}{(1-\sigma')^{N+ps}} \int_0^t \iint_{\hat{B}\times B^c} \frac{|u(y,\tau)|^{p-1}}{|y-x_0|^{N+ps}} \Big|(u_{\bar{h}}(x,\tau)+\nu)^{\frac{p-2}{p}} - (u(x,\tau)+\nu)^{\frac{p-2}{p}}\Big|\,dx\,dy\,d\tau \\
&= \frac{\gamma t^{\frac{1}{p}}}{(1-\sigma')^{N+ps}}(H_9+H_{10}).
\end{aligned}
$$

We deal with H_9 by Hölder's inequality and Lagrange's rule:

$$
\begin{aligned}
H_9 &\leqslant \Big[\int_0^t\int_{\hat{B}} u^p(x,\tau)\,dx\,d\tau\Big]^{\frac{p-1}{p}} \Big[\int_0^t\int_{\hat{B}} \Big|(u_{\bar{h}}(x,\tau)+\nu)^{\frac{p-2}{p}} - (u(x,\tau)+\nu)^{\frac{p-2}{p}}\Big|^p\,dx\,d\tau\Big]^{\frac{1}{p}} \int_{B^c}\frac{dy}{|y-x_0|^{N+ps}} \\
&\leqslant \frac{\gamma}{\rho^{ps}}\|u\|^{p-1}_{L^p(\hat{B}\times(0,t))}\Big[\int_0^t\int_{\hat{B}} \max\big\{(u_{\bar{h}}(x,\tau)+\nu)^{-\frac{2}{p}},\ (u(x,\tau)+\nu)^{-\frac{2}{p}}\big\}^p |u_{\bar{h}}(x,\tau)-u(x,\tau)|^p\,dx\,d\tau\Big]^{\frac{1}{p}} \\
&\leqslant \frac{\gamma}{\nu^{\frac{2}{p}}\rho^{ps}}\|u\|^{p-1}_{L^p(\hat{B}\times(0,t))}\|u_{\bar{h}}-u\|_{L^1(\hat{B}\times(0,t))}.
\end{aligned}
$$

The first multiplier above is bounded, while the second tends to 0 as $h \to 0$ by Proposition A.1 (ii) (with $q = 1$). So we have

$$
\lim_{h\to 0} H_9 = 0.
$$

For H_{10} we use a different approach, separating the space integration variables. For all $\tau \in (0, t)$ we have

$$
\begin{aligned}
&\iint_{\hat{B}\times B^c} \frac{|u(y,\tau)|^{p-1}}{|y-x_0|^{N+ps}} \Big|(u_{\bar{h}}(x,\tau)+\nu)^{\frac{p-2}{p}} - (u(x,\tau)+\nu)^{\frac{p-2}{p}}\Big|\,dx\,dy \\
&\leqslant \gamma\Big[\int_{\hat{B}} \max\big\{(u_{\bar{h}}(x,\tau)+\nu)^{-\frac{2}{p}},\,(u(x,\tau)+\nu)^{-\frac{2}{p}}\big\}|u_{\bar{h}}(x,\tau)-u(x,\tau)|\,dx\Big]\int_{B^c}\frac{|u(y,\tau)|^{p-1}}{|y-x_0|^{N+ps}}\,dy \\
&\leqslant \frac{\gamma}{\nu^{\frac{2}{p}}\rho^{ps}}\Big[\int_{\hat{B}}|u_{\bar{h}}(x,\tau)-u(x,\tau)|\,dx\Big]\Big[\rho^{ps}\int_{B^c}\frac{|u(y,\tau)|^{p-1}}{|y-x_0|^{N+ps}}\,dy\Big].
\end{aligned}
$$

Now integrate in time and recall (1.4):

$$H_{10} \leqslant \frac{\gamma}{\nu^{\frac{2}{p}}\rho^{ps}}\|u_{\bar{h}}-u\|_{L^1(\hat{B}\times(0,t))}\mathrm{Tail}(u, x_0, \rho, 0, t)^{p-1}.$$

