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Research paper

Blow-up and boundedness in a chemotaxis system with flux-limited diffusion and logistic source

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ABSTRACT

In this paper we consider radially symmetric solutions of the parabolic–elliptic cross-diffusion system with flux limitation term,

$$\begin{cases} u_t = \nabla \cdot \left(\frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}} \right) - \chi \nabla \cdot (u \nabla v) + \lambda u - \mu u^k, & x \in \Omega,\ t > 0, \\ 0 = \Delta v - m(t) + u, \quad m(t) = \frac{1}{|\Omega|} \int_\Omega u(x,t)dx, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega \end{cases}$$

under no-flux boundary conditions, where $\Omega = B_R(0) \subset \mathbb{R}^N$ $(N \geq 1)$ is a ball, χ, λ, μ are positive constants and $k > 1$. Under suitable conditions on the data, we prove that the solution is global in time. If $N \geq 3$, under conditions on the data, we prove that the solution $u(x,t)$ blows up in L^∞-norm at finite time T_{max}. Moreover for some $p > 1$ we prove that the solution blows up also in L^p-norm and a lower bound of the blow-up time is derived.

1. Introduction

The chemotaxis phenomenon, the directed movement of cells and organisms in response to chemical gradients, has attracted significant interests due to its important role in various scientific fields like biology, biotechnology, chemistry, medicine, etc.

In 1970 Keller and Segel [1,2] introduced an important mathematical model, which describes the movement of a microbiological population under the influence of a chemical substance produced by the biological organism themself: this phenomenon is known as chemotaxis and the model is formulated as

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), \\ \tau v_t = \Delta v + u - v, \end{cases} \tag{1.1}$$

where $\tau \in \{0, 1\}$, $u = u(x,t)$ is the cell (organism) density at position x and time t, $v = v(x,t)$ is the density of the chemoattractant and χ represents the chemotactic sensitivity.

The seminal papers by Keller and Segel have spawned a vast literature on their model of chemotaxis and on related systems of equations. In the last decades different variants of the original Keller–Segel system have been introduced and investigated. This paper presents an analytical study of a flux-limited chemotaxis model, which extends the classical Keller–Segel model to the natural case where the population cells moves with a finite speed of propagation.

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In the case of linear diffusion, there are a number of results on global existence and finite-time blow-up of solutions to (1.1). In particular, in [3] Winkler studied the problem with a source term of logistic type,

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \lambda u - \mu u^k, \\ 0 = \Delta v - m(t) + u, \quad m(t) = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u(x,t)dx \end{cases} \tag{1.2}$$

under homogeneous Neumann boundary conditions, in a ball $\Omega \subset \mathbb{R}^N$, with $\chi > 0$, $\lambda \geq 0$, $\mu \geq 0$ and $k > 1$. It is shown that if $N \geq 5$ and $k < \frac{3}{2} + \frac{1}{2N-2}$ then there exist initial data such that the smooth solution of the problem (1.2) blows up in finite time. Also in [4] Winkler proved that in low-dimensional spatial settings, including three-dimensional cases, chemotactic cross-diffusion can significantly counterbalance even superlinear logistic-type dampening. Recently, in [5] Fuest completed the analysis of finite-time blow-up of the solution of (1.2) for increasing values of k in four-dimensional and higher-dimensional settings and obtained corresponding results up to (and for spatial dimension $N \geq 5$ even including) the optimal exponent $k = 2$. In [6] the region of exponents allowing for finite-time blow-up in (1.2) has been further widened to $k < \frac{4}{3}$ and $k < \frac{3}{2}$ for $N = 3$ and $N = 4$, respectively.

In the case of nonlinear diffusion like porous media type, in presence of the source term of logistic type and with nonlinear production, in [7,8] Tanaka obtained conditions for k and the data to have boundedness and finite-time blow-up. In [9] the authors considered a degenerate chemotaxis model with porous media type diffusion and a nonlinear source which satisfies natural or Hadamard growth condition and proved the Hölder regularity of bounded solutions for the parabolic–parabolic case as well as for the parabolic–elliptic case. Recent results on Hölder regularity in degenerate Keller-Segel systems are present in [10].

Inspired by various biological phenomena, in [11,12] Bellomo and Winkler gave a first step toward a qualitative theory for a new class of chemotaxis by introducing the chemotaxis system with flux limitation,

$$\begin{cases} u_t = \nabla\cdot\left(\dfrac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\right) - \chi\nabla\cdot\left(\dfrac{u\nabla v}{\sqrt{1+|\nabla v|^2}}\right), & x\in\Omega,\ t>0, \\ 0 = \Delta v - m_0 + u, & x\in\Omega,\ t>0, \\ \left(\dfrac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}} - \chi\dfrac{u\nabla v}{\sqrt{1+|\nabla v|^2}}\right)\cdot\nu = 0, & x\in\partial\Omega,\ t>0, \\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.3}$$

where $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geq 1$, $m_0 = \frac{1}{|\Omega|}\int_\Omega u_0$ and $\chi > 0$. The introduction of these types of terms is based on the assumption that particles do not move (diffuse) arbitrarily in a space but rather move along specific pathways, such as the borders of cells. Additionally, this new approach eliminates non-physical diffusion, resulting in the population moving with a finite speed of propagation, which is a key characteristic of the model. In particular in [11] Bellomo and Winkler proved existence of classical solutions, extensible in time up to a maximal $T_{max} \in (0,\infty]$ which has the property that if $T_{max} < \infty$ then $\limsup_{t\to T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$. Moreover, by means of suitable a priori estimates it has been shown that the above solutions are global and bounded when either $N \geq 2$ and $\chi < 1$, *or* $N = 1$, $\chi > 0$ and $\int_\Omega u_0 < m_c$, with $m_c := \frac{1}{\sqrt{\chi^2-1}}$ if $\chi > 1$ and $m_c := \infty$ if $\chi \leq 1$. Also in [12] Bellomo and Winkler proved that these conditions are essentially optimal and derived complementary results on the occurrence of blow-up in finite time with respect to the norm of u in $L^\infty(\Omega)$.

In [13] Mizukami et al. studied more general problems of the form

$$\begin{cases} u_t = \nabla\cdot\left(\dfrac{u^p\nabla u}{\sqrt{u^2+|\nabla u|^2}}\right) - \chi\nabla\cdot\left(\dfrac{u^q\nabla v}{\sqrt{1+|\nabla v|^2}}\right), & x\in\Omega,\ t>0, \\ 0 = \Delta v - m_0 + u, & x\in\Omega,\ t>0, \\ \left(\dfrac{u^p\nabla u}{\sqrt{u^2+|\nabla u|^2}} - \chi\dfrac{u^q\nabla v}{\sqrt{1+|\nabla v|^2}}\right)\cdot\nu = 0, & x\in\partial\Omega,\ t>0, \\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.4}$$

with $p, q \geq 1$ and established local existence and extensibility criterion ruling out gradient blow-up when $p, q \geq 1$, and moreover derived global existence and boundedness of solutions when $p > q + 1 - \frac{1}{N}$. In [14] Chiyoda et al. extended the result of [12] and proved existence of blow-up solutions under some conditions for χ and u_0 when $1 \leq p \leq q$.

Captivated by this new class of Keller–Segel models, in [15] Marras et al. examined how the solutions of the problem evolved over time in presence of the flux limitation term $k_f\nabla\cdot\left(\frac{u\nabla v}{(1+|\nabla v|^2)^\alpha}\right)$ with $k_f > 0$, $\alpha > 0$. This term describes gradient-dependent limitation of cross diffusion fluxes. They also considered the logistic source term $g(u) = \lambda u - \mu u^k$ for varying $k \in (1,2]$.

Previously, Negreanu and Tello in [16] obtained uniform boundedness in $L^\infty(\Omega)$ and existence of global in time solutions of the parabolic–elliptic Keller–Segel problem with chemotaxis term of the form $-\nabla\cdot\left(\chi u|\nabla v|^{p-2}\nabla v\right)$, and moreover for the one dimensional case they proved that there exist infinitely many non-constant steady-states for $p \in (1,2)$ for any $\chi > 0$ and a given positive mass. In this connection, some qualitative properties of weak solutions to several problems with such chemotaxis term were studied by [17–19].

More recently in [20] the authors studied a Keller–Segel type system, where the nonlinear diffusion term is $\nabla\cdot\left(\frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\right)$ and the drift term is $-\chi\nabla\cdot\left(\frac{u\nabla v}{(1+|\nabla v|^2)^\alpha}\right)$, $\alpha > 0$. They showed that the solution blows up in L^∞-norm at a finite time T_{max} and for some $p > 1$ it blows up also in L^p-norm.

Table 1
The case $k = 2$.

