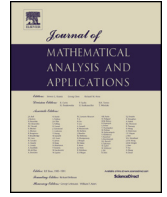




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Polyharmonic curves in semi-Riemannian manifolds [☆]

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ABSTRACT

Let (M_t^m, g) be a semi-Riemannian manifold of dimension m with a non-degenerate metric of index t , $m \geq 2$, $1 \leq t \leq m - 1$. The main aim of this paper is to investigate the existence of Frenet curves in (M_t^m, g) which are polyharmonic of order r , shortly, r -harmonic. We shall focus primarily on the cases that the ambient space is a semi-Riemannian space form $N_t^m(c)$ of sectional curvature c , a ruled Lorentzian surface or a suitable, possibly warped, product space. We shall obtain existence, non-existence and classification results.

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1. Introduction and statement of the results

Let (M_t^m, g) be an m -dimensional semi-Riemannian (or, pseudo-Riemannian) manifold, where g is a non-degenerate metric of index t , $1 \leq t \leq m - 1$. If $\gamma : I \rightarrow M_t^m$ is a smooth curve defined on an open real interval I , then we say that γ has one of the three causal characters, i.e., *space-like*, *light-like (null)* or *time-like* if $g(\gamma', \gamma') > 0$, $g(\gamma', \gamma') = 0$ or $g(\gamma', \gamma') < 0$ respectively on I .

In this paper we shall study curves γ which are either space-like or time-like. Then, to avoid trivialities arising from re-parametrization, we shall always assume that γ is parametrized with respect to the arc length s , i.e.,

$$\langle \gamma'(s), \gamma'(s) \rangle = g(\gamma'(s), \gamma'(s)) = \epsilon_1,$$

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where $\epsilon_1 = 1$ if γ is space-like and $\epsilon_1 = -1$ if γ is time-like. From now on, we shall use the notation $T = \gamma'(s)$.

The main topic of this paper is the study of polyharmonic curves of order r , shortly *r-harmonic curves*, $r \geq 2$. We recall that γ is *r-harmonic* if and only if its *r-tension field* $\tau_r(\gamma)$ vanishes. This condition, up to an irrelevant sign $(-1)^{r-1}$, is expressed by (see [2,15,17-19])

$$\tau_r(\gamma) = \nabla_T^{2r-1}T + \sum_{\ell=0}^{r-2} (-1)^\ell R(\nabla_T^{2r-3-\ell}T, \nabla_T^\ell T) T = 0, \tag{1.1}$$

where $\nabla_T^0 T = T$, $\nabla_T^k T = \nabla_T(\nabla_T^{k-1}T)$ and R is the semi-Riemannian curvature operator of the ambient space.

We point out that any geodesic is trivially *r-harmonic* for all $r \geq 1$. Therefore, we say that γ is a *proper r-harmonic curve* if it is *r-harmonic* and *not* a geodesic.

Several papers on the biharmonic case in this context can be found in the literature. By way of example, we cite [4,6-12]. More precisely, the classification of proper biharmonic curves in the flat 3-dimensional Lorentz-Minkowski space was achieved in [6,7]. Other related results for biharmonic curves in non flat Lorentz space forms were obtained by Sasahara in [22]. Further complementary results were proved in [11,12] in the special case that the ambient space is the 3-dimensional Minkowski space.

An interesting feature of these results is the fact that, differently from the Riemannian case, proper biharmonic curves do exist also in the *flat* Minkowski space.

By contrast, in the literature we found no paper dealing with *r-harmonic curves* in a semi-Riemannian manifold when $r \geq 3$.

Thus, our aim is to start the investigation of this open case.

We shall restrict our attention to the study of the so-called *n-Frenet curves*. We shall follow the approach of [23,25], where the authors focus on the study of helices of order $n \geq 3$.

Definition 1.1 (*Frenet curves (helices) of order n*). Let $(M_t^m, \langle \cdot, \cdot \rangle)$ be an m -dimensional semi-Riemannian manifold of index t , $1 \leq t \leq m - 1$. Let $\gamma : I \rightarrow M_t^m$ be a *nonnull* curve parametrized by the arc length s . We say that γ is a *Frenet curve of order n*, $n \leq m$, if it admits a *Frenet frame field* $\{F_1, \dots, F_n\}$ along γ which verifies:

$$\begin{cases} \nabla_T F_1 = \epsilon_2 k_1 F_2 \\ \nabla_T F_i = -\epsilon_{i-1} k_{i-1} F_{i-1} + \epsilon_{i+1} k_i F_{i+1} & (1 < i < n) \\ \nabla_T F_n = -\epsilon_{n-1} k_{n-1} F_{n-1}, \end{cases} \tag{1.2}$$

where $F_1 = T$ and

$$\epsilon_j = \langle F_j, F_j \rangle (= \pm 1) \quad (1 \leq j \leq n) \quad \text{and} \quad \langle F_i, F_j \rangle = 0 \quad (i \neq j),$$

and the functions $k_1(s), \dots, k_{n-1}(s)$ are positive. The function $k_i(s)$ is called the *i-th curvature* of the curve.

In the special case that M_t^m is oriented and $n = m \geq 2$ we allow that $k_{n-1}(s)$ can assume any real value because in this case F_n can be defined by requiring that $\{F_1, \dots, F_n\}$ is a positively oriented orthonormal base of $T_{\gamma(s)}M_t^m$. Also, we say that an *n-Frenet curve* γ is *full* if $k_{n-1}(s)$ never vanishes. Finally, we say that an *n-Frenet curve* is a *helix* if its curvatures $k_1(s), \dots, k_{n-1}(s)$ are constant functions which will be denoted $\kappa_1, \dots, \kappa_{n-1}$.

Remark 1.2. System (1.2) can be written in matrix form as follows:

$$\begin{bmatrix} \nabla_T F_1 \\ \nabla_T F_2 \\ \vdots \\ \nabla_T F_n \end{bmatrix} = \Omega \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix}$$

where

$$\Omega = \begin{bmatrix} 0 & \epsilon_2 k_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\epsilon_1 k_1 & 0 & \epsilon_3 k_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\epsilon_2 k_2 & 0 & \epsilon_4 k_3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\epsilon_{n-2} k_{n-2} & 0 & \epsilon_n k_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\epsilon_{n-1} k_{n-1} & 0 \end{bmatrix}$$

We point out that the matrix Ω is skew-symmetric if and only if

$$\epsilon_1 = \epsilon_2 = \cdots = \epsilon_n$$

which essentially corresponds to the Riemannian case.

Remark 1.3. In the notation of Definition 1.1, we must have $0 \leq \#\{\epsilon_j : \epsilon_j = -1\} \leq t$ and $0 \leq \#\{\epsilon_j : \epsilon_j = 1\} \leq (m - t)$.

First, let us discuss 2-Frenet curves. In this context, we simply write $k(s)$ to denote the first and unique curvature $k_1(s)$. For biharmonic curves ($r = 2$) into a 2-dimensional Lorentzian space form $N_1^2(c)$ we have the following comprehensive and well-known characterization (see, for example, [22]).

Theorem 1.4. *Let γ be a space-like proper biharmonic curve parametrized by the arc length s into a 2-dimensional Lorentzian space form $N_1^2(c)$. Then $k(s)$ is a constant, say $k(s) = \kappa > 0$. Moreover, biharmonicity is equivalent to*

$$\kappa^2 + c = 0. \tag{1.3}$$

In particular, if $c \geq 0$, then there exists no space-like proper biharmonic curve in $N_1^2(c)$.

In the general case of r -harmonic curves ($r > 3$) it is not automatic that the curvature $k(s)$ is constant. Thus, as a first, preliminary result, we shall prove:

Theorem 1.5. *Let $r \geq 2$. Let γ be a space-like curve parametrized by the arc length s into a 2-dimensional Lorentzian space form $N_1^2(c)$. Let us assume that γ has constant, non-zero curvature $k(s)$, say $k(s) = \kappa > 0$. Then γ is proper r -harmonic if and only if*

$$\kappa^2 + (r - 1)c = 0. \tag{1.4}$$

In particular, there exists no space-like proper r -harmonic curve when $c \geq 0$.

Remark 1.6. It is interesting to compare Theorem 1.5 with the corresponding result for the Riemannian case. Indeed, we recall that (see [2] and [16]) it was proved that the version of (1.4) in the case of curves into a 2-dimensional Riemannian space form $N^2(c)$ is

$$\kappa^2 - (r - 1)c = 0.$$

Things do change if we consider 2-Frenet helices in a higher dimensional semi-Riemannian space form. The main reason which explains the appearance of significant differences is the fact that in this case ϵ_1 and ϵ_2 may have the same sign. Then, in this context, the corresponding version of Theorem 1.5 is:

Theorem 1.7. *Let $r \geq 2$. Let γ be a 2-Frenet helix into an m -dimensional semi-Riemannian space form $N_t^m(c)$, $m \geq 3$, $1 \leq t \leq m - 1$. Then γ is proper r -harmonic if and only if*

$$\kappa^2 - \epsilon_2 (r - 1)c = 0. \quad (1.5)$$

The case of triharmonic curves appears to be special. Indeed, unlike the case where $r > 3$, 3-harmonicity forces $k(s)$ to be a constant. More precisely, we prove:

Theorem 1.8. *Let γ be a triharmonic 2-Frenet curve into an m -dimensional semi-Riemannian space form $N_t^m(c)$, $m \geq 2$, $1 \leq t \leq m - 1$. Then $k(s)$ is a constant function.*

This result leads us to the classification of proper triharmonic 2-Frenet curves in semi-Riemannian space forms. Specifically, we shall prove the following:

Corollary 1.9. *Let γ be a proper triharmonic 2-Frenet curve in a semi-Riemannian space form $N_t^m(c)$, $m \geq 2$, $1 \leq t \leq m - 1$. Then necessarily $\epsilon_2 c > 0$, where $\epsilon_2 = \langle N, N \rangle$, and the curvature of γ is a constant given by*

$$\kappa = \sqrt{2\epsilon_2 c}.$$

Remark 1.10. A 2-Frenet helix as in Corollary 1.9 necessarily lies on a totally geodesic $N_{t'}^2(c)$, $0 \leq t' \leq 2$. Since r -harmonicity is not affected by multiplication of the ambient metric by -1 , there are only two relevant cases, i.e., $t' = 0$ and $t' = 1$. If $t' = 0$, then $\epsilon_1 = \epsilon_2 = 1$, $c > 0$ and the curve is a triharmonic circle in an Euclidean sphere. The geometrically interesting case is $t' = 1$. In this case, $\epsilon_1 \epsilon_2 = -1$ and, considering the natural embeddings of $S_1^2(c)$ in \mathbb{R}_1^3 or $\mathbb{H}_1^2(c)$ in \mathbb{R}_2^3 , the curve in both cases is the intersection of $N_1^2(c)$ with a plane.

In Theorem 1.8 we have used in an essential way the fact that the 2-Frenet curves lie in an m -dimensional semi-Riemannian manifold with constant sectional curvature.

Therefore, a natural problem is to investigate the existence of r -harmonic, n -Frenet curves with *non constant* curvatures in a semi-Riemannian space such that its sectional curvature is not constant.

In this direction, let $\gamma : I \rightarrow \mathbb{R}_1^3$ be a curve parametrized by the arc length s that satisfies the Frenet equations.

$$\begin{cases} \nabla_T T = \epsilon_2 k_\gamma(s) N \\ \nabla_T N = -\epsilon_1 k_\gamma(s) T + \epsilon_3 \tau_\gamma(s) B \\ \nabla_T B = -\epsilon_2 \tau_\gamma(s) N. \end{cases} \quad (1.6)$$

Then we associate to γ a ruled surface S_γ immersed in \mathbb{R}_1^3 and locally described by the parametrization

$$X(s, v) = \gamma(s) + v N(s), \quad (1.7)$$

where $s \in I$ and $v \in (-\delta, \delta)$ for some $\delta > 0$.