As soon as $h \to 0$, the first term tends to 0 by Proposition A.1 *(ii)* ($q = 1$) and the second is bounded, hence

$$\lim_{h\to 0} H_{10} = 0.$$

Therefore, we have

$$\lim_{h\to 0} H_6 = 0. \tag{A.6}$$

Now (A.5) and (A.6) imply (A.4). This, in turn, leads to

$$\limsup_{\varepsilon,h\to 0} H_2 \geqslant -\gamma\int_0^t\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{G(x,y,\tau)F(x,y,\tau)}{|x-y|^{N+ps}}\,dx\,dy\,d\tau. \tag{A.7}$$

Finally, we use both (A.2) and (A.7) and pass to the limit in (A.1) to conclude. □

Next we give a proof of inequality (5.2) in Lemma 5.1:

Lemma A.3 *Let u, r, t_0, σ, ξ, B, $\tilde{B}$, $\hat{B}$ be as in Lemma 5.1. Then, there exists $\gamma > 1$ depending on the data and r, s.t.%*

$$
\begin{aligned}
0 \geqslant\ & \frac{1}{r}\int_{\tilde{B}} u^r(x,\tau)\,dx\Big|_0^{t_0} \\
&+\frac{1}{\gamma}\int_0^{t_0}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x,\tau)-u(y,\tau)|^{p-2}(u(x,\tau)-u(y,\tau))}{|x-y|^{N+ps}} \\
&\Big[u^{r-1}(x,\tau)\xi^p(x)-u^{r-1}(y,\tau)\xi^p(y)\Big]\,dx\,dy\,d\tau.
\end{aligned}
$$

Proof The argument closely follows that of Lemma A.2, so we will omit some steps. Fix $\varepsilon \in (0, t_0/2)$ and define the Lipschitz mapping $\psi_\varepsilon : [0, T] \to \mathbb{R}$ as in Lemma A.2, with t replaced by t_0. Also fix $h \in (0, 1)$, $\nu > 0$, and set for all $(x, \tau) \in \mathbb{R}^N \times (0, T)$

$$\varphi_h^\varepsilon(x,\tau) = (u_{\bar{h}}(x,\tau)+\nu)^{r-1}\xi^p(x)\psi_\varepsilon(\tau),$$

where the parameter ν is included in order to avoid singularities in the case $1 < r < 2$. By Proposition A.1, the function $\varphi_h^\varepsilon \in W_0^{1,2}(0, t_0, L^2(\tilde{B})) \cap L^p(0, t_0, W_0^{s,p}(\tilde{B}))$ can be chosen as a test for (1.2) in $\tilde{B} \times (0, t_0)$ (see Definition 2.1), producing

$$0 = -\int_0^{t_0} \int_{\tilde{B}} u(x,\tau) \frac{\partial \varphi_h^\varepsilon}{\partial \tau}(x,\tau)\, dx\, d\tau \tag{A.8}$$
$$+ \int_0^{t_0} \iint_{\mathbb{R}^N \times \mathbb{R}^N} |u(x,\tau)$$
$$- u(y,\tau)|^{p-2}(u(x,\tau) - u(y,\tau))(\varphi_h^\varepsilon(x,\tau) - \varphi_h^\varepsilon(y,\tau))K(x,y,\tau)\, dx\, dy\, d\tau$$
$$= H_1 + H_2.$$

First we split H_1 as follows:

$$H_1 = -\int_0^{t_0} \int_{\tilde{B}} u_{\bar{h}}(x,\tau) \frac{\partial \varphi_h^\varepsilon}{\partial \tau}(x,\tau)\, dx\, d\tau + \int_0^{t_0} \int_{\tilde{B}} (u_{\bar{h}}(x,\tau) - u(x,\tau)) \frac{\partial \varphi_h^\varepsilon}{\partial \tau}(x,\tau)\, dx\, d\tau$$
$$= H_3 + H_4.$$

