Z. Angew. Math. Phys. (2026) 77:256

https://doi.org/10.1007/s00033-026-02895-6

Zeitschrift für angewandte Mathematik und Physik ZAMP

Blow-up in a degenerate attraction-repulsion chemotaxis system with flux limitation

Monica Marras, Stella Vernier-Piro and Tomomi Yokota

Abstract. In this paper we consider radially symmetric solutions of the cross-diffusion system with flux limitation,

$$\begin{cases} u_t = \nabla \cdot \Big(\dfrac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}} \Big) - \nabla \cdot \Big(\dfrac{u\nabla(\chi v - \xi w)}{(1+|\nabla v|^2+|\nabla w|^2)^q} \Big), \\ 0 = \Delta v + \alpha u - \beta v, \\ 0 = \Delta w + \gamma u - \delta w \end{cases}$$

in $\Omega \times (0, \infty)$, with Ω a ball in $\mathbb{R}^N$, $N \geq 1$ under no flux boundary conditions and positive initial data. If $N \geq 3$, under conditions on the data, we prove that the solution blows up in L^∞-norm at finite time T_{max}. Moreover for all $p > N$ we prove that the solution blows up also in L^p-norm and a lower bound of blow-up time is derived.

Mathematics Subject Classification. Primary: 35B44; Secondary: 35Q92, 92C17.

Keywords. Finite-time blow-up, Chemotaxis, Flux limitation.

1. Introduction

Chemotaxis refers to the movement of cells or organisms in response to chemical stimuli. This phenomenon plays a crucial role in biological processes such as immune response, wound healing, and bacterial aggregation. Cells typically migrate toward higher concentrations of attractant chemicals or away from repellents, enabling them to navigate complex environments. In 1970, Keller and Segel in [7] introduced the following foundational mathematical model for chemotaxis which describes the dynamics of cell density $u(x,t)$ and chemical concentration $v(x,t)$,

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v + \alpha u - \beta v, & x \in \Omega,\ t > 0. \end{cases} \tag{1.1}$$

Originally formulated to capture the aggregation behavior of slime molds, this system has a cornerstone in the study of nonlinear partial differential equations and pattern formation in biological systems. At its core, the Keller–Segel model couples the dynamics of cell density with the concentration of a chemoattractant, leading to rich phenomena such as blow-up, travelling waves, and spatial clustering. These models help researchers to understand pattern formation, population dynamics, and the conditions under which cells aggregate or disperse.

Recent interdisciplinary research has begun to explore the relevance of chemotactic models like Keller–Segel in the context of neurodegenerative diseases, particularly Alzheimer's disease. In Alzheimer's pathology, microglial cells (immune cells of the brain) exhibit chemotactic behavior as they respond to amyloid-beta plaques. By adapting the Keller–Segel framework to describe microglial migration and interaction with neurotoxic agents, researchers can gain insight into the spatial-temporal dynamics of inflammation

Published online: 01 September 2026

and plaque formation. This mathematical perspective offers a promising avenue for understanding disease progression and potentially guiding therapeutic strategies.

In 2003, Luca et al. in [11] introduced the following attraction-repulsion system to model the clustering behavior of microglial cells $u(x,t)$ observed in Alzheimer's disease,

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w), & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ \tau w_t = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0. \end{cases} \tag{1.2}$$

In this framework, microglial migration is influenced by competing chemical signals–chemoattractants like β-amyloid and chemorepellents such as TNF-α (here represented by the functions $v(x,t)$ and $w(x,t)$).

Over the past few years, the problem (1.2) has attracted considerable attention and has been the subject of extensive mathematical and biological analysis. In [17], Tao–Wang considered the system (1.2) and studied the global solvability, boundedness, blow-up, existence of non-trivial stationary solutions and asymptotic behavior of the system on various parameter regimes. Particularly, in the two-dimensional setting, they proved that the system with $\tau = 0$ is globally well-posed in high dimensions if repulsion prevails over attraction in the sense that $\chi\alpha - \xi\gamma < 0$, and that the system with $\tau = 1$ is globally well-posed in two dimensions if repulsion dominates over attraction in the sense that $\chi\alpha - \xi\gamma < 0$ and $\beta = \delta$. Moreover they derived finite-time blow-up under the conditions that $\int_\Omega u_0(x)|x-x_0|^2\,dx$ is sufficiently small for $x_0 \in \Omega$ and $\chi\alpha - \xi\gamma > 0$, $\delta = \beta$ and $\int_\Omega u_0(x)\,dx > \frac{8\pi}{\chi\alpha-\xi\gamma}$. Finite-time blow-up in the system with $\tau = 1$ was established by Lankeit [8] even in the case $\beta \neq \delta$. In [10], Li–Li extended Tao–Wang's blow up result to the conditions $\chi\alpha - \xi\gamma > 0$, $\delta \geq \beta$ and $\int_\Omega u_0(x)\,dx > \frac{8\pi}{\chi\alpha-\xi\gamma}$ or $\chi\alpha\delta - \xi\gamma\beta > 0$, $\delta < \beta$ and $\int_\Omega u_0(x)\,dx > \frac{8\pi}{\chi\alpha\delta-\xi\gamma\beta}$. In [5], Chiyo et al. analyzed, in a ball $\Omega \subset \mathbb{R}^N$, $N \geq 3$, finite-time blow-up behavior of the attraction–repulsion chemotaxis system (1.2) by incorporating logistic degradation and established a lower bound for the blow-up time.

Mathematical models of chemotaxis with flux-limited diffusion arise naturally in the description of tumor invasion processes, in particular glioblastoma progression, where cancer cells migrate under the combined influence of biochemical cues and interactions with the surrounding tissue, leading to the formation of structured invasion fronts whose dynamics require saturated flux mechanisms to ensure finite propagation speed (see, e.g., Conte, Casas-Tintò and Soler [6]). In this framework, the cell density evolves under the interplay of chemoattractant and chemorepellent signals, giving rise to complex pattern formation phenomena. Such modeling approaches can be viewed as extensions of the Keller–Segel model; recent contributions include Xu [19], who investigated asymptotic behavior and pattern formation in flux-limited settings, and Zhigun [20], who established existence of global entropy solutions for degenerate chemotaxis systems with saturated diffusion.

Inspired by the interesting applications in the medical field, this paper aims to study the behavior in time of solutions to the following variant of attraction–repulsion model (1.2):

$$\begin{cases} u_t = \nabla\cdot\Big(\dfrac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\Big) - \nabla\cdot\Big(\dfrac{u\nabla(\chi v - \xi w)}{(1+|\nabla v|^2+|\nabla w|^2)^q}\Big), & x\in\Omega,\ t>0,\\ 0 = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0,\\ \dfrac{u\frac{\partial u}{\partial\nu}}{\sqrt{u^2+|\nabla u|^2}} = \dfrac{\partial v}{\partial\nu} = \dfrac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), & x\in\Omega \end{cases} \tag{1.3}$$

in $\Omega \times (0,\infty)$, with Ω a ball in $\mathbb{R}^N$, $N \geq 1$, where χ, ξ, α, β, γ, δ are positive, and q is nonnegative. The initial data u_0 satisfies

$$u_0 \in C^2(\overline{\Omega}), \text{ radially symmetric and positive in } \overline{\Omega}. \tag{1.4}$$

The novelty of this model is the presence not only of the flux limitation in the drift term but also the presence of the flux diffusion term. The choice to study such type problem with flux limitation term is due to the fact that these terms assume particles follow preferred paths (like cell boundaries) rather than diffusing randomly. This approach removes non-physical diffusion, ensuring finite-speed propagation, a core feature of the model (cf. [4]).

Originally, these flux limitation terms were introduced by Bellomo–Winkler in [1] and [2] where, by introducing the model

$$\begin{cases} u_t = \nabla\cdot\Big(\dfrac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\Big) - \chi\nabla\cdot(\dfrac{u\nabla v}{\sqrt{1+|\nabla v|^2}}), & x\in\Omega,\ t>0,\\ 0 = \Delta v - m_0 + u, & x\in\Omega,\ t>0,\\ \Big(\frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}} - \chi\frac{u\nabla v}{\sqrt{1+|\nabla v|^2}}\Big)\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), & x\in\Omega \end{cases} \tag{1.5}$$

in a ball $\Omega \subset \mathbb{R}^N$, $N \geq 1$, $m_0 := \frac{1}{|\Omega|}\int\limits_\Omega u_0$, under some conditions on $\chi > 0$ and $\int\limits_\Omega u_0$, they showed local existence with extensibility criterion and global existence of bounded radial solutions in [1], and in [2] the authors established the existence of an initial datum such that the corresponding solution blows up in finite time. Thereafter, in [15] Mizukami et al. analyzed the more general problem

$$\begin{cases} u_t = \nabla\cdot\Big(\dfrac{u^{\mathsf{p}}\nabla u}{\sqrt{u^2+|\nabla u|^2}}\Big) - \chi\nabla\cdot\Big(\dfrac{u^{\mathsf{q}}\nabla v}{\sqrt{1+|\nabla v|^2}}\Big), & x\in\Omega,\ t>0,\\ 0 = \Delta v - m_0 + u, & x\in\Omega,\ t>0,\\ \Big(\frac{u^{\mathsf{p}}\nabla u}{\sqrt{u^2+|\nabla u|^2}} - \chi\frac{u^{\mathsf{q}}\nabla v}{\sqrt{1+|\nabla v|^2}}\Big)\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), & x\in\Omega \end{cases} \tag{1.6}$$

with $\mathsf{p},\mathsf{q} \geq 1$, and established local existence and extensibility criterion ruling out gradient blow-up when $\mathsf{p},\mathsf{q} \geq 1$, and moreover derived global existence and boundedness of solutions when $\mathsf{p} > \mathsf{q} + 1 - \frac{1}{N}$. Chiyoda et al. in [3] considered the problem (1.6) and extended the result of [2] by proving the existence of blow-up solutions under some conditions on χ and u_0 with $1 \leq \mathsf{p} \leq \mathsf{q}$.

The principal purpose of the present work is to investigate on the blow-up phenomenon for the solution of (1.3).

The main results are summarized as follows.

Theorem 1.1. (Finite-time blow-up in L^∞-norm) *Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geq 3$, $R > 0$ and let $\alpha,\ \beta,\ \delta,\ \gamma,\ \chi,\ \xi > 0$. Assume that*

$$\chi\alpha - \xi\gamma > 0 \quad \textit{and} \quad 0 \leq q < \frac{1}{N-1}.$$

Then for all $m > 0$ and $m_1 \in (0,m)$ we can find $r_1 \in (0,R)$ such that:
if u_0 complies with (1.4) and

$$\int\limits_\Omega u_0\,dx = m, \tag{1.7}$$

but

$$\int\limits_{B_{r_1}(0)} u_0\,dx \geq m_1, \tag{1.8}$$

there exist $T_{max} < \infty$ and a classical solution (u, v, w) of (1.3),

$$\begin{cases} u \in C^{2,1}(\overline{\Omega} \times [0, T_{max})), \\ v, w \in C^{2,0}(\overline{\Omega} \times [0, T_{max})), \end{cases}$$

which blows up at T_{max} in the sense that

$$\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty, \tag{1.9}$$

where $C^{2,k}$ means C^2 in x and C^k in t $(k = 0, 1)$.

Remark 1.1. Compared to the blow-up results for (1.5) and (1.6), the ideal exponent for q in the main results should be $1/2$, which however can not be covered in the main results.

