

Quantum Mechanics of the Extended Snyder Model

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Abstract: We investigate a quantum mechanical harmonic oscillator based on the extended Snyder model. This realization of the Snyder model is constructed as a quantum phase space generated by D spatial coordinates and $D(D - 1)/2$ tensorial degrees of freedom, together with their conjugated momenta. The coordinates obey nontrivial commutation relations and generate a noncommutative geometry, which admits nicer properties than the usual realization of the model, in particular giving rise to an associative star product. The spectrum of the harmonic oscillator is studied through the introduction of creation and annihilation operators. Some physical consequences of the introduction of the additional degrees of freedom are discussed.

Keywords: noncommutative geometry, extended Snyder model, harmonic oscillator

1. Introduction

The Snyder model [1] is known to be the first published proposal of quantized spacetime. It is based on an algebra generated by spacetime coordinates and Lorentz generators, that allows nontrivial commutation relations between position operators without breaking the Lorentz invariance.

Although in its time it did not attract much attention (see however [2–10] for early investigations), its relevance increased when new models, as Moyal space or κ -Poincaré algebra [11,12], and methods related to noncommutative geometry [13] were introduced. In particular, its formulation in terms of Hopf algebras was investigated in [14,15]. In that paper the coproduct and the star product were calculated for the algebra generated by the noncommuting position operators, while the generators of the Lorentz algebra were considered in the standard way as functions of the position and momenta operators.

However, in Snyder algebra the commutation relations of the position coordinates do not close, since they give rise to Lorentz generators, and therefore the structure obtained in [14,15] is not strictly a Hopf algebra, since it is nonassociative. A way to obtain an associative Hopf algebra was proposed in [16], where the Lorentz generators were considered as independent generators of the Snyder algebra, associated with antisymmetric tensorial operators.

This idea was then developed in a series of papers [17–19] using methods of realization of quantum phase spaces in terms of Heisenberg algebra [14–20]; the algebra that included the tensorial generators was named extended Snyder algebra, to distinguish it from the standard realization of the Snyder model in terms of vectorial degrees of freedom only (called standard Snyder model in the following). There were also generalizations including κ -Poincaré deformations [18,19].

Although the extended Snyder framework solves the mathematical problem related to the definition of a proper Hopf algebra, the physical interpretation of the antisymmetric degrees of freedom is not obvious.

In this paper, we shall attempt to investigate a quantum mechanical problem based on the Euclidean version of the model and inspired by an analogous one introduced in



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the context of Moyal space, where the objects of noncommutativity were promoted to antisymmetric operators [21]. In particular, we study the harmonic oscillator in this theory, with the aim of understanding in a simple case the physical implications of the addition of the tensorial degrees of freedom, comparing the results with those obtained in [22–24] for the standard Snyder model.

Of course, our investigation is only preliminary and more complicated situations should be addressed in order to understand if the extended Snyder model has a counterpart in physical reality or can be simply catalogued as a mathematical curiosity. In particular, the next step should be the investigation of interacting oscillators, which already presents nontrivial problems.

2. The Snyder Model

We recall that the original D -dimensional Euclidean Snyder model is defined by the commutation relations

$$\begin{aligned} [\hat{x}_i, \hat{x}_j] &= i\beta^2 M_{ij}, & [M_{ij}, \hat{x}_k] &= i(\delta_{ik}\hat{x}_j - \delta_{jk}\hat{x}_i), \\ [M_{ij}, M_{kl}] &= i(\delta_{ik}M_{jl} - \delta_{il}M_{jk} - \delta_{jk}M_{il} + \delta_{jl}M_{ik}), \end{aligned} \tag{1}$$

and β is a real constant that can be identified with the noncommutative Snyder parameter, which is usually assumed to be of the scale of the Planck length L_p . Of course, in the Euclidean case the Lorentz algebra is replaced by the algebra of rotations in D dimensions. For $\beta = 0$, the commutation relations (1) reduce to those of the standard rotation algebra acting on commutative coordinates. One can then extend the algebra to phase space adding momenta \hat{p}_i conjugated to \hat{x}_i , that, in order to obey the Jacobi identities, must satisfy

$$[\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}_i, \hat{p}_j] = i(\delta_{ij} + \beta^2 \hat{p}_i \hat{p}_j), \quad [M_{ij}, \hat{p}_k] = i(\delta_{ik}\hat{p}_j - \delta_{jk}\hat{p}_i). \tag{2}$$

The standard realization of the Snyder model is obtained by considering the Lorentz generators as dependent variables, functions of the \hat{x}_i and \hat{p}_i , and looking for suitable representations of \hat{x}_i and \hat{p}_i on a canonical quantum phase space of coordinates x_i, p_i with $M_{ij} = x_i p_j - x_j p_i$, so that the algebra (1) and (2) is satisfied. In this way, it is possible to construct a coalgebra structure [14,15] and to define a star product. However, since the algebra generated by the \hat{x}_i does not close, the resulting coproduct is not coassociative and the star product is not associative [14,15].