Repeated integration by parts and the definition of ψ_ε allows for the following rephrasing of H_3:

$$H_3 = \frac{1}{r\varepsilon} \int_{t_0-\varepsilon}^{t_0} \int_{\tilde{B}} (u_{\bar{h}}(x,\tau) + \nu)^r \xi^p(x)\, dx\, d\tau - \frac{1}{r\varepsilon} \int_0^\varepsilon \int_{\tilde{B}} (u_{\bar{h}}(x,\tau) + \nu)^r \xi^p(x)\, dx\, d\tau.$$

Letting $\varepsilon, h, \nu \to 0$ gives

$$\lim_{\varepsilon,h,\nu\to 0} H_3 = \frac{1}{r} \int_{\tilde{B}} u^r(x,t_0)\xi^p(x)\, dx - \frac{1}{r} \int_{\tilde{B}} u^r(x,0)\xi^p(x)\, dx.$$

To estimate H_4, we apply Proposition A.1 (i):

$$H_4 = \int_0^{t_0} \int_{\tilde{B}} (u_{\bar{h}}(x,\tau) - u(x,\tau)) \frac{\partial}{\partial \tau}(u_{\bar{h}}(x,\tau) + \nu)^{r-1} \xi^p(x)\psi_\varepsilon(\tau)\, dx\, d\tau$$
$$+ \int_0^{t_0} \int_{\tilde{B}} (u_{\bar{h}}(x,\tau) - u(x,\tau))(u_{\bar{h}}(x,\tau) + \nu)^{r-1} \xi^p(x)\psi_\varepsilon'(\tau)\, dx\, d\tau$$
$$= (r-1) \int_0^{t_0} \int_{\tilde{B}} \frac{(u_{\bar{h}}(x,\tau) - u(x,\tau))^2}{h} (u_{\bar{h}}(x,\tau) + \nu)^{r-2} \xi^p(x)\psi_\varepsilon(\tau)\, dx\, d\tau$$
$$+ \frac{1}{\varepsilon} \int_0^\varepsilon \int_{\tilde{B}} (u_{\bar{h}}(x,\tau) - u(x,\tau))(u_{\bar{h}}(x,\tau) + \nu)^{r-1} \xi^p(x)\, dx\, d\tau$$
$$- \frac{1}{\varepsilon} \int_{t_0-\varepsilon}^{t_0} \int_{\tilde{B}} (u_{\bar{h}}(x,\tau) - u(x,\tau))(u_{\bar{h}}(x,\tau) + \nu)^{r-1} \xi^p(x)\, dx\, d\tau.$$

The first integral above is non-negative, while the others tend to 0 as $h \to 0$, due to convergence of $u_{\bar{h}}$ to u in $L^1(\tilde{B} \times (0, t_0))$ (Proposition A.1 (ii) with $q = 1$). In conclusion we have

$$\liminf_{\varepsilon,h,\nu\to 0} H_4 \geqslant 0,$$

which along with the previous relation gives

$$\limsup_{\varepsilon,h,\nu\to 0} H_1 \geqslant \frac{1}{r} \int_{\tilde{B}} u^r(x,\tau)\xi^p(x)\, dx \Big|_0^{t_0}. \tag{A.9}$$

Now for the diffusive term H_2, with a similar argument as in Lemma A.2. First we define G as in Lemma A.2, then we set for all $(x, y, \tau) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, T)$

$$F_h(x,y,\tau) = (u_{\bar{h}}(x,\tau) + \nu)^{r-1}\xi^p(x) - (u_{\bar{h}}(x,\tau) + \nu)^{r-1}\xi^p(y),$$

$$F(x,y,\tau) = (u(x,\tau)+\nu)^{r-1}\xi^p(x) - (u(x,\tau)+\nu)^{r-1}\xi^p(y).$$

Clearly

$$\int_0^{t_0}\iint_{\hat{B}\times\hat{B}} \frac{|G(x,y,\tau)|^{\frac{p}{p-1}}}{|x-y|^{N+ps}}\,dx\,dy\,d\tau \leqslant \|u\|^p_{L^p(0,t_0,W^{s,p}(\hat{B}))},$$