Theorem 1.2. (Finite-time blow-up in L^p-norm) *Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geq 3$ and $R > 0$. Then, a classical solution (u, v, w) of (1.3), provided by Theorem 1.1, is such that for all $p > N$,*

$$\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^p(\Omega)} = \infty.$$

The analysis of blow-up solutions for the system (1.3) proceeds by examining the behavior of solutions as they approach the blow-up time T_{max}. The principal aim is to establish a guaranteed interval $(0, T)$, $(T < T_{max})$, of existence of the solutions to (1.3); to this end, we define, for all $p > 1$, the auxiliary function

$$\Psi(t) := \frac{1}{p}\|u(\cdot, t)\|^p_{L^p(\Omega)} \quad \text{with} \quad \Psi_0 := \Psi(0) = \frac{1}{p}\|u_0\|^p_{L^p(\Omega)}, \tag{1.10}$$

and we determine a lower estimate of the blow-up time T_{max}.

Theorem 1.3. (A lower bound of the blow-up time T_{max}) *Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geq 3$, $R > 0$ and let Ψ be defined in (1.10). Then, for all $p > N$ and some positive constants B_1, B_2, B_3, B_4, the blow-up time T_{max} for (1.3), provided by Theorem 1.2, satisfies the estimate*

$$T_{max} \geq T := \int\limits_{\Psi_0}^{\infty} \frac{d\eta}{B_1\eta + B_2\eta^{\gamma_1} + B_3\eta^{\gamma_2} + B_4\eta^{\gamma_3}}, \tag{1.11}$$

with $\gamma_1 := \frac{p+1}{p}$, $\gamma_2 := \frac{p+1-N}{p-N}$, $\gamma_3 := \frac{(p+1)[(p+1+\epsilon_1)-N(1+\epsilon_1)]}{p(p+1+\epsilon_1)-N(1+\epsilon_1)(p+1)}$, $\epsilon_1 > 0$.

When $q = 0$, the problem (1.3) simplifies and Theorems 1.1, 1.2 and 1.3 remain valid as detailed in the corollary that follows.

Corollary 1.1. *Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geq 3$, $R > 0$ and let $\alpha,\ \beta,\ \delta,\ \gamma,\ \chi,\ \xi > 0$. Assume that*

$$\chi\alpha - \xi\gamma > 0.$$

Then Theorems 1.1, 1.2 and 1.3 hold for

$$\begin{cases} u_t = \nabla\cdot\Big(\frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\Big) - \chi\nabla\cdot\big(u\nabla v\big) + \xi\nabla\cdot\big(u\nabla w\big), & x\in\Omega,\ t>0,\\ 0 = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0,\\ \frac{u\frac{\partial u}{\partial\nu}}{\sqrt{u^2+|\nabla u|^2}} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), & x\in\Omega. \end{cases} \tag{1.12}$$

By setting $w=0$ in (1.3), the problem is reduced to a problem of two equations and Theorems 1.1, 1.2 and 1.3 are preserved as stated in the following corollary.

Corollary 1.2. *Let $\Omega = B_R(0)\subset\mathbb{R}^N$, $N\geq 3$, $R>0$ and let $\alpha,\ \beta,\ \delta,\ \gamma,\ \chi>0$. Assume that*

$$0\leq q<\frac{1}{N-1}.$$

Then Theorems 1.1, 1.2 and 1.3 hold for

$$\begin{cases} u_t = \nabla\cdot\Big(\frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\Big) - \chi\nabla\cdot\Big(\frac{u\nabla v}{(1+|\nabla v|^2)^q}\Big), & x\in\Omega,\ t>0,\\ 0 = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ \frac{u\frac{\partial u}{\partial\nu}}{\sqrt{u^2+|\nabla u|^2}} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), & x\in\Omega. \end{cases} \tag{1.13}$$

This paper is organized as follows. In Sect. 2 we give the proof of finite-time blow-up in L^∞-norm (Theorem 1.1) by using a moment-like functional. Finite-time blow-up in L^p-norm (Theorem 1.2) and lower bound of the blow-up time T_{max} (Theorem 1.3) are established in Sect. 3.

2. Blow-up in L^∞

The central goal of this section is to prove that the solution of (1.3) blows up in finite time in $L^\infty(\Omega)$-norm.

Firstly we state the following basic lemma for the local existence.

Lemma 2.1. *Let $\Omega = B_R(0)\subset\mathbb{R}^N$, $N\geq 3$ and $R>0$, $q\geq 0$. Assume that u_0 complies (1.4) and (1.7). Then there exist $T_{max}\in(0,\infty]$ and a triplet (u,v,w) of positive radially symmetric functions $u\in C^{2,1}(\overline{\Omega}\times[0,T_{max}))$ and $v,w\in C^{2,0}(\overline{\Omega}\times[0,T_{max}))$ which solve (1.3) classically in $\Omega\times(0,T_{max})$, and moreover u satisfies that*

$$\text{if}\quad T_{max}<\infty,\quad \text{then}\quad \limsup_{t\nearrow T_{max}}\|u(\cdot,t)\|_{L^\infty(\Omega)}=\infty.$$

Proof. The claim can be proved by a pointwise lower estimate for u and by a uniform estimate for $|\nabla u|$ almost in the same way as in [15]. □

In order to study the blow up phenomena of the solution to (1.3) we introduce radial coordinates with $s = r^N \in [0, R^N]$, $t \in [0, T_{max})$ defining the functions

$$U(s,t) := \int_0^{s^{\frac{1}{N}}} \rho^{N-1} u(\rho,t) d\rho, \tag{2.1}$$

$$V(s,t) := \int_0^{s^{\frac{1}{N}}} \rho^{N-1} v(\rho,t) d\rho,$$

$$W(s,t) := \int_0^{s^{\frac{1}{N}}} \rho^{N-1} w(\rho,t) d\rho.$$

Thus we have

$$\begin{aligned} U_s(s,t) &= \frac{1}{N} u(s^{\frac{1}{N}}, t) \geq 0, \quad U_{ss}(s,t) = \frac{1}{N^2} s^{\frac{1}{N}-1} u_r(s^{\frac{1}{N}}, t), \\ V_s(s,t) &= \frac{1}{N} v(s^{\frac{1}{N}}, t) \geq 0, \quad V_{ss}(s,t) = \frac{1}{N^2} s^{\frac{1}{N}-1} v_r(s^{\frac{1}{N}}, t), \\ W_s(s,t) &= \frac{1}{N} w(s^{\frac{1}{N}}, t) \geq 0, \quad W_{ss}(s,t) = \frac{1}{N^2} s^{\frac{1}{N}-1} w_r(s^{\frac{1}{N}}, t). \end{aligned} \tag{2.2}$$

From the second and third Eqs. in (1.3) we deduce

$$\begin{aligned} \frac{1}{r^{N-1}} \left(r^{N-1} v_r(r,t) \right)_r &= -\alpha u(r,t) + \beta v(r,t), \\ \frac{1}{r^{N-1}} \left(r^{N-1} w_r(r,t) \right)_r &= -\gamma u(r,t) + \delta w(r,t) \end{aligned} \tag{2.3}$$

and by integrating (2.3) and using (2.1) (with $s = r^N$) we obtain

$$\begin{aligned} r^{N-1} v_r &= -\alpha \int_0^r \rho^{N-1} u d\rho + \beta \int_0^r \rho^{N-1} v d\rho = -\alpha U(r^N, t) + \beta V(r^N, t), \\ r^{N-1} w_r &= -\gamma \int_0^r \rho^{N-1} u d\rho + \delta \int_0^r \rho^{N-1} w d\rho = -\gamma U(r^N, t) + \delta W(r^N, t) \end{aligned} \tag{2.4}$$

for all $r \in (0, R)$, and $t \in (0, T_{max})$. The radial version of the first Eq. in (1.3) is

$$\begin{aligned} U_t(s,t) &= \int_0^{s^{\frac{1}{N}}} r^{N-1} u_t(r,t) dr \\ &= \int_0^{s^{\frac{1}{N}}} \left(r^{N-1} \frac{u u_r}{\sqrt{u^2 + u_r^2}} \right)_r dr - \int_0^{s^{\frac{1}{N}}} \left(r^{N-1} u \frac{\chi v_r - \xi w_r}{\left(1 + v_r^2 + w_r^2\right)^q} \right)_r dr \\ &= s^{1-\frac{1}{N}} \frac{u u_r}{\sqrt{u^2 + u_r^2}} - s^{1-\frac{1}{N}} \frac{u(\chi v_r - \xi w_r)}{\left(1 + v_r^2 + w_r^2\right)^q}. \end{aligned}$$

By using (2.2) and (2.3) we obtain

$$
\begin{aligned}
U_t(s,t) = &\frac{N^2 s^{2-\frac{2}{N}} U_s U_{ss}}{\sqrt{U_s^2 + N^2 s^{2-\frac{2}{N}} U_{ss}^2}} \\
&- NU_s \frac{[\chi(-\alpha U + \beta V) - \xi(-\gamma U + \delta W)]}{\left(1 + s^{\frac{2}{N}-2}[(-\alpha U + \beta V)^2 + (-\gamma U + \delta W)^2]\right)^q} \\
= &\frac{N^2 s^{2-\frac{2}{N}} U_s U_{ss}}{\sqrt{U_s^2 + N^2 s^{2-\frac{2}{N}} U_{ss}^2}} \\
&+ N(\chi\alpha - \xi\gamma) \frac{UU_s}{\left(1 + s^{\frac{2}{N}-2}([-\alpha U + \beta V)^2 + (-\gamma U + \delta W)^2]\right)^q} \\
&- \chi N\beta \frac{VU_s}{\left(1 + s^{\frac{2}{N}-2}[(-\alpha U + \beta V)^2 + (-\gamma U + \delta W)^2]\right)^q} \\
&+ \xi N\delta \frac{WU_s}{\left(1 + s^{\frac{2}{N}-2}[(-\alpha U + \beta V)^2 + (-\gamma U + \delta W)^2]\right)^q}.
\end{aligned} \tag{2.5}
$$

In order to obtain our blow-up result we derive a basic differential inequality describing the evolution of the following moment-like functional ϕ with $s_0 \in (0, R^N)$ and $a \in (0,1)$:

$$
\phi(t) := \int_0^{s_0} s^{-a}(s_0 - s)U(s,t)ds, \quad t \in [0, T_{max}). \tag{2.6}
$$

To this end we start with the following lemma.

Lemma 2.2. *Let $N \geq 3$, $R > 0$ and assume that u_0 satisfies (1.4). Then for any choice $a \in (0, 1 - \frac{1}{N})$ and $s_0 \in (0, R^N)$, the function $\phi : [0, T_{max}) \to \mathbb{R}$ defined in (2.6) belongs to $C^0([0, T_{max})) \cap C^1((0, T_{max}))$ and satisfies*

$$
\begin{aligned}
\phi'(t) \geq &-Ns_0 \int_0^{s_0} s^{-\frac{1}{N}-a} U ds + N(\chi\alpha - \xi\gamma)\bar{C} \int_0^{s_0} s^{-a+(2-\frac{2}{N})q}(s_0 - s)UU_s ds \\
&- \chi N\beta s_0(a+1) \int_0^{s_0} s^{-a-1} VU ds,
\end{aligned} \tag{2.7}
$$

with $\bar{C} = \frac{1}{\left(R^{2N-2} + \frac{m^2}{\omega_N^2}[\alpha^2 + \gamma^2]\right)^q}$.

Proof. Since $u \in C^0(\overline{\Omega} \times [0, T_{max}))$ and $u_t \in C^0(\overline{\Omega} \times (0, T_{max}))$, a standard argument based on the dominated convergence theorem ensures that indeed ϕ belongs to $C^0([0, T_{max}))$ and to $C^1((0, T_{max}))$.