To avoid this problem, one can define an extended Snyder algebra, by promoting the M_{ij} to $D(D - 1)/2$ noncommutative tensorial degrees of freedom $\hat{x}_{ij} = -\hat{x}_{ji}$ to be added to the D position operators \hat{x}_i as independent generators of the algebra [16]. The total number of independent degrees of freedom of the extended algebra is therefore $D(D + 1)/2$, and the D -dimensional Euclidean extended Snyder algebra takes the form

$$\begin{aligned} [\hat{x}_i, \hat{x}_j] &= i\lambda\beta^2 \hat{x}_{ij}, & [\hat{x}_{ij}, \hat{x}_k] &= i\lambda(\delta_{ik}\hat{x}_j - \delta_{jk}\hat{x}_i), \\ [\hat{x}_{ij}, \hat{x}_{kl}] &= i\lambda(\delta_{ik}\hat{x}_{jl} - \delta_{il}\hat{x}_{jk} - \delta_{jk}\hat{x}_{il} + \delta_{jl}\hat{x}_{ik}), \end{aligned} \tag{3}$$

where we have introduced a deformation parameter λ , which in natural units is dimensionless. Note that we assume that the coordinates \hat{x}_i have dimension of length, while the tensorial coordinates \hat{x}_{ij} are dimensionless, like the M_{ij} , although in this formalism the \hat{x}_{ij} are no longer identified with the rotation generators.

Again, one may extend the algebra to phase space, by introducing the momenta \hat{p}_i and $\hat{p}_{ij} = -\hat{p}_{ji}$, conjugated to the \hat{x}_i and \hat{x}_{ij} , respectively. This can be done in several inequivalent ways compatible with the Jacobi identities, that correspond to different realizations of the model [17]. At order λ there exists a one-parameter family of realizations [17]. The realization (5) corresponds to the case $c_1 = 0$ of ref. [17]. For the moment, we consider the so-called Weyl realization, for which at leading order in λ ,

$$\begin{aligned}
 [\hat{p}_i, \hat{p}_j] &= [\hat{p}_{ij}, \hat{p}_{kl}] = [\hat{p}_i, \hat{p}_{jk}] = 0, & [\hat{x}_i, \hat{p}_{jk}] &= i\frac{\lambda\beta^2}{2}(\delta_{ik}\hat{p}_j - \delta_{jk}\hat{p}_i), & [\hat{x}_i, \hat{p}_j] &= i\left(\delta_{ij} + \frac{\lambda}{2}\hat{p}_{ij}\right), \\
 [\hat{x}_{ij}, \hat{p}_k] &= i\frac{\lambda}{2}(\delta_{ik}\hat{p}_j - \delta_{jk}\hat{p}_i), & [\hat{x}_{ij}, \hat{p}_{kl}] &= i\left(\delta_{ik}\delta_{jl} + \frac{\lambda}{2}(\delta_{ik}\hat{p}_{jl} - \delta_{il}\hat{p}_{jk}) - (k \leftrightarrow l)\right).
 \end{aligned}
 \tag{4}$$

However, we remark that a realization closer to (2) is given by what we may call classical realization, defined so that the commutation relations (2) hold at order λ^2 (in particular, $[\hat{x}_i, \hat{p}_j] = i(\delta_{ij} + \lambda^2\beta^2\hat{p}_i\hat{p}_j)$). In this case, the full set of commutation relations at order λ is given by [17]

$$\begin{aligned}
 [\hat{p}_i, \hat{p}_j] &= [\hat{p}_{ij}, \hat{p}_{kl}] = [\hat{p}_i, \hat{p}_{jk}] = 0, & [\hat{x}_i, \hat{p}_{jk}] &= i\frac{\lambda\beta^2}{2}(\delta_{ik}\hat{p}_j - \delta_{jk}\hat{p}_i), & [\hat{x}_i, \hat{p}_j] &= i\delta_{ij}, \\
 [\hat{x}_{ij}, \hat{p}_k] &= i\lambda(\delta_{ik}\hat{p}_j - \delta_{jk}\hat{p}_i), & [\hat{x}_{ij}, \hat{p}_{kl}] &= i\left(\delta_{ik}\delta_{jl} + \frac{\lambda}{2}(\delta_{ik}\hat{p}_{jl} - \delta_{il}\hat{p}_{jk}) - (k \leftrightarrow l)\right).
 \end{aligned}
 \tag{5}$$