while by Proposition A.1 we have a.e. in $\mathbb{R}^N\times\mathbb{R}^N\times(0,T)$

$$\lim_{h\to 0} F_h(x,y,\tau) = F(x,y,\tau).$$

Also, by (K_2), we have for some $\gamma>1$ depending on the data

$$\liminf_{\varepsilon\to 0} H_2 \geqslant \frac{1}{\gamma}\int_0^{t_0}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(x,y,\tau)F_h(x,y,\tau)}{|x-y|^{N+ps}}\,dx\,dy\,d\tau.$$

By symmetry and the properties of ξ we have

$$\begin{aligned}
&\int_0^{t_0}\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(x,y,\tau)}{|x-y|^{N+ps}}\big[F_h(x,y,\tau)-F(x,y,\tau)\big]\,dx\,dy\,d\tau\\
&= \int_0^{t_0}\iint_{\hat{B}\times\hat{B}} \frac{G(x,y,\tau)}{|x-y|^{N+ps}}\big[F_h(x,y,\tau)-F(x,y,\tau)\big]\,dx\,dy\,d\tau\\
&\quad + 2\int_0^{t_0}\iint_{\tilde{B}\times\hat{B}^c} \frac{G(x,y,\tau)}{|x-y|^{N+ps}}\big[(u_{\bar{h}}(x,\tau)+\nu)^{r-1} - (u(x,\tau)+\nu)^{r-1}\big]\xi^p(x)\,dx\,dy\,d\tau\\
&= H_5 + H_6.
\end{aligned}$$

We first consider H_5, noting that by boundedness of ξ

$$\begin{aligned}
&\int_0^{t_0}\iint_{\hat{B}\times\hat{B}} \frac{|F_h(x,y,\tau)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau\\
&\leqslant \gamma\int_0^{t_0}\iint_{\hat{B}\times\hat{B}} \frac{|(u_{\bar{h}}(x,\tau)+\nu)^{r-1} - (u_{\bar{h}}(y,\tau)+\nu)^{r-1}|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau\\
&\quad + \gamma\int_0^{t_0}\iint_{\hat{B}\times\hat{B}} (u_{\bar{h}}(y,\tau)+\nu)^{p(r-1)}\frac{|\xi^p(x)-\xi^p(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau\\
&= H_7 + H_8.
\end{aligned}$$

Recalling that u is locally bounded in the present framework, we set

$$K_\nu = \begin{cases} \nu^{r-2} & \text{if } 1<r<2\\ \big(\|u\|_{L^\infty(\hat{B}\times(0,t_0))}+\nu\big)^{r-2} & \text{if } r\geqslant 2. \end{cases} \tag{A.10}$$

To estimate H_7, we apply Lagrange's rule and Proposition A.1 (iii):

$$\begin{aligned}
H_7 &\leqslant \gamma\int_0^{t_0}\iint_{\hat{B}\times\hat{B}} \max\big\{(u_{\bar{h}}(x,\tau)+\nu)^{r-2},\ (u_{\bar{h}}(y,\tau)+\nu)^{r-2}\big\}^p\\
&\qquad \frac{|u_{\bar{h}}(x,\tau)-u_{\bar{h}}(y,\tau)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau\\
&\leqslant \gamma K_\nu^p\int_0^{t_0}\iint_{\hat{B}\times\hat{B}} \frac{|u_{\bar{h}}(x,\tau)-u_{\bar{h}}(y,\tau)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau\\
&\leqslant \gamma K_\nu^p\|u\|^p_{L^p(0,t_0,W^{s,p}(\hat{B}))}.
\end{aligned}$$