Following the step in [18] (and [5]), differentiating $\phi(t)$ in t and using the Eq. (2.5), we have

$$
\phi'(t) = \int_0^{s_0} s^{-a}(s_0 - s)U_t(s,t)ds \tag{2.8}
$$

$$
\begin{aligned}
&= \int_0^{s_0} s^{-a}(s_0-s)\frac{N^2 s^{2-\frac{2}{N}}U_sU_{ss}}{\sqrt{U_s^2+N^2s^{2-\frac{2}{N}}U_{ss}^2}}ds\\
&\quad + N(\chi\alpha-\xi\gamma)\int_0^{s_0} s^{-a}(s_0-s)\frac{UU_s}{\left(1+s^{\frac{2}{N}-2}([(-\alpha U+\beta V)^2+(-\gamma U+\delta W)^2]\right)^q}ds\\
&\quad - \chi N\beta\int_0^{s_0} s^{-a}(s_0-s)\frac{VU_s}{\left(1+s^{\frac{2}{N}-2}[(-\alpha U+\beta V)^2+(-\gamma U+\delta W)^2]\right)^q}ds\\
&\quad + \int_0^{s_0} s^{-a}(s_0-s)\xi N\delta\frac{WU_s}{\left(1+s^{\frac{2}{N}-2}[(-\alpha U+\beta V)^2+(-\gamma U+\delta W)^2]\right)^q}ds
\end{aligned}
$$

for all $t\in(0,T_{max})$. Neglecting the positive fourth term on the right-hand side of (2.8) and noting that $\frac{N^2s^{2-\frac{2}{N}}U_sU_{ss}}{\sqrt{U_s^2+N^2s^{2-\frac{2}{N}}U_{ss}^2}} \geq -Ns^{1-\frac{1}{N}}U_s$, we obtain

$$
\begin{aligned}
\phi'(t) &\geq -N\int_0^{s_0} s^{1-\frac{1}{N}-a}(s_0-s)U_s ds \qquad (2.9)\\
&\quad + N(\chi\alpha-\xi\gamma)\int_0^{s_0} s^{-a}(s_0-s)\frac{UU_s}{\left(1+s^{\frac{2}{N}-2}([(-\alpha U+\beta V)^2+(-\gamma U+\delta W)^2]\right)^q}ds\\
&\quad - \chi N\beta\int_0^{s_0} s^{-a}(s_0-s)\frac{VU_s}{\left(1+s^{\frac{2}{N}-2}[(-\alpha U+\beta V)^2+(-\gamma U+\delta W)^2]\right)^q}ds\\
&= I_1+I_2+I_3.
\end{aligned}
$$

In I_1, noting that $1-\frac{1}{N}-a>0$ because $a\in(0,1-\frac{1}{N})$, we integrate by parts to derive

$$
\begin{aligned}
I_1 &= -N\int_0^{s_0} s^{1-\frac{1}{N}-a}(s_0-s)U_s ds \qquad (2.10)\\
&= -N[s^{1-\frac{1}{N}-a}(s_0-s)U]_0^{s_0} + N\int_0^{s_0}\frac{d}{ds}\left(s^{1-\frac{1}{N}-a}(s_0-s)\right)U ds\\
&= N(1-\frac{1}{N}-a)\int_0^{s_0} s^{-\frac{1}{N}-a}(s_0-s)U ds - N\int_0^{s_0} s^{1-\frac{1}{N}-a}U ds\\
&\geq -N\int_0^{s_0} s^{1-\frac{1}{N}-a}U ds \geq -Ns_0\int_0^{s_0} s^{-\frac{1}{N}-a}U ds.
\end{aligned}
$$

To estimate I_3 we firstly take into account that $\frac{1}{\left(1+s^{\frac{2}{N}-2}[(-\alpha U+\beta V)^2+(-\gamma U+\delta W)^2]\right)^q} \leq 1$ and then we integrate by parts, with some $\varepsilon > 0$, to have

$$
\begin{aligned}
I_3 &\geq -\chi N\beta \int_0^{s_0} s^{-a}(s_0-s)VU_s ds \qquad (2.11)\\
&= \chi N\beta \liminf_{\varepsilon\to 0^+}[\varepsilon^{-a}(s_0-\varepsilon)V(\varepsilon,t)U(\varepsilon,t)]\\
&\quad - \chi N\beta a \int_0^{s_0} s^{-a-1}(s_0-s)VUds - \chi N\beta \int_0^{s_0} s^{-a}VUds\\
&\quad + \chi N\beta \int_0^{s_0} s^{-a}(s_0-s)V_sUds\\
&\geq -\chi N\beta a s_0 \int_0^{s_0} s^{-a-1}VUds - \chi N\beta \int_0^{s_0} s^{-a}VUds\\
&\geq -\chi N\beta s_0(a+1) \int_0^{s_0} s^{-a-1}VUds,
\end{aligned}
$$

due to the fact that U, V and V_s are all nonnegative. To estimate the term I_2 in (2.8) we observe that

$$
\begin{aligned}
&\frac{1}{\left(1+s^{\frac{2}{N}-2}[(-\alpha U+\beta V)^2+(-\gamma U+\delta W)^2]\right)^q}\\
&\geq \frac{(s^{2-\frac{2}{N}})^q}{\left(R^{2N-2}+\frac{m^2}{\omega_N^2}(\alpha^2+\gamma^2)\right)^q} = \bar{C}s^{(2-\frac{2}{N})q}.
\end{aligned}
$$

In fact, from the second and third Eqs. in (1.3) we can deduce that

$$
\begin{aligned}
&\int_\Omega v(\cdot,t)dx = \frac{\alpha}{\beta}\int_\Omega u(\cdot,t)dx = \frac{\alpha}{\beta}\int_\Omega u_0dx = \frac{\alpha}{\beta}m, \qquad (2.12)\\
&\int_\Omega w(\cdot,t)dx = \frac{\gamma}{\delta}\int_\Omega u(\cdot,t)dx = \frac{\gamma}{\delta}\int_\Omega u_0dx = \frac{\gamma}{\delta}m.
\end{aligned}
$$

Since u, v and w are nonnegative, for all $s \in (0, R^N)$ *and* $t \in (0, T_{max})$ we have

$$
\begin{aligned}
&0 \leq U(s,t) \leq U(R^N,t) = \frac{1}{\omega_N}\int_\Omega u(\cdot,t)dx = \frac{m}{\omega_N}, \qquad (2.13)\\
&0 \leq V(s,t) \leq V(R^N,t) = \frac{1}{\omega_N}\int_\Omega v(\cdot,t)dx = \frac{\alpha}{\beta}\frac{m}{\omega_N},\\
&0 \leq W(s,t) \leq W(R^N,t) = \frac{1}{\omega_N}\int_\Omega w(\cdot,t)dx = \frac{\gamma}{\delta}\frac{m}{\omega_N}.
\end{aligned}
$$

By using (2.13) we can see that $-\alpha\frac{m}{\omega_N} \le -\alpha U \le -\alpha U + \beta V \le \beta V \le \alpha\frac{m}{\omega_N}$ and $-\gamma\frac{m}{\omega_N} \le -\gamma U \le -\gamma U + \delta W \le \delta W \le \gamma\frac{m}{\omega_N}$, and thus,

$$(-\alpha U + \beta V)^2 \le \alpha^2 \frac{m^2}{\omega_N^2}, \qquad (-\gamma U + \delta W)^2 \le \gamma^2 \frac{m^2}{\omega_N^2}. \tag{2.14}$$

From (2.14), since $s \le R^N$, we arrive at

$$\begin{aligned} &\frac{1}{\left(1 + s^{\frac{2}{N}-2}[(-\alpha U + \beta V)^2 + (-\gamma U + \delta W)^2]\right)^q} \\ &\ge \frac{1}{\left(1 + s^{\frac{2}{N}-2}\frac{m^2}{\omega_N^2}[\alpha^2+\gamma^2]\right)^q} \\ &= \frac{s^{(2-\frac{2}{N})q}}{\left(s^{2-\frac{2}{N}} + \frac{m^2}{\omega_N^2}[\alpha^2+\gamma^2]\right)^q} \\ &\ge \frac{s^{(2-\frac{2}{N})q}}{\left(R^{2N-2} + \frac{m^2}{\omega_N^2}[\alpha^2+\gamma^2]\right)^q} = \bar{C} s^{(2-\frac{2}{N})q}. \end{aligned} \tag{2.15}$$

To estimate I_2 we apply (2.15) to obtain

$$\begin{aligned} I_2 &= N(\chi\alpha - \xi\gamma)\int_0^{s_0} s^{-a}(s_0 - s)\frac{UU_s}{\left(1 + s^{\frac{2}{N}-2}([-\alpha U + \beta V)^2 + (-\gamma U + \delta W)^2]\right)^q} ds \\ &\ge N(\chi\alpha - \xi\gamma)\bar{C}\int_0^{s_0} s^{-a+(2-\frac{2}{N})q}(s_0 - s)UU_s ds. \end{aligned} \tag{2.16}$$

By substituting (2.10), (2.11) and (2.16) into (2.8) we obtain (2.7). □

We reduce each term on the right-hand side of (2.7) to powers of

$$\psi(t) := \int_0^{s_0} s^{-a+\theta}(s_0 - s)U(s,t)U_s(s,t)ds, \quad \theta := 2\Big(1 - \frac{1}{N}\Big)q. \tag{2.17}$$

To control the first and third terms of (2.7) we use the following lemmata.

Lemma 2.3. *Let $\gamma_1 \in (1,2)$ and $\gamma_2 \in (0,1)$. Then there exists $C = C(\gamma_1, \gamma_2)$ such that if $s_0 > 0$ then*

$$\int_0^s \int_\sigma^{s_0} \xi^{-\gamma_1}(s_0 - \xi)^{-\gamma_2} d\xi d\sigma \le C s_0^{-\gamma_2} s^{2-\gamma_1} \quad \text{for all } s \in (0, s_0). \tag{2.18}$$

For the proof see Lemma 4.6 in [18].

Lemma 2.4. *Let $N \ge 3$, $R > 0$, $a \in (0,2)$ and $0 \le \theta < a$ such that*

$$a < \frac{2}{\theta+1}\Big(1 - \frac{1}{N}\Big) - \frac{1}{N}\frac{\theta}{\theta+1}. \tag{2.19}$$

Then there exists $C_1 = C_1(R, m, a, \theta, \omega_N) > 0$, such that if u_0 fulfils (1.4) with $m = \int_\Omega u_0$, then for $s_0 \in (0, R^N)$,

$$s_0\int_0^{s_0} s^{-a-\frac{1}{N}}U(s,t)ds \le C_1 s_0^{2-a-\frac{1}{N}+\frac{a-\theta-1}{\theta+2}}\left\{\int_0^{s_0} s^{-a+\theta}(s_0-s)UU_s ds\right\}^{\frac{1}{\theta+2}} \tag{2.20}$$

for all $t\in(0,T_{max})$.