BlowLLL-up	Open	Boundedness
$0 < \mu < (1-\frac{2}{N})(1-\frac{1}{N})\chi$	$(1-\frac{2}{N})(1-\frac{1}{N})\chi \le \mu \le (1-\frac{1}{N})\chi$	$\mu > (1-\frac{1}{N})\chi$

The purpose of this paper is to extend blow-up results to the parabolic–elliptic cross-diffusion chemotaxis system with flux limitation term,

$$\begin{cases} u_t = \nabla\cdot\left(\dfrac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\right) - \chi\nabla\cdot(u\nabla v) + \lambda u - \mu u^k, & x\in\Omega,\ t>0,\\ 0 = \Delta v - m(t) + u, \quad m(t) = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u(x,t)dx, & x\in\Omega,\ t>0,\\ \dfrac{u\nabla u\cdot\nu}{\sqrt{u^2+|\nabla u|^2}} = \nabla v\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.5}$$

where $\Omega = B_R(0)\subset\mathbb{R}^N$ ($N\ge 1$) is a ball, $R>0$, with the positive constants χ,λ,μ and $k>1$. The initial data u_0 satisfies

$$u_0 \in C^2(\overline{\Omega}),\ \text{radially symmetric and positive in } \overline{\Omega}. \tag{1.6}$$

The scheme of this paper is the following: in Section 2 we present some preliminary lemmas. In Section 3 we prove that the solution blows up in $L^\infty(\Omega)$-norm in finite time T_{max} in the cases

i) $1<k<2,\quad \mu>0,\ \chi>0;$

ii) $k=2,\quad 0<\mu<\left(1-\dfrac{2}{N}\right)\left(1-\dfrac{1}{N}\right)\chi.$

Under the hypothesis of blow-up in $L^\infty(\Omega)$-norm in finite time, Sections 4 and 5 are devoted to studying blow-up in $L^p(\Omega)$-norm with some $p>1$ and to deriving also a lower bound of the blow-up time T_{max}. Finally in Section 6 global existence and boundedness of solutions are obtained in the cases

1) $k>2,\quad \mu>0,\ \chi>0;$

2) $k=2,\quad \mu>\left(1-\dfrac{1}{N}\right)\chi.$

As displayed in Table 1, we leave a region open in the case $k=2$.

The main results are the following.

Theorem 1.1 (*Finite-Time Blow-Up in L^∞-Norm*). *Let $\Omega = B_R(0)\subset\mathbb{R}^N$, $N\ge 3$, $R>0$. Moreover suppose that either one of the following two conditions is satisfied:*

i) $1<k<2,\quad \mu>0,\ \chi>0;$ (1.7)

ii) $k=2,\quad 0<\mu<(1-\frac{2}{N})(1-\frac{1}{N})\chi.$ (1.8)

Then for all $m_0>0$ there exist a radially decreasing initial data u_0 which complies with (1.6) and

$$\frac{1}{|\Omega|}\int_\Omega u_0\,dx = m_0 \tag{1.9}$$

such that (1.5) possesses a classical positive solution (u,v) in $\Omega\times(0,T_{max})$ for some $T_{max}\in(0,\infty)$, which blows up at T_{max} in the sense that

$$\limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{1.10}$$

Theorem 1.2 (*Finite-Time Blow-Up in L^p-Norm*). *Let $\Omega = B_R(0)\subset\mathbb{R}^N$, $N\ge 3$ and $R>0$. Then, the classical solution (u,v) of (1.5), provided by Theorem 1.1, is such that for all $p>N$,*

$$\limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^p(\Omega)} = \infty.$$

Theorem 1.3 (*A Lower Bound of the Blow-Up Time T_{max}*). *Assume that Theorem 1.2 holds. Then there exist positive constants $\tilde a_1,\tilde a_2$ such that there exists a safe interval $[0,T)$ of existence of $u(x,t)$ with*

$$T_{max} \ge \frac{p-N}{\tilde a_1}\ln\left[1+\frac{\tilde a_1}{\tilde a_2}\Psi_0^{-\frac{1}{p-N}}\right] := T,$$

where T is a lower bound of the blow-up time T_{max} and $\Psi_0 := \frac{1}{p}\|u_0\|^p_{L^p(\Omega)}$.

Theorem 1.4 (*Global Existence and Boundedness*). *Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geq 1$ and $R > 0$. Assume that either one of the following two conditions is satisfied:*

1) $k > 2, \quad \mu > 0, \ \chi > 0;$

2) $k = 2, \quad \mu > (1 - \frac{1}{N})\chi > 0.$

Then for all initial data u_0 fulfilling (1.6), the system (1.5) possesses a global classical positive solution (u, v) in $\Omega \times (0, \infty)$, which is bounded in the sense that

$$\sup_{t\in(0,\infty)} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty.$$

2. Preliminaries

We start with a result on local existence of classical solutions to (1.5). From the blow-up criterion (see [13, Lemma 2.1]) we have

$$\text{if } T_{max} < \infty, \text{ then either } \liminf_{t\nearrow T_{max}} \inf_{x\in\Omega} u(x,t) = 0 \text{ or } \limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} = \infty.$$

This includes the possibility of extinction and gradient blow-up of solutions. However, the following lemma provides a simple criterion that rules out this possibility.

Lemma 2.1. *Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geq 1$ and $R > 0$. Then for all initial data u_0 fulfilling (1.6), there exist $T_{max} \in (0, \infty]$ and a pair (u, v) of positive radially symmetric functions $u \in C^{2,1}(\overline{\Omega} \times [0, T_{max}))$ and $v \in C^{2,0}(\overline{\Omega} \times [0, T_{max}))$ which solve (1.5) classically in $\Omega \times (0, T_{max})$, and moreover u satisfies that*

$$\text{if} \quad T_{max} < \infty, \quad \text{then} \quad \limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

Proof. The claim can be proved by a pointwise lower estimate for u and by a uniform estimate for $|\nabla u|$ almost in the same way as in the proof of [13, Theorem 1.1]. □

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded and smooth domain, and $\lambda > 0$, $\mu > 0$, $k > 1$. Then for any solution (u, v) of (1.5) we have*

$$\int_\Omega u\,dx \leq \overline{m} \quad \text{for all } t \in (0, T_{max}), \tag{2.1}$$

with

$$\overline{m} = \max\Big\{ \int_\Omega u_0\,dx, \ \Big(\frac{\lambda}{\mu}\,|\Omega|^{k-1}\Big)^{\frac{1}{k-1}} \Big\}. \tag{2.2}$$

Proof. From the first equation in (1.5) we obtain

$$\frac{d}{dt}\int_\Omega u\,dx = \lambda\int_\Omega u\,dx - \mu\int_\Omega u^k\,dx \leq \lambda\int_\Omega u\,dx - \mu|\Omega|^{1-k}\Big(\int_\Omega u\,dx\Big)^k, \tag{2.3}$$

where in the last term we used Hölder's inequality: $\int_\Omega u\,dx \leq |\Omega|^{\frac{k-1}{k}} \big(\int_\Omega u^k\,dx\big)^{\frac{1}{k}}$. From (2.3) we infer that $z = \int_\Omega u dx$ satisfies

$$\begin{cases} z'(t) \leq \lambda z(t) - \overline{\mu} z^k(t), \quad \overline{\mu} = \mu|\Omega|^{1-k}, \quad \text{for all } t \in [0, T_{max}), \\ z(0) = z_0. \end{cases}$$

Upon an ODE comparison argument this entails that $z(t) \leq \overline{m}$ for all $t \in (0, T_{max})$. This clearly proves the lemma. □

From now on we will consider radially symmetric solutions of (1.5) so that we utilize the notations $u(r,t)$ and $v(r,t)$ instead of $u(x,t)$ and $v(x,t)$, respectively, with $r = |x| \in (0, R)$. In this setting we prove, in the following lemma, that u satisfies a parabolic equation.

Lemma 2.3. *Let $u_0 \in C^2(\overline{\Omega})$ be positive, radially symmetric as well as radially decreasing in $\overline{\Omega}$. Then*

$$u_r \leq 0 \quad \text{in } (0, R) \times (0, T_{max}).$$

Proof. Following the steps in [11, Lemmas 2.2 and 2.3] (with $\chi\nabla\cdot(u\nabla v)$ instead of $\chi\nabla\cdot(\frac{u\nabla v}{\sqrt{1+|\nabla v|^2}})$), we can write

$$u_t = \frac{u^3 u_{rr}}{\sqrt{u^2+u_r^2}^3} + \frac{u_r^4}{\sqrt{u^2+u_r^2}^3} + \frac{N-1}{r}\frac{uu_r}{\sqrt{u^2+u_r^2}} \tag{2.4}$$

$$- \chi u_r v_r - \chi u(m(t) - u) + \lambda u - \mu u^k.$$

By differentiation of (2.4) to respect to r we obtain

$$\begin{aligned} u_{rt} = & \frac{u^3 u_{rrr}}{\sqrt{u^2+u_r^2}^3} + 3\frac{u^2 u_r^3 u_{rr}}{\sqrt{u^2+u_r^2}^5} - 3\frac{u^3 u_r u_{rr}^2}{\sqrt{u^2+u_r^2}^5} + 4\frac{u^2 u_r^3 u_{rr}}{\sqrt{u^2+u_r^2}^5} \\ & + \frac{u_r^5 u_{rr}}{\sqrt{u^2+u_r^2}^5} - 3\frac{u u_r^5}{\sqrt{u^2+u_r^2}^5} - \frac{N-1}{r^2}\frac{u u_r}{\sqrt{u^2+u_r^2}} \\ & + \frac{N-1}{r}\frac{u^3 u_{rr}}{\sqrt{u^2+u_r^2}^3} + \frac{N-1}{r}\frac{u_r^4}{\sqrt{u^2+u_r^2}^3} - \chi u_r v_{rr} \\ & - \chi u_{rr} v_r - \chi m(t) u_r + 2\chi u u_r + \lambda u_r - \mu k u^{k-1} u_r. \end{aligned} \tag{2.5}$$