The extended Snyder algebra admits realizations in terms of an extended Heisenberg algebra [17], obtained by adding tensorial degrees of freedom $x_{ij} = -x_{ji}$ to the standard Heisenberg algebra, as

$$\begin{aligned}
 [x_i, x_j] &= [p_i, p_j] = [x_{ij}, x_{kl}] = [p_{ij}, p_{kl}] = 0, \\
 [x_i, p_j] &= i\delta_{ij}, & [x_{ij}, p_{kl}] &= i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \\
 [x_i, x_{jk}] &= [x_i, p_{jk}] = [x_{ij}, x_k] = [x_{ij}, p_k] = 0,
 \end{aligned}
 \tag{6}$$

where p_i and p_{ij} are momenta canonically conjugate to x_i and x_{ij} respectively, and $p_{ij} = -p_{ji}$.

A simplification of the formalism can be obtained noticing that for $\beta \neq 0$ the algebra (3) is isomorphic to $so(D + 1)$, so that it is convenient to define new variables [17]

$$\hat{x}_i = \beta\hat{x}_{i,D+1}, \quad \hat{p}_i = \frac{\hat{p}_{i,D+1}}{\beta},
 \tag{7}$$

such that the algebra (3) takes the form

$$[\hat{x}_{\mu\nu}, \hat{x}_{\rho\sigma}] = i\lambda(\delta_{\mu\rho}\hat{x}_{\nu\sigma} - \delta_{\mu\sigma}\hat{x}_{\nu\rho} - \delta_{\nu\rho}\hat{x}_{\mu\sigma} - \delta_{\nu\sigma}\hat{x}_{\mu\rho}),
 \tag{8}$$

with Greek indices running from 1 to $N + 1$.

The same can be done for the extended Heisenberg algebra (6), which becomes

$$[x_{\mu\nu}, x_{\rho\sigma}] = [p_{\mu\nu}, p_{\rho\sigma}] = 0, \quad [x_{\mu\nu}, p_{\rho\sigma}] = i(\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}),
 \tag{9}$$

The algebra (3) can then be realized in terms of the extended Heisenberg algebra as a power series: in the Weyl realization one has at first order in λ ,

$$\hat{x}_{\mu\nu} = x_{\mu\nu} + \frac{\lambda}{2}(x_{\mu\alpha}p_{\nu\alpha} - x_{\nu\alpha}p_{\mu\alpha}),
 \tag{10}$$

while $\hat{p}_{\mu\nu} = p_{\mu\nu}$. In terms of components

$$\begin{aligned}
 \hat{x}_i &= x_i + \frac{\lambda}{2}(x_k p_{ik} - \beta^2 x_{ik} p_k) \\
 \hat{x}_{ij} &= x_{ij} + \frac{\lambda}{2}(x_i p_j + x_{ik} p_{jk} - (i \leftrightarrow j)).
 \end{aligned}
 \tag{11}$$

It may be interesting to consider the symmetries of the extended Snyder algebra. The algebra (3) and (4) is covariant under the action of the group $SO\left(\frac{(D-1)D(D+1)(D+2)}{8}\right)$ generated by $L_{\mu\nu,\rho\sigma} = x_{\mu\nu}p_{\rho\sigma} - x_{\rho\sigma}p_{\mu\nu}$. However, from a physical standpoint it is more interesting to consider its subgroup corresponding to rotations of the D -dimensional space, with generators

$$M_{ij} = x_i p_j - x_j p_i + x_{ik} p_{jk} - x_{jk} p_{ik}.
 \tag{12}$$

acting as

$$[M_{ij}, \hat{x}_k] = i(\delta_{ik}\hat{x}_j - \delta_{jk}\hat{x}_i), \quad [M_{ij}, \hat{x}_{jk}] = i(\delta_{ik}\hat{x}_{jl} - \delta_{il}\hat{x}_{jk} - \delta_{jk}\hat{x}_{il} + \delta_{jl}\hat{x}_{ik}). \tag{13}$$