Our main result in this context is a semi-Riemannian version of a result of [17]:

Theorem 1.11. *There exist a curve $\gamma : I \rightarrow \mathbb{R}_1^3$ parametrized by the arc length s , which verifies the Frenet equations (1.6) with $\epsilon_3 = 1$, and a small $\delta > 0$ such that the associated S_γ is a Lorentz surface and $\gamma(s) = X(s, 0)$ is a proper triharmonic curve in S_γ with non constant curvature $k(s)$.*

Now, we turn our attention to the study of 3-Frenet curves, where the two relevant curvatures will be denoted $k(s), \tau(s)$ as it is usual in the literature. In this setting, our first result is:

Theorem 1.12. *Let γ be a 3-Frenet helix parametrized by the arc length s into an m -dimensional semi-Riemannian space form $N_t^m(c)$, $m \geq 3$, $1 \leq t \leq m - 1$. Then*

- (i) γ is a proper biharmonic curve if and only if $\epsilon_2 (\epsilon_1 \kappa^2 + \epsilon_3 \tau^2) = c \epsilon_1$;
- (ii) γ is a proper triharmonic curve if and only if $\epsilon_2 (\epsilon_1 \kappa^2 + \epsilon_3 \tau^2)^2 = c (\epsilon_1 \epsilon_3 \tau^2 + 2\kappa^2)$;
- (iii) γ is a proper r -harmonic curve, $r \geq 4$, if and only if the following condition holds:

$$(\epsilon_1 \kappa^2 + \epsilon_3 \tau^2)^{r-3} \left[\epsilon_2 (\epsilon_1 \kappa^2 + \epsilon_3 \tau^2)^2 - c (\epsilon_1 \epsilon_3 \tau^2 + (r - 1)\kappa^2) \right] = 0. \tag{1.8}$$

According to Theorem 1.12 the existence of proper r -harmonic 3-Frenet helices depends strongly on the sign of $\epsilon_1, \epsilon_2, \epsilon_3$ and c . Therefore, we discuss now some relevant cases.

First, we focus on the geometrically significant Lorentzian case $N_1^3(c)$. In this space necessarily we have $\epsilon_3 = -\epsilon_1 \epsilon_2$. Our first corollary shows that in the flat case the condition of r -harmonicity depends on the casual character of the normal vector field N . More precisely, we have

Corollary 1.13. *Let γ be a 3-Frenet helix parametrized by the arc length s into \mathbb{R}_1^3 . Then γ is proper r -harmonic, $r \geq 2$, if and only if its normal vector field N is space-like and $\kappa^2 = \tau^2$.*

Remark 1.14. A 3-Frenet helix in \mathbb{R}_1^3 with $\epsilon_2 = 1$ and verifying $\kappa^2 = \tau^2$ is proper r -harmonic for all $r \geq 2$. This property is consistent with the fact that the ambient space is flat. Explicit examples of curves which verify the condition $\kappa^2 = \tau^2$ are given in [22, Proposition 3.4].

Next, we examine biharmonicity in a non flat Lorentzian case. Then we obtain:

Corollary 1.15. *Let γ be a 3-Frenet helix parametrized by the arc length s into a 3-dimensional Lorentzian space form $N_1^3(c)$ with $c \neq 0$.*

- (i) If $c < 0$ and the normal vector field N is space-like, then γ is a proper biharmonic curve if and only if $\tau^2 - \kappa^2 = -c$.
- (ii) If $c < 0$ and the normal vector field N is time-like, then γ is a proper biharmonic curve if and only if $\tau^2 + \kappa^2 = -c$.
- (iii) If $c > 0$ and the normal vector field N is space-like, then γ is a proper biharmonic curve if and only if $\kappa^2 = \tau^2 + c$.
- (iv) If $c > 0$ and the normal vector field N is time-like, then γ cannot be proper biharmonic.

Finally, here is a consequence of Theorem 1.12 in the non flat case.

Corollary 1.16. *Assume that $r \geq 3$. Let γ be a 3-Frenet helix parametrized by the arc length s into a non flat semi-Riemannian space form $N_1^3(c)$.*

- (i) If $c > 0$ and the normal vector field N is space-like, then γ is a proper r -harmonic curve if and only if $\kappa^2 = \tau^2$ for $r > 3$ or, for $r \geq 3$, if one of the following cases holds:

- (a) $\kappa^2 \geq c(r-1)$ and $\tau^2 = \frac{2\kappa^2 - c \pm \sqrt{c^2 + 4c(r-2)\kappa^2}}{2}$;
- (b) $\kappa^2 < c(r-1)$ and $\tau^2 = \frac{2\kappa^2 - c + \sqrt{c^2 + 4c(r-2)\kappa^2}}{2}$.
- (ii) If $c > 0$ and the normal vector field N is time-like, then γ cannot be proper r -harmonic for any $r \geq 3$.
- (iii) If $c < 0$ and the normal vector field N is space-like, the γ is a proper r -harmonic curve if and only if $\kappa^2 = \tau^2$ for $r > 3$ or

$$\kappa^2 \leq -\frac{c}{4(r-2)} \text{ and } \tau^2 = \frac{2\kappa^2 - c \pm \sqrt{c^2 + 4c(r-2)\kappa^2}}{2}$$

for $r \geq 3$.

- (iv) If $c < 0$ and the normal vector field N is time-like, then γ is a proper r -harmonic curve, $r \geq 3$, if and only if:

$$\kappa^2 \leq -c(r-1) \text{ and } \tau^2 = \frac{-2\kappa^2 - c + \sqrt{c^2 - 4c(r-2)\kappa^2}}{2}.$$

Remark 1.17. It is interesting to point out that Corollary 1.16 guarantees the existence of r -harmonic helices in a semi-Riemannian space form $N_1^3(c)$ with $c < 0$, regardless of the character of the normal vector field N . This should be compared with the corresponding Riemannian case, where proper r -harmonic helices do not exist when the ambient space has constant negative curvature.

Next, let us turn our attention to n -Frenet curves with $n \geq 4$. Then our first result is:

Theorem 1.18. Let γ be an n -Frenet curve parametrized by the arc length s into an m -dimensional semi-Riemannian manifold (M_t^m, g) , $1 \leq t \leq m-1$, $m \geq n \geq 4$. If

$$\langle R(F_1, F_2)F_1, W \rangle = 0 \quad \forall W \in (\text{Span}\{F_1, \dots, F_n\})^\perp, \quad (1.9)$$

then γ is proper biharmonic if and only if:

$$\begin{cases} k_1 = \kappa_1 = \text{constant} > 0 \\ \epsilon_1 k_1^2 + \epsilon_3 k_2^2 = \langle R(F_2, F_1)F_1, F_2 \rangle \\ k_2' = -\langle R(F_2, F_1)F_1, F_3 \rangle \\ \epsilon_3 k_2 k_3 = -\langle R(F_2, F_1)F_1, F_4 \rangle \\ \langle R(F_2, F_1)F_1, F_i \rangle = 0 \quad 5 \leq i \leq n. \end{cases} \quad (1.10)$$

Since (1.9) is automatically satisfied when the ambient space is a semi-Riemannian space form, we have the following immediate consequence:

Corollary 1.19. Let γ be an n -Frenet curve parametrized by the arc length s into an m -dimensional semi-Riemannian space form $N_t^m(c)$, $1 \leq t \leq m-1$, $m \geq n \geq 4$. Then γ is proper biharmonic if and only if:

$$\begin{cases} k_1 = \kappa_1 = \text{constant} > 0 \\ k_2 = \kappa_2 = \text{constant} > 0 \\ \epsilon_1 k_1^2 + \epsilon_3 k_2^2 = c \epsilon_1 \epsilon_2 \\ \epsilon_3 k_2 k_3 = 0. \end{cases} \quad (1.11)$$

In particular, there exists no proper biharmonic, full n -Frenet curve in $N_t^m(c)$, $m \geq n \geq 4$.

Remark 1.20. The last sentence of Corollary 1.19 displays a phenomenon analogous to the Riemannian case (compare with [19, Theorem 1.9]). Moreover, as in the Riemannian case, 3-Frenet biharmonic curves in a semi-Riemannian space-form are necessarily helices.

Therefore, condition (i) of Theorem 1.12 provides the complete classification of proper biharmonic Frenet curves in a semi-Riemannian space form.

Specializing to the triharmonic case we obtain:

Theorem 1.21. *Let γ be a n -Frenet helix parametrized by the arc length s into $N_t^n(c)$, $1 \leq t \leq n - 1$.*

(i) *If $n = 4$, then γ is triharmonic if and only if:*

$$\begin{aligned} (\epsilon_1 \kappa_1^2 + \epsilon_3 \kappa_2^2)^2 + \epsilon_2 \epsilon_4 \kappa_2^2 \kappa_3^2 &= c\epsilon_1\epsilon_2 (2\epsilon_1 \kappa_1^2 + \epsilon_3 \kappa_2^2) \\ \epsilon_2(\epsilon_1 \kappa_1^2 + \epsilon_3 \kappa_2^2) + \epsilon_3\epsilon_4 \kappa_3^2 &= c\epsilon_1. \end{aligned}$$

(ii) *If $n = 5$, then γ is triharmonic if and only if:*

$$\begin{aligned} (\epsilon_1 \kappa_1^2 + \epsilon_3 \kappa_2^2)^2 + \epsilon_2 \epsilon_4 \kappa_2^2 \kappa_3^2 &= c\epsilon_1\epsilon_2 (2\epsilon_1 \kappa_1^2 + \epsilon_3 \kappa_2^2) \\ \epsilon_2(\epsilon_1 \kappa_1^2 + \epsilon_3 \kappa_2^2) + \epsilon_4(\epsilon_3 \kappa_3^2 + \epsilon_5 \kappa_4^2) &= c\epsilon_1. \end{aligned}$$

Remark 1.22. The Riemannian version of Theorem 1.21 was first obtained by Maeta in [16, Proposition 5.9]. Maeta’s result corresponds to the condition of Theorem 1.21 with $\epsilon_j = 1$ for all j .

As a consequence of Theorem 1.21 we are able to point out *an important difference with respect to the Riemannian case*. More precisely, we know (see [16] and [19, Theorem 5.1]) that there exists no proper triharmonic n -Frenet helix in a Riemannian space form $N^n(c)$, $n \geq 4$. By contrast, as an application of Theorem 1.21, we have:

Example 1.23. There exists a space-like proper triharmonic 5-Frenet helix in S_1^5 with the following features:

$$\epsilon_3 = -1, \epsilon_j = 1 \quad \text{if } j \neq 3; \quad \kappa_1 = \kappa_2 = \kappa_3 = 1, \kappa_4 = \sqrt{2}.$$

Also, with a similar analysis:

Example 1.24. There exists a space-like proper triharmonic 4-Frenet helix in S_2^4 with the following features:

$$\epsilon_2 = \epsilon_3 = -1, \epsilon_1 = \epsilon_4 = 1; \quad \kappa_1 = \sqrt{2}, \kappa_2 = 2, \kappa_3 = 1.$$

Now, we turn our attention to more general ambient spaces. First, we establish the following result:

Proposition 1.25. *Let γ be a 3-Frenet helix in any semi-Riemannian manifold M_t^m , where $m \geq 3$ and $1 \leq t \leq m - 1$. If the curvatures κ, τ of the helix γ verify*

$$\epsilon_1 \kappa^2 + \epsilon_3 \tau^2 = 0, \tag{1.12}$$

then γ is proper r -harmonic for all $r \geq 4$.

To continue our investigation in ambient spaces with non constant sectional curvature we focus on the Lorentzian product space $(\mathbb{R} \times N^{m-1}(c), g_{\text{prod}})$, where $(N^{m-1}(c), g)$ is a Riemannian $(m - 1)$ -dimensional space form with $c \neq 0$, $m \geq 4$ and

$$g_{\text{prod}} = -dt^2 + g.$$

In this context, we introduce a family of curves which will play an interesting role. More precisely, let $\alpha(s)$ be a 3-Frenet helix in $N^{m-1}(c)$. Note that here s is the arc length parameter in $N^{m-1}(c)$. Then we consider curves γ in $(\mathbb{R} \times N^{m-1}(c), g_{\text{prod}})$ of the type

$$\gamma(s) = (d_1 s, \alpha(d_2 s)) \quad (1.13)$$

where, to insure that γ is parametrized by the arc length, we require that the constants d_1, d_2 satisfy

$$\epsilon_1 + d_1^2 > 0 \quad \text{and} \quad d_2 = \sqrt{\epsilon_1 + d_1^2}. \quad (1.14)$$

Our main result in this setting is:

Theorem 1.26. *Assume $c \neq 0, m \geq 4$. Let γ be a 3-Frenet curve in $(\mathbb{R} \times N^{m-1}(c), g_{\text{prod}})$ of the type (1.13) and assume that*

$$\epsilon_3 (\tau_\alpha^2 - \epsilon_1 d_1^2 \kappa_\alpha^2) > 0, \quad (1.15)$$

where k_α, τ_α are the curvatures of the helix α in $N^{m-1}(c)$. Then γ is a helix which does not verify (1.12). Moreover, for all $r \geq 2$, γ is proper r -harmonic in $(\mathbb{R} \times N^{m-1}(c), g_{\text{prod}})$ if and only if α is proper r -harmonic in $N^{m-1}(c)$.