Taking into account the facts that by the radial symmetry $u_r(0,\cdot) = 0$, $u_r(R,\cdot) \le 0$ and that $u_{0r} \le 0$ by the radially decreasing of u_0, then, fixing $T \in (0, T_{max})$, we see that the function $\varphi = u_r$ belongs to $C^0([0,R]\times[0,T])$ as well as to $C^{2,1}((0,R)\times(0,T))$ and satisfies the parabolic problem

$$\begin{cases} \varphi_t = c_1(r,t)\varphi_{rr} + c_2(r,t)\varphi_r + c_3(r,t)\varphi & \text{in } (0,R)\times(0,T), \\ \varphi \le 0 & \text{on } \{0,R\}\times(0,T), \\ \varphi(\cdot,0) \le 0 & \text{in } (0,R), \end{cases} \tag{2.6}$$

with

$$\begin{aligned} c_1(r,t) &= \frac{u^3}{\sqrt{u^2+u_r^2}^3}, \\ c_2(r,t) &= 3\frac{u^2 u_r^3}{\sqrt{u^2+u_r^2}^5} - 3\frac{u^3 u_r u_{rr}}{\sqrt{u^2+u_r^2}^5} + 4\frac{u^2 u_r^3}{\sqrt{u^2+u_r^2}^5} + \frac{u_r^5}{\sqrt{u^2+u_r^2}^5} \\ &\quad + \frac{N-1}{r}\frac{u^3}{\sqrt{u^2+u_r^2}^3} - \chi v_r, \\ c_3(r,t) &= -3\frac{u u_r^4}{\sqrt{u^2+u_r^2}^5} - \frac{N-1}{r^2}\frac{u}{\sqrt{u^2+u_r^2}} + \frac{N-1}{r}\frac{u_r^3}{\sqrt{u^2+u_r^2}^3} \\ &\quad + 2\chi u - \chi v_{rr} - \chi m(t) + \lambda - \mu k u^{k-1}. \end{aligned}$$

We can obtain $\sup_{(r,t)\in(0,R)\times[0,T]} c_3(r,t) < \infty$.

In fact, neglecting the negative terms in $c_3(r,t)$ and taking into account the facts that $-v_{rr} \le u$ and that $\frac{N-1}{r}\frac{u_r^3}{\sqrt{u^2+u_r^2}^3} \le \frac{N-1}{r}\frac{|u_r|}{\sqrt{u^2+u_r^2}}$, we can write

$$c_3(r,t) \le \frac{N-1}{r}\frac{|u_r|}{\sqrt{u^2+u_r^2}} + 3\chi u + \lambda.$$

Since $u \in C^{2,1}(\overline{\Omega}\times[0,T])$, $T < T_{max}$ and $u_0 > 0$, $u > 0$ in $\overline{\Omega}$, there exists a constant $C_0 > 0$ such that $u(r,t) \ge \inf_{[0,R]\times[0,T]} u(r,t) \ge C_0$. Moreover from the mean value theorem there exists $\theta \in (0,1)$ such that $u_r(r,t) = u_r(0,t) + r u_{rr}(r\theta,t) = r u_{rr}(r\theta,t)$ with $|u_{rr}| \le M$ with some constant $M > 0$ and we can write

$$\frac{N-1}{r}\frac{|u_r|}{\sqrt{u^2+u_r^2}} \le (N-1)\frac{|u_{rr}(r\theta,t)|}{u} \le (N-1)\frac{M}{C_0}.$$

Thus we obtain

$$c_3(r,t) \le (N-1)\frac{M}{C_0} + 3\chi u + \lambda$$

and since u is bounded in $[0,R]\times[0,T]$, we have $\sup_{(r,t)\in(0,R)\times[0,T]} c_3(r,t) < \infty$. By the application of a comparison principle (see [21, Proposition 52.4]) we conclude that $\varphi = u_r \le 0$. □

3. Blow-up in $L^\infty(\Omega)$

In this section we prove that the solution (u,v) of (1.5) blows up in $L^\infty(\Omega)$-norm in the finite time T_{max}. To this end, we transform the equations in (1.5) to scalar parabolic equations. Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \ge 3$, $R > 0$ and $u_0 \in C^2(\overline{\Omega})$ radially symmetric with respect $x = 0$.

Following the steps in [15,20] we introduce the mass accumulation function

$$w(s,t) := \int_0^{s^{\frac{1}{N}}} \rho^{N-1}u(\rho,t)d\rho, \quad s = r^N \in [0,R^N], \quad t \in [0,T_{max}). \tag{3.1}$$

Then we have

$$w_s(s,t) = \frac{1}{N}u(s^{\frac{1}{N}},t) \geq 0, \quad w_{ss}(s,t) = \frac{1}{N^2}s^{\frac{1}{N}-1}u_r(s^{\frac{1}{N}},t).$$

From the second equation in (1.5) we deduce

$$\frac{1}{r^{N-1}}\left(r^{N-1}v_r(r,t)\right)_r = m(t) - u(r,t), \tag{3.2}$$

and by integrating (3.2),

$$r^{N-1}v_r(r,t) = m(t)\int_0^r \rho^{N-1}d\rho - \int_0^r \rho^{N-1}u(\rho,t)d\rho = \frac{m(t)r^N}{N} - w. \tag{3.3}$$

By integration in the radial version of the first equation in (1.5) we obtain

$$\begin{aligned} w_t(s,t) &= \int_0^{s^{\frac{1}{N}}} r^{N-1}u_t(r,t)dr \\ &= \int_0^{s^{\frac{1}{N}}} \left(r^{N-1}\frac{uu_r}{\sqrt{u^2+u_r^2}}\right)_r dr - \chi\int_0^{s^{\frac{1}{N}}}\left(r^{N-1}u(r,t)v_r\right)_r dr \\ &\quad + \lambda\int_0^{s^{\frac{1}{N}}} r^{N-1}u(r,t)dr - \mu\int_0^{s^{\frac{1}{N}}} r^{N-1}u^k dr \\ &= s^{1-\frac{1}{N}}\frac{u\,u_r}{\sqrt{u^2+u_r^2}} - \chi s^{1-\frac{1}{N}}uv_r + \lambda w - \mu\int_0^{s^{\frac{1}{N}}} r^{N-1}u^k dr. \end{aligned}$$

Since $u = Nw_s$ and $u_r = N^2 s^{1-\frac{1}{N}}w_{ss}$, by using (3.3) we derive

$$\begin{aligned} w_t(s,t) &= s^{1-\frac{1}{N}}\frac{Nw_sN^2s^{1-\frac{2}{N}}w_{ss}}{\sqrt{N^2w_s^2 + N^4s^{2-\frac{2}{N}}w_{ss}^2}} - \chi Nw_s\left(\frac{m(t)}{N}s - w\right) \\ &\quad + \lambda w - \mu N^{k-1}\int_0^s w_s(\sigma,t)^k d\sigma \\ &= N^2\frac{s^{2-\frac{2}{N}}w_sw_{ss}}{\sqrt{w_s^2 + N^2s^{2-\frac{2}{N}}w_{ss}^2}} + \chi Nww_s - \chi m(t)sw_s \\ &\quad + \lambda w - \mu N^{k-1}\int_0^s w_s(\sigma,t)^k d\sigma \end{aligned}$$

for $s \in (0,R^N)$ and $t \in (0,T_{max})$.

Taking account of (2.1), we obtain $m(t) \leq \frac{1}{|\Omega|}\overline{m}$ with $\overline{m}$ defined in (2.2). Moreover, since $w_s > 0$, we can conclude that w satisfies the parabolic problem

$$\begin{cases} w_t \geq N^2\dfrac{s^{2-\frac{2}{N}}w_sw_{ss}}{\sqrt{w_s^2 + N^2s^{2-\frac{2}{N}}w_{ss}^2}} - \dfrac{\chi\overline{m}}{|\Omega|}sw_s + \chi Nww_s \\ \qquad -\mu N^{k-1}\displaystyle\int_0^s w_s(\sigma,t)^k d\sigma, \\ w(0,t) = 0, \qquad w(R^N,t) = \dfrac{m(t)R^N}{N}, \quad t \in (0,T_{max}), \\ w(s,0) = w_0(s), \quad s \in (0,R^N) \end{cases} \tag{3.4}$$

with

$$w_0(s) = \int_0^{s^{\frac{1}{N}}} \rho^{N-1}u_0(\rho)d\rho, \quad s \in [0,R^N].$$

In order to prove that the solution u blows up in finite time, we introduce the functional

$$y(t) := \int_0^{R^N} s^{-a}w^b(s,t)ds$$

with $a > 0$ and $b \in (0,1)$ and we show that $y(t)$ satisfies an integral inequality (see (3.31), (3.33)). To this end we first prove the following lemmata.

Lemma 3.1. *For all $s \in (0, R^N]$ and $t \in (0, T_{max})$,*

$$w_s(s,t) \le \frac{w(s,t)}{s} \le w_s(0,t). \tag{3.5}$$

Proof. By Lemma 2.3, following the steps in [5, Lemma 3.3], we arrive to (3.5). □

Lemma 3.2. *Let $\Omega = B_R(0) \subset \mathbb{R}^N$ with some $R > 0$. Let $u_0 \in C^2(\overline{\Omega})$ be radial, and let (u, v) denote the solution of (1.5) in $\Omega \times (0, T_{max})$. Then for all $a > 0$, $b \in (0,1)$, the function w defined in (3.1) satisfies*

$$\begin{aligned}\frac{1}{b}\int_0^{R^N} s^{-a} w^b(s,t)ds \ge \frac{1}{b}\int_0^{R^N} s^{-a} w_0^b(s)ds - a_1 t - a_2\int_0^t\int_0^{R^N} s^{-\frac{1}{N}-a} w^b ds d\tau \\ - a_3\int_0^t\int_0^{R^N} s^{-a} w^b ds d\tau + a_4\int_0^t\int_0^{R^N} s^{-a-1} w^{b+1} ds d\tau \\ - \mu N^{k-1}\int_0^t\int_0^{R^N} s^{-a} w^{b-1}\Big(\int_0^s w_s(\sigma,t)^k d\sigma\Big) ds d\tau,\end{aligned} \tag{3.6}$$

where a_1, a_2, a_3, a_4 are constants defined in (3.12).