Proof. Following the steps of [16, Lemma 4.4], for $s\in(0,s_0]$ we introduce the function $\Psi(s) := \frac{1}{\theta+2}s^{-a+\theta}(s_0-s)\varphi^{\theta+2}(s)$ with $\theta\in[0,a)$, $\varphi\in C^1([0,s_0])$ is nonnegative, $\varphi(0)=0$ and $\varphi'(s)\ge 0$ for all $s\in(0,s_0)$. We have

$$\Psi(s) = \int_0^s \Psi'(\sigma)d\sigma \le \int_0^s \sigma^{-a+\theta}(s_0-\sigma)\varphi^{\theta+1}(\sigma)\varphi'(\sigma)d\sigma.$$

From the definition of Ψ we obtain

$$\varphi(s) \le (\theta+2)^{\frac{1}{\theta+2}} s^{\frac{a-\theta}{\theta+2}}(s_0-s)^{-\frac{1}{\theta+2}}\left\{\int_0^{s_0} \sigma^{-a+\theta}(s_0-\sigma)\varphi^{\theta+1}(\sigma)\varphi'(\sigma)d\sigma\right\}^{\frac{1}{\theta+2}}. \tag{2.21}$$

By using this inequality with $\varphi(s)=U(s,t)$ and $U(s,t)\le c_1 := \frac{m}{\omega_N}$ we obtain

$$\begin{aligned}
& s_0\int_0^{s_0} s^{-a-\frac{1}{N}}U(s,t)ds \\
&\le s_0 c_1^{\frac{\theta}{\theta+2}}(\theta+2)^{\frac{1}{\theta+2}}\int_0^{s_0} s^{-a-\frac{1}{N}+\frac{a-\theta}{\theta+2}}(s_0-s)^{-\frac{1}{\theta+2}}ds\cdot\left\{\int_0^{s_0} s^{-a+\theta}(s_0-s)UU_s ds\right\}^{\frac{1}{\theta+2}} \\
&= C_1 s_0^{2-a-\frac{1}{N}+\frac{a-\theta-1}{\theta+2}}\left\{\int_0^{s_0} s^{-a+\theta}(s_0-s)UU_s ds\right\}^{\frac{1}{\theta+2}},
\end{aligned}$$

where $C_1 = c_1^{\frac{\theta}{\theta+2}}(\theta+2)^{\frac{1}{\theta+2}}B$ and $B = B(1-a-\frac{1}{N}+\frac{a-\theta}{\theta+1}, 1-\frac{1}{\theta+2})$ is the Euler's beta function. We note that $B<\infty$ because a satisfies (2.19) and $1-\frac{1}{\theta+2}>0$. $\square$

Lemma 2.5. *Let* $N\ge 3$, $R>0$, $a\in(0,1-\frac{1}{N})$ *and* $0\le\theta<a$ *such that*

$$\frac{\theta}{\theta+2}(a+2) < \frac{2}{N}. \tag{2.22}$$

Then there exists $C_2 = C_2(R,m,a,\theta)>0$ *such that for all* u_0 *fulfilling (1.4) and each* $s_0\in(0,R^N)$ *we have*

$$\begin{aligned}
& s_0\int_0^{s_0} s^{-a-1}UV ds \\
&\le C_2 s_0^{1+\frac{2}{N}-a} + C_2 s_0^{1+\frac{2}{N}-a+\frac{2(a-\theta-1)}{\theta+2}}\left\{\int_0^{s_0} s^{-a+\theta}(s_0-s)UU_s ds\right\}^{\frac{2}{\theta+2}}.
\end{aligned} \tag{2.23}$$

Proof. For the proof we follow the step in the proof of Lemmas 4.8 and 4.9 in [16].

Since $V\le\frac{\alpha}{\beta}\frac{m}{\omega_N}$, from (2.4) we can write

$$r^{N-1}v_r = -\alpha U(r^N,t) + \beta V(r^N,t) \ge -\alpha U(r^N,t),$$

thus

$$r^{N-1}v_r = -\alpha U(r^N,t) + \beta V(r^N,t) \le \beta V(r^N,t) \le 2c_2 = \alpha\frac{m}{\omega_N},$$

and

$$v(r,t) \leq r^{2-N}\frac{c_2}{N-2}$$

and consequently, since $s^{\frac{1}{N}} = r$

$$v(s^N, t) \leq \tilde{c}_2 s^{\frac{2}{N}-1},$$

with $\tilde{c}_2 = \frac{c_2}{N-2}$. Taking into account of (2.2) and $v_r(s^N, t) \geq -\alpha s^{\frac{1}{N}-1}U(s,t)$, we have

$$V_{ss}(s,t) = \frac{1}{N^2}s^{\frac{1}{N}-1}v_r(s^{\frac{1}{N}}, t) \geq -\frac{\alpha}{N^2}s^{\frac{2}{N}-2}U(s,t),$$

and hence

$$V_s(s,t) = V_s(s_0,t) - \int_s^{s_0} V_{ss}(\sigma,t)d\sigma \leq \frac{1}{N}v(s_0^{\frac{1}{N}}, t) + \frac{\alpha}{N^2}\int_s^{s_0}\sigma^{\frac{2}{N}-2}U(\sigma,t)d\sigma,$$

with $s_0 \in (0, R^N)$. According to the latest estimates and $v \leq \tilde{c}_2 s^{\frac{2}{N}-1}$, we find that

$$V(s,t) = \int_0^s V_s(\sigma,t)d\sigma \leq \frac{\tilde{c}_2}{N}s_0^{\frac{2}{N}-1}s + \frac{\alpha}{N^2}\int_0^s\int_\sigma^{s_0}\xi^{\frac{2}{N}-2}U(\xi,t)d\xi d\sigma.$$

In the last term we apply the inequality (2.21) with $\varphi = U$ to get

$$\begin{aligned} V(s,t) \leq{}& \frac{\tilde{c}_2}{N}s_0^{\frac{2}{N}-1}s \\ &+ \frac{\alpha}{N^2}\Big\{\int_0^{s_0} s^{-a+\theta}(s_0-s)UU_s ds\Big\}^{\frac{1}{\theta+2}}\int_0^s\int_\sigma^{s_0}\xi^{\frac{2}{N}-2+\frac{a-\theta}{\theta+2}}(s_0-\xi)^{-\frac{1}{\theta+2}}d\xi d\sigma \\ \leq{}& \frac{\tilde{c}_2}{N}s_0^{\frac{2}{N}-1}s + \tilde{c}_3 s_0^{-\frac{1}{\theta+2}}s^{\frac{2}{N}+\frac{a-\theta}{\theta+2}}\Big\{\int_0^{s_0} s^{-a+\theta}(s_0-s)UU_s ds\Big\}^{\frac{1}{\theta+2}}, \end{aligned}$$

where in the last step we have used (2.18) with $\gamma_1 = -\frac{2}{N} + 2 - \frac{a-\theta}{\theta+2} \in (1,2)$ and $\gamma_2 = \frac{1}{\theta+2} \in (0,1)$ and with $\tilde{c}_3 = \frac{\alpha}{N^2}C(a,\theta,N)$.

Considering the last inequality and applying inequality (2.21) twice with $\varphi = U$, we get

$$\begin{aligned} &s_0\int_0^{s_0} s^{-a-1}UV ds \\ &\leq \frac{\tilde{c}_2}{N}s_0^{\frac{2}{N}}\int_0^{s_0} s^{-a}U ds + \tilde{c}_3 s_0^{\frac{\theta+1}{\theta+2}}\int_0^{s_0} s^{-a-1+\frac{2}{N}+\frac{a-\theta}{\theta+2}}U ds \cdot \Big\{\int_0^{s_0} s^{-a+\theta}(s_0-s)UU_s ds\Big\}^{\frac{1}{\theta+2}} \\ &\leq \frac{\tilde{c}_2}{N}s_0^{\frac{2}{N}}\int_0^{s_0} s^{-a}U ds + \tilde{c}_3\widetilde{B}s_0^{1+\frac{2}{N}-a+\frac{2(a-\theta-1)}{\theta+2}}\Big\{\int_0^{s_0} s^{-a+\theta}(s_0-s)UU_s ds\Big\}^{\frac{2}{\theta+2}}, \end{aligned}$$

where $\widetilde{B} := B(-a + \frac{2}{N} + \frac{2(a-\theta)}{\theta+2}, 1 - \frac{1}{\theta+2}) < \infty$ by virtue of (2.22).

Finally, in view of (2.13) and since $a < 1$ we obtain (2.23). □

By replacing (2.20) and (2.23) in (2.7) with $(2 - \frac{2}{N})q = \theta$ we obtain

$$\phi'(t) \geq -NC_1 s_0^{2-\frac{1}{N}-a+\frac{a-\theta-1}{\theta+2}}\Big\{\int_0^{s_0} s^{-a+\theta}(s_0-s)UU_s ds\Big\}^{\frac{1}{\theta+2}} \tag{2.24}$$

$$+ N(\chi\alpha - \xi\gamma)\bar{C}\int_0^{s_0} s^{-a+\theta}(s_0 - s)UU_s ds - \chi N\beta(a+1)C_2 s_0^{1+\frac{2}{N}-a}$$
$$- \chi N\beta(a+1)C_2 s_0^{1+\frac{2}{N}-a+\frac{2(a-\theta-1)}{\theta+2}}\left\{\int_0^{s_0} s^{-a+\theta}(s_0 - s)UU_s ds\right\}^{\frac{2}{\theta+2}}.$$

We are ready to write a first order differential inequality for ϕ.

Lemma 2.6. *Let $N \geq 3$, $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$ and assume that $\chi\alpha - \xi\gamma > 0$. If*

$$0 \leq \theta < \frac{2}{N},$$

then there exists $a \in (0, 1 - \frac{1}{N})$ satisfying $a > \theta$, (2.19) and (2.22) such that for all $m > 0$ there exist $d_1,\ d_2 > 0$ such that whenever u_0 fulfills (1.4), (1.7), for any $s_0 \in (0, R^N)$ the function ϕ satisfies the differential inequality

$$\phi'(t) \geq d_1 s_0^{a(\theta+1)-(\theta+3)}\phi^{\theta+2}(t) - d_2 s_0^{1+\frac{2}{N}-a} \quad \textit{for all } t \in (0, T_{max}). \tag{2.25}$$

Proof. We observe that if $0 \leq \theta < \frac{2}{N}$ and $N \geq 3$ then $\theta < 1 - \frac{1}{N}$ and we can take $\theta < a < 1 - \frac{1}{N}$. Thus we easily get (2.19). In fact, we have $(1 - \frac{1}{N}) - [\frac{2}{\theta+1}(1 - \frac{1}{N}) - \frac{1}{N}\frac{\theta}{\theta+1}] = \frac{1}{\theta+1}[\theta - (1 - \frac{1}{N})] < 0$. Moreover we can choose a satisfying (2.22). In fact, $\theta < \frac{2}{N}\frac{\theta+2}{\theta} - 2$ implies $N\theta^2 + 2(N-1)\theta - 4 < 0$ i.e. $0 \leq \theta < \frac{2}{N}$. Back to (2.24), and ψ defined in (2.17), we can write

$$\phi'(t) \geq N(\chi\alpha - \xi\gamma)\bar{C}\psi(t) - NC_1 s_0^{2-\frac{1}{N}-a+\frac{a-\theta-1}{\theta+2}}\psi^{\frac{1}{\theta+2}}(t)$$
$$- \chi N\beta(a+1)C_2 s_0^{1+\frac{2}{N}-a+\frac{2(a-\theta-1)}{\theta+2}}\psi^{\frac{2}{\theta+2}}(t) - \chi N\beta(a+1)C_2 s_0^{1+\frac{2}{N}-a}.$$

Since $0 \leq \theta < \frac{2}{N}$, we observe that $1 + \frac{2}{N} - a < 2 - \frac{1}{N} - a + \frac{a-\theta-1}{\theta+2}$ and $1 + \frac{2}{N} - a < 1 + \frac{2}{N} - a + \frac{2(a-\theta-1)}{\theta+2}$. Thus we can write

$$\begin{aligned}
\phi'(t) &\geq N(\chi\alpha - \xi\gamma)\bar{C}\psi(t) - NC_1 s_0^{2-\frac{1}{N}-a+\frac{a-\theta-1}{\theta+2}}\psi^{\frac{1}{\theta+2}}(t) \\
&\quad - \chi N\beta(a+1)C_2 s_0^{1+\frac{2}{N}-a+\frac{2(a-\theta-1)}{\theta+2}}\psi^{\frac{2}{\theta+2}}(t) - \chi N\beta(a+1)C_2 s_0^{1+\frac{2}{N}-a} \\
&\geq [N(\chi\alpha - \xi\gamma)\bar{C} - \epsilon_1 - \epsilon_2]\psi(t) \\
&\quad - s_0^{1+\frac{2}{N}-a}\left[c_1(\epsilon_1)s_0^{\frac{1}{N(\theta+1)}(2N-4-3\theta)} + c_2(\epsilon_2)s_0^{(\frac{2}{N}-\theta)\frac{2}{\theta}} + \chi N\beta(a+1)C_2\right] \\
&\geq A_1\psi(t) - A_2 s_0^{1+\frac{2}{N}-a}
\end{aligned} \tag{2.26}$$

with

$$\begin{aligned}
A_{1:} &= N(\chi\alpha - \xi\gamma)\bar{C} - \epsilon_1 - \epsilon_2 \\
A_2 &:= c_1(\epsilon_1)R^{\frac{1}{(\theta+1)}(2N-4-3\theta)} + c_2(\epsilon_2)R^{(2-\theta N)\frac{2}{\theta}} + \chi N\beta(a+1)C_2,
\end{aligned} \tag{2.27}$$

where in (2.26) we used Young inequality and then $s_0 \leq R^N$.