As an immediate consequence of Theorem 1.26 we have the following explicit construction of r -harmonic 3-Frenet curves in $(\mathbb{R} \times N^{m-1}(c), g_{\text{prod}})$.

Corollary 1.27. *Let $\alpha(s)$ be a proper r -harmonic 3-Frenet helix in $N^{m-1}(c)$, $r \geq 2$. Then, choosing $d_1, d_2 \neq 0$ such that $d_1^2 = d_2^2 + 1$, the curve $\gamma(s)$ defined in (1.13) is a time-like, proper r -harmonic helix in $(\mathbb{R} \times N^{m-1}(c), g_{\text{prod}})$.*

In the last section of the paper we shall analyze another important case where the sectional curvature of the ambient space is not constant. More precisely, let J be a real open interval. Then a *Robertson-Walker space-time* is a semi-Riemannian manifold $\mathcal{RW}_1^m = (J \times N^{m-1}(c), g_f)$, where g_f is a Lorentzian metric defined by

$$g_f = -dt^2 + f^2(t)g, \quad (1.16)$$

where f is a smooth positive function on J . These space-times, which are largely used in theoretical physics and in the study of cosmological phenomena, appeared for the first time in the context of harmonic maps in [21].

Our main results in this context are:

Theorem 1.28. *Let $\alpha(s), s \in I$, be a geodesic in $N^{m-1}(c)$ and consider a curve $\gamma : I \rightarrow \mathcal{RW}_1^m$, $\gamma(s) = (t_0, \alpha(ds))$, where t_0 is any fixed real value in the interval J and $d = 1/f(t_0)$. Assume that $r \geq 2$. Then $\gamma(s)$ is a proper r -harmonic curve in \mathcal{RW}_1^m if and only if*

$$f'(t_0) \neq 0 \quad \text{and} \quad f'^2(t_0) + (r-1)f(t_0)f''(t_0) = 0.$$

Theorem 1.29. *Let $\gamma : I \rightarrow \mathcal{RW}_1^m$ be a curve parametrized by the arc length of the form $\gamma(s) = (t_0, \alpha(s))$, where t_0 is a fixed value on J such that $f'(t_0) = 0$. Then $\gamma(s)$ is r -harmonic in \mathcal{RW}_1^m if and only if $\beta(s) = \alpha(f(t_0)s)$ is r -harmonic in $N^{m-1}(c)$.*

As a final remark, we recall that the quantity

$$q = -\frac{f(t)f''(t)}{f'(t)^2}$$

is called the *deceleration parameter* of the Robertson-Walker space-time \mathcal{RW}_1^m . Relevant cases of physical interest are those corresponding to constant positive values of q which are achieved by taking

$$f(t) = t^\lambda, \quad 0 < \lambda < 1$$

which gives

$$q = -\frac{\lambda - 1}{\lambda}.$$

Taking $\lambda = (r - 1)/r$, as an immediate consequence of Theorem 1.28 we have:

Corollary 1.30. *Let us fix $r \geq 2$ and consider a Robertson-Walker space-time \mathcal{RW}_1^m with constant positive deceleration parameter*

$$q = \frac{1}{r-1} \quad \text{i.e., } f(t) = t^{\frac{r-1}{r}}.$$

Let $\alpha(s), s \in I$, be a geodesic in $N^{m-1}(c)$ and consider any curve $\gamma : I \rightarrow \mathcal{RW}_1^m$ of the type $\gamma(s) = (t_0, \alpha(s/f(t_0)))$, where t_0 is any positive fixed real value. Then γ is proper r -harmonic.

Our paper is organized as follows.

In order to make this article reasonably self-contained we have recalled in Section 2 some relevant preliminary notions. In Section 3 we shall prove the stated results concerning 2-Frenet curves in space forms. In Section 4 we shall prove the existence of a proper triharmonic curve with non constant curvature $k(s)$ in a ruled Lorentzian surface S_γ , as stated in Theorem 1.11. In Section 5 we shall focus on 3-Frenet curves and we shall prove the corresponding results stated in this introduction. Section 6 contains the proofs of the stated results concerning n -Frenet curves, $n \geq 4$. In Section 7 we shall prove Proposition 1.25 and Theorem 1.26. Last, in Section 8 we shall prove the stated results concerning r -harmonic curves in Robertson-Walker space-times.

2. Preliminaries

A basic reference for semi-Riemannian geometry is the classical book of O'Neill (see [20]), but for the specific topics treated in this paper we also refer to [3,5,13,14,22].

Let (M_t^m, g) be a semi-Riemannian manifold of dimension m with a *non-degenerate* metric of index t . Since in this paper we are not interested in the Riemannian case we shall always assume that $1 \leq t \leq m-1$. We recall that non-degenerate means that the only vector $X \in T_p M$ satisfying $g_p(X, Y) = 0$ for all $Y \in T_p M$ is $X = 0$, for any $p \in M$. A local *orthonormal* frame field of (M_t^m, g) is a set of local vector fields $\{e_i\}_{i=1}^m$ such that $g(e_i, e_j) = \varepsilon_i \delta_{ij}$, with $\varepsilon_1 = \dots = \varepsilon_t = -1$, $\varepsilon_{t+1} = \dots = \varepsilon_m = 1$.

Note that we shall write

$$\|X\| = \sqrt{|g(X, X)|}$$

for any vector field X tangent to M . Also, the notation $\langle X, Y \rangle$ for $g(X, Y)$ will be used throughout this paper.

In this article we shall focus primarily on the case that the ambient space is a semi-Riemannian space form. Therefore, we now fix terminology and notations concerning these ambient spaces.

First, the m -dimensional pseudo-Euclidean space with index t is denoted by $\mathbb{R}_t^m = (\mathbb{R}^m, \langle, \rangle)$, where

$$\langle x, y \rangle = - \sum_{i=1}^t x_i y_i + \sum_{i=t+1}^m x_i y_i.$$

The m -dimensional semi-Riemannian sphere, denoted by $\mathbb{S}_t^m(c)$ is defined as follows:

$$\mathbb{S}_t^m(c) = \left\{ x \in \mathbb{R}_t^{m+1} : \langle x, x \rangle = \frac{1}{c} \right\} \quad (c > 0). \quad (2.1)$$

$\mathbb{S}_t^m(c)$, with the induced metric from \mathbb{R}_t^{m+1} , is a complete semi-Riemannian manifold with index t and constant positive sectional curvature c .

The m -dimensional semi-Riemannian hyperbolic space, denoted by $\mathbb{H}_t^m(c)$, is defined by

$$\mathbb{H}_t^m(c) = \left\{ x \in \mathbb{R}_{t+1}^{m+1} : \langle x, x \rangle = \frac{1}{c} \right\} \quad (c < 0). \quad (2.2)$$

$\mathbb{H}_t^m(c)$, with the induced metric from \mathbb{R}_{t+1}^{m+1} , is a complete semi-Riemannian manifold with index t and constant negative sectional curvature c .

A semi-Riemannian space form $N_t^m(c)$ refers to one of the three spaces \mathbb{R}_t^m , $\mathbb{S}_t^m(c)$, $\mathbb{H}_t^m(c)$. We shall write \mathbb{S}_t^m and \mathbb{H}_t^m for $\mathbb{S}_t^m(1)$ and $\mathbb{H}_t^m(-1)$ respectively.

The flat ($c = 0$) semi-Riemannian space \mathbb{R}_t^m is called *Minkowski space*, while $\mathbb{S}_t^m(c)$ and $\mathbb{H}_t^m(c)$ are known as *de Sitter space* and *anti-de Sitter space* respectively. When the index is $t = 1$, these spaces are also referred to as *Lorentz space forms*. We also point out that $\mathbb{S}_t^m(c)$ is diffeomorphic to $\mathbb{R}^t \times \mathbb{S}^{m-t}$, while $\mathbb{H}_t^m(c)$ is diffeomorphic to $\mathbb{S}^t \times \mathbb{R}^{m-t}$. In particular, $\mathbb{S}_{m-1}^m(c)$ and $\mathbb{H}_1^m(c)$ are not simply connected.

In this paper we shall adopt the following notation and sign convention for the Riemannian curvature tensor field R of the ambient space M_t^m :

$$\begin{aligned} R(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \\ R(X, Y, Z, W) &= \langle R(X, Y)W, Z \rangle. \end{aligned}$$

In particular, the sectional curvature tensor field of $N_t^m(c)$ is described by the following simple expression:

$$R^{N(c)}(X, Y)Z = c (\langle Y, Z \rangle X - \langle X, Z \rangle Y) \quad \forall X, Y, Z \in C(TN(c)). \quad (2.3)$$

In this paper we shall establish several algebraic conditions which are equivalent to the r -harmonicity of an n -Frenet helix. In the case of curves with non constant curvatures, algebraic conditions will be replaced by a system of differential equations. A common feature to all these cases is the dependence on the ϵ_i 's associated to the Frenet frame.

Therefore, it is vital for us to know that our results are supported by the following semi-Riemannian version of the Fundamental Theorem of curves. More specifically, any time that we prove that certain curvature conditions associated with appropriate ϵ_i 's verify the r -harmonicity equation we shall be able to conclude that there exists a curve with those requisites.

Theorem 2.1 (*The Fundamental Theorem of Curves in Semi-Riemannian Geometry*). *Let $M = (M_t^m, \langle, \rangle)$ be an m -dimensional semi-Riemannian manifold of index $1 \leq t \leq m - 1$. Given smooth, positive curvature functions $k_1(s), \dots, k_{n-1}(s)$ defined on an open real neighborhood of $s_0 \in \mathbb{R}$, an initial point $p_0 \in M$ and an*

initial orthonormal frame $\{T_0, F_{2,0}, \dots, F_{n,0}\}$ at p_0 there exists an n -Frenet curve $\gamma(s)$ in M parametrized by the arc length s such that

- (i) $\gamma(s_0) = p_0$;
- (ii) The n -Frenet frame field $\{T, F_2, \dots, F_n\}$ of γ coincides with $\{T_0, F_{2,0}, \dots, F_{n,0}\}$ at $s = s_0$;
- (iii) The given functions $k_1(s), \dots, k_{n-1}(s)$ are the curvatures of $\gamma(s)$.

This result can be obtained adapting the classical argument of the Riemannian case (see [24]) but, since we were not able to find a precise reference in the literature, for the sake of completeness we have added its proof in Appendix A at the end of the paper.

Remark 2.2. The construction of the Frenet frame field $\{F_1, \dots, F_n\}$ is based on the orthonormalization of the set of vector fields

$$\{\nabla_T^0 T = T, \nabla_T T, \dots, \nabla_T^{n-1} T\}.$$

Here we point out that the correct generalization of the Gram-Schmidt algorithm in a semi-Riemannian context uses the idea of a *non-degenerate basis*. Indeed, if V is an n -dimensional vector space endowed with a non-degenerate scalar product g , an ordered basis $\{E_1, \dots, E_n\}$ for V is said to be non-degenerate if, for each $k = 1, \dots, n$, the scalar product g restricts to a non-degenerate scalar product on the subspace spanned by $\{E_1, \dots, E_k\}$.

Now, if V has an ordered non-degenerate basis $\{v_1, \dots, v_n\}$, then the Gram-Schmidt algorithm applied to $\{v_1, \dots, v_n\}$ produces an orthonormal basis $\{E_1, \dots, E_n\}$ with the property that $\text{span}\{E_1, \dots, E_k\} = \text{span}\{v_1, \dots, v_k\}$ for each $k = 1, \dots, n$.

3. r -harmonic 2-Frenet curves

Here we prove the results stated in Section 1 concerning 2-Frenet curves in semi-Riemannian space forms.