Proof. Following the proof of [15, Lemma 2.3] we multiply the first equation in (3.4) by $(s+\epsilon)^{-a} w^{b-1}(s,\tau)$, $\epsilon > 0$, and integrating from 0 to R^N, we obtain

$$\begin{aligned}\frac{1}{b}\frac{d}{dt}\int_0^{R^N} (s+\epsilon)^{-a} w^b(s,t)ds &\ge N^2\int_0^{R^N} \frac{s^{2-\frac{2}{N}}(s+\epsilon)^{-a} w^{b-1} w_s w_{ss}}{\sqrt{w_s^2 + N^2 s^{2-\frac{2}{N}} w_{ss}^2}} ds \\ &+ \chi N\int_0^{R^N} (s+\epsilon)^{-a} w^b w_s ds - \frac{\chi\overline{m}}{|\Omega|}\int_0^{R^N} s(s+\epsilon)^{-a} w^{b-1} w_s ds \\ &- \mu N^{k-1}\int_0^{R^N} (s+\epsilon)^{-a} w^{b-1}\Big(\int_0^s w_s(\sigma,t)^k d\sigma\Big) ds \\ &=: I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{3.7}$$

To estimate I_1 in (3.7) we observe that $\frac{w_s\, w_{ss}}{\sqrt{w_s^2 + N^2 s^{2-\frac{2}{N}} w_{ss}^2}} \ge -\frac{w_s |w_{ss}|}{N s^{1-\frac{1}{N}} |w_{ss}|} = -\frac{1}{N} s^{\frac{1}{N}-1} w_s$ and thus we obtain

$$I_1 \ge -N\int_0^{R^N} s^{1-\frac{1}{N}} (s+\epsilon)^{-a} w^{b-1} w_s ds. \tag{3.8}$$

Upon integration by parts of (3.8) we arrive to

$$\begin{aligned}I_1 &\ge -\frac{N}{b} s^{1-\frac{1}{N}} (s+\epsilon)^{-a} w^b\Big|_0^{R^N} + \frac{N}{b}\int_0^{R^N} \frac{d}{ds}\Big[s^{1-\frac{1}{N}}(s+\epsilon)^{-a}\Big] w^b ds \\ &= -\frac{N}{b}\Big[R^{N-1}(R^N+\epsilon)^{-a}\frac{m^b(t) R^{Nb}}{N^b}\Big] \\ &\quad + \frac{N}{b}\Big(1 - \frac{1}{N}\Big)\int_0^{R^N} \Big[s^{-\frac{1}{N}}(s+\epsilon)^{-a}\Big] w^b ds \\ &\quad - a\frac{N}{b}\int_0^{R^N} \Big[s^{1-\frac{1}{N}}(s+\epsilon)^{-a-1}\Big] w^b ds \\ &\ge -A_1(\epsilon) - A_2\int_0^{R^N} \Big[s^{-\frac{1}{N}}(s+\epsilon)^{-a}\Big] w^b ds,\end{aligned} \tag{3.9}$$

where $A_1(\epsilon) := \frac{N}{b}\Big[R^{N-1}(R^N+\epsilon)^{-a}\frac{\overline{m}^b R^{Nb}}{N^b|\Omega|^b}\Big]$ since $m(t) \le \frac{\overline{m}}{|\Omega|}$, $A_2 := \frac{N}{b}\Big[a - (1 - \frac{1}{N})\Big]$.

Similarly, for the term I_2 we obtain

$$\begin{aligned}I_2 &= \chi N\int_0^{R^N} (s+\epsilon)^{-a} w^b w_s ds \\ &\ge \frac{\chi N}{b+1}[(R^N+\epsilon)^{-a} w^{b+1}(R^N)] + a\frac{\chi N}{b+1}\int_0^{R^N} (s+\epsilon)^{-a-1} w^{b+1} ds \\ &\ge a\frac{\chi N}{b+1}\int_0^{R^N} (s+\epsilon)^{-a-1} w^{b+1} ds.\end{aligned} \tag{3.10}$$

Similarly, we estimate I_3, by integrating by parts we obtain

$$\begin{aligned} I_3 &= -\frac{\chi\overline{m}}{|\Omega|}\int_0^{R^N} s(s+\epsilon)^{-a}w^{b-1}w_s ds \\ &= -\frac{\chi\overline{m}}{b|\Omega|}\left[s(s+\epsilon)^{-a}w^b\right]_0^{R^N} + \frac{\chi\overline{m}}{b|\Omega|}\int_0^{R^N}\frac{d}{ds}\left[s(s+\epsilon)^{-a}\right]w^b ds \\ &\geq -A_3(\epsilon) + \frac{\chi\overline{m}}{b|\Omega|}\int_0^{R^N}(s+\epsilon)^{-a}w^b ds - a\frac{\chi\overline{m}}{b|\Omega|}\int_0^{R^N} s(s+\epsilon)^{-a-1}w^b ds \\ &\geq -A_3(\epsilon) - A_4\int_0^{R^N}(s+\epsilon)^{-a}w^b ds, \end{aligned} \tag{3.11}$$

where $A_3(\epsilon) := \frac{\chi\overline{m}^{b+1}}{b|\Omega|^{b+1}}\frac{R^{N(b+1)}}{N^b}(R^N+\epsilon)^{-a}$ since $m(t) \leq \frac{\overline{m}}{|\Omega|}$, and $A_4 := \frac{\chi\overline{m}}{b|\Omega|}(a-1)$.

Substituting (3.9), (3.10) and (3.11) in (3.7), we see that

$$\begin{aligned} \frac{1}{b}\frac{d}{dt}\int_0^{R^N}(s+\epsilon)^{-a}w^b(s,t)ds &\geq -(A_1(\epsilon)+A_3(\epsilon)) - A_2\int_0^{R^N}\left[s^{-\frac{1}{N}}(s+\epsilon)^{-a}\right]w^b ds \\ &- A_4\int_0^{R^N}(s+\epsilon)^{-a}w^b ds + a\frac{\chi N}{b+1}\int_0^{R^N}(s+\epsilon)^{-a-1}w^{b+1}ds \\ &- \mu N^{k-1}\int_0^{R^N}(s+\epsilon)^{-a}w^{b-1}\Big(\int_0^s w_s(\sigma,t)^k d\sigma\Big)ds. \end{aligned}$$

Integrating from 0 to $t \in (0, T_{max})$, we have

$$\begin{aligned} \frac{1}{b}\int_0^{R^N}(s+\epsilon)^{-a}w^b(s,t)ds - \frac{1}{b}\int_0^{R^N}(s+\epsilon)^{-a}w_0^b(s)ds &\geq -(A_1(\epsilon)+A_3(\epsilon))t \\ &- A_2\int_0^t\int_0^{R^N}\left[s^{-\frac{1}{N}}(s+\epsilon)^{-a}\right]w^b ds d\tau - A_4\int_0^t\int_0^{R^N}(s+\epsilon)^{-a}w^b ds d\tau \\ &+ a\frac{\chi N}{b+1}\int_0^t\int_0^{R^N}(s+\epsilon)^{-a-1}w^{b+1}ds d\tau \\ &- \mu N^{k-1}\int_0^t\int_0^{R^N}(s+\epsilon)^{-a}w^{b-1}\Big(\int_0^s w_s(\sigma,t)^k d\sigma\Big)ds d\tau. \end{aligned}$$

Now, from the monotone convergence theorem, taking $\epsilon \searrow 0$, we arrive at

$$\begin{aligned} \frac{1}{b}\int_0^{R^N} s^{-a}w^b(s,t)ds &\geq \frac{1}{b}\int_0^{R^N} s^{-a}w_0^b(s)ds - a_1 t - a_2\int_0^t\int_0^{R^N} s^{-\frac{1}{N}-a}w^b ds d\tau \\ &- a_3\int_0^t\int_0^{R^N} s^{-a}w^b ds d\tau + a_4\int_0^t\int_0^{R^N} s^{-a-1}w^{b+1}ds d\tau \\ &- \mu N^{k-1}\int_0^t\int_0^{R^N} s^{-a}w^{b-1}\Big(\int_0^s w_s(\sigma,t)^k d\sigma\Big)ds d\tau, \end{aligned}$$

with

$$\begin{cases} a_1 = \frac{N}{b}\left[R^{N(b+1-a)-1}\frac{\overline{m}^b}{N^b|\Omega|^b}\right] + \frac{\chi\overline{m}^{b+1}}{b|\Omega|^{b+1}}\frac{R^{N(b+1-a)}}{N^b}, \\ a_2 = A_2 := \frac{N}{b}\left[a-(1-\frac{1}{N})\right], \quad a_3 = A_4 := \frac{\chi\overline{m}}{b|\Omega|}(a-1), \\ a_4 = a\frac{\chi N}{b+1}. \end{cases} \tag{3.12}$$

This proves the lemma. □

Under the assumption of Lemma 3.2 we prove the following lemma.