For H_8, we act mainly on ξ:

$$\begin{aligned} H_8 &\leqslant \gamma \int_0^{t_0} \iint_{\hat{B}\times\hat{B}} (u_{\bar{h}}(y,\tau)+\nu)^{p(r-1)} \max\left\{\xi^{p-1}(x),\ \xi^{p-1}(y)\right\}^p \\ &\quad \frac{|\xi(x)-\xi(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau \\ &\leqslant \frac{\gamma}{\sigma^p\rho^p} \int_0^{t_0} \iint_{\hat{B}\times\hat{B}} \frac{(u_{\bar{h}}(y,\tau)+\nu)^{p(r-1)}}{|x-y|^{N+ps-p}}\,dx\,dy\,d\tau \\ &\leqslant \frac{\gamma}{\sigma^p\rho^p}\Big[\int_0^{t_0}\int_{\hat{B}} (u_{\bar{h}}(y,\tau)+\nu)^{p(r-1)}\,dy\,d\tau\Big]\Big[\sup_{y\in\hat{B}}\int_{\hat{B}}\frac{dx}{|x-y|^{N+ps-p}}\Big] \\ &\leqslant \frac{\gamma t_0\rho^{N-ps}}{\sigma^p}\big[\|u\|_{L^\infty(\hat{B}\times(0,t_0))}+\nu\big]^{p(r-1)}. \end{aligned}$$

Combining the previous bounds, we can find $\gamma > 0$ depending on ν, but not on h, s.t.

$$\int_0^{t_0}\iint_{\hat{B}\times\hat{B}} \frac{|F_h(x,y,\tau)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau \leqslant \gamma.$$

By reflexivity, passing to a sequence $h_n \to 0$, we get

$$\limsup_{h\to 0}\ H_5 \geqslant 0. \tag{A.11}$$

To estimate H_6, we first note that for all $(x, y) \in \tilde{B} \times \hat{B}^c$

$$|x-y| \geqslant \frac{\sigma}{2(1+\sigma)}|y-x_0|.$$

Using Lagrange's rule, subadditivity and the constant K_ν defined in (A.10), we have:

$$\begin{aligned} |H_6| &\leqslant \gamma \int_0^{t_0}\iint_{\tilde{B}\times\hat{B}^c} \frac{|G(x,y,\tau)|}{|x-y|^{N+ps}}\Big|(u_{\bar{h}}(x,\tau)+\nu)^{r-1}-(u(x,\tau)+\nu)^{r-1}\Big|\,dx\,dy\,d\tau \\ &\leqslant \gamma K_\nu \Big[\int_0^{t_0}\int_{\tilde{B}} |u_{\bar{h}}(x,\tau)-u(x,\tau)|\,dx\,d\tau\Big] \\ &\quad \Big[\sup_{(x,\tau)\in\tilde{B}\times(0,t_0)} \int_{\hat{B}^c}\frac{|u(x,\tau)-u(y,\tau)|^{p-1}}{|x-y|^{N+ps}}\,dy\Big] \\ &\leqslant \frac{\gamma K_\nu}{\sigma^{N+ps}}\|u_{\bar{h}}-u\|_{L^1(\tilde{B}\times(0,t_0))} \\ &\quad \Big[\|u\|^{p-1}_{L^\infty(\tilde{B}\times(0,t_0))}\int_{\hat{B}^c}\frac{dy}{|y-x_0|^{N+ps}} + \sup_{0<\tau<t_0}\int_{\hat{B}^c}\frac{|u(y,\tau)|^{p-1}}{|y-x_0|^{N+ps}}\,dy\Big] \\ &\leqslant \frac{\gamma K_\nu}{\sigma^{N+ps}\rho^{ps}}\|u_{\bar{h}}-u\|_{L^1(\tilde{B}\times(0,t_0))} \\ &\quad \Big[\|u\|^{p-1}_{L^\infty(\tilde{B}\times(0,t_0))} + \mathrm{Tail}(u,x_0,(1+\sigma)\rho,0,t_0)^{p-1}\Big] \end{aligned}$$