3. The Harmonic Oscillator

To test the dynamics of the model defined by the algebra (4), we consider an isotropic harmonic oscillator. We start by defining an Hamiltonian invariant under the extended Snyder symmetry, namely

$$H = \frac{1}{4} \sum_{\mu\nu} \left(\frac{\hat{p}_{\mu\nu}^2}{M} + M\omega^2 \hat{x}_{\mu\nu}^2 \right). \tag{14}$$

where M has dimension of length. Substituting the realization (10) we obtain at leading order in λ ,

$$H = \frac{1}{4} \sum_{\mu\nu} \left[\frac{p_{\mu\nu}^2}{M} + M\omega^2 \left(x_{\mu\nu}^2 + \frac{\lambda^2}{2} x_{\mu\rho} p_{\nu\rho} (x_{\mu\sigma} p_{\nu\sigma} - x_{\nu\sigma} p_{\mu\sigma}) \right) \right], \tag{15}$$

Notice that terms of order λ in the Hamiltonian vanish. The factor $\frac{1}{4}$ is chosen because antisymmetric degrees of freedom are counted twice.

To discuss the spectrum, it is useful to introduce creation and annihilation operators,

$$a_{\mu\nu} = \sqrt{\frac{M\omega}{2}} \left(x_{\mu\nu} + i \frac{p_{\mu\nu}}{M\omega} \right), \quad a_{\mu\nu}^\dagger = \sqrt{\frac{M\omega}{2}} \left(x_{\mu\nu} - i \frac{p_{\mu\nu}}{M\omega} \right), \tag{16}$$

satisfying $a_{\mu\nu} = -a_{\nu\mu}$, $a_{\mu\nu}^\dagger = -a_{\nu\mu}^\dagger$, and

$$[a_{\mu\nu}, a_{\rho\sigma}^\dagger] = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\nu\rho}\delta_{\mu\sigma}. \tag{17}$$

It is then convenient to split the Hamiltonian in a free part H_0 and an interaction term V , with $H = H_0 + \lambda^2 V$, such that

$$H_0 = \frac{1}{4} \sum_{\mu\nu} \left(\frac{p_{\mu\nu}^2}{M} + M\omega^2 x_{\mu\nu}^2 \right), \quad V = \frac{M\omega^2}{8} \left(x_{\mu\rho} p_{\nu\rho} (x_{\mu\sigma} p_{\nu\sigma} - x_{\nu\sigma} p_{\mu\sigma}) \right). \tag{18}$$

One has then

$$H_0 = \frac{\omega}{4} \sum_{\mu\nu} (a_{\mu\nu} a_{\mu\nu}^\dagger + a_{\mu\nu}^\dagger a_{\mu\nu}) = \frac{\omega}{2} \left(\sum_{\mu \neq \nu} N_{\mu\nu} + \frac{D(D+1)}{2} \right), \tag{19}$$

with $N_{\mu\nu} = a_{\mu\nu}^\dagger a_{\mu\nu}$, so that $N_{\mu\nu} = N_{\nu\mu}$, $N_{\mu\mu} = 0$, and

$$V = -\frac{M\omega^2}{16} \sum_{\mu\nu} \sum_{\rho\sigma} (a_{\mu\rho} a_{\nu\rho}^\dagger - a_{\mu\rho}^\dagger a_{\nu\rho}) (a_{\mu\sigma}^\dagger a_{\nu\sigma} - a_{\nu\sigma}^\dagger a_{\mu\sigma}), \tag{20}$$

In (20) we have retained only the terms that can contribute to the leading-order corrections to the energy. After some manipulations, the interaction term reduces to

$$V = \frac{M\omega^2}{8} \left(\sum_{\mu \neq \nu} \sum_{\rho} N_{\mu\rho} N_{\nu\rho} + (D-1) \sum_{\mu \neq \nu} N_{\mu\nu} \right). \tag{21}$$

The free part (19) of the Hamiltonian describes an harmonic oscillator in canonical extended spacetime, which we will call a canonical extended oscillator. Its energy spectrum is, as could be expected, that of a harmonic oscillator with $D(D+1)/2$ degrees of freedom.