Lemma 3.3. *Assume that the conditions of Lemma 3.2 are satisfied. Suppose that $N \geq 3$ and*

$$\frac{1}{N-1} < b < 1, \quad 1 < a < (b+1)\Big(1-\frac{1}{N}\Big). \tag{3.13}$$

Case (i) *If*

$$1 < k < 2, \quad \mu > 0, \ \chi > 0, \tag{3.14}$$

then

$$\int_0^{R^N} s^{-a}w^b(s,t)ds \geq \int_0^{R^N} s^{-a}w_0^b ds + b_1\int_0^t\int_0^{R^N} s^{-a}w^b ds d\tau \tag{3.15}$$

$$- b_2 \int_0^t \Big(\int_0^{R^N} s^{-a} w^b ds \Big)^{\frac{b+1}{b}} d\tau - b_3 t,$$

where $b_1,\ b_2$ *and* b_3 *are constants defined in* (3.26).

Case (ii) *If*

$$k = 2, \quad \mu < \frac{a(a-1)}{b+1}\chi, \tag{3.16}$$

then

$$\begin{aligned} \int_0^{R^N} s^{-a} w^b(s,t) ds \geq & \int_0^{R^N} s^{-a} w_0^b ds + b_1 \int_0^t \int_0^{R^N} s^{-a} w^b ds d\tau \\ & - \bar{b}_2 \int_0^t \Big(\int_0^{R^N} s^{-a} w^b ds \Big)^{\frac{b+1}{b}} d\tau - \bar{b}_3 t, \end{aligned} \tag{3.17}$$

where $b_1,\ \bar{b}_2$ *and* $\bar{b}_3$ *are constants defined respectively in* (3.26) *and* (3.29).

Proof. We rewrite (3.6):

$$\begin{aligned} \frac{1}{b} \int_0^{R^N} s^{-a} w^b(s,t) ds \geq & \frac{1}{b} \int_0^{R^N} s^{-a} w_0^b(s) ds - a_1 t \\ & - a_2 \int_0^t \int_0^{R^N} s^{-\frac{1}{N}-a} w^b ds d\tau - a_3 \int_0^t \int_0^{R^N} s^{-a} w^b ds d\tau \\ & + a_4 \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1} ds d\tau \\ & - \mu N^{k-1} \int_0^t \int_0^{R^N} s^{-a} w^{b-1} \Big(\int_0^s w_s(\sigma,t)^k d\sigma \Big) ds d\tau \\ = & \frac{1}{b} \int_0^{R^N} s^{-a} w_0^b(s) ds - a_1 t + J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{3.18}$$

In view of (3.12) we observe that the coefficients $a_1,\ a_2, a_3,\ a_4$ are positive constants.

Estimate of $J_1 = -a_2 \int_0^t \int_0^{R^N} s^{-\frac{1}{N}-a} w^b ds d\tau$

Using the Hölder and Young inequalities, we have

$$\begin{aligned} & a_2 \int_0^{R^N} s^{-\frac{1}{N}-a} w^b ds = a_2 \int_0^{R^N} s^{-\frac{1}{N}-a+(a+1)\frac{b}{b+1}} s^{-(a+1)\frac{b}{b+1}} w^b ds \\ & \leq \Big(a_2^{b+1} \int_0^{R^N} s^{(-\frac{1}{N}-a)(b+1)+(a+1)b} ds \Big)^{\frac{1}{b+1}} \Big(\int_0^{R^N} s^{-a-1} w^{b+1} ds \Big)^{\frac{b}{b+1}} \\ & \leq C(\delta_1) \int_0^{R^N} s^{\gamma_1} ds + \frac{b}{b+1} \delta_1 \int_0^{R^N} s^{-a-1} w^{b+1} ds \\ & = C_1(\delta_1) + \tilde{\delta}_1 \int_0^{R^N} s^{-a-1} w^{b+1} ds, \end{aligned} \tag{3.19}$$

where $\gamma_1 := b(1-\frac{1}{N}) - a - \frac{1}{N} > -1$, $\tilde{\delta}_1 = \frac{b}{b+1}\delta_1$, $\delta_1 > 0$, $C_1(\delta_1) = C(\delta_1)\frac{R^{N(\gamma_1+1)}}{\gamma_1+1}$, $C(\delta_1) = \frac{1}{\delta_1^b}\frac{1}{b+1} a_2^{b+1}$. From (3.19) we deduce

$$J_1 = -a_2 \int_0^t \int_0^{R^N} s^{-\frac{1}{N}-a} w^b ds d\tau \geq -C_1(\delta_1) t - \tilde{\delta}_1 \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1} ds d\tau. \tag{3.20}$$

Estimate of $J_3 = a_4 \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1} ds d\tau$

The Hölder inequality leads to

$$\begin{aligned} \int_0^{R^N} s^{-a} w^b ds = & \int_0^{R^N} s^{-a+(a+1)\frac{b}{b+1}} s^{-(a+1)\frac{b}{b+1}} w^b ds \\ & \leq \Big(\int_0^{R^N} s^{-a(b+1)+(a+1)b} ds \Big)^{\frac{1}{b}} \Big(\int_0^{R^N} s^{-a-1} w^{b+1} ds \Big)^{\frac{b}{b+1}} \\ & = \tilde{c}^{-1} \Big(\int_0^{R^N} s^{-a-1} w^{b+1} ds \Big)^{\frac{b}{b+1}}, \end{aligned}$$

with $\tilde{c}^{-1} = \big(\frac{R^{N(b-a+1)}}{b-a+1}\big)^{\frac{1}{b}}$. From the last inequality we deduce

$$\int_0^{R^N} s^{-a-1} w^{b+1} ds \geq \tilde{c}^{\frac{b+1}{b}} \Big(\int_0^{R^N} s^{-a} w^b ds \Big)^{\frac{b+1}{b}}.$$

Thus for J_3 we can write

$$J_3 = a_4 \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1} ds d\tau \geq a_4 \tilde{c}^{\frac{b+1}{b}} \int_0^t \Big(\int_0^{R^N} s^{-a} w^b ds \Big)^{\frac{b+1}{b}} d\tau. \tag{3.21}$$

Estimate of J_4

We observe that a first estimate of J_4 is the following:

Using Fubini's theorem we can rewrite the term J_4 in (3.18) as

$$\begin{aligned} J_4 &= -\mu N^{k-1} \int_0^t \int_0^{R^N} s^{-a} w^{b-1} \Big(\int_0^s w_s^k(\sigma, t) d\sigma \Big) ds d\tau \\ &= -\mu N^{k-1} \int_0^t \int_0^{R^N} \Big(\int_\sigma^{R^N} s^{-a} w^{b-1} ds \Big) w_s^k(\sigma, t) d\sigma d\tau. \end{aligned}$$

Since $b \in (0,1)$ and $w_s > 0$ and then w^{b-1} is decreasing and $-w^{b-1}(s,t) \geq -w(\sigma,t)^{b-1}$ for $s \geq \sigma$, we see that

$$\begin{aligned} J_4 &\geq -\mu N^{k-1} \int_0^t \int_0^{R^N} \Big(\int_\sigma^{R^N} s^{-a} ds \Big) w^{b-1}(\sigma, t) w_s^k(\sigma, t) d\sigma d\tau \\ &= -\frac{\mu N^{k-1}}{1-a} \int_0^t \int_0^{R^N} \Big(R^{N(1-a)} - \sigma^{1-a} \Big) w^{b-1}(\sigma, t) w_s^k(\sigma, t) d\sigma d\tau. \end{aligned}$$

Since $a > 1$ by virtue of (3.13), we can neglect the positive term $\frac{\mu N^{k-1}}{a-1} \int_0^t \int_0^{R^N} R^{N(1-a)} w^{b-1}(\sigma, t) w_s^k(\sigma, t) d\sigma d\tau$ to have

$$J_4 \geq -\frac{\mu N^{k-1}}{a-1} \int_0^t \int_0^{R^N} s^{1-a} w^{b-1}(s,t) w_s^k(s,t) ds d\tau.$$

Now, we apply (3.5) to obtain

$$J_4 \geq -\frac{\mu N^{k-1}}{a-1} \int_0^t \int_0^{R^N} s^{1-k-a} w^{b+k-1}(s,t) ds d\tau. \tag{3.22}$$

Proof of Case (i)

As (3.13) and (3.14) hold, by the Hölder and Young inequalities in (3.22) we obtain

$$\begin{aligned} J_4 &\geq -\frac{\mu N^{k-1}}{a-1} \int_0^t \int_0^{R^N} s^{\gamma_3} s^{-(a+1)\frac{b+k-1}{b+1}} w^{b+k-1}(s,t) ds d\tau \qquad (3.23) \\ &\geq -\int_0^t \Big\{ \Big[\delta_2^{-\frac{b+k-1}{2-k}} \Big(\frac{\mu N^{k-1}}{a-1} \Big)^{\frac{b+1}{2-k}} \int_0^{R^N} s^{\gamma_3 \frac{b+1}{2-k}} ds \Big]^{\frac{2-k}{b+1}} \Big(\delta_2 \int_0^{R^N} s^{-a-1} w^{b+1} ds \Big)^{\frac{b+k-1}{b+1}} \Big\} d\tau \\ &\geq -\widetilde{C}(\delta_2) \int_0^t \int_0^{R^N} s^{\gamma_3 \frac{b+1}{2-k}} ds d\tau - \frac{b+k-1}{b+1} \delta_2 \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1} ds d\tau \\ &= -C_2(\delta_2) t - \tilde{\delta}_2 \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1} ds d\tau \end{aligned}$$

with $\delta_2 > 0$ and $\gamma_3 := 1 - k - a + (a+1)\frac{b+k-1}{b+1}$ and $C_2(\delta_2) = \widetilde{C}(\delta_2) \frac{R^{N\gamma_3(\frac{b+1}{2-k}+1)}}{\gamma_3(\frac{b+1}{2-k}+1)}$, $\widetilde{C}(\delta_2) = \frac{2-k}{b+1} \Big(\frac{\mu N^{k-1}}{a-1} \Big)^{\frac{b+1}{2-k}} \delta_2^{-\frac{b+k-1}{2-k}}$ and $\tilde{\delta}_2 = \frac{b+k-1}{b+1} \delta_2$. Here we observe that $\gamma_3 \frac{b+1}{2-k} := \frac{(1-k-a)(b+1)+(a+1)(b+k-1)}{2-k} = -a + b > -1$, because $a < (b+1)(1 - \frac{1}{N}) < b+1$.