As a preliminary work, we study *nonnull* r -harmonic curves in a semi-Riemannian surface $(M_1^2, \langle, \rangle)$. Let $\gamma : I \rightarrow (M_1^2, \langle, \rangle)$ be a *non-degenerate curve*, i.e., a curve such that

$$\langle \nabla_T T, \nabla_T T \rangle \neq 0$$

on I . Then we can define

$$\epsilon_2 = \text{sign}(\langle \nabla_T T, \nabla_T T \rangle) (= \pm 1)$$

and the principal unit normal vector field N together with its associated positive curvature function $k(s)$:

$$\nabla_T T = \epsilon_2 k(s) N.$$

Then γ is a 2-Frenet curve and we have the Frenet system of equations

$$\begin{cases} \nabla_T T = \epsilon_2 k(s) N \\ \nabla_T N = -\epsilon_1 k(s) T \end{cases}$$

where $\langle T, T \rangle = \epsilon_1$, $\langle N, N \rangle = \epsilon_2$, $\epsilon_1 \epsilon_2 = -1$. In this context we also have (see [20])

$$K_M = \frac{\langle R(T, N)N, T \rangle}{\epsilon_1 \epsilon_2} = -\langle R(T, N)N, T \rangle,$$

where K_M denotes the Gaussian curvature of the semi-Riemannian surface $(M_1^2, \langle \cdot, \cdot \rangle)$. A first routine computation leads us to the following proposition.

Proposition 3.1. *Let γ be a non-degenerate curve parametrized by the arc length s into an m -dimensional semi-Riemannian manifold $(M_t^m, \langle \cdot, \cdot \rangle)$. Then its bitension and tritension fields are given respectively by:*

$$\tau_2(\gamma) = [-3\epsilon_1\epsilon_2 k(s)k'(s)]T + [\epsilon_2 (K_M(s)\epsilon_1 k(s) + k''(s) - \epsilon_1\epsilon_2 k(s)^3)]N; \quad (3.1)$$

$$\begin{aligned} \tau_3(\gamma) = & [-5\epsilon_1\epsilon_2 k^{(3)}(s)k(s) + 10k(s)^3 k'(s) - 10\epsilon_1\epsilon_2 k'(s)k''(s)]T \\ & + [\epsilon_2 K_M(s)(\epsilon_1 k''(s) - 2\epsilon_2 k(s)^3) + \epsilon_2 k^{(4)}(s) - 10\epsilon_1 k(s)^2 k''(s) \\ & - 15\epsilon_1 k(s)k'(s)^2 + \epsilon_2 k(s)^5]N, \end{aligned} \quad (3.2)$$

where, with a slight abuse of notation, we have written $K_M(s)$ for $K_M(\gamma(s))$.

Next, let us assume that $k(s)$ is a constant function, say $k(s) = \kappa$. Then it is easy to deduce from the Frenet equations that for all $\ell \geq 0$ we have:

$$\begin{cases} \nabla_T^{2\ell} T = (-\epsilon_1 \epsilon_2)^\ell \kappa^{2\ell} T \\ \nabla_T^{2\ell+1} T = (-\epsilon_1 \epsilon_2)^\ell \epsilon_2 \kappa^{2\ell+1} N. \end{cases} \quad (3.3)$$

Remark 3.2. When $m = 2$, $-\epsilon_1\epsilon_2 = 1$. However, we have preferred to write $-\epsilon_1\epsilon_2$ in (3.3) because this makes the formula applicable in the more general context $m \geq 3$, as we shall see in the proof of Theorem 1.7, for instance.

Proof of Theorem 1.4. The result follows immediately from (3.1) with $K_M(s) = c = \text{constant}$. \square

Proof of Theorem 1.5. We have to compute $\tau_r(\gamma)$. Using (1.1), (2.3) and (3.3) it is straightforward to compute

$$\tau_r(\gamma) = C [\kappa^2 - \epsilon_2 (r-1)c]N,$$

where

$$C = \epsilon_2 \kappa^{2r-3}. \quad \square$$

Proof of Theorem 1.8. The formula (3.2) with $K_M(s) = c = \text{constant}$ is valid in this context. Explicit integration of the tangential component of the 3-tension fields leads us to

$$k''(s) = \frac{2}{5}\epsilon_1\epsilon_2 k(s)^3 + \frac{c_1}{k(s)^2} \quad (3.4)$$

and

$$k'^2(s) = \frac{\epsilon_1\epsilon_2}{5}k^4(s) - \frac{2c_1}{k(s)} + c_2 \quad (3.5)$$

where c_1, c_2 are integration constants. From (3.4) and (3.5) it is easy to deduce that

$$k^{(4)}(s) = \frac{2(5c_1 + 2\epsilon_1\epsilon_2k(s)^5)(-35c_1 + 15c_2k(s) + 6\epsilon_1\epsilon_2k(s)^5)}{25k(s)^5}.$$

Finally, substituting these expressions into the normal component of the 3-tension field we find that $k(s)$ must be a root of a not identically zero polynomial and so $k(s)$ must be a constant as required. \square

Finally, putting together Theorem 1.7 and Theorem 1.8, we easily deduce the classification Corollary 1.9.

4. Proof of Theorem 1.11

We recall that here the problem is to investigate the existence of r -harmonic curves with *non constant* curvatures. To a given curve $\gamma : I \rightarrow \mathbb{R}_1^3$, parametrized by the arc length s and verifying the Frenet equations

$$\begin{cases} \nabla_T T = \epsilon_2 k_\gamma(s) N \\ \nabla_T N = -\epsilon_1 k_\gamma(s) T + \epsilon_3 \tau_\gamma(s) B \\ \nabla_T B = -\epsilon_2 \tau_\gamma(s) N, \end{cases} \tag{4.1}$$

we associate a ruled surface S_γ immersed in \mathbb{R}_1^3 and locally described by the parametrization

$$X(s, v) = \gamma(s) + v N(s), \tag{4.2}$$

where $s \in I$ and $v \in (-\delta, \delta)$ for some $\delta > 0$.

Now, let us find under which conditions the curve $\gamma(s) = X(s, 0)$ is a triharmonic curve into the ruled surface S_γ and S_γ is a Lorentz surface of \mathbb{R}_1^3 . A local frame field for the tangent plane of S_γ is given by the coordinate vector fields

$$X_s = (1 - \epsilon_1 k_\gamma(s)v) T + \epsilon_3 \tau_\gamma(s)v B; \quad X_v = N.$$

Thus the coefficients of the first fundamental form of S_γ are

$$\begin{aligned} E &= \langle X_s, X_s \rangle = \epsilon_1(1 - \epsilon_1 k_\gamma(s)v)^2 + \epsilon_3 \tau_\gamma^2 v^2; \\ F &= \langle X_s, X_v \rangle = \langle (1 - \epsilon_1 k_\gamma(s)v) T + \epsilon_3 \tau_\gamma(s)v B, N \rangle = 0; \\ G &= \langle X_v, X_v \rangle = \epsilon_2. \end{aligned} \tag{4.3}$$

If we denote by N^S the normal vector field to the surface S_γ in \mathbb{R}_1^3 we have

$$N^S = \frac{X_s \wedge X_v}{\|X_s \wedge X_v\|} = \frac{-\epsilon_1 \epsilon_3 \tau_\gamma(s)v T + \epsilon_3(1 - \epsilon_1 k_\gamma(s)v) B}{\sqrt{|\epsilon_1(\tau_\gamma(s)v)^2 + \epsilon_3(1 - \epsilon_1 k_\gamma(s)v)^2|}},$$

where we have used that $T \wedge N = \epsilon_3 B$. Restricting the unit normal N^S along the curve $\gamma(s) = X(s, 0)$ we obtain $N^S(s) = \epsilon_3 B(s)$. Therefore, S_γ is a Lorentzian surface provided that $\epsilon_3 = 1$ and $\delta > 0$ is sufficiently small.

Next, if we denote by ∇^S the Levi-Civita connection of S_γ and by $\epsilon^{N^S} = \langle N^S, N^S \rangle$, we have

$$\nabla_T T = \nabla_T^S T + \epsilon^{N^S} \langle \nabla_T T, N^S \rangle N^S = \nabla_T^S T$$

because, from (4.1), $\langle \nabla_T T, N^S \rangle = \langle \epsilon_2 k_\gamma(s) N, \epsilon_3 B \rangle = 0$. Thus $k(s) = k_\gamma(s)$.

Now we compute the Gaussian curvature K_S of the surface S_γ along γ . Since $F = 0$, we can use the standard formula (see [20, p.81])

$$K_S = -\frac{1}{\sqrt{|EG|}} \left(\text{sign}(E) \left(\frac{(\sqrt{|G|})_s}{\sqrt{|E|}} \right)_s + \text{sign}(G) \left(\frac{(\sqrt{|E|})_v}{\sqrt{|G|}} \right)_v \right)$$

to obtain

$$K_S = -\epsilon_2 \left(\frac{2k_\gamma^2 + 2\epsilon_1\epsilon_3\tau_\gamma^2}{2} - \frac{4k_\gamma^2}{4} \right) = -\epsilon_1\epsilon_2\epsilon_3\tau_\gamma^2 = \tau_\gamma^2.$$

According to (3.2), we see that the condition of triharmonicity is equivalent to the following system of equations:

$$\begin{cases} k_\gamma^{(3)}k_\gamma + 2k_\gamma^3k'_\gamma + 2k'_\gamma k''_\gamma = 0; \\ (\epsilon_1 k''_\gamma - 2\epsilon_2 k_\gamma^3)\tau_\gamma^2 + k_\gamma^{(4)}(s) + 10k_\gamma^2 k''_\gamma + 15k_\gamma(k'_\gamma)^2 + k_\gamma^5 = 0. \end{cases} \quad (4.4)$$

Since we look for curves γ which are *proper* triharmonic, we can assume $k_\gamma > 0$. Then, multiplying the first equation of (4.4) by k_γ , we deduce

$$(k''_\gamma k_\gamma^2)' + \frac{2}{5}(k_\gamma^5)' = 0.$$

Now integration yields

$$5k''_\gamma k_\gamma^2 + 2k_\gamma^5 = c_1$$

for some constant $c_1 \in \mathbb{R}$. Since we are interested in non constant solutions, we also assume that $k'_\gamma \neq 0$. Then, after multiplying by $2k'_\gamma k_\gamma^{-2}$, we deduce that:

$$(5(k'_\gamma)^2)' + (k_\gamma^4)' + \left(\frac{2c_1}{k_\gamma}\right)' = 0.$$

Integrating again, we obtain

$$5(k'_\gamma)^2 + k_\gamma^4 + \frac{2c_1}{k_\gamma} = c_2 \quad (4.5)$$

for some constant $c_2 \in \mathbb{R}$. Now, in order to reduce technicalities, we can assume $c_1 = 0$, $c_2 = 1$ and denote by $\bar{k}_\gamma(s)$ a non constant positive solution of (4.5) defined on an interval around $s = s_0$ with initial condition $\bar{k}(s_0) < (1/2)^{1/4}$. Taking derivatives we easily obtain

$$10\bar{k}'_\gamma \bar{k}''_\gamma + 4\bar{k}_\gamma^3 \bar{k}'_\gamma = 0$$

and, due to the fact that $\bar{k}'_\gamma \neq 0$ and $\bar{k}''_\gamma = -\frac{2}{5}\bar{k}_\gamma^3$, we have

$$\epsilon_1 \bar{k}''_\gamma - 2\epsilon_2 \bar{k}_\gamma^3 = -\frac{2}{5}(\epsilon_1 + 5\epsilon_2)\bar{k}_\gamma^3 \neq 0.$$

Therefore, after a straightforward manipulation, we find that the second equation of the triharmonicity system (4.4) becomes

$$\tau_\gamma^2(s) = \frac{63(1 - 2\bar{k}_\gamma^4)}{10\bar{k}_\gamma^2(\epsilon_1 + 5\epsilon_2)}. \quad (4.6)$$

Thus, choosing $\epsilon_1 = -1$ and $\epsilon_2 = 1$ there exists an open neighborhood of s_0 on which the right-hand side of (4.6) is positive. This ensures the existence of a function $\bar{\tau}_\gamma(s)$ defined in an open neighborhood of $s = s_0$ and such that (4.4) is verified. Then the conclusion of Theorem 1.11 follows immediately from the Fundamental Existence Theorem 2.1. In details, given $p_0 \in \mathbb{R}_1^3$ and the two non constant functions $\bar{k}_\gamma(s)$ and $\bar{\tau}_\gamma(s)$ as described above, we can choose $\{T_0, N_0, B_0\}$ such that $\langle T_0, T_0 \rangle = \epsilon_1 = -1$, $\langle N_0, N_0 \rangle = \epsilon_2 = 1$ and $\langle B_0, B_0 \rangle = \epsilon_3 = 1$. From Theorem 2.1 there exists a curve $\gamma(s)$ parametrized by arc length with Frenet Frame field coinciding with $\{T_0, N_0, B_0\}$ in s_0 and curvature functions coinciding with the given functions $\bar{k}_\gamma(s)$ and $\bar{\tau}_\gamma(s)$. Then, for a suitable open neighborhood I of s_0 and $\delta > 0$ sufficiently small the surface S_γ is Lorentzian and the curve $\gamma(s) = X(s, 0)$ is a triharmonic curve into S_γ with non constant curvature $k(s) = \bar{k}_\gamma(s)$.