By choosing ϵ_1 and ϵ_2 such that $N(\chi\alpha - \xi\gamma)\bar{C} - \epsilon_1 - \epsilon_2 = A_0 > 0$ and with $A_2(R, N, \theta)$ defined in (2.27) with ϵ_1 and ϵ_2 fixed, we arrive at the differential inequality

$$\phi'(t) \geq A_0\psi(t) - A_2 s_0^{1+\frac{2}{N}-a}. \tag{2.28}$$

By using [16, Lemma 4.5] (since (2.19) holds, we have also $a < \frac{2}{1+\theta}$) we can estimate $\phi(t) = \int_0^{s_0} s^{-a}(s_0 - s)U(s,t)ds$ as

$$\phi(t) \le C_3^{-\frac{1}{\theta+2}} s_0^{2-a+\frac{a-\theta-1}{\theta+2}} \psi^{\frac{1}{\theta+2}}(t)$$

with $C_3 = C_3(R, m, N, \theta, a) > 0$. This implies that

$$\psi(t) \ge C_3 s_0^{a(1+\theta)-\theta-3} \phi^{\theta+2}(t). \tag{2.29}$$

By replacing (2.29) in (2.28) we have

$$\phi'(t) \ge A_0 C_3 s_0^{a(\theta+1)-(\theta+3)} \phi^{\theta+2}(t) - A_2 s_0^{1+\frac{2}{N}-a}.$$

Thus we arrive at the desired inequality with $d_1 := C_3 A_o$ and $d_2 := A_2$. □

We are ready to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We note that $0 \le q < \frac{1}{N-1}$ implies $0 \le \theta = 2(1 - \frac{1}{N})q < \frac{2}{N}$. Hence by Lemma 2.6 we can take $a \in (0, 1 - \frac{1}{N})$ such that for all $s_0 \in (0, R^N)$,

$$\phi'(t) \ge d_1 s_0^{a(\theta+1)-(\theta+3)} \phi^{\theta+2}(t) - d_2 s_0^{1+\frac{2}{N}-a}.$$

Next we let

$$r_1 := \left(\frac{s_0}{4}\right)^{\frac{1}{N}} \in (0, R)$$

and suppose (1.8).

We show that $T_{max} \le \frac{1}{2}$ by contradiction. Assume that $T_{max} > \frac{1}{2}$. Then, by (1.8) for all $m_1 \in (0, m)$ we have $U(s, 0) \ge U(\frac{s_0}{4}, 0) \ge \frac{m_1}{\omega_N}$, $s \in (\frac{s_0}{4}, R^N)$. This yields

$$\phi(0) = \int_0^{s_0} s^{-a}(s_0 - s)U(s,0)ds \ge \int_{\frac{s_0}{4}}^{\frac{s_0}{2}} s^{-a}(s_0 - s)U(s,0)ds \tag{2.30}$$

$$\ge \int_{\frac{s_0}{4}}^{\frac{s_0}{2}} \left(\frac{s_0}{2}\right)^{-a} \frac{s_0}{2} \frac{m_1}{\omega_N} ds = 2^{a-3} \frac{m_1}{\omega_N} s_0^{2-a},$$

and hence for s_0 sufficiently small we obtain

$$\frac{\frac{d_1}{2} s_0^{a(1+\theta)-(\theta+3)} \phi^{\theta+2}(0)}{d_2 s_0^{1+\frac{2}{N}-a}} \ge \frac{2^{(a-3)(\theta+2)} d_1 m_1^{\theta+2} s_0^{\theta-\frac{2}{N}}}{2 d_2 \omega_N^{\theta+2}} \ge 1.$$

Then it follows that

$$d_1 s_0^{a(1+\theta)-(\theta+3)} \phi^{\theta+2}(0) - d_2 s_0^{1+\frac{2}{N}-a} \ge \frac{d_1}{2} s_0^{a(1+\theta)-(\theta+3)} \phi^{\theta+2}(0).$$

Hence by the ODE comparison,

$$\phi'(t) \ge \frac{d_1}{2} s_0^{a(1+\theta)-(\theta+3)} \phi^{\theta+2}(t) \quad \text{for all } t \in (0, T_{max}),$$

from which we obtain

$$t \le \frac{2^{(3-a)(\theta+1)+1} \omega_N^{\theta+1}}{(\theta+1) d_1 m_1^{\theta+1}} s_0^{1-\theta} < \frac{1}{2}.$$

Therefore we conclude that $T_{max} \le \frac{1}{2}$, which contradicts the assumptions. The blow-up property immediately results from Lemma 2.1. □

3. Blow-up in L^p and lower bound of blow-up time

The first aim of this section is to prove, by contradiction, that if $u(x,t)$ blows up in $L^\infty(\Omega)$-norm then it blows up also in $L^p(\Omega)$-norm for all $p > N$.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a bounded and smooth domain. Let (u, v, w) be a classical solution of system (1.3). Then if there exists $C > 0$ such that for some $p_0 > N$,*

$$\|u(\cdot,t)\|_{L^{p_0}(\Omega)} \leq C,$$

then, for some $\hat{C} > 0$,

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \hat{C},$$

for any $t \in (0, T_{max})$.

Proof. Let us consider the L^p-norm of $u(\cdot,t)$ in (1.3) for all $p > \max\{p_0, 2N\}$. Using the first equation of (1.3) and then integrating by parts, we obtain

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx &= \int_\Omega u^{p-1} u_t dx \\
&= -\int_\Omega \nabla u^{p-1} \frac{u\nabla u}{\sqrt{u^2 + |\nabla u|^2}} dx + \chi \int_\Omega \nabla u^{p-1} \frac{u\nabla v}{(1 + |\nabla v|^2 + |\nabla w|^2)^q} dx \\
&\quad - \xi \int_\Omega \nabla u^{p-1} \frac{u\nabla w}{(1 + |\nabla v|^2 + |\nabla w|^2)^q} dx \\
&= -(p-1)\int_\Omega \frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2 + |\nabla u|^2}} dx + \chi(p-1)\int_\Omega u^{p-1} \frac{\nabla u \cdot \nabla v}{(1 + |\nabla v|^2 + |\nabla w|^2)^q} dx \\
&\quad - \xi(p-1)\int_\Omega u^{p-1} \frac{\nabla u \cdot \nabla w}{(1 + |\nabla v|^2 + |\nabla w|^2)^q} dx.
\end{aligned}$$

We use the following inequality proved in [1, Lemma 6.1]:

$$\int_\Omega u^{p-1}|\nabla u| dx \leq \int_\Omega \frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2 + |\nabla u|^2}} dx + \int_\Omega u^p dx.$$

Thus we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx \leq -(p-1)\int_\Omega u^{p-1}|\nabla u| dx + (p-1)\int_\Omega u^p dx + I, \tag{3.1}$$

for all $t \in (0, T_{max})$, where

$$I := \chi(p-1)\int_\Omega u^{p-1} \frac{\nabla u \cdot \nabla v}{(1 + |\nabla v|^2 + |\nabla w|^2)^q} dx - \xi(p-1)\int_\Omega u^{p-1} \frac{\nabla u \cdot \nabla w}{(1 + |\nabla v|^2 + |\nabla w|^2)^q} dx.$$

Integrating by parts and using the boundary conditions and the second equation of (1.3), we obtain

$$I = \chi\frac{p-1}{p}\int_\Omega \frac{\nabla u^p \cdot \nabla v}{(1 + |\nabla v|^2 + |\nabla w|^2)^q} dx - \xi(p-1)\int_\Omega \frac{\nabla u^p \cdot \nabla w}{(1 + |\nabla v|^2 + |\nabla w|^2)^q} dx \tag{3.2}$$

$$
\begin{aligned}
&= -\chi\frac{p-1}{p}\int_\Omega \frac{u^p\Delta v}{(1+|\nabla v|^2+|\nabla w|^2)^q}dx + q\chi\frac{p-1}{p}\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2+|\nabla w|^2)}{(1+|\nabla v|^2+|\nabla w|^2)^{q+1}}dx\\
&\quad + \xi\frac{p-1}{p}\int_\Omega \frac{u^p\Delta w}{(1+|\nabla v|^2+|\nabla w|^2)^q}dx - q\xi\frac{p-1}{p}\int_\Omega u^p\frac{\nabla w\cdot\nabla(|\nabla v|^2+|\nabla w|^2)}{(1+|\nabla v|^2+|\nabla w|^2)^{q+1}}dx\\
&= -\chi\frac{p-1}{p}\int_\Omega \frac{u^p(\beta v-\alpha u)}{(1+|\nabla v|^2+|\nabla w|^2)^q}dx + q\chi\frac{p-1}{p}\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2+|\nabla w|^2)}{(1+|\nabla v|^2+|\nabla w|^2)^{q+1}}dx\\
&\quad + \xi\frac{p-1}{p}\int_\Omega \frac{u^p(\delta w-\gamma u)}{(1+|\nabla v|^2+|\nabla w|^2)^q}dx - q\xi\frac{p-1}{p}\int_\Omega u^p\frac{\nabla w\cdot\nabla(|\nabla v|^2+|\nabla w|^2)}{(1+|\nabla v|^2+|\nabla w|^2)^{q+1}}dx\\
&\le (\chi\alpha-\xi\gamma)\frac{p-1}{p}\int_\Omega u^{p+1}dx + q\chi\frac{p-1}{p}\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2+|\nabla w|^2)}{(1+|\nabla v|^2+|\nabla w|^2)^{q+1}}dx\\
&\quad + \xi\delta\frac{p-1}{p}\int_\Omega u^p w dx - q\xi\frac{p-1}{p}\int_\Omega u^p\frac{\nabla w\cdot\nabla(|\nabla v|^2+|\nabla w|^2)}{(1+|\nabla v|^2+|\nabla w|^2)^{q+1}}dx\\
&=: I_1+I_2+I_3+I_4,
\end{aligned}
$$

where in the last step we neglected the negative term $-\beta\chi\frac{p-1}{p}\int_\Omega \frac{u^p v}{(1+|\nabla v|^2+|\nabla w|^2)^q}dx$ and used the inequality $\frac{1}{(1+|\nabla v|^2+|\nabla w|^2)^q}\le 1$ as $q>0$.