Now we put 3.20 in (3.18) to have

$$\begin{aligned} \int_0^{R^N} s^{-a} w^b(s,t)\, ds &\geq \int_0^{R^N} s^{-a} w_0^b(s) ds - a_3 b \int_0^t \int_0^{R^N} s^{-a} w^b ds d\tau \qquad (3.24) \\ &\quad + b(a_4 - \tilde{\delta}) \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1} ds d\tau - b(a_1 + C_1(\delta_1) + C_2(\delta_3)) t \end{aligned}$$

with $\tilde{\delta} = \tilde{\delta}_1 + \tilde{\delta}_2$. Since $a > 1$, $b < 1$ and $k < 2$, we can choose $\tilde{\delta} > 0$ such that $a_4 - \tilde{\delta} = a\frac{\chi N}{b+1} - \frac{b}{b+1}\delta_1 - \frac{b+k-1}{b+1}\delta_2 > \frac{\chi N}{2} - \delta_1 - \delta_2 > 0$, a possible choice of δ_1 and δ_2 is $\delta_1 = \delta_2 = \frac{\chi N}{8}$. Then in (3.24) we apply (3.21) to achieve

$$\begin{aligned} \int_0^{R^N} s^{-a} w^b(s,t) ds &\geq \int_0^{R^N} s^{-a} w_0^b(s) ds - b_1 \int_0^t \int_0^{R^N} s^{-a} w^b ds d\tau \qquad (3.25) \\ &\quad + b_2 \int_0^t \Big(\int_0^{R^N} s^{-a} w^b ds \Big)^{\frac{b+1}{b}} d\tau - b_3 t, \end{aligned}$$

with

$$\begin{cases} b_1 := a_3 b > 0, \quad b_2 := b(a_4 - \tilde{\delta}) \tilde{c}^{\frac{b+1}{b}}, \\ b_3 := (a_1 + C_1(\delta_1) + C_2(\delta_2)) b. \end{cases} \tag{3.26}$$

Proof of Case (ii)

Taking into account of (3.20) and (3.22) with $k = 2$ we rewrite (3.6) as

$$\begin{aligned}\int_0^{R^N} s^{-a} w^b(s,t)\,ds &\geq \int_0^{R^N} s^{-a} w_0^b(s) ds - b(a_1 + C_1(\delta_1))t \\ &- b\tilde{\delta}_1 \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1} ds d\tau - b a_3 \int_0^t \int_0^{R^N} s^{-a} w^b ds d\tau \\ &+ b a_4 \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1} ds d\tau - b\frac{\mu N}{a-1} \int_0^t \int_0^{R^N} s^{-1-a} w^{b+1} ds d\tau \\ &= \int_0^{R^N} s^{-a} w_0^b(s) ds - b(a_1 + C_1(\delta_1))t - b a_3 \int_0^t \int_0^{R^N} s^{-a} w^b ds d\tau \\ &+ b\Big(a_4 - \frac{\mu N}{a-1} - \tilde{\delta}_1\Big) \int_0^t \int_0^{R^N} s^{-a-1} w^{b+1} ds d\tau. \end{aligned} \tag{3.27}$$

Since (3.16) holds and $0 < b < 1$, we can choose $\tilde{\delta}_1$ such that

$$a_4 - \frac{\mu N}{a-1} - \tilde{\delta}_1 = a\frac{\chi N}{b+1} - \frac{\mu N}{a-1} - \frac{b}{b+1}\delta_1 > a\frac{\chi N}{b+1} - \frac{\mu N}{a-1} - \delta_1 > 0.$$

For example, we can take $\delta_1 = \frac{1}{2}\big(a\frac{\chi N}{b+1} - \frac{\mu N}{a-1}\big)$. Then, using (3.21) in (3.27), we arrive to

$$\begin{aligned}\int_0^{R^N} s^{-a} w^b(s,t)\,ds &\geq \int_0^{R^N} s^{-a} w_0^b(s) ds - b_1 \int_0^t \int_0^{R^N} s^{-a} w^b ds d\tau \\ &+ \overline{b}_2 \int_0^t \Big(\int_0^{R^N} s^{-a} w^b ds\Big)^{\frac{b+1}{b}} d\tau - \overline{b}_3 t, \end{aligned} \tag{3.28}$$

with

$$\begin{cases} b_1 = b a_3, \\ \overline{b}_2 := b\big(a_4 - \frac{\mu N}{a-1} - \tilde{\delta}_1\big)\tilde{c}^{\frac{b+1}{b}}, \\ \overline{b}_3 := b(a_1 + C_1(\delta_1)). \end{cases} \tag{3.29}$$

This proves the lemma. □

We are now in the position to complete the proof of finite-time blow-up in $L^\infty(\Omega)$.

Proof of Theorem 1.1. We divide the proof into two cases, by relying on the two estimates (3.25) and (3.28) for Cases (i) and (ii), respectively.

Case (i)

If we assume (1.7), the inequality (3.25) is of the following type:

$$y(t) \geq b_0 - b_1 \int_0^t y(\tau) d\tau + b_2 \int_0^t y^{1+\gamma}(\tau) d\tau - b_3 t, \quad \forall\ t \in (0, T_{max}), \tag{3.30}$$

with

$$\begin{aligned} y(t) &:= \int_0^{R^N} s^{-a} w^b(s,t) ds, \\ b_0 &:= \int_0^{R^N} s^{-a} w_0^b(s) ds, \\ \gamma &:= \frac{1}{b} \end{aligned}$$

and b_1, b_2, b_3 are defined in (3.26). We observe that there exists $\epsilon_1 \in (0, \frac{b_2}{b_1})$ such that

$$y(t) \leq \epsilon_1 y^{1+\gamma}(t) + c(\epsilon_1).$$

Using this inequality in (3.30), we have

$$y(t) \geq b_0 + B_1 \int_0^t y^{1+\gamma}(\tau) d\tau - B_2 t, \quad \forall\ t \in (0, T_{max}), \tag{3.31}$$

with $B_1 := b_2 - b_1\epsilon_1$, $B_2 := b_1 c(\epsilon_1) + b_3$. Let us introduce the function

$$z(t) := b_0 - \Big(\frac{B_2}{B_1}\Big)^{\frac{1}{1+\gamma}} + B_1 \int_0^t y^{1+\gamma} d\tau - B_2 t,$$

which satisfies

$$z'(t) = B_1 y^{1+\gamma}(t) - B_2 \geq B_1\Big(z(t) + \Big(\frac{B_2}{B_1}\Big)^{\frac{1}{1+\gamma}}\Big)^{1+\gamma} - B_2 \geq B_1 z^{1+\gamma}(t), \tag{3.32}$$

$$z(0) = b_0 - \Big(\frac{B_2}{B_1}\Big)^{\frac{1}{1+\gamma}}.$$

From the hypothesis on the initial data we obtain $z(0) > 0$. In fact, let us introduce the nonnegative function

$$\psi_\varepsilon(s) := \frac{m_0}{N}\frac{R^N + \varepsilon}{s + \varepsilon}s, \quad s \in [0, R^N],\ \varepsilon > 0,$$

which satisfies $\psi_\varepsilon \nearrow \frac{m_0 R^N}{N}$ as $\varepsilon \searrow 0$. From the monotone convergence theorem, since $a > 1$, we obtain

$$\int_0^{R^N} s^{-a}\psi_\varepsilon^b(s)ds \to \infty \quad as\ \varepsilon \searrow 0.$$

Finally, for some sufficiently small $\varepsilon > 0$ we define $w_0(s) := \psi_\varepsilon(s)$, $s \in [0, R^N]$ and we easily obtain $z(0) > 0$.

We confirm that $z(t)$ is positive. We observe that w_0 belongs to $C^\infty([0, R^N])$ with $w_0(0) = 0$, $w_0(R^N) = \frac{m_0 R^N}{N}$ and $w_{0s}(s) > 0$ for all $s \in [0, R^N]$. As a consequence $u_0(x) := N w_{0s}(|x|^N)$ for $x \in \overline{\Omega}$ is radially symmetric, smooth and positive in $\overline{\Omega}$ with $\frac{1}{|\Omega|}\int_\Omega u_0 dx = m_0$. Then by comparison theorem, $z(t)$ is positive.

Integrating (3.32) from 0 to t we have

$$-z^{-\gamma}(0) \leq z^{-\gamma}(t) - z^{-\gamma}(0) \leq -\gamma B_2 t, \quad t \in (0, T_{max}),$$

and $t \leq \frac{z^{-\gamma}(0)}{\gamma B_2}$. If $t \to T_{max}$ then $z \to +\infty$ with $T_{max} \leq \frac{z^{-\gamma}(0)}{\gamma B_2}$. The proof of Theorem 1.1 in the Case (i) is completed by the blow-up criterion in Lemma 2.1.