(where again we have used boundedness of u). By Proposition A.1 (ii) (with $q = 1$) the latter tends to 0 as $h \to 0$, hence

$$\lim_{h\to 0} H_6 = 0. \tag{A.12}$$

By (A.11) and (A.12) we obtain

$$\limsup_{h\to 0}\int_0^{t_0}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{G(x,y,\tau)}{|x-y|^{N+ps}}\big[F_h(x,y,\tau)-F(x,y,\tau)\big]\,dx\,dy\,d\tau\geqslant 0,$$

which in turn implies

$$\limsup_{\varepsilon,h,\nu\to 0} H_2\geqslant\frac{1}{\gamma}\int_0^{t_0}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x,\tau)-u(y,\tau)|^{p-2}(u(x,\tau)-u(y,\tau))}{|x-y|^{N+ps}}\Big[u^{r-1}(x,\tau)\xi^p(x)-u^{r-1}(y,\tau)\xi^p(y)\Big]\,dx\,dy\,d\tau. \tag{A.13}$$

Passing to the limit in (A.8) and using (A.9) and (A.13), we get

$$\begin{aligned}0\geqslant{}&\frac{1}{r}\int_{\tilde B}u^r(x,\tau)\xi^p(x)\,dx\Big|_0^{t_0}\\&+\frac{1}{\gamma}\int_0^{t_0}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x,\tau)-u(y,\tau)|^{p-2}(u(x,\tau)-u(y,\tau))}{|x-y|^{N+ps}}\\&\Big[u^{r-1}(x,\tau)\xi^p(x)-u^{r-1}(y,\tau)\xi^p(y)\Big]\,dx\,dy\,d\tau,\end{aligned}$$

which concludes the proof. □

We finally include the formal derivation of inequalities (6.1) and (6.3) in the proof of Theorem 1.5:

Lemma A.4 *Let Ω, u_0, u, and q be as in Theorem 1.5:*

(*i*) *if $1<p<p_c$, then there exists $\gamma>0$ depending on N,p,s s.t. for all $\psi\in W_0^{1,2}(0,T)$, $\psi\geqslant 0$ in $(0,T)$*

$$\int_0^T\Big[-\|u(\cdot,\tau)\|^q_{L^q(\Omega)}\psi'(\tau)+\frac{C_1}{\gamma}\|u(\cdot,\tau)\|^{p+q-2}_{L^q(\Omega)}\psi(\tau)\Big]\,d\tau\leqslant 0;$$

(*ii*) *if $p_c\leqslant p<2$, then there exists $\gamma>0$ depending on N,p,s s.t. for all $\psi\in W_0^{1,2}(0,T)$, $\psi\geqslant 0$ in $(0,T)$*

$$\int_0^T\Big[-\|u(\cdot,\tau)\|^2_{L^2(\Omega)}\psi'(\tau)+\frac{C_1}{\gamma}|\Omega|^{-\frac{\lambda_2}{2N}}\|u(\cdot,\tau)\|^p_{L^2(\Omega)}\psi(\tau)\Big]\,d\tau\leqslant 0.$$

Proof First recall that u is locally essentially bounded in Ω_T. We consider (*i*). We already know that $q>2$. Fix $h\in(0,1)$ and set for all $(x,\tau)\in\mathbb{R}^N\times(0,T)$

$$\varphi_h(x,\tau)=u_{\bar h}^{q-1}(x,\tau)\psi(\tau),$$

so that $\varphi_h\in W_0^{1,2}(0,T,L^2(\Omega))\cap L^p(0,T,W_0^{s,p}(\Omega))$ can be used as a test in Definition 2.2 (*i*), giving

$$\begin{aligned}0={}&\int_\Omega u(x,\tau)\varphi_h(x,\tau)\,dx\Big|_0^T-\int_0^T\int_\Omega u(x,\tau)\frac{\partial\varphi_h}{\partial\tau}(x,\tau)\,dx\,d\tau\\&+\int_0^T\iint_{\mathbb{R}^N\times\mathbb{R}^N}|u(x,\tau)-u(y,\tau)|^{p-2}(u(x,\tau)\\&-u(y,\tau))(\varphi_h(x,\tau)-\varphi_h(y,\tau))K(x,y,\tau)\,dx\,dy\,d\tau\\={}&H_1+H_2.\end{aligned}\tag{A.14}$$