To estimate I_2 and I_4 of (3.2) we consider the radial coordinates and we have

$$
\begin{aligned}
I_2 &= q\chi\frac{p-1}{p}\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2+|\nabla w|^2)}{(1+|\nabla v|^2+|\nabla w|^2)^{q+1}}dx\\
&= 2q\chi\frac{p-1}{p}\omega_N\int_0^R u^p\frac{v_r^2 v_{rr}}{(1+v_r^2+w_r^2)^{q+1}}r^{N-1}dr\\
&\quad + 2q\chi\frac{p-1}{p}\omega_N\int_0^R u^p\frac{v_r w_r w_{rr}}{(1+v_r^2+w_r^2)^{q+1}}r^{N-1}dr\\
&=: I_{21}+I_{22}
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
I_4 &= -q\xi\frac{p-1}{p}\int_\Omega u^p\frac{\nabla w\cdot\nabla(|\nabla v|^2+|\nabla w|^2)}{(1+|\nabla v|^2+|\nabla w|^2)^{q+1}}dx\\
&= -2q\xi\frac{p-1}{p}\omega_N\int_0^R u^p\frac{w_r^2 w_{rr}}{(1+v_r^2+w_r^2)^{q+1}}r^{N-1}dr
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
& -2q\xi\frac{p-1}{p}\omega_N\int_0^R u^p\frac{v_rw_rv_{rr}}{(1+v_r^2+w_r^2)^{q+1}}r^{N-1}dr\\
&:= I_{41}+I_{42}.
\end{aligned}
$$

In I_{21} of (3.3), noting that $v_{rr}=-\alpha u+\beta v-\frac{N-1}{r}v_r\le \beta v+\alpha\frac{N-1}{r^N}\int_0^r\rho^{N-1}ud\rho$ and $\frac{v_r^2}{(1+v_r^2+w_r^2)^{q+1}}\le 1$, we have

$$
\begin{aligned}
I_{21}&\le 2q\chi\frac{p-1}{p}\omega_N\int_0^R u^p\frac{v_r^2\big[\beta v+\alpha\frac{N-1}{r^N}\int_0^r\rho^{N-1}ud\rho\big]}{(1+v_r^2+w_r^2)^{q+1}}r^{N-1}dr \qquad (3.5)\\
&\le 2q\beta\chi\frac{p-1}{p}\omega_N\int_0^R u^pvr^{N-1}dr\\
&\quad +2q\alpha\chi\frac{p-1}{p}(N-1)\omega_N\int_\Omega u^p\frac{1}{r}\Big(\int_0^r\rho^{N-1}ud\rho\Big)dr.
\end{aligned}
$$

In I_{41} of (3.4), we similarly see that $w_{rr}=-\gamma u+\delta w-\frac{N-1}{r}w_r\ge -\gamma u-\delta\frac{N-1}{r^N}\int_0^r\rho^{N-1}wd\rho$ and $\frac{w_r^2}{(1+v_r^2+w_r^2)^{q+1}}\le 1$ to obtain

$$
\begin{aligned}
I_{41}&=-2q\xi\frac{p-1}{p}\omega_N\int_0^R u^p\frac{w_r^2w_{rr}}{(1+v_r^2+w_r^2)^{q+1}}r^{N-1}dr \qquad (3.6)\\
&\le 2q\xi\frac{p-1}{p}\omega_N\int_0^R u^p\frac{w_r^2\big[\gamma u+\delta\frac{N-1}{r^N}\int_0^r\rho^{N-1}wd\rho\big]}{(1+v_r^2+w_r^2)^{q+1}}r^{N-1}dr\\
&\le 2q\xi\gamma\frac{p-1}{p}\omega_N\int_0^R u^{p+1}r^{N-1}dr\\
&\quad +2q\xi\delta\frac{p-1}{p}(N-1)\omega_N\int_0^R u^p\Big(\frac{1}{r}\int_0^r\rho^{N-1}wd\rho\Big)dr.
\end{aligned}
$$

From (3.5) and (3.6), we can write

$$
\begin{aligned}
I_{21}+I_{41}&\le 2q\beta\chi\frac{p-1}{p}\omega_N\int_0^R u^pvr^{N-1}dr \qquad (3.7)\\
&\quad +2q\alpha\chi\frac{p-1}{p}(N-1)\omega_N\int_\Omega u^p\frac{1}{r}\Big(\int_0^r\rho^{N-1}ud\rho\Big)dr
\end{aligned}
$$

$$+ 2q\xi\gamma\frac{p-1}{p}\omega_N \int_0^R u^{p+1} r^{N-1} dr$$

$$+ 2q\xi\delta\frac{p-1}{p}(N-1)\omega_N \int_0^R u^p \Big(\frac{1}{r}\int_0^r \rho^{N-1} w d\rho\Big) dr$$

$$= 2q\beta\chi\frac{p-1}{p}\int_\Omega u^p v dx + 2q\xi\gamma\frac{p-1}{p}\int_\Omega u^{p+1} dx$$

$$+ 2q\alpha\chi\frac{p-1}{p}(N-1)\omega_N \int_\Omega u^p \frac{1}{r}\Big(\int_0^r \rho^{N-1} u d\rho\Big) dr$$

$$+ 2q\xi\delta\frac{p-1}{p}(N-1)\omega_N \int_0^R u^p \Big(\frac{1}{r}\int_0^r \rho^{N-1} w d\rho\Big) dr.$$

We now estimate $I_{22} + I_{42}$:

$$I_{22} + I_{42} = 2q\frac{p-1}{p}\omega_N(\chi-\xi)\int_0^R u^p \frac{v_r w_r w_{rr}}{(1+v_r^2+w_r^2)^{q+1}} r^{N-1} dr \tag{3.8}$$

$$\leq 2q\frac{p-1}{p}\omega_N|\chi-\xi|\int_0^R u^p \frac{|v_r w_r w_{rr}|}{v_r^2+w_r^2} r^{N-1} dr$$

$$\leq q\frac{p-1}{p}\omega_N|\chi-\xi|\int_0^R u^p |w_{rr}| r^{N-1} dr$$

$$\leq \bar{c}_1\omega_N\int_0^R u^p\Big(\gamma u + \delta w + \gamma\frac{N-1}{r^N}\int_0^r \rho^{n-1} u d\rho + \delta\frac{N-1}{r^N}\int_0^r \rho^{n-1} w d\rho\Big) r^{N-1} dr$$

$$= \bar{c}_1\gamma\int_\Omega u^{p+1} dx + \bar{c}_1\delta\int_\Omega u^p w dx + \bar{c}_1\gamma(N-1)\omega_N\int_0^R u^p\Big(\frac{1}{r}\int_0^r \rho^{n-1} u d\rho\Big) dr$$

$$+ \bar{c}_1\delta(N-1)\omega_N\int_0^R u^p\Big(\frac{1}{r}\int_0^r \rho^{n-1} w d\rho\Big) dr$$

with $\bar{c}_1 := q\frac{p-1}{p}|\chi-\xi|$.

Following (4.6) in [12], we can write

$$\omega_N\int_0^R u^p \frac{1}{r}\Big(\int_0^r \rho^{N-1} u d\rho\Big) dr$$

$$\leq \left(\frac{1}{N}\right)^{\frac{p}{p+1}}\Big(\int_\Omega u^{p+1}dx\Big)^{\frac{1}{p+1}}\omega_N^{\frac{p}{p+1}}\Big(\int_0^R u^{p+1+\epsilon}r^{N-1}dr\Big)^{\frac{p}{p+1+\epsilon}}\Big(\int_0^R r^{\frac{\epsilon Np}{p+1}-1}dr\Big)^{\frac{1+\epsilon}{p+1+\epsilon}}$$

$$= c_1\Big(\int_\Omega u^{p+1}dx\Big)^{\frac{1}{p+1}}\Big(\int_\Omega u^{p+1+\epsilon_1}dx\Big)^{\frac{p}{p+1+\epsilon_1}}$$

for all $\epsilon_1 > 0$ and $c_1 = c_1(\epsilon_1, N, p)$. By using the Young inequality we obtain

$$\omega_N\int_0^R u^p\frac{1}{r}\Big(\int_0^r \rho^{N-1}ud\rho\Big)dr \tag{3.9}$$

$$\leq c_1\frac{1}{p+1}\epsilon_3\int_\Omega u^{p+1}dx + C_1(\epsilon_3)\Big(\int_\Omega u^{p+1+\epsilon_1}dx\Big)^{\frac{p+1}{p+1+\epsilon_1}}$$

with $\epsilon_3 > 0$ and $C_1(\epsilon_3) = c_1\frac{p}{p+1}\epsilon_3^{-\frac{1}{p}}$. Similarly we infer that

$$\omega_N\int_0^R u^p\frac{1}{r}\Big(\int_0^r \rho^{N-1}wd\rho\Big)dr$$

$$\leq \omega_N\int_0^R u^p\frac{1}{r}\Big(\int_0^r \rho^{N-1}d\rho\Big)^{\frac{p}{p+1}}\Big(\int_0^r w^{p+1}\rho^{N-1}d\rho\Big)^{\frac{1}{p+1}}dr$$

$$\leq \left(\frac{1}{N}\right)^{\frac{p}{p+1}}\Big(\int_\Omega w^{p+1}dx\Big)^{\frac{1}{p+1}}\omega_N^{\frac{p}{p+1}}\int_0^R u^p r^{\frac{Np}{p+1}-1}dr$$

$$\leq \left(\frac{1}{N}\right)^{\frac{p}{p+1}}\Big(\int_\Omega w^{p+1}dx\Big)^{\frac{1}{p+1}}\omega_N^{\frac{p}{p+1}}\Big(\int_0^R u^{p+1+\epsilon_2}r^{N-1}dr\Big)^{\frac{p}{p+1+\epsilon_2}}\Big(\int_0^R r^{\frac{\epsilon_2 Np}{p+1}-1}dr\Big)^{\frac{1+\epsilon_2}{p+1+\epsilon_2}}$$

$$= c_2\Big(\int_\Omega w^{p+1}dx\Big)^{\frac{1}{p+1}}\Big(\int_\Omega u^{p+1+\epsilon_2}dx\Big)^{\frac{p}{p+1+\epsilon_2}},$$

with $\epsilon_2 > 0$, $c_2 = c_2(\epsilon_2, N, p)$.

Taking into account that, from the third and fourth equations, we obtain

$$\int_\Omega v^{p+1}dx \leq \Big(\frac{\alpha}{\beta}\Big)^{p+1}\int_\Omega u^{p+1}dx, \tag{3.10}$$

$$\int_\Omega w^{p+1}dx \leq \Big(\frac{\gamma}{\delta}\Big)^{p+1}\int_\Omega u^{p+1}dx \tag{3.11}$$

(in fact, $\delta\int_\Omega w^{p+1}dx - \gamma\int_\Omega uw^p dx = \int_\Omega w^p\Delta w dx \leq 0$, and thus $\delta\int_\Omega w^{p+1}dx \leq \gamma\int_\Omega uw^p dx \leq \gamma\big(\int_\Omega u^{p+1}dx\big)^{\frac{1}{p+1}}\big(\int_\Omega w^{p+1}dx\big)^{\frac{p}{p+1}}$).

Hence we invoke (3.11) and the Young inequality to see that

$$\omega_N \int_0^R u^p \frac{1}{r}\Big(\int_0^r \rho^{N-1} w d\rho\Big) dr \tag{3.12}$$

$$\leq c_2 \frac{\gamma}{\delta}\Big(\int_\Omega u^{p+1} dx\Big)^{\frac{1}{p+1}} \Big(\int_\Omega u^{p+1+\epsilon_2} dx\Big)^{\frac{p}{p+1+\epsilon_2}}$$

$$\leq c_2 \frac{1}{p+1}\epsilon_4 \frac{\gamma}{\delta}\int_\Omega u^{p+1} dx + C_2(\epsilon_4)\Big(\int_\Omega u^{p+1+\epsilon_2} dx\Big)^{\frac{p+1}{p+1+\epsilon_2}},$$

for all $\epsilon_4 > 0$ and $C_2(\epsilon_4) := c_2 \frac{p}{p+1}\epsilon_4^{-\frac{1}{p}}\frac{\gamma}{\delta}$ (we can choose $\epsilon_4 = \epsilon_3$).