Case (ii)

If $a \to (b+1)(1 - \frac{1}{N})$ and $b \to 1$, then

$$\frac{a(a-1)}{b+1} \to \Big(1 - \frac{2}{N}\Big)\Big(1 - \frac{1}{N}\Big).$$

Therefore, if we assume (1.8), that is, $\mu < (1 - \frac{2}{N})(1 - \frac{1}{N})\chi$, then we can take a, b such that (3.13) and (3.16) hold.

Taking into account of (3.29), we can rewrite the inequality (3.28) as

$$y(t) \geq b_0 - b_1\int_0^t y(\tau)d\tau + \overline{b}_2\int_0^t y^{1+\gamma}(\tau)d\tau - \overline{b}_3 t, \quad \forall\ t \in (0, T_{max}), \tag{3.33}$$

with $y(t) := \int_0^{R^N} s^{-a}w^b(s,t)ds$, $b_0 := \int_0^{R^N} s^{-a}w_0^b(s)ds$. The proof of Case (ii) is similar to the proof of Case (i). □

4. Blow-up in $L^p(\Omega)$

Lemma 4.1. *Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geq 1$ and $R > 0$. Assume that u_0 is nonnegative and $u_0 \in C^2(\overline{\Omega})$. Let (u, v) be a classical solution of the system (1.5). Then if there exists $C > 0$ such that for some $p_0 > N$,*

$$\|u(\cdot,t)\|_{L^{p_0}(\Omega)} \leq C \quad \textit{for all } t \in (0, T_{max}),$$

then, for some $\hat{C} > 0$,

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \hat{C} \quad \textit{for any } t \in (0, T_{max}).$$

Proof. Following the step in [20, Lemma 3.1] we can consider the $L^p(\Omega)$-norm, $p > 1$, of $u(x,t)$ of (1.5) to obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx = \int_\Omega u^{p-1}u_t dx &= -\int_\Omega \nabla u^{p-1}\frac{u\nabla u}{\sqrt{u^2 + |\nabla u|^2}}dx \\ &\quad + \chi(p-1)\int_\Omega u^{p-1}\nabla u\nabla v dx + \lambda\int_\Omega u^p dx - \mu\int_\Omega u^{p+k-1}dx \\ &= -(p-1)\int_\Omega \frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2 + |\nabla u|^2}}dx - \chi\frac{p-1}{p}m(t)\int_\Omega u^p dx \\ &\quad + \chi\frac{p-1}{p}\int_\Omega u^{p+1}dx + \lambda\int_\Omega u^p dx - \mu\int_\Omega u^{p+k-1}dx.\end{aligned}$$

Here we neglect the second and the last negative terms, we arrive to

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx \leq -(p-1)\int_\Omega \frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2 + |\nabla u|^2}}dx + \chi\frac{p-1}{p}\int_\Omega u^{p+1}dx + \lambda\int_\Omega u^p dx.$$

By virtue of the inequality $\int_\Omega u^{p-1}|\nabla u|dx \le \int_\Omega \frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2+|\nabla u|^2}}dx + \int_\Omega u^p dx$ proved in [11, Lemma 6.1], we have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx &\le -(p-1)\int_\Omega u^{p-1}|\nabla u|dx + (p-1)\int_\Omega u^p dx \\ &\quad + \chi\frac{p-1}{p}\int_\Omega u^{p+1}dx + \lambda\int_\Omega u^p dx \\ &\le -\frac{p-1}{p}\int_\Omega |\nabla u^p|dx + \mathcal{A}\int_\Omega u^p dx + \mathcal{B}\int_\Omega u^{p+1}dx\end{aligned} \tag{4.1}$$

for all $t \in (0, T_{max})$, where

$$\begin{cases}\mathcal{A} := p - 1 + \lambda, \\ \mathcal{B} := \chi\frac{p-1}{p}.\end{cases} \tag{4.2}$$

Following the step of the proof in [20, Lemma 3.1] as a particular case we obtain

$$\frac{d}{dt}\int_\Omega u^p dx + \int_\Omega u^p dx \le c_1 p^{1+N}\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^2 + c_2 p\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^{2\frac{p+1}{p}}.$$

Defining

$$p := p_j = p_0 2^j$$

for nonnegative integers j and introducing

$$M_j := \sup_{t\in(0,T)}\int_\Omega u^{p_j}dx$$

with $j \ge 1$ and $T \in (0, T_{max})$ fixed, we have

$$\int_\Omega u^{p_j}dx = M_j \le \max\Big\{\int_\Omega u_0^{p_j}dx,\ c_1 p_j^{1+N} M_{j-1}^2 + c_2 p_j M_{j-1}^{2\frac{p_j+1}{p_j}}\Big\} \quad \text{for all } j \ge 1.$$

By induction and limiting procedure as in [22, p. 714] we obtain

$$\limsup_{j\to\infty} M_j^{\frac{1}{p_j}} \le c_3$$

for some $c_3 > 0$. In view of the definition of M_j, this proves

$$\sup_{t\in(0,T)}\|u(\cdot,t)\|_{L^\infty(\Omega)} \le c_3.$$

Thus we obtain that if $\|u(\cdot,t)\|_{L^{p_0}(\Omega)}$ is bounded for some $p_0 > N$ then $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded. □

Proof of Theorem 1.2. Taking into account of Lemma 4.1, if $\|u(\cdot,t)\|_{L^{p_0}(\Omega)} \le C$ for some $p_0 > N$, then also $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded. This is a contradiction since Theorem 1.1 holds. Therefore $\limsup_{t\nearrow T_{max}}\|u(\cdot,t)\|_{L^p(\Omega)} = \infty$ for all $p > N$. □

5. Lower bound of the blow-up time T_{max}

Throughout this section we assume that Theorem 1.2 holds. We want to obtain a safe interval $[0, T]$ of existence of the solution of (1.5), with T a lower bound of the blow-up time T_{max}. To this end, we construct a first order differential inequality for $\Psi(t) := \frac{1}{p}\|u(\cdot,t)\|_{L^p(\Omega)}^p$ and by integration we get the lower bound.

Proof of Theorem 1.3. We differentiate $\Psi(t)$ and recall (4.1) in Section 4:

$$\Psi' \le -\frac{p-1}{p}\int_\Omega |\nabla u^p|dx + \mathcal{A}\int_\Omega u^p dx + \mathcal{B}\int_\Omega u^{p+1}dx, \tag{5.1}$$

for all $t \in (0, T_{max})$, with $\mathcal{A}$ and $\mathcal{B}$ defined in (4.2).

In (5.1) we reduce $\int_\Omega u^{p+1}dx$ in term of $\int_\Omega u^p dx$. To this end, under the condition $p > N$, firstly we use the Gagliardo–Nirenberg inequality, and then we apply the Young inequality. We get

$$\begin{aligned}\int_\Omega u^{p+1}dx = \|u^p\|^{\frac{p+1}{p}}_{L^{\frac{p+1}{p}}(\Omega)} &\le C_{GN}\|\nabla u^p\|^{\frac{p+1}{p}\theta_1}_{L^1(\Omega)}\|u^p\|^{\frac{p+1}{p}(1-\theta_1)}_{L^1(\Omega)} + C_{GN}\|u^p\|^{\frac{p+1}{p}}_{L^1(\Omega)} \\ &\le \epsilon_1\int_\Omega |\nabla u^p|dx + c(\epsilon_1)\Big(\int_\Omega u^p dx\Big)^{\frac{p+1-N}{p-N}} + C_{GN}\Big(\int_\Omega u^p dx\Big)^{\frac{p+1}{p}},\end{aligned} \tag{5.2}$$

where $\theta_1 = \frac{N}{p+1} \in (0,1)$ and $c(\epsilon_1) = \epsilon_1^{-\frac{N}{p-N}} C_{GN}^{\frac{1}{p-N}}$, $\epsilon_1 > 0$ to be fixed later.

Substituting (5.2) in (5.1), we obtain

$$\Psi' \le -\Big(\frac{p-1}{p} - \mathcal{B}\epsilon_1\Big)\int_\Omega |\nabla u^p| dx + \mathcal{A}\int_\Omega u^p dx + \mathcal{B}c(\epsilon_1)\Big(\int_\Omega u^p dx\Big)^{\frac{p+1-N}{p-N}} + \Big(\mathcal{B}C_{GN}\Big)\Big(\int_\Omega u^p dx\Big)^{\frac{p+1}{p}}$$

for $t \in (0, T_{max})$. Recalling that $\mathcal{B} = \chi\frac{p-1}{p}$ in (4.2), we eliminate the first term by choosing $\epsilon_1 = \frac{1}{\chi}$ to have

$$\Psi' \le \mathcal{A}\int_\Omega u^p dx + \tilde{c}_1\Big(\int_\Omega u^p dx\Big)^{\frac{p+1}{p}} + \tilde{c}_2\Big(\int_\Omega u^p dx\Big)^{\frac{p+1-N}{p-N}}, \quad t \in (0, T_{max}),$$

with $\tilde{c}_1 = \mathcal{B}C_{GN}$, $\tilde{c}_2 = \mathcal{B}c(\frac{1}{\chi})$.