5. r -harmonic 3-Frenet curves

In this section we shall prove Theorem 1.12 and his corollaries.

Proof of Theorem 1.12. First, it is important for us to establish a version in the semi-Riemannian context of Lemma 2.1 of [2]. Indeed, we prove:

Lemma 5.1. *Let $\gamma(s)$ be a 3-Frenet helix into a semi-Riemannian manifold $(M_t^m, \langle \cdot, \cdot \rangle)$, $m \geq 3, 1 \leq t \leq m - 1$. Then, for all $\ell \geq 0$, we have:*

$$\begin{cases} \nabla_T^{2\ell} T = \mathcal{A}_\ell T + \mathcal{B}_\ell B \\ \nabla_T^{2\ell+1} T = \mathcal{C}_\ell N, \end{cases} \tag{5.1}$$

where

$$\begin{cases} \mathcal{A}_0 = 1 \\ \mathcal{A}_\ell = (-1)^\ell \epsilon_1 \epsilon_2^\ell \kappa^2 (\epsilon_1 \kappa^2 + \epsilon_3 \tau^2)^{\ell-1} & \text{if } \ell \geq 1 \\ \mathcal{B}_0 = 0 \\ \mathcal{B}_\ell = -(-1)^\ell \epsilon_3 \epsilon_2^\ell \kappa \tau (\epsilon_1 \kappa^2 + \epsilon_3 \tau^2)^{\ell-1} & \text{if } \ell \geq 1 \\ \mathcal{C}_\ell = (-1)^\ell \epsilon_2^{\ell+1} \kappa (\epsilon_1 \kappa^2 + \epsilon_3 \tau^2)^\ell & \text{if } \ell \geq 0. \end{cases} \tag{5.2}$$

Proof. The proof of this lemma can be easily carried out separating the two cases $\nabla_T^{2\ell}$ and $\nabla_T^{2\ell+1}$. In both cases a simple induction argument which uses the Frenet equations enables us to obtain the required result. \square

As an application, we can now compute explicitly the condition of r -harmonicity for 3-Frenet helices in any semi-Riemannian space form.

Using the expression (2.3) for the Riemannian curvature tensor field R we can rewrite (1.1) as follows:

$$\tau_r(\gamma) = \nabla_T^{2r-1} T + c \sum_{\ell=0}^{r-2} (-1)^\ell (\langle \nabla_T^\ell T, T \rangle \nabla_T^{2r-3-\ell} T - \langle \nabla_T^{2r-3-\ell} T, T \rangle \nabla_T^\ell T) = 0. \tag{5.3}$$

In the case that $r = 2$, using (5.1) we can easily compute the bitension field of γ :

$$\begin{aligned} \tau_2(\gamma) &= \nabla_T^3 T + c(\langle T, T \rangle \nabla_T T - \langle \nabla_T T, T \rangle T) \\ &= -\epsilon_2^2 \kappa (\epsilon_1 \kappa^2 + \epsilon_3 \tau^2) N + c \epsilon_1 \epsilon_2 \kappa N \\ &= -\epsilon_2 \kappa (\epsilon_2 (\epsilon_1 \kappa^2 + \epsilon_3 \tau^2) - c \epsilon_1) N \end{aligned}$$

from which (i) follows immediately.

Now, let $r = 3$. Then the expression (5.3) becomes:

$$\begin{aligned}\tau_3(\gamma) &= \nabla_T^5 T + c(\langle T, T \rangle \nabla_T^3 T - \langle \nabla_T^3 T, T \rangle T - \langle \nabla_T T, T \rangle \nabla_T^2 T + \langle \nabla_T^2 T, T \rangle \nabla_T T) \\ &= \epsilon_2^3 \kappa (\epsilon_1 \kappa^2 + \epsilon_3 \tau^2)^2 N - c \epsilon_1 \epsilon_2^2 \kappa (\epsilon_1 \kappa^2 + \epsilon_3 \tau^2) N - c \epsilon_2^2 \kappa^3 N \\ &= -\kappa \left[-\epsilon_2 (\epsilon_1 \kappa^2 + \epsilon_3 \tau^2)^2 + c (\epsilon_1 \epsilon_3 \tau^2 + 2\kappa^2) \right] N\end{aligned}$$

from which (ii) follows immediately.

Now we deal with the general case $r \geq 4$. First, we suppose that $r = 2s$ is even. Using (2.3) into (1.1) and separating even and odd indices we obtain:

$$\begin{aligned}\tau_{2s}(\gamma) &= \mathcal{C}_{2s-1} N \\ &+ c \left[\sum_{j=0}^{s-1} \langle \nabla_T^{2j} T, T \rangle \nabla_T^{2(2s-j-2)+1} T - \langle \nabla_T^{2(2s-j-2)+1} T, T \rangle \nabla_T^{2j} T \right] \\ &- c \left[\sum_{j=1}^{s-1} \langle \nabla_T^{2(j-1)+1} T, T \rangle \nabla_T^{2(2s-j-1)} T - \langle \nabla_T^{2(2s-j-1)} T, T \rangle \nabla_T^{2(j-1)+1} T \right].\end{aligned}$$

Next, using Lemma 5.1 we find:

$$\begin{aligned}\tau_{2s}(\gamma) &= \mathcal{C}_{2s-1} N + c \sum_{j=0}^{s-1} \langle \nabla_T^{2j} T, T \rangle \nabla_T^{2(2s-j-2)+1} T \\ &+ c \sum_{j=1}^{s-1} \langle \nabla_T^{2(2s-j-1)} T, T \rangle \nabla_T^{2(j-1)+1} T \\ &= \left[\mathcal{C}_{2s-1} + c \epsilon_1 \sum_{j=0}^{s-1} \mathcal{A}_j \mathcal{C}_{2s-j-2} + c \epsilon_1 \sum_{j=1}^{s-1} \mathcal{C}_{j-1} \mathcal{A}_{2s-1-j} \right] N.\end{aligned}$$

Finally, using the explicit expressions for \mathcal{A}_ℓ , \mathcal{C}_ℓ given in (5.2) and performing a straightforward simplification we find

$$\tau_{2s}(\gamma) = \epsilon_2 \kappa \left[(\epsilon_1 \kappa^2 + \epsilon_3 \tau^2)^{2s-3} \left(-\epsilon_2 (\epsilon_1 \kappa^2 + \epsilon_3 \tau^2)^2 + c [(2s-1)\kappa^2 + \epsilon_1 \epsilon_3 \tau^2] \right) \right] N$$

from which the conclusion (iii) follows immediately. The case that r is odd is similar and so we omit further details. \square

Proof of Corollary 1.13. Since $c = 0$, we deduce from (5.3) that γ is proper r -harmonic if and only if $\epsilon_1 \kappa^2 + \epsilon_3 \tau^2 = 0$. Therefore, there exist solutions only if ϵ_1 and ϵ_3 have opposite sign. Since the index of the metric is 1, it follows that $\epsilon_2 = 1$ and so the normal vector field N is space-like. Finally, the r -harmonicity condition clearly reduces to $\kappa^2 = \tau^2$. \square

Proof of Corollary 1.16. Essentially, the proof consists in the analysis of (1.8).

Case (i) – By assumption $\epsilon_2 = 1$. Then $\epsilon_1 \epsilon_3 = -1$. It follows immediately that if $\kappa^2 = \tau^2$ then the curve γ is r -harmonic for all $r > 3$.

Next, we study the equation $-\epsilon_2 (\epsilon_1 \kappa^2 + \epsilon_3 \tau^2)^2 + c (\epsilon_1 \epsilon_3 \tau^2 + (r-1)\kappa^2) = 0$ when $r \geq 3$. This equation in our case is equivalent to

$$\tau^4 + (c - 2\kappa^2)\tau^2 + (\kappa^4 - c(r - 1)\kappa^2) = 0,$$

that is a second order degree equation in τ^2 . Now, a routine analysis shows that there exist positive roots if and only if either (a) or (b) holds.

Case (ii) – It is immediate.

Cases (iii) and (iv) – The computations are similar to Case (i), so we omit further details. \square

6. n -Frenet curves, $n \geq 4$

Starting from Theorem 1.18, in the introduction we stated several results concerning n -Frenet curves with $n \geq 4$. This section is devoted to their proofs. First we establish:

Proposition 6.1. *Let γ be an n -Frenet curve parametrized by the arc length s into an m -dimensional semi-Riemannian manifold (M_t^m, g) , $m \geq n \geq 4$. Then its bitension field is given by:*

$$\begin{aligned} \tau_2(\gamma) = & \epsilon_2(-3\epsilon_1 k_1 k_1' F_1 + (k_1'' - \epsilon_1 \epsilon_2 k_1^3 - \epsilon_2 \epsilon_3 k_1 k_2^2)F_2 + \\ & + \epsilon_3(2k_1' k_2 + k_1 k_2') F_3 + \epsilon_3 \epsilon_4 k_1 k_2 k_3 F_4 + k_1 R(F_2, F_1)F_1). \end{aligned} \tag{6.1}$$

Proof. We need to compute the first three covariant derivatives of T . Using the Frenet equations (1.2) we obtain

$$\begin{aligned} \nabla_T T &= \epsilon_2 k_1 F_2; \\ \nabla_T^2 T &= \epsilon_2 \nabla_T(k_1 F_2) = \epsilon_2 (k_1' F_2 + k_1 (-\epsilon_1 k_1 F_1 + \epsilon_3 k_2 F_3)) \\ &= \epsilon_2(-\epsilon_1 k_1^2 F_1 + k_1' F_2 + \epsilon_3 k_1 k_2 F_3); \\ \nabla_T^3 T &= \epsilon_2(-\epsilon_1 \nabla_T(k_1^2 F_1) + \nabla_T(k_1' F_2) + \epsilon_3 \nabla_T(k_1 k_2 F_3)) \\ &= \epsilon_2(-\epsilon_1 2k_1 k_1' F_1 - \epsilon_1 \epsilon_2 k_1^3 F_2 + k_1'' F_2 + k_1' (-\epsilon_1 k_1 F_1 \\ &+ \epsilon_3 k_2 F_3) + \epsilon_3 (k_1' k_2 + k_1 k_2') F_3 + \epsilon_3 k_1 k_2(-\epsilon_2 k_2 F_2 + \epsilon_4 k_3 F_4)) \\ &= \epsilon_2 \left(-3\epsilon_1 k_1 k_1' F_1 + (k_1'' - \epsilon_1 \epsilon_2 k_1^3 - \epsilon_2 \epsilon_3 k_1 k_2^2)F_2 \right. \\ & \left. + \epsilon_3(2k_1' k_2 + k_1 k_2') F_3 + \epsilon_3 \epsilon_4 k_1 k_2 k_3 F_4 \right). \end{aligned}$$

Thus, replacing in (1.1) and simplifying we obtain

$$\begin{aligned} \tau_2(\gamma) &= \nabla_T^3 T + R(\nabla_T T, T)T \\ &= \epsilon_2(-3\epsilon_1 k_1 k_1' F_1 + (k_1'' - \epsilon_1 \epsilon_2 k_1^3 - \epsilon_2 \epsilon_3 k_1 k_2^2)F_2 + \\ &+ \epsilon_3(2k_1' k_2 + k_1 k_2') F_3 + \epsilon_3 \epsilon_4 k_1 k_2 k_3 F_4) + \epsilon_2 k_1 R(F_2, F_1)F_1 \end{aligned}$$

and so we have the required expression. \square

Proof of Theorem 1.18. Since the Frenet frame field $\{F_1, \dots, F_n\}$ is an orthonormal frame and (1.9) holds, we can decompose the bitension field as $\tau_2(\gamma) = \sum_{i=1}^n \epsilon_i \langle \tau_2(\gamma), F_i \rangle F_i$.