We focus on the evolutive term H_1. Noting that $\varphi_h(\cdot,0)=\varphi_h(\cdot,T)=0$, we have

$$H_1 = -\int_0^T\int_\Omega u_{\bar h}(x,\tau)\frac{\partial\varphi_h}{\partial\tau}(x,\tau)\,dx\,d\tau + \int_0^T\int_\Omega (u_{\bar h}(x,\tau)-u(x,\tau))\frac{\partial\varphi_h}{\partial\tau}(x,\tau)\,dx\,d\tau$$
$$= H_3 + H_4.$$

Using Proposition A.1 (i) and the integration by parts formula (twice), we have

$$\begin{aligned} H_3 &= \int_0^T\int_\Omega \frac{\partial u_{\bar h}}{\partial\tau}(x,\tau)u_{\bar h}^{q-1}(x,\tau)\psi(\tau)\,dx\,d\tau \\ &= \frac{1}{q}\int_0^T\int_\Omega \frac{\partial u_{\bar h}^q}{\partial\tau}(x,\tau)\psi(\tau)\,dx\,d\tau \\ &= -\frac{1}{q}\int_0^T\int_\Omega u_{\bar h}^q(x,\tau)\psi'(\tau)\,dx\,d\tau. \end{aligned}$$

Therefore, by Proposition A.1 (ii) we have

$$\lim_{h\to 0} H_3 = -\frac{1}{q}\int_0^T\int_\Omega u^q(x,\tau)\psi'(\tau)\,dx\,d\tau.$$

Besides, recalling that $q>2$ and applying Proposition A.1 (i), we get

$$\begin{aligned} H_4 &= (q-1)\int_0^T\int_\Omega (u_{\bar h}(x,\tau)-u(x,\tau))u_{\bar h}^{q-2}(x,\tau)\frac{\partial u_{\bar h}}{\partial\tau}(x,\tau)\psi(\tau)\,dx\,d\tau \\ &\quad + \int_0^T\int_\Omega (u_{\bar h}(x,\tau)-u(x,\tau))u_{\bar h}^{q-1}(x,\tau)\psi'(\tau)\,dx\,d\tau \\ &= (q-1)\int_0^T\int_\Omega \frac{(u_{\bar h}(x,\tau)-u(x,\tau))^2}{h}u_{\bar h}^{q-2}(x,\tau)\psi(\tau)\,dx\,d\tau \\ &\quad + \int_0^T\int_\Omega u_{\bar h}^q(x,\tau)\psi'(\tau)\,dx\,d\tau - \int_0^T\int_\Omega u_{\bar h}^{q-1}(x,\tau)u(x,\tau)\psi'(\tau)\,dx\,d\tau. \end{aligned}$$

The first integral is non-negative, while the second and third integrals converge to the same limit as $h\to 0$ by Proposition A.1 (ii), so we have

$$\liminf_{h\to 0} H_4 \geqslant 0.$$

Combining the estimates above, we get

$$\liminf_{h\to 0} H_1 \geqslant -\frac{1}{q}\int_0^T \|u(\cdot,\tau)\|_{L^q(\Omega)}^q\psi'(\tau)\,d\tau. \tag{A.15}$$

We deal with the diffusive term H_2 as in Lemma A.2 (this case is in fact simpler). Define G as in Lemma A.2, and set for all $(x,y,t)\in\mathbb{R}^N\times\mathbb{R}^N\times(0,T)$

$$F_h(x,y,t) = (u_{\bar h}^{q-1}(x,t)-u_{\bar h}^{q-1}(y,t))\psi(t),$$

$$F(x,y,t) = (u^{q-1}(x,t)-u^{q-1}(y,t))\psi(t).$$

As usual we have $F_h(x,y,t)\to F(x,y,t)$ a.e. in $\mathbb{R}^N\times\mathbb{R}^N\times(0,T)$, as $h\to 0$. Also, by positivity and (K_2), we have

$$H_2 \geqslant C_1\int_0^T\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{G(x,y,\tau)F_h(x,y,\tau)}{|x-y|^{N+ps}}\,dx\,dy\,d\tau.$$