By using the inequality $Y^{\frac{p+1}{p}} \le Y + Y^{\frac{p+1-N}{p-N}}$, valid for all $Y \ge 0$, we arrive to

$$\begin{aligned}\Psi' &\le \mathcal{A}(p\Psi) + \tilde{c}_1\Big(p\Psi\Big)^{\frac{p+1}{p}} + \tilde{c}_2\Big(p\Psi\Big)^{\frac{p+1-N}{p-N}} \\ &\le (\mathcal{A}+\tilde{c}_1)(p\Psi) + (\tilde{c}_1+\tilde{c}_2)\Big(p\Psi\Big)^{\frac{p+1-N}{p-N}}, \quad t \in [0, T_{max}),\end{aligned}$$

i.e.

$$\Psi' - \tilde{a}_1\Psi \le \tilde{a}_2\Psi^{\frac{p+1-N}{p-N}}, \qquad t \in [0, T_{max}), \tag{5.3}$$

with $\tilde{a}_1 := p(\mathcal{A}+\tilde{c}_1)$ and $\tilde{a}_2 := p^{\frac{p+1-N}{p-N}}(\tilde{c}_1+\tilde{c}_2)$.

Integrating (5.3) from 0 to t, $t \in [0, T_{max})$, we obtain

$$\begin{aligned}\Psi^{-\frac{1}{p-N}} e^{\frac{\tilde{a}_1}{p-N}t} - \Psi_0^{-\frac{1}{p-N}} &= \int_0^t \Big(\Psi^{-\frac{1}{p-N}} e^{\frac{\tilde{a}_1}{p-N}\tau}\Big)' d\tau \\ &\ge -\frac{\tilde{a}_2}{p-N}\int_0^t e^{\frac{\tilde{a}_1}{p-N}\tau} d\tau,\end{aligned}$$

and thus

$$\frac{1}{\Psi^{\frac{1}{p-N}}} \ge \Big(\Psi_0^{-\frac{1}{p-N}} - \frac{\tilde{a}_2}{\tilde{a}_1} e^{\frac{\tilde{a}_1}{p-N}t} + \frac{\tilde{a}_2}{\tilde{a}_1}\Big) e^{-\frac{\tilde{a}_1}{p-N}t},$$

If $t \to T_{max}$, then $\Psi(t) \to \infty$ and we obtain

$$\Big(\Psi_0^{-\frac{1}{p-N}} + \frac{\tilde{a}_2}{\tilde{a}_1}\Big) e^{-\frac{\tilde{a}_1}{p-N}T_{max}} - \frac{\tilde{a}_2}{\tilde{a}_1} \le 0$$

and hence

$$T_{max} \ge \frac{p-N}{\tilde{a}_1}\ln\Big[1 + \frac{\tilde{a}_1}{\tilde{a}_2}\Psi_0^{-\frac{1}{p-N}}\Big] =: T$$

where T is a lower bound of the blow-up time T_{max} so that, we find a safe interval $[0, T)$ of existence of $u(x,t)$. □

6. Global existence and boundedness

In order to prove global existence and boundedness we derive the estimate for $\int_\Omega u^p dx$ $(p > 1)$. From the first equation of (1.5) and integration by parts we have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx &= -\int_\Omega \nabla u^{p-1} \cdot \frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}} dx + \chi\int_\Omega \nabla u^{p-1}\cdot u\nabla v dx \\ &\quad + \lambda\int_\Omega u^p dx - \mu\int_\Omega u^{p+k-1} dx \\ &= -(p-1)\int_\Omega \frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2+|\nabla u|^2}} dx + \chi\frac{p-1}{p}\int_\Omega \nabla u^p\cdot\nabla v dx \\ &\quad + \lambda\int_\Omega u^p dx - \mu\int_\Omega u^{p+k-1} dx.\end{aligned}$$

Neglecting the negative term $-(p-1)\int_\Omega \frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2+|\nabla u|^2}} dx$, integrating by parts once more and using the second equation of (1.5)

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx \le -\chi\frac{p-1}{p}\int_\Omega u^p \Delta v dx + \lambda\int_\Omega u^p dx - \mu\int_\Omega u^{p+k-1} dx$$

$$= -\chi\frac{p-1}{p}\int_\Omega u^p(m(t)-u)dx + \lambda\int_\Omega u^p dx - \mu\int_\Omega u^{p+k-1}dx$$
$$\le \chi\frac{p-1}{p}\int_\Omega u^{p+1}dx + \lambda\int_\Omega u^p dx - \mu\int_\Omega u^{p+k-1}dx,$$

where in the last step we neglected the negative term $-\chi\frac{p-1}{p}\int_\Omega u^p m(t)dx$.

The proof of Theorem 1.4 is divided into two cases: $k > 2$ and $k = 2$.

6.1. Proof of Theorem 1.4. Case (1) $k > 2$

Let $k > 2$. Since $\frac{p-1}{p} \le 1$ and $p < p+1 < p+k-1$, we see from the Young inequality ($\chi u^{p+1} \le c(\epsilon_1)u^p + \epsilon_1 u^{p+k-1}$, $\epsilon_1 > 0$) that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx \le \chi\int_\Omega u^{p+1}dx + \lambda\int_\Omega u^p dx - \mu\int_\Omega u^{p+k-1}dx$$
$$\le (\lambda + c(\epsilon_1))\int_\Omega u^p dx - (\mu-\epsilon_1)\int_\Omega u^{p+k-1}dx,$$

where $c(\epsilon_1) > 0$ is a constant independent of p. Taking $\epsilon_1 = \frac{\mu}{2}$, we obtain

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx \le c_1\int_\Omega u^p dx - \frac{\mu}{2}\int_\Omega u^{p+k-1}dx$$

for some constant $c_1 > 0$. From the Hölder inequality we infer that

$$\int_\Omega u^p dx \le \left(\int_\Omega u^{p+k-1}dx\right)^{\frac{p}{p+k-1}} |\Omega|^{\frac{k-1}{p+k-1}},$$

which implies

$$\int_\Omega u^{p+k-1}dx \ge \left(\int_\Omega u^p dx\right)^{\frac{p+k-1}{p}} |\Omega|^{-\frac{k-1}{p}}.$$

Hence we have

$$\frac{d}{dt}\int_\Omega u^p dx \le c_1 p\int_\Omega u^p dx - c_2 p|\Omega|^{-\frac{k-1}{p}}\left(\int_\Omega u^p dx\right)^{\frac{p+k-1}{p}}$$
$$= c_2 p\left(\int_\Omega u^p dx\right)\left[\frac{c_1}{c_2} - \left(\frac{1}{|\Omega|}\int_\Omega u^p dx\right)^{\frac{k-1}{p}}\right]$$

for some constant $c_2 > 0$. Thus comparison arguments yield

$$\int_\Omega u^p dx \le \max\left\{\int_\Omega u_0^p dx,\ |\Omega|\left(\frac{c_1}{c_2}\right)^{\frac{p}{k-1}}\right\}.$$

This implies that

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le \max\left\{\|u_0\|_{L^p(\Omega)},\ |\Omega|^{\frac{1}{p}}\left(\frac{c_1}{c_2}\right)^{\frac{1}{k-1}}\right\}.$$

Letting $p \to \infty$, we can conclude that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \max\left\{\|u_0\|_{L^\infty(\Omega)},\ \left(\frac{c_1}{c_2}\right)^{\frac{1}{k-1}}\right\},$$

which means that $\limsup_{t\nearrow T_{max}}\|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$. Therefore applying Lemma 2.1 leads to the conclusion. □

6.2. Proof of Theorem 1.4. Case (2) $k = 2$

Let $k = 2$. Since $p+k-1 = p+1$, we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx \le \chi\frac{p-1}{p}\int_\Omega u^{p+1}dx + \lambda\int_\Omega u^p dx - \mu\int_\Omega u^{p+k-1}dx$$
$$= \lambda\int_\Omega u^p dx - \left(\mu - \frac{p-1}{p}\chi\right)\int_\Omega u^{p+1}dx.$$

If we assume that $\mu > \left(1 - \frac{1}{N}\right)\chi = \frac{N-1}{N}\chi$, we can take p close to N such that $p > N$ and $\mu > \frac{p-1}{p}\chi$. With this fixed $p > N$, we obtain

$$\frac{d}{dt}\int_\Omega u^p dx \le c_3 \int_\Omega u^p dx - c_4 \int_\Omega u^{p+1} dx$$

for some constants $c_3, c_4 > 0$. From the Hölder inequality we infer that

$$\int_\Omega u^p dx \le \left(\int_\Omega u^{p+1} dx\right)^{\frac{p}{p+1}} |\Omega|^{\frac{1}{p+1}} \quad \text{i.e.} \quad \int_\Omega u^{p+1} dx \ge \left(\int_\Omega u^p dx\right)^{\frac{p+1}{p}} |\Omega|^{-\frac{1}{p}}.$$

Hence it follows that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p dx &\le c_3 \int_\Omega u^p dx - c_5 \left(\int_\Omega u^p dx\right)^{\frac{p+1}{p}} \\ &= c_5 \left(\int_\Omega u^p dx\right)\left[\frac{c_3}{c_5} - \left(\int_\Omega u^p dx\right)^{\frac{1}{p}}\right]\end{aligned}$$

for some constant $c_5 > 0$. Thus comparison arguments imply

$$\int_\Omega u^p dx \le \max\left\{\int_\Omega u_0^p dx,\ \left(\frac{c_3}{c_5}\right)^p\right\},$$

which means that

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le \max\left\{\|u_0\|_{L^p(\Omega)},\ \frac{c_1}{c_2}\right\}$$

with $p > N$. Taking account of Lemma 4.1, we see that $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded. This proves that $\limsup_{t \nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$. Thanks to Lemma 2.1, we arrive at the conclusion. □

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No data was used for the research described in the article.

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