Also we observe from the Hölder inequality that $\int_\Omega u^p v dx \leq \Big(\int_\Omega u^{p+1} dx\Big)^{\frac{p}{p+1}}\Big(\int_\Omega v^{p+1} dx\Big)^{\frac{1}{p+1}}$ and from (3.10) we arrive at

$$\int_\Omega u^p v dx \leq \frac{\alpha}{\beta}\int_\Omega u^{p+1} dx. \tag{3.13}$$

Similarly we estimate the term $\int_\Omega u^p w dx$:

$$\int_\Omega u^p w dx \leq \frac{\gamma}{\delta}\int_\Omega u^{p+1} dx. \tag{3.14}$$

By using the estimates (3.9), (3.12), (3.13) and (3.14) in (3.7) and (3.8) we have

$$I_{21} + I_{41} \leq 2q\alpha\chi\frac{p-1}{p}\int_\Omega u^{p+1} dx + 2q\xi\gamma\frac{p-1}{p}\int_\Omega u^{p+1} dx \tag{3.15}$$

$$+ 2q\chi\alpha\frac{p-1}{p}(N-1)\Big[c_1\frac{1}{p+1}\epsilon_3\int_\Omega u^{p+1} dx + C_1(\epsilon_3)\Big(\int_\Omega u^{p+1+\epsilon_1} dx\Big)^{\frac{p+1}{p+1+\epsilon_1}}\Big]$$

$$+ 2q\xi\delta\frac{p-1}{p}(N-1)\Big[c_2\frac{1}{p+1}\epsilon_4\frac{\gamma}{\delta}\int_\Omega u^{p+1} dx + C_2(\epsilon_4)\Big(\int_\Omega u^{p+1+\epsilon_2} dx\Big)^{\frac{p+1}{p+1+\epsilon_2}}\Big]$$

and

$$I_{22} + I_{42} \leq 2\bar{c}_1\gamma\int_\Omega u^{p+1} dx \tag{3.16}$$

$$+ \bar{c}_1\gamma(N-1)\Big[c_1\frac{1}{p+1}\epsilon_3\int_\Omega u^{p+1} dx + C_1(\epsilon_3)\Big(\int_\Omega u^{p+1+\epsilon_1} dx\Big)^{\frac{p+1}{p+1+\epsilon_1}}\Big]$$

$$+ \bar{c}_1\delta(N-1)\Big[c_2\frac{1}{p+1}\epsilon_4\frac{\gamma}{\delta}\int_\Omega u^{p+1} dx + C_2(\epsilon_4)\Big(\int_\Omega u^{p+1+\epsilon_2} dx\Big)^{\frac{p+1}{p+1+\epsilon_2}}\Big].$$

Combining (3.3)–(3.6), (3.15) and (3.16) with (3.2) in (3.1), we arrive at

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx &\le -(p-1)\int_\Omega u^{p-1}|\nabla u|dx + (p-1)\int_\Omega u^p dx + A_1\int_\Omega u^{p+1}dx \\ &\quad + \tilde{A}_2\Big(\int_\Omega u^{p+1+\epsilon_1}dx\Big)^{\frac{p+1}{p+1+\epsilon_1}} + \tilde{A}_3\Big(\int_\Omega u^{p+1+\epsilon_2}dx\Big)^{\frac{p+1}{p+1+\epsilon_2}} \\ &= -\frac{p-1}{p}\int_\Omega |\nabla u^p|dx + (p-1)\int_\Omega u^p dx + A_1\int_\Omega u^{p+1}dx \\ &\quad + A_2\Big(\int_\Omega u^{p+1+\epsilon_1}dx\Big)^{\frac{p+1}{p+1+\epsilon_1}},\end{aligned} \tag{3.17}$$

where we have chosen $\epsilon_2 = \epsilon_1$ and

$$\begin{aligned}A_1 &:= \frac{p-1}{p}\Big[\chi\alpha + 2q\xi\gamma + 2q\chi\alpha + 2q\frac{(N-1)}{p+1}(\alpha\chi c_1\epsilon_3 + \gamma\xi c_2\epsilon_4)\Big] \\ &\quad + 2\bar{c}_1\gamma + \bar{c}_1\gamma\frac{(N-1)}{p+1}(\epsilon_3 c_1 + \epsilon_4 c_2), \\ \tilde{A}_2 &:= 2q\alpha\chi\frac{p-1}{p}(N-1)C_1(\epsilon_3) + \bar{c}_1\gamma C_1(\epsilon_3), \\ \tilde{A}_3 &:= 2q\xi\delta\frac{p-1}{p}(N-1)C_2(\epsilon_4) + \bar{c}_1\delta C_2(\epsilon_4), \\ A_2 &:= \tilde{A}_2 + \tilde{A}_3.\end{aligned} \tag{3.18}$$

In order to estimate $\int_\Omega u^p dx$, $\int_\Omega u^{p+1}dx$ and $\Big(\int_\Omega u^{p+1+\epsilon_1}dx\Big)^{\frac{p+1}{p+1+\epsilon_1}}$ we use the version of the Gagliardo-Nirenberg inequality proved by Li-Lankeit (see Lemma 2.3 in [9]), where, compared to the standard version, the authors include the possibility that the exponent can be less than 1 (here we used the exponent $\frac{1}{2}$). We have

$$\begin{aligned}\int_\Omega u^p dx = \|u^p\|_{L^1(\Omega)} &\le C_{GN}\|\nabla u^p\|^{\theta}_{L^1(\Omega)}\|u^p\|^{1-\theta}_{L^{\frac{1}{2}}(\Omega)} + C_{GN}\|u^p\|_{L^{\frac{1}{2}}(\Omega)} \\ &\le \epsilon_5\int_\Omega |\nabla u^p|dx + c(\epsilon_5)\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^2,\end{aligned} \tag{3.19}$$

where $\theta := \frac{N}{N+1} \in (0,1)$ and $\epsilon_5 > 0$ to be chosen later. Similarly,

$$\begin{aligned}\int_\Omega u^{p+1}dx = \|u^p\|^{\frac{p+1}{p}}_{L^{\frac{p+1}{p}}(\Omega)} &\le C_{GN}\|\nabla u^p\|^{\frac{p+1}{p}\bar{\theta}}_{L^1(\Omega)}\|u^p\|^{\frac{p+1}{p}(1-\bar{\theta})}_{L^{\frac{1}{2}}(\Omega)} + C_{GN}\|u^p\|^{\frac{p+1}{p}}_{L^{\frac{1}{2}}(\Omega)} \\ &\le \epsilon_6\int_\Omega |\nabla u^p|dx + c(\epsilon_6)\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^{2\frac{p+1-N}{p-2N}} + C_{GN}\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^{2\frac{p+1}{p}},\end{aligned} \tag{3.20}$$

where $\bar{\theta} := \frac{1+\frac{1}{p+1}}{1+\frac{1}{N}} = \frac{N(p+2)}{(N+1)(p+1)} \in (0,1)$ and $\epsilon_6 > 0$ to be chosen later. We observe that, since $p > \max\{p_0, 2N\}$ we get $\frac{p+1}{p}\bar{\theta} = \frac{N(p+2)}{p(N+1)} \in (0,1)$, $\frac{p+1}{p}(1-\bar{\theta}) = \frac{p+1-N}{p(N+1)} \in (0,1)$. Finally, for sufficiently small $\epsilon_1 > 0$

$$\Big(\int_\Omega u^{p+1+\epsilon_1}dx\Big)^{\frac{p+1}{p+1+\epsilon_1}} = \|u^p\|^{\frac{p+1}{p}}_{L^{\frac{p+1+\epsilon_1}{p}}(\Omega)} \tag{3.21}$$

$$\le C_{GN}\|\nabla u^p\|^{\frac{p+1}{p}\tilde{\theta}}_{L^1(\Omega)}\|u^p\|^{\frac{p+1}{p}(1-\tilde{\theta})}_{L^{\frac{1}{2}}(\Omega)} + C_{GN}\|u^p\|^{\frac{p+1}{p}}_{L^{\frac{1}{2}}(\Omega)}$$

$$\le \epsilon_7\int_\Omega |\nabla u^p|dx + c(\epsilon_7)\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^{\sigma} + C_{GN}\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^{2\frac{p+1}{p}},$$

where $\tilde{\theta} := \frac{1+\frac{1+\epsilon_1}{p+1+\epsilon_1}}{1+\frac{1}{N}} = \frac{N(p+2+2\epsilon_1)}{(N+1)(p+1+\epsilon_1)} \in (0,1)$, $\sigma = \frac{2(p+1)[(p+1-N)-(N-1)\epsilon_1]}{(p+1)(p-2N)-((N-1)p+2N)\epsilon_1}$ and $\epsilon_7 > 0$ to be chosen later. We observe that, since $p > \max\{p_0, 2N\}$ we have $\tilde{\theta}\frac{p+1}{p} \in (0,1)$, and $(1-\tilde{\theta})\frac{p+1}{p} \in (0,1)$. Replacing (3.19), (3.20) and (3.21) in (3.17) and noticing $2 < 2\frac{p+1}{p} < 2\frac{p+1-N}{p-2N} < \sigma$, we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx \le -\Big(\frac{p-1}{p} - (p-1)\epsilon_5 - A_1\epsilon_6 - A_2\epsilon_7\Big)\int_\Omega |\nabla u^p|dx \tag{3.22}$$

$$+ c_1(\epsilon_5,\epsilon_6,\epsilon_7)\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^2 + c_2(\epsilon_5,\epsilon_6,\epsilon_7)\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^{\sigma}.$$

By choosing $\epsilon_5, \epsilon_6, \epsilon_7$ small enough and noting from (3.19) that

$$\int_\Omega |\nabla u^p|dx \ge \bar{c}_1\int_\Omega u^p dx - \bar{c}_2\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^2,$$

we arrive at

$$\frac{d}{dt}\int_\Omega u^p dx + \int_\Omega u^p dx \le \tilde{c}_1 p\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^2 + \tilde{c}_2 p\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^{\sigma}.$$

The proof is completed by applying a standard Moser iteration argument (see e.g. [13, Theorem 1.2], hence if $\|u(\cdot,t)\|_{L^{p_0}(\Omega)} \le C$ for some $p_0 > N$, then also $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded. □

Proof of Theorem 1.2. Thanks to Lemma 3.1, we infer that if $\|u(\cdot,t)\|_{L^{p_0}(\Omega)} \le C$ for some $p_0 > N$, then $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded. We have a contradiction since Theorem1.1 holds. Hence $\limsup_{t\nearrow T_{max}}$ $\|u(\cdot,t)\|_{L^p(\Omega)} = \infty$ for all $p > N$. □

The next aim is to prove Theorem 1.3. Following the steps in [14], assume that Theorem 1.2 holds. We want to obtain a safe interval $[0,T]$ of existence of the solution to (1.3), with T a lower bound of the blow-up time T_{max}.

Proof of Theorem 1.3. We differentiate $\Psi(t)$ and recall (3.17):

$$\Psi'(t) \le -\frac{p-1}{p}\int_\Omega |\nabla u^p|dx + (p-1)\int_\Omega u^p dx + A_1\int_\Omega u^{p+1}dx \tag{3.23}$$

$$+ A_2\Big(\int_\Omega u^{p+1+\epsilon_1}dx\Big)^{\frac{p+1}{p+1+\epsilon_1}}$$

for all $t \in (0, T_{max})$, with A_1 and A_2 defined in (3.18). In (3.23) we reduce $\int_\Omega u^{p+1}dx$ and $\int_\Omega u^{p+1+\epsilon_1}dx$ in term of $\int_\Omega u^p dx$. To this end, under the condition $p > N$, firstly we use the Gagliardo–Nirenberg inequality, and then we apply the Young inequality. For the term $\int_\Omega u^{p+1}dx$ we have

$$\begin{aligned}\int_\Omega u^{p+1}dx = \|u^p\|^{\frac{p+1}{p}}_{L^{\frac{p+1}{p}}(\Omega)} &\le C_{GN}\|\nabla u^p\|^{\frac{p+1}{p}\theta_1}_{L^1(\Omega)}\|u^p\|^{\frac{p+1}{p}(1-\theta_1)}_{L^1(\Omega)} + C_{GN}\|u^p\|^{\frac{p+1}{p}}_{L^1(\Omega)} \\ &\le \varepsilon_1\int_\Omega|\nabla u^p|dx + c(\varepsilon_1)\Big(\int_\Omega u^p dx\Big)^{\frac{p+1-N}{p-N}} + C_{GN}\Big(\int_\Omega u^p dx\Big)^{\frac{p+1}{p}},\end{aligned} \tag{3.24}$$

where $\theta_1 := \frac{N}{p+1} \in (0,1)$ and $\varepsilon_1 > 0$ to be fixed later. We proceed in the same way to estimate the term $\int_\Omega u^{p+1+\epsilon_1}dx$ and we arrive at

$$\begin{aligned}\Big(\int_\Omega u^{p+1+\epsilon_1}dx\Big)^{\frac{p+1}{p+1+\epsilon_1}} &= \|u^p\|^{\frac{p+1}{p}}_{L^{\frac{p+1+\epsilon_1}{p}}(\Omega)} \\ &\le C_{GN}\|\nabla u^p\|^{\theta_2\frac{p+1}{p}}_{L^1(\Omega)}\|u^p\|^{(1-\theta_2)\frac{p+1}{p}}_{L^1(\Omega)} + C_{GN}\|u^p\|^{\frac{p+1}{p}}_{L^1(\Omega)},\end{aligned} \tag{3.25}$$

where $\theta_2 := \frac{N(1+\epsilon_1)}{p+1+\epsilon_1} \in (0,1)$ and $\theta_2\frac{p+1}{p} \in (0,1)$ for all $p > N$ and sufficiently small $\epsilon_1 > 0$.