From (6.1) and using the fact that $\langle R(F_2, F_1)F_1, F_1 \rangle = 0$ we find that

$$\langle \tau_2(\gamma), F_1 \rangle = -3\epsilon_2 k_1 k_1' = 0.$$

Now, since γ is proper biharmonic, we obtain that the first condition in (1.10) must hold. Therefore, we can rewrite the bitension field as follows:

$$\tau_2(\gamma) = \epsilon_2 [(-\epsilon_1 \epsilon_2 k_1^3 - \epsilon_2 \epsilon_3 k_1 k_2^2) F_2 + \epsilon_3 k_1 k_2' F_3 + \epsilon_3 \epsilon_4 k_1 k_2 k_3 F_4 + k_1 R(F_2, F_1) F_1]$$

Now, it is easy to compute all the other components of $\tau_2(\gamma)$:

$$\begin{aligned} \langle \tau_2(\gamma), F_2 \rangle &= \epsilon_2 k_1 (- (\epsilon_1 k_1^2 + \epsilon_3 k_2^2) + \langle R(F_2, F_1) F_1, F_2 \rangle); \\ \langle \tau_2(\gamma), F_3 \rangle &= \epsilon_2 k_1 (k_2' + \langle R(F_2, F_1) F_1, F_3 \rangle); \\ \langle \tau_2(\gamma), F_4 \rangle &= \epsilon_2 k_1 (\epsilon_3 k_2 k_3 + \langle R(F_2, F_1) F_1, F_4 \rangle); \\ \langle \tau_2(\gamma), F_i \rangle &= \epsilon_2 k_1 \langle R(F_2, F_1) F_1, F_i \rangle, \quad 5 \leq i \leq n, \end{aligned}$$

as required to end the proof of Theorem 1.18. \square

A computation similar to Proposition 6.1, using just (1.1) and (1.2), leads us to:

Proposition 6.2. *Let γ be an n -Frenet helix parametrized by the arc length s into an m -dimensional semi-Riemannian manifold (M_t^m, g) , $1 \leq t \leq m - 1$, $m \geq n \geq 6$. Then its tritension field is given by:*

$$\begin{aligned} \tau_3(\gamma) &= -\epsilon_2 \kappa_1 \{ -\epsilon_2 [\epsilon_2 (\epsilon_1 \kappa_1^2 + \epsilon_3 \kappa_2^2)^2 + \epsilon_3^2 \epsilon_4 \kappa_2^2 \kappa_3^2] F_2 + \\ &\quad + \epsilon_3 \epsilon_4 \kappa_2 \kappa_3 [\epsilon_2 (\epsilon_1 \kappa_1^2 + \epsilon_3 \kappa_2^2) + \epsilon_4 (\epsilon_3 \kappa_3^2 + \epsilon_5 \kappa_4^2)] F_4 \\ &\quad - \epsilon_3 \epsilon_4 \epsilon_5 \epsilon_6 \kappa_2 \kappa_3 \kappa_4 \kappa_5 F_6 + \epsilon_2 (2\epsilon_1 \kappa_1^2 + \epsilon_3 \kappa_2^2) R(F_2, F_1) F_1 \\ &\quad + \epsilon_2 \epsilon_3 \kappa_1 \kappa_2 R(F_3, F_2) F_1 - \epsilon_3 \epsilon_4 \kappa_2 \kappa_3 R(F_4, F_1) F_1 \}. \end{aligned}$$

Proof. From (1.1), the tritension field can be written as

$$\tau_3(\gamma) = \nabla_T^5 T + R(\nabla_T^3 T, T)T - R(\nabla_T^2 T, \nabla_T T)T. \quad (6.2)$$

Thus we need to compute the first five covariant derivatives of T :

$$\begin{aligned} \nabla_T T &= \epsilon_2 \kappa_1 F_2; \\ \nabla_T^2 T &= \epsilon_2 \kappa_1 (-\epsilon_1 \kappa_1 F_1 + \epsilon_3 \kappa_2 F_3); \\ \nabla_T^3 T &= -\epsilon_2 \kappa_1 [\epsilon_2 (\epsilon_1 \kappa_1^2 + \epsilon_3 \kappa_2^2) F_2 - \epsilon_3 \epsilon_4 \kappa_2 \kappa_3 F_4]; \\ \nabla_T^4 T &= -\epsilon_2 \kappa_1 [-\epsilon_1 \epsilon_2 \kappa_1 (\epsilon_1 \kappa_1^2 + \epsilon_3 \kappa_2^2) F_1 \\ &\quad + \epsilon_3 \kappa_2 (\epsilon_2 (\epsilon_1 \kappa_1^2 + \epsilon_3 \kappa_2^2) + \epsilon_3 \epsilon_4 \kappa_3^2) F_3 - \epsilon_3 \epsilon_4 \epsilon_5 \kappa_2 \kappa_3 \kappa_4 F_5]; \\ \nabla_T^5 T &= -\epsilon_2 \kappa_1 [-\epsilon_2 (\epsilon_2 (\epsilon_1 \kappa_1^2 + \epsilon_3 \kappa_2^2)^2 + \epsilon_3^2 \epsilon_4 \kappa_2^2 \kappa_3^2) F_2 \\ &\quad + \epsilon_3 \epsilon_4 \kappa_2 \kappa_3 (\epsilon_2 (\epsilon_1 \kappa_1^2 + \epsilon_3 \kappa_2^2) + \epsilon_4 (\epsilon_3 \kappa_3^2 + \epsilon_5 \kappa_4^2)) F_4 \\ &\quad - \epsilon_3 \epsilon_4 \epsilon_5 \epsilon_6 \kappa_2 \kappa_3 \kappa_4 \kappa_5 F_6]. \end{aligned}$$

Using these expressions in (6.2) it is straightforward to end the proof. \square

Now, we examine more in detail the case that the ambient is a semi-Riemannian space form. As a consequence of Proposition 6.2, inspection of the component of the tritension field along F_6 suggests that there are no proper solutions if $n \geq 6$. Therefore, it is geometrically natural to restrict our attention to the

case that $m = n = 5$. In this case the computations are equivalent to those in Proposition 6.2 with the additional assumption $\kappa_5 = 0$. Then, using (2.3) and simplifying, we easily deduce Theorem 1.21.

As for Examples 1.23 and 1.24 it is easy to check that all the conditions of Theorem 1.21 are verified with $c = 1$. Then the conclusion follows from the Fundamental Existence Theorem 2.1.

7. Proof of Proposition 1.25 and Theorem 1.26

Proof of Proposition 1.25. The assumption (1.12), together with the expressions given in Lemma 5.1, enable us to conclude that

$$\nabla_T^j T = 0$$

for all $j \geq 3$. Then direct inspection of the tension field (1.1) shows that all its terms identically vanish when $r \geq 4$. \square

The proof of Theorem 1.26 is more laborious and so we carry out some preliminary work.

First, we observe that in the ambient space $(\mathbb{R} \times N^{m-1}(c), g_{\text{prod}})$ the curvature terms of the r -tension field (1.1) are always tangent to the factor $N^{m-1}(c)$. It follows readily from Lemma 5.1 that, if (1.12) does not hold, then a *necessary condition* for a helix γ to be proper r -harmonic, $r \geq 2$, is the fact that its normal vector field N is itself tangent to $N^{m-1}(c)$.

This can be achieved only if $\epsilon_2 = 1$ and the component of γ along the factor \mathbb{R} is a linear function of s . Therefore, without loss of generality since vertical translations are isometries, in the family (1.13) we have assumed that this component of γ is equal to $d_1 s$. Moreover, we observe that all the conclusions of Theorem 1.26 are trivial if $d_1 = 0$. Therefore, from now on, we shall assume $d_1 \neq 0$. To fix notation, we write as follows the Frenet equations verified by α in $N^{m-1}(c)$:

$$\begin{cases} \tilde{\nabla}_{T_\alpha} T_\alpha = \kappa_\alpha N_\alpha \\ \tilde{\nabla}_{T_\alpha} N_\alpha = -\kappa_\alpha T_\alpha + \tau_\alpha B_\alpha \\ \tilde{\nabla}_{T_\alpha} B_\alpha = -\tau_\alpha N_\alpha, \end{cases} \tag{7.1}$$

where here $\tilde{\nabla}$ denotes the covariant derivative in $N^{m-1}(c)$. Also, we write the Frenet field of γ as follows:

$$T = \dot{\gamma} = d_1 \partial_t + d_2 \dot{\alpha}(d_2 s) = d_1 \partial_t + d_2 T_\alpha, \quad N = \tilde{N}, \quad B = -B_t \partial_t + \tilde{B},$$

where \tilde{N}, \tilde{B} are tangent to the factor $N^{m-1}(c)$. Note that

$$\epsilon_1 = \langle T, T \rangle = -d_1^2 + d_2^2$$

forces the assumptions (1.14) on the constants d_1, d_2 .

Proof of Theorem 1.26. First, we establish

Lemma 7.1. *The curve γ verifies the Frenet equations*

$$\begin{cases} \nabla_T T = \epsilon_2 \kappa N \\ \nabla_T N = -\epsilon_1 \kappa T + \epsilon_3 \tau B \\ \nabla_T B = -\epsilon_2 \tau N, \end{cases} \tag{7.2}$$

with $\epsilon_2 = 1$ and

$$\kappa = (\epsilon_1 + d_1^2)\kappa_\alpha, \quad \tau = \sqrt{\epsilon_3 (\epsilon_1 + d_1^2) (\tau_\alpha^2 - \epsilon_1 d_1^2 \kappa_\alpha^2)}. \quad (7.3)$$

Moreover, γ does not verify (1.12).

Proof. We compute

$$\nabla_T T = \nabla_{d_1 \partial_t + d_2 T_\alpha} (d_1 \partial_t + d_2 T_\alpha) = d_2^2 \tilde{\nabla}_{T_\alpha} T_\alpha = d_2^2 \kappa_\alpha N_\alpha$$

from which we deduce $N = N_\alpha$, $\epsilon_2 = 1$ and κ as in (7.3). Similarly

$$\nabla_T N = d_2 \tilde{\nabla}_{T_\alpha} N_\alpha = -d_2 \kappa_\alpha T_\alpha + d_2 \tau_\alpha B_\alpha.$$

Next, equalling the right-hand side to $-\epsilon_1 \kappa T + \epsilon_3 \tau B$ and considering the components tangent to \mathbb{R} and $N^{m-1}(c)$ respectively we find:

$$\begin{aligned} \text{(i)} \quad & \epsilon_1 \kappa d_1 + \epsilon_3 \tau B_t = 0 \\ \text{(ii)} \quad & (\epsilon_1 \kappa d_2 - \kappa/d_2) T_\alpha + d_2 \tau_\alpha B_\alpha - \epsilon_3 \tau \tilde{B} = 0 \end{aligned}$$

which imply

$$\begin{aligned} \text{(i)} \quad & B_t = -\epsilon_1 \epsilon_3 (\kappa/\tau) d_1 \\ \text{(ii)} \quad & \tilde{B} = (\epsilon_3/\tau) \left((\epsilon_1 \kappa d_2 - \kappa/d_2) T_\alpha + d_2 \tau_\alpha B_\alpha \right) \end{aligned} \quad (7.4)$$

We use these facts to compute the expression of τ :

$$\begin{aligned} \tau &= \langle \nabla_T N, B \rangle \\ &= \langle -d_2 \kappa_\alpha T_\alpha + d_2 \tau_\alpha B_\alpha, -B_t \partial_t + \tilde{B} \rangle \\ &= \langle -d_2 \kappa_\alpha T_\alpha + d_2 \tau_\alpha B_\alpha, \frac{\epsilon_3}{\tau} \left((\epsilon_1 \kappa d_2 - \kappa/d_2) T_\alpha + d_2 \tau_\alpha B_\alpha \right) \rangle \\ &= \frac{\epsilon_3}{\tau} \left(d_2^2 \tau_\alpha^2 - \kappa^2 \left(\epsilon_1 - \frac{1}{d_2^2} \right) \right). \end{aligned} \quad (7.5)$$

Next, using $d_2^2 = \epsilon_1 + d_1^2$ and the assumption (1.15) we easily deduce the expression for τ given in (7.3).