The fractional Poincaré inequality implies

$$\int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|G(x,y,t)|^{\frac{p}{p-1}}}{|x-y|^{N+ps}}\,dx\,dy\,d\tau \leqslant \|u\|^p_{L^p(0,T,W_0^{s,p}(\Omega))}.$$

We use boundedness of ψ, $q>2$, and Proposition A.1 (ii) (iii) to get

$$\begin{aligned}
&\int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|F_h(x,y,\tau)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau\\
&\leqslant \gamma \int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u_{\bar h}^{q-1}(x,\tau)-u_{\bar h}^{q-1}(y,\tau)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau\\
&\leqslant \gamma \int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \max\left\{u_{\bar h}^{q-2}(x,\tau),\ u_{\bar h}^{q-2}(y,\tau)\right\}^p \frac{|u_{\bar h}(x,\tau)-u_{\bar h}(y,\tau)|^p}{|x-y|^{N+ps}}\,dx\,dy\,d\tau\\
&\leqslant \gamma \|u\|^{p(q-2)}_{L^\infty(\Omega_T)}\|u\|^p_{L^p(0,T,W_0^{s,p}(\Omega))}.
\end{aligned}$$

By reflexivity, along a sequence $h_n\to 0$ we have

$$\begin{aligned}
\lim_n \int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} &\frac{G(x,y,\tau)F_{h_n}(x,y,\tau)}{|x-y|^{N+ps}}\,dx\,dy\,d\tau\\
&= \int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{G(x,y,\tau)F(x,y,\tau)}{|x-y|^{N+ps}}\,dx\,dy\,d\tau.
\end{aligned}$$

Passing to the limit in H_2, using Lemma 2.7 (i), and Sobolev's inequality (2.1), we get

$$\begin{aligned}
\limsup_{h\to 0} H_2 \geqslant C_1 \int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} &\frac{|u(x,\tau)-u(y,\tau)|^{p-2}(u(x,\tau)-u(y,\tau))}{|x-y|^{N+ps}}(u^{q-1}(x,\tau)\\
&- u^{q-1}(y,\tau))\psi(\tau)\,dx\,dy\,d\tau\\
&\geqslant \frac{C_1}{\gamma}\int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \left|u^{\frac{p+q-2}{p}}(x,\tau)-u^{\frac{p+q-2}{p}}(y,\tau)\right|^p \frac{dx\,dy}{|x-y|^{N+ps}}\psi(\tau)d\tau\\
&\geqslant \frac{C_1}{\gamma}\int_0^T \left[\int_\Omega u^{\frac{p+q-2}{p}p_s^*}(x,\tau)\,dx\right]^{\frac{p}{p_s^*}}\psi(\tau)\,d\tau\\
&= \frac{C_1}{\gamma}\int_0^T \|u(\cdot,\tau)\|^{p+q-2}_{L^q(\Omega)}\psi(\tau)\,d\tau,
\end{aligned} \tag{A.16}$$

where we have used the relation

$$\left(\frac{p+q-2}{p}\right)p_s^* = q.$$

Passing to the limit as $h\to 0$ in (A.14) and using (A.15) and (A.16), we arrive at

$$0 \geqslant -\frac{1}{q}\int_0^T \|u(\cdot,\tau)\|^q_{L^q(\Omega)}\psi'(\tau)\,dx\,d\tau + \frac{C_1}{\gamma}\int_0^T \|u(\cdot,\tau)\|^{p+q-2}_{L^q(\Omega)}\psi(\tau)\,d\tau,$$

which yields (i) with a new constant $\gamma>0$ depending on N, p, s. The argument for (ii) is entirely analogous, just replacing q with 2 and taking care of the measure-type multiplier. □

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