The application of the Young inequality in (3.25) leads to

$$\Big(\int_\Omega u^{p+1+\epsilon_1}dx\Big)^{\frac{p+1}{p+1+\epsilon_1}} \le \varepsilon_2\int_\Omega|\nabla u^p|dx + c(\varepsilon_2)\Big(\int_\Omega u^p dx\Big)^b + \bar{c}_3\Big(\int_\Omega u^p dx\Big)^{\frac{p+1}{p}} \tag{3.26}$$

with $b := \frac{(p+1)[(p+1+\epsilon_1)-N(1+\epsilon_1)]}{p(p+1+\epsilon_1)-N(1+\epsilon_1)(p+1)} > \frac{p+1}{p}$ and $\varepsilon_2 > 0$ to be fixed later. Substituting (3.24) and (3.26) into (3.23), we obtain

$$\begin{aligned}\Psi'(t) \le& -\Big(\frac{p-1}{p} - A_1\varepsilon_1 - A_2\varepsilon_2\Big)\int_\Omega|\nabla u^p|dx + (p-1)\int_\Omega u^p dx \\ &+ A_1c(\varepsilon_1)\Big(\int_\Omega u^p dx\Big)^{\frac{p+1-N}{p-N}} + \Big(A_1C_{GN} + A_2\bar{c}_3\Big)\Big(\int_\Omega u^p dx\Big)^{\frac{p+1}{p}} \\ &+ A_2c(\varepsilon_2)\Big(\int_\Omega u^p dx\Big)^b, \qquad t \in (0, T_{max}).\end{aligned}$$

To eliminate the first term we choose ε_1 and ε_2 small and we arrive at

$$\begin{aligned}\Psi'(t) \le (p-1)\int_\Omega u^p dx &+ A_1c(\varepsilon_1)\Big(\int_\Omega u^p dx\Big)^{\frac{p+1-N}{p-N}} + \Big(A_1C_{GN} + A_2\bar{c}_3\Big)\Big(\int_\Omega u^p dx\Big)^{\frac{p+1}{p}} \\ &+ A_2c(\varepsilon_2)\Big(\int_\Omega u^p dx\Big)^b, \quad t \in (0, T_{max}),\end{aligned}$$

which can be rewritten as

$$\Psi'(t) \le B_1\Psi(t) + B_2\Psi^{\frac{p+1}{p}}(t) + B_3\Psi^{\frac{p+1-N}{p-N}}(t) + B_4\Psi^b(t), \quad t \in (0, T_{max}). \tag{3.27}$$

Integrating (3.27) from 0 to T_{max}, we arrive at the desired lower bound (1.11) with $\gamma_1 := \frac{p+1}{p}$, $\gamma_2 := \frac{p+1-N}{p-N}$ and $\gamma_3 := b$. □

Remark 3.1. Since $\Psi(t)$ blows up at finite time T_{max}, there exists a time $t_1 \in (0, T_{max})$ such that $\Psi(t) \geq \Psi(0) = \Psi_0$ for all $t > t_1$. Thus, since $1 < \frac{p+1-N}{p-N} < \frac{p+1}{p} < b$, we have

$$\Psi(t) \leq \Psi^b(t)\Psi_0^{1-b},$$

$$\Psi^{\frac{p+1-N}{p-N}}(t) \leq \Psi^b(t)\Psi_0^{\frac{p+1-N}{p-N}-b},$$

$$\Psi^{\frac{p+1}{p}}(t) \leq \Psi^b(t)\Psi_0^{\frac{p+1}{p}-b},$$

from which we obtain

$$\Psi'(t) \leq K\Psi^b(t), \tag{3.28}$$

with $K = (p-1)(p\Psi_0)^{1-b} + A_1 c(\varepsilon_1)(p\Psi_0)^{\frac{p+1-N}{p-N}-b} + \Big(A_1 C_{GN} + A_2 \bar{c}_3\Big)(p\Psi_0)^{\frac{p+1}{p}-b} + A_2 c(\varepsilon_2)p^b$. Integrating (3.28) from 0 to t we arrive at

$$\Psi^{1-b}(t) - \Psi_0^{1-b} \geq -K(b-1)t.$$

If $t \to T_{max}$, then $\Psi^{1-b}(t) \to 0$ and we get the following explicit lower bound of T_{max}:

$$T_{max} \geq \frac{\Psi_0^{1-b}}{(b-1)K} =: T.$$

Acknowledgements

The authors are very grateful to the anonymous referees for his/her helpful suggestions and comments. M. Marras and S. Vernier-Piro are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). This work was done while T. Yokota visited Università di Cagliari in 2025.

Author contributions All authors contributed to the manuscript equally

Funding Open access funding provided by Università degli Studi di Cagliari within the CRUI-CARE Agreement. M. Marras is partially supported by the research project "Partial Differential Equations and their role in understanding natural phenomena", CUP F23C25000080007, funded by Fondazione Banco di Sardegna annuality (2023); by the grant PRIN n. PRIN-2022ZXZTN2: *Nonlinear differential problems with applications to real phenomena*; by the project National Recovery and Resilience Plan (NRRP), Mission 4 Component 2 Investment 1.5 - Call for tender No.3277 published on December 30, 2021 by the Italian Ministry of University and Research (MUR) funded by the European Union – NextGenerationEU. Project Code ECS0000038 – Project Title eINS Ecosystem of Innovation for Next Generation Sardinia – CUP F53C22000430001- Grant Assignment Decree No. 1056 adopted on June 23, 2022 by the Italian Ministry of University and Research (MUR). T. Yokota is partially supported by JSPS KAKENHI Grant Number JP25K00917.

Data Availability Statement No datasets were generated or analysed during the current study.

References

[1] Bellomo, N., Winkler, M.: A degenerate chemotaxis system with flux limitation: maximally extended solutions and absence of gradient blow-up. Comm. Partial Differ. Equ. **42**(3), 436–473 (2017)
[2] Bellomo, N., Winkler, M.: Finite-time blow-up in a degenerate chemotaxis system with flux limitation. Trans. Amer. Math. Soc. Ser. **4**, 31–67 (2017)
[3] Chiyoda, Y., Mizukami, M., Yokota, T.: Finite-time blow-up in a quasilinear degenerate chemotaxis system with flux limitation. Acta Appl. Math. **167**, 231–259 (2020)
[4] Chiyo, Y., Hasegawa, K., Shohei, K., Yokota, T.: Boundedness in a one-dimensional quasilinear attraction-repulsion chemotaxis system with flux limitation. Appl. Anal. **104**, 3474–3487 (2025)
[5] Chiyo, Y., Tanaka, Y., Marras, M., Yokota, T.: Blow-up phenomena in a parabolic-elliptic-elliptic attraction-repulsion chemotaxis system with superlinear logistic degradation. Nonlinear Anal. **212**, 112550 (2021)
[6] Conte, M., Casas-Tintò, S., Soler, J.: Modeling invasion patterns in the glioblastoma battlefield. PLoS Comput. Biol. **17**(1), e1008632 (2021)
[7] Keller, E.F., Segel, L.A.: Initiation of slime mold aggregation viewed as an instability. J. Theoret. Biol. **26**, 399–415 (1970)
[8] Lankeit, J.: Finite-time blow-up in the three-dimensional fully parabolic attraction-dominated attraction-repulsion chemotaxis system. J. Math. Anal. Appl. **504**, 125409 (2021)
[9] Li, Y., Lankeit, J.: Boundedness in a chemotaxis-haptotaxis model with nonlinear diffusion. Nonlinearity **29**, 1564–1595 (2016)
[10] Li, Y., Li, Y.: Blow-up of nonradial solutions to attraction-repulsion chemotaxis system in two dimensions. Nonlinear Anal. Real World Appl. **30**, 170–183 (2016)
[11] Luca, M., Chavezross, A., Edelsteinkeshet, L., Mogilner, A.: Chemotactic signaling, microglia, and Alzheimer's disease senile plaques: is there a connection? Bull. Math. Biol. **65**(4), 693–730 (2003)
[12] Marras, M., Vernier-Piro, S., Yokota, T.: Blow-up phenomena for a chemotaxis system with flux limitation. J. Math. Anal. Appl. **515**, 126376 (2022)
[13] Marras, M., Vernier-Piro, S., Yokota, T.: Behavior in time of solutions to a degenerate chemotaxis system with flux limitation. Nonlinear Anal. Real World Appl. **81**, 104215 (2025)
[14] Marras, M., Vernier-Piro, S., Yokota, T.: Blow-up and boundedness in a chemotaxis system with flux-limited diffusion and logistic source. Nonlinear Anal. **261**, 113868 (2025)
[15] Mizukami, M., Ono, T., Yokota, T.: Extensibility criterion ruling out gradient blow-up in a quasilinear degenerate chemotaxis system with flux limitation. J. Differ. Equ. **267**(9), 5115–5164 (2019)
[16] Tanaka, Y., Yokota, T.: Blow-up in a parabolic-elliptic Keller-Segel system with density-dependent sublinear sensitivity and logistic source. Math. Meth. Appl. Sci. **43**, 7372–7396 (2020)
[17] Tao, Y., Wang, Z.-A.: Competing effects of attraction vs. repulsion in chemotaxis. Math. Models Methods Appl. Sci. **23**(1), 1–36 (2013)
[18] Winkler, M.: Finite-time blow-up in low-dimensional Keller-Segel systems with logistic-type superlinear degradation. Z. Angew. Math. Phys. **69**, 40 (2018)
[19] Xu, X.: A refined asymptotic result of one-dimensional flux limited Keller-Segel models. J. Biol. Dyn. **17**(1), 2203165 (2023)
[20] Zhigun, A.: Global entropy solutions to a degenerate parabolic-parabolic chemotaxis system for flux-limited dispersal. Nonlinear Anal. Real World Appl. **79**, 104132 (2024)

Monica Marras
Dipartimento di Matematica e Informatica
Università degli Studi di Cagliari
via Ospedale 72
09124 Cagliari
Italy
e-mail: mmarras@unica.it

Stella Vernier-Piro
Facoltà di Ingegneria e Architettura
Università degli Studi di Cagliari
Viale Merello 92
09123 Cagliari
Italy
e-mail: svernier@unica.it

Tomomi Yokota
Department of Mathematics
Tokyo University of Science
1-3, Kagurazaka
Shinjuku-ku Tokyo162-8601
Japan
e-mail: yokota@rs.tus.ac.jp

(Received: November 30, 2025; revised: August 4, 2026; accepted: August 7, 2026)