Finally, a simple computation yields

$$\epsilon_1 \kappa^2 + \epsilon_3 \tau^2 = (d_1^2 + \epsilon_1) (\kappa_\alpha^2 + \tau_\alpha^2) \neq 0.$$

Thus γ does not verify condition (1.12), as required to end the lemma. \square

To deal with the curvature terms of the r -tension field $\tau_r(\gamma)$ it is useful to compute some scalar products, denoted $\langle \cdot, \cdot \rangle_g$, with respect to the metric g of $N^{m-1}(c)$.

More precisely, we make the following slight abuse of notation: writing vectors as $X = -X_t \partial_t + \tilde{X}$, $Y = -Y_t \partial_t + \tilde{Y}$, we shall consider

$$\langle X, Y \rangle_g = \langle \tilde{X}, \tilde{Y} \rangle_g.$$

We have:

$$\begin{aligned} \langle T, T \rangle_g &= \epsilon_1 + d_1^2 \\ \langle T, N \rangle_g &= 0 \\ \langle T, B \rangle_g &= \epsilon_1 \epsilon_3 d_1^2 \frac{\kappa}{\tau} \end{aligned} \tag{7.6}$$

where, for the third scalar product, we have used (7.4)(i) and

$$0 = \langle T, B \rangle = d_1 B_t + \langle T, B \rangle_g.$$

We also point out that in our product space, with a notation as above,

$$R(X, Y)Z = c \left(\langle \tilde{Y}, \tilde{Z} \rangle_g \tilde{X} - \langle \tilde{X}, \tilde{Z} \rangle_g \tilde{Y} \right).$$

Now, we can proceed to the computation of $\tau_r(\gamma)$ as given in (1.1).

First we assume that r is even, say $r = 2s$. We use Lemma 5.1 and split the sum into even and odd values for the index ℓ . Computing in a fashion similar to the proof of Theorem 1.12 we manage to express the $2s$ -tension field using the functions $\mathcal{A}_\ell, \mathcal{B}_\ell, \mathcal{C}_\ell$ introduced in Lemma 5.1. More precisely, we obtain:

$$\begin{aligned} \tau_{2s}(\gamma) &= \mathcal{C}_{2s-1}N + c \sum_{j=0}^{s-1} \mathcal{C}_{2s-j-2} (\langle T, T \rangle_g \mathcal{A}_j + \langle T, B \rangle_g \mathcal{B}_j)N \\ &\quad + c \sum_{j=1}^{s-1} \mathcal{C}_{j-1} (\langle T, T \rangle_g \mathcal{A}_{2s-1-j} + \langle T, B \rangle_g \mathcal{B}_{2s-1-j})N \\ &= \left[-(-1)^{(2s)} \kappa (\kappa^2 \epsilon_1 + \epsilon_3 \tau^2)^{(-3+2s)} \left(\kappa^4 + 2\kappa^2 \epsilon_1 \epsilon_3 \tau^2 + \tau^4 \right. \right. \\ &\quad \left. \left. - c(\kappa^2(-1 + 2s + d_1^2 \epsilon_1) + (d_1^2 + \epsilon_1) \epsilon_3 \tau^2) \right) \right] N \end{aligned} \tag{7.7}$$

where for the various simplifications we have also used (7.6). We have proved in Lemma 7.1 that in our family of curves condition (1.12) never holds. Therefore the condition of proper r -harmonicity, when r is even, is equivalent to

$$\kappa^4 + 2\kappa^2 \epsilon_1 \epsilon_3 \tau^2 + \tau^4 - c(\kappa^2(r - 1 + d_1^2 \epsilon_1) + (d_1^2 + \epsilon_1) \epsilon_3 \tau^2) = 0. \tag{7.8}$$

A similar computation shows that the same condition rules r -harmonicity also when r is odd. Finally, we replace the explicit expressions (7.3) for κ and τ into (7.8). Then we see that the r -harmonicity equation (7.8) becomes equivalent to

$$(\kappa_\alpha^2 + \tau_\alpha^2)^2 - c((r - 1)\kappa_\alpha^2 + \tau_\alpha^2) = 0.$$

But the latter is precisely the condition of r -harmonicity for the 3-Frenet helix α in $N^{m-1}(c)$ (see [2]), as required to end the proof of Theorem 1.26. \square

8. r -harmonic curves in Robertson-Walker space time

In this section we study r -harmonic curves into the Robertson-Walker Lorentzian manifold $\mathcal{RW}_1^m = (J \times N^{m-1}(c), g_f)$, where the metric g_f was introduced in (1.16). In particular, we shall prove Theorems 1.28 and 1.29.

A vector field X on \mathcal{RW}_1^m can be decomposed as

$$X = (-X_t, \tilde{X}) = -X_t \partial_t + \tilde{X},$$

where \tilde{X} is a vector field tangent to $N^{m-1}(c)$ and $X_t = g_f(X, \partial_t) = \langle X, \partial_t \rangle$.

Preliminarily, we observe that the action of the Levi-Civita connection in a Robertson-Walker semi-Riemannian manifold is described in the following proposition.

Proposition 8.1 (see [20]). *Let \tilde{X}, \tilde{Y} be two vector fields tangent to $N^{m-1}(c)$ and ∂_t a unit vector field tangent to J . Then the Levi-Civita connection ∇ of the Robertson-Walker space-time \mathcal{RW}_1^m verifies the following properties:*

- (i) $\nabla_{\partial_t} \partial_t = 0$;
- (ii) $\nabla_{\partial_t} \tilde{X} = \nabla_{\tilde{X}} \partial_t = \frac{f'}{f} \tilde{X}$;
- (iii) $\nabla_{\tilde{X}} \tilde{Y} = \tilde{\nabla}_{\tilde{X}} \tilde{Y} + \langle \tilde{X}, \tilde{Y} \rangle \frac{f'}{f} \partial_t$, where $\tilde{\nabla}$ is the Levi-Civita connection of $N^{m-1}(c)$ and $\langle \tilde{X}, \tilde{Y} \rangle = g_f(\tilde{X}, \tilde{Y}) = f^2 g(\tilde{X}, \tilde{Y})$.

Next, following [20] again, we recall the following properties of the Riemannian curvature tensor of \mathcal{RW}_1^m :

Proposition 8.2. *Let \tilde{X}, \tilde{Y} and \tilde{Z} be vector fields tangent to $N^{m-1}(c)$ and ∂_t a unit vector field tangent to J . Then the Riemannian curvature operator R of the Robertson-Walker space-time \mathcal{RW}_1^m satisfies*

- (i) $R(\tilde{X}, \tilde{Y})\tilde{Z} = \left(\frac{(f')^2 + c}{f^2} \right) (\langle \tilde{Y}, \tilde{Z} \rangle \tilde{X} - \langle \tilde{X}, \tilde{Z} \rangle \tilde{Y})$;
- (ii) $R(\tilde{X}, \partial_t)\partial_t = -\left(\frac{f''}{f} \right) \tilde{X}$;
- (iii) $R(\tilde{X}, \tilde{Y})\partial_t = 0$;
- (iv) $R(\tilde{X}, \partial_t)\tilde{Y} = -\left(\frac{f''}{f} \right) \langle \tilde{X}, \tilde{Y} \rangle \partial_t$.

Our first result in this context is:

Proposition 8.3. *Let γ be a 2-Frenet helix parametrized by the arc length s into an m -dimensional Robertson-Walker space-time \mathcal{RW}_1^m such that its normal vector field N is equal to $\pm \partial_t$. Then*

$$\tau_r(\gamma) = -\kappa^{2r-3} \left(\kappa^2 + (r-1) \frac{f''}{f} \right) N.$$

Proof. We assume $N = \partial_t$, $\epsilon_2 = -1$ (the case $N = -\partial_t$, $\epsilon_2 = -1$ is analogous). This implies that T must be space-like and $\epsilon_1 = 1$. Moreover, T is tangent to $N^{m-1}(c)$. Thus, using Proposition 8.2,

$$R(N, T)T = -R(T, N)T = \left(\frac{f''}{f} \right) \langle T, T \rangle \partial_t = \frac{f''}{f} N. \quad (8.1)$$

Now assume $r = 2$. From (1.1) and using (3.3), (8.1) we obtain

$$\tau_2(\gamma) = \nabla_T^3 T + R(\nabla_T T, T)T = -\kappa^3 N - \kappa R(N, T)T = -\kappa \left(\kappa^2 + \frac{f''}{f} \right) N.$$

In the case that $r = 3$, the expression (1.1) becomes:

$$\begin{aligned} \tau_3(\gamma) &= \nabla_T^5 T + R(\nabla_T^3 T, T)T - R(\nabla_T^2 T, \nabla_T T)T \\ &= -\kappa^5 N - \kappa^3 R(N, T)T + \kappa^3 R(T, N)T = -\kappa^3 \left(\kappa^2 + 2 \frac{f''}{f} \right) N \end{aligned}$$

Now we find the general expression of the r -tension field for $r \geq 4$. Assume $r = 2s$ is even. Then, separating even and odd indices in (1.1) and using (3.3) and (8.1):

$$\begin{aligned} \tau_{2s}(\gamma) &= \nabla_T^{2(2s-1)+1} T + \sum_{j=0}^{s-1} R\left(\nabla_T^{2(2s-j-2)+1} T, \nabla_T^{2j} T\right) T \\ &\quad - \sum_{j=1}^{s-1} R\left(\nabla_T^{2(2s-j-1)} T, \nabla_T^{2(j-1)+1} T\right) T \\ &= -\kappa^{2(2s-1)+1} N - \sum_{j=0}^{s-1} \kappa^{2(2s-2)+1} R(N, T) T + \sum_{j=1}^{s-1} \kappa^{2(2s-2)+1} R(T, N) T \\ &= -\kappa^{2(2s-2)+1} \left(\kappa^2 + (2s-1) \frac{f''}{f}\right) N. \end{aligned}$$

The case that r is odd follows the same argument. \square

Proof of Theorem 1.28. The unit tangent vector field of γ is given by $\gamma' = T = (0, d\alpha')$. We observe that T is tangent to $N^{m-1}(c)$ and $\epsilon_1 = 1$. From the first Frenet equation and Proposition 8.1 we deduce

$$\epsilon_2 k(s) N = \nabla_T T = d^2 \tilde{\nabla}_{\alpha'} \alpha' + \frac{f'(t_0)}{f(t_0)} \partial_t. \tag{8.2}$$

Now, since α is a geodesic on $N^{m-1}(c)$, $\tilde{\nabla}_{\alpha'} \alpha' = 0$. Then (8.2) implies $N = \pm \partial_t$, $\epsilon_2 = -1$ and

$$k(s)^2 = \kappa^2 = \left(\frac{f'(t_0)}{f(t_0)}\right)^2.$$

Combining again Frenet equations (1.2) and Proposition 8.1 we deduce:

$$-\kappa T + \epsilon_3 \tau(s) B = \nabla_T N = -\left|\frac{f'(t_0)}{f(t_0)}\right| T.$$

From this equation we obtain $\tau = 0$. This means that γ is a 2-Frenet helix.

The curve γ is a geodesic if and only if $\kappa = 0$, that is $f'(t_0) = 0$. Thus, from Proposition 8.3, γ is proper r -harmonic if and only if

$$f'(t_0) \neq 0 \quad \text{and} \quad \kappa^2 + (r-1) \frac{f''(t_0)}{f(t_0)} = 0.$$

From this condition it is immediate to conclude the proof. \square

Proof of Theorem 1.29. The tangent vector field of γ is given by $\gamma' = T = (0, \alpha')$. Thus T is tangent to $N^{m-1}(c)$ and $\epsilon_1 = 1$. From Proposition 8.1 and the hypothesis $f'(t_0) = 0$ we obtain $\nabla_T T = \tilde{\nabla}_{\alpha'} \alpha'$.

Now, let $\beta(s) = \alpha(f(t_0)s)$. Then $T_\beta = f(t_0) \alpha'$ and

$$g(T_\beta, T_\beta) = f(t_0)^2 g(\alpha', \alpha') = \langle T, T \rangle = 1,$$

that is β is parametrized by the arc length on $N^{m-1}(c)$. Next,

$$\nabla_T T = \tilde{\nabla}_{\alpha'} \alpha' = \frac{1}{f(t_0)^2} \tilde{\nabla}_{T_\beta} T_\beta \tag{8.3}$$

and by an induction argument

$$\nabla_T^j T = \frac{1}{f(t_0)^{j+1}} \tilde{\nabla}_{T_\beta}^j T_\beta \tag{8.4}$$

for all $j \geq 1$. Let \tilde{X}, \tilde{Y} and \tilde{Z} be vector fields tangent to $N^{m-1}(c)$. From Proposition 8.2 and the assumption $f'(t_0) = 0$ we deduce that along the curve γ we have:

$$\begin{aligned} R(\tilde{X}, \tilde{Y})\tilde{Z} &= \frac{c}{f(t_0)^2} (\langle \tilde{Y}, \tilde{Z} \rangle \tilde{X} - \langle \tilde{X}, \tilde{Z} \rangle \tilde{Y}) \\ &= c(g(\tilde{Y}, \tilde{Z})\tilde{X} - g(\tilde{X}, \tilde{Z})\tilde{Y}) \\ &= R^{N^{m-1}(c)}(\tilde{X}, \tilde{Y})\tilde{Z}. \end{aligned} \tag{8.5}$$

Finally, using (8.4) and (8.5) in the expression of r -tension field (1.1),

$$\begin{aligned} \tau_r(\gamma) &= \nabla_T^{2r-1} T + \sum_{\ell=0}^{r-2} (-1)^\ell R(\nabla_T^{2r-3-\ell} T, \nabla_T^\ell T) T \\ &= \frac{1}{f(t_0)^{2r}} \tilde{\nabla}_{T_\beta}^{2r-1} T_\beta + \sum_{\ell=0}^{r-2} (-1)^\ell R\left(\frac{1}{f(t_0)^{2r-2-\ell}} \tilde{\nabla}_{T_\beta}^{2r-3-\ell} T_\beta, \frac{1}{f(t_0)^{\ell+1}} \tilde{\nabla}_{T_\beta}^\ell T_\beta\right) T \\ &= \frac{1}{f(t_0)^{2r}} \tilde{\nabla}_{T_\beta}^{2r-1} T_\beta + \frac{1}{f(t_0)^{2r}} \sum_{\ell=0}^{r-2} (-1)^\ell R^{N^{m-1}(c)}\left(\tilde{\nabla}_{T_\beta}^{2r-3-\ell} T_\beta, \tilde{\nabla}_{T_\beta}^\ell T_\beta\right) T_\beta \\ &= \frac{1}{f(t_0)^{2r}} \tau_r(\beta) \end{aligned}$$

and so the conclusion of the proof follows readily. \square

Appendix A. The fundamental theorem of curves

Theorem A.1 (*The Fundamental Theorem of Curves in Semi-Riemannian Geometry*). Let $M = (M_t^m, \langle \cdot, \cdot \rangle)$ be an m -dimensional semi-Riemannian manifold of index $1 \leq t \leq m - 1$. Given smooth, positive curvature functions $k_1(s), \dots, k_{n-1}(s)$ defined on an open real neighborhood of $s_0 \in \mathbb{R}$, an initial point $p_0 \in M$ and an initial orthonormal frame $\{T_0, F_{2,0}, \dots, F_{n,0}\}$ at p_0 there exists an n -Frenet curve $\gamma(s)$ in M parametrized by the arc length s such that

- (i) $\gamma(s_0) = p_0$;
- (ii) The n -Frenet frame field $\{T, F_2, \dots, F_n\}$ of γ coincides with $\{T_0, F_{2,0}, \dots, F_{n,0}\}$ at $s = s_0$;
- (iii) The given functions $k_1(s), \dots, k_{n-1}(s)$ are the curvatures of $\gamma(s)$.

Proof. We follow the approach of [14] and [24]. Let $p_0 \in M$ and let $(U, \varphi) = (U, x^1, \dots, x^m)$ be a local chart of M_t^m around the point p_0 . Consider in the chart (U, φ) a given curve $\alpha(s) = (\alpha^1(s), \dots, \alpha^m(s)) = (x^1 \circ \alpha, \dots, x^m \circ \alpha)$ and let $\{F_1^\alpha, \dots, F_n^\alpha\}$ be the Frenet frame field along α . Then, with respect to the coordinate frame field $\{\frac{\partial}{\partial x^i}\}_{i=1}^m$ we have

$$T = \alpha'(s) = \dot{\alpha}^j(s) \frac{\partial}{\partial x^j}$$

and

$$F_i^\alpha = w_i^j \frac{\partial}{\partial x^j}, \quad i = 1, \dots, n$$

Note that $w_1^j = \dot{\alpha}^j$. Moreover,

$$\nabla_T F_i^\alpha = \left(\dot{w}_i^j + \Gamma_{rs}^j \dot{\alpha}^r w_i^s \right) \frac{\partial}{\partial x^j},$$

where the Γ_{rs}^j 's are the Christoffel symbols of the Levi-Civita connection in the local chart (U, φ) . Using the Frenet Equations (1.2) we obtain that the w_i^j 's satisfy the following system with $1 \leq j \leq m$:

$$\begin{cases} \dot{\alpha}^j(s) = w_1^j(s) \\ \dot{w}_1^j + \Gamma_{rs}^j \dot{\alpha}^r w_1^s = \epsilon_2 k_1^\alpha w_2^j \\ \dot{w}_i^j + \Gamma_{rs}^j \dot{\alpha}^r w_i^s = -\epsilon_{i-1} k_{i-1}^\alpha w_{i-1}^j + \epsilon_{i+1} k_{i+1}^\alpha w_{i+1}^j, \quad 1 < i < n \\ \dot{w}_n^j + \Gamma_{rs}^j \dot{\alpha}^r w_n^s = -\epsilon_{n-1} k_{n-1}^\alpha w_{n-1}^j \end{cases} \tag{A.1}$$

If we put $A(s) = (\alpha^1(s), \dots, \alpha^m(s))$ and $W_i(s) = (w_i^1(s), \dots, w_i^m(s)), 1 \leq i \leq n$, then (A.1) can be rewritten as follows:

$$\begin{cases} A'(s) &= W_1(s), \\ W_i'(s) &= G_i(A(s), W_1(s), \dots, W_n(s)), \quad i = 1, \dots, n \end{cases} \tag{A.2}$$

where the G_i 's are functions which depend only on the curvatures $k_1^\alpha, \dots, k_{n-1}^\alpha$, the values of $\epsilon_1 \dots \epsilon_n$ and the Christoffel symbols. By way of summary, for a given curve $\alpha(s)$ in the local chart (U, φ) , system (A.1) or, equivalently, system (A.2) holds.

Now, let us suppose to have $(n - 1)$ positive differentiable functions $k_1(s), \dots, k_{n-1}(s)$ and an initial orthonormal frame $\{T_0, F_{2,0}, \dots, F_{n,0}\}$ at p_0 such that

$$\epsilon_1 = \langle T_0, T_0 \rangle, \quad \epsilon_2 = \langle F_{2,0}, F_{2,0} \rangle, \quad \dots \quad \epsilon_n = \langle F_{n,0}, F_{n,0} \rangle. \tag{A.3}$$

We consider the system of differential equations in the variables $C(s), V_1(s), \dots, V_n(s)$:

$$\begin{cases} C'(s) &= V_1(s), \\ V_i'(s) &= G_i(C(s), V_1(s), \dots, V_n(s)), \quad i = 1, \dots, n \end{cases} \tag{A.4}$$

where G_1, \dots, G_n are formally the functions defined in (A.2), but they depend on the given functions $k_1, \dots, k_{n-1}, \epsilon_1 \dots \epsilon_n$ and the Christoffel symbols. If we choose the initial conditions

$$(C(s_0), V_1(s_0), V_2(s_0), \dots, V_n(s_0)) = (\varphi(p_0), t_0, f_{2,0}, \dots, f_{n,0}),$$

where $t_0, f_{2,0}, \dots, f_{n,0} \in \mathbb{R}^m$ are the components, with respect to the coordinate frame field, of $T_0, F_{2,0}, \dots, F_{n,0}$ respectively, there exists a unique solution $(c(s), v_1(s), \dots, v_n(s))$ of (A.4), in a neighborhood of s_0 , such that

$$(c(s_0), v_1(s_0), v_2(s_0), \dots, v_n(s_0)) = (\varphi(p_0), t_0, f_{2,0}, \dots, f_{n,0}).$$

Define $\gamma(s) = \varphi^{-1} \circ c(s)$. Then $\gamma(s)$ is a curve on M_t^m such that $\gamma(s_0) = p_0$.

Let us define the vector fields $f_i = v_i^j \frac{\partial}{\partial x_j}, i = 1, \dots, n$, along γ . Then the frame field $\{f_1, f_2, \dots, f_n\}$

coincides with $\{T_0, F_{2,0}, \dots, F_{n,0}\}$ at the point s_0 and satisfy (A.4) by construction. To finish the proof, we have to show that k_1, \dots, k_{n-1} are the curvatures of the curve $\gamma(s)$. For this, it is sufficient to prove that $\{f_1, \dots, f_n\}$ is an orthonormal frame field of the type (A.3) along the curve γ . If the matrix Ω defined in Remark 1.2 were skew-symmetric, then the conclusion could be rapidly obtained by using the method of [1, Theorem 6, p. 124].

In our case, generalizing the method of [14], we consider the following auxiliary system of $n(n+1)/2$ differential equations in the variables $X_{ij}(s), 1 \leq i \leq j \leq n$:

$$\left\{ \begin{array}{ll} X'_{11} &= 2\epsilon_2 k_1 X_{12} \\ X'_{ii} &= -2\epsilon_{i-1} k_{i-1} X_{i-1i} + 2\epsilon_{i+1} k_i X_{i+1i} & 1 < i < n \\ X'_{nn} &= -2\epsilon_{n-1} k_{n-1} X_{n-1n} \\ X'_{1i} &= \epsilon_2 k_1 X_{2i} - \epsilon_{i-1} k_{i-1} X_{1i-1} + \epsilon_{i+1} k_i X_{1i+1} & 1 < i < n \\ X'_{1n} &= \epsilon_2 k_1 X_{2n} - \epsilon_{n-1} k_{n-1} X_{1n-1} \\ X'_{ij} &= -\epsilon_{i-1} k_{i-1} X_{i-1j} + \epsilon_{i+1} k_i X_{i+1j} \\ &\quad - \epsilon_{j-1} k_{j-1} X_{ij-1} + \epsilon_{j+1} k_j X_{ij+1} & 1 < i < j < n \\ X'_{in} &= -\epsilon_{i-1} k_{i-1} X_{i-1n} + \epsilon_{i+1} k_i X_{i+1n} \\ &\quad - \epsilon_{n-1} k_{n-1} X_{in-1} & 1 < i < n, \end{array} \right. \quad (\text{A.5})$$

with initial conditions at the point $s = s_0$

$$X_{ii}(s_0) = \epsilon_i, \quad 1 \leq i \leq n; \quad X_{ij}(s_0) = 0, \quad i < j.$$

Since $\{f_1, \dots, f_n\}$ satisfy (A.4), then $Y_{ij} = \langle f_i, f_j \rangle, 1 \leq i \leq j \leq n$, is a solution of (A.5). Moreover, since $\{f_1, f_2, \dots, f_n\}$ coincide with the orthonormal frame $\{T_0, F_{2,0}, \dots, F_{n,0}\}$ at the point s_0 , then

$$\begin{aligned} Y_{11}(s_0) &= \langle T_0, T_0 \rangle = \epsilon_1 \\ Y_{ii}(s_0) &= \langle F_{i0}, F_{i0} \rangle = \epsilon_i \quad 2 \leq i \leq n \\ Y_{ij}(s_0) &= \langle F_{i0}, F_{j0} \rangle = 0 \quad i < j, \end{aligned}$$

that is the solution $\{Y_{ij}\}_{1 \leq i \leq j \leq n}$ satisfy the initial conditions at the point s_0 . On the other hand, the $n(n+1)/2$ constant functions given by

$$Z_{ii} = \epsilon_i, \quad 1 \leq i \leq n; \quad Z_{ij} = 0, \quad i < j$$

are also solutions of (A.5). Thus, by the uniqueness of the solution, we deduce that $Y_{ij} = Z_{ij}, 1 \leq i \leq j \leq n$, that is

$$\langle f_i, f_i \rangle = \epsilon_i, \quad 1 \leq i \leq n; \quad \langle f_i, f_j \rangle = 0, \quad i \neq j.$$

This means that $\{f_1, \dots, f_n\}$ is an orthonormal frame along γ , as required to end the proof. \square

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