In fact, defining the occupation numbers $n_{\mu\nu}$ such that $N_{\mu\nu} | \dots, n_{\mu\nu}, \dots \rangle = n_{\mu\nu} | \dots, n_{\mu\nu}, \dots \rangle$, the energy corresponding to a set of occupation numbers $\{n_{\mu\nu}\}$ is at order 0 (i.e., for the canonical extended oscillator)

$$E_{\{n_{\mu\nu}\}} = \frac{\omega}{2} \left(\sum_{\mu \neq \nu} n_{\mu\nu} + \frac{D(D+1)}{2} \right), \tag{22}$$

while the leading-order corrections due to the Snyder structure are given by

$$\Delta E_{\{n_{\mu\nu}\}} = \langle \{n_{\mu\nu}\} | \lambda^2 V | \{n_{\mu\nu}\} \rangle = \frac{\lambda^2 M \omega^2}{8} \left(\sum_{\mu \neq \nu} \sum_{\rho} n_{\mu\rho} n_{\nu\rho} + (D-1) \sum_{\mu \neq \nu} n_{\mu\nu} \right). \tag{23}$$

Hence, while the canonical extended oscillator has a standard spectrum depending only on the quantum number $n = \sum_{\mu < \nu} n_{\mu\nu}$, the Snyder extended oscillator has eigenvalues that depend on combinations of all the quantum numbers $n_{\mu\nu}$. The order of magnitude of the corrections to the energy spectrum is the same as in the standard Snyder oscillator [22–24] for $M = \beta^2 m / \lambda^2$, with m the mass of the vectorial degrees of freedom.

4. Noncovariant Formalism

To better understand the physics, it is however useful to separate vector and tensor degrees of freedom, studying the model from a D -dimensional point of view. Then the Hamiltonian can be written as

$$H = \frac{1}{2} \sum_i \left(\frac{\hat{p}_i^2}{m} + m\omega^2 \hat{x}_i^2 \right) + \frac{1}{4} \sum_{ij} \left(\frac{\hat{p}_{ij}^2}{M} + M\omega^2 \hat{x}_{ij}^2 \right), \tag{24}$$

where we have identified $m = M\beta^{-2}$ with the mass of the vectorial degrees of freedom (i.e. standard position coordinates). Note that in the limit $\beta \rightarrow 0$, m diverges if M is finite. However, as we shall see, the energy spectrum is regular and goes to the canonical one for $\beta \rightarrow 0$.

Then the free Hamiltonian reads

$$H_0 = \frac{1}{2} \sum_i \left(\frac{p_i^2}{m} + m\omega^2 x_i^2 \right) + \frac{1}{4} \sum_{ij} \left(\frac{p_{ij}^2}{M} + M\omega^2 x_{ij}^2 \right), \tag{25}$$

and the interaction term becomes

$$V = \frac{M\omega^2}{8} \sum_{ij} \left(x_i p_j (x_i p_j - x_j p_i) + x_{ik} p_{jk} (x_{il} p_{jl} - x_{jl} p_{il}) + 2x_i p_j (x_{ik} p_{jk} - x_{jk} p_{ik}) + \beta^{-2} x_j p_{ij} x_k p_{ik} + \beta^2 x_{ij} p_j x_{ik} p_k - x_i p_j p_{jk} x_{ik} - p_j x_i x_{ik} p_{jk} \right) \tag{26}$$

Defining, for $\beta \neq 0$,

$$a_i = \sqrt{\frac{M\omega}{2}} \left(\frac{x_{4i}}{\beta} + i \frac{\beta p_{4i}}{M\omega} \right), \quad a_i^\dagger = \sqrt{\frac{M\omega}{2}} \left(\frac{x_{4i}}{\beta} - i \frac{\beta p_{4i}}{M\omega} \right), \tag{27}$$

the free Hamiltonian takes the form

$$H_0 = \omega \left(\sum_i a_i^\dagger a_i + \frac{D}{2} + \frac{1}{2} \sum_{i \neq j} a_{ij}^\dagger a_{ij} + \frac{D(D-1)}{4} \right), \tag{28}$$

with spectrum

$$E_{\{n_i, n_{ij}\}} = \omega \left(\sum_i n_i + \frac{1}{2} \sum_{i \neq j} n_{ij} + \frac{D(D+1)}{4} \right), \tag{29}$$

where n_i, n_{ij} are the occupation numbers for the vector and tensor degrees of freedom, respectively. The leading-order corrections due to the Snyder structure arising from (24) are instead

$$\Delta E_{\{n_i, n_{ij}\}} = \frac{\lambda^2 \beta^2 m \omega^2}{8} \left[\sum_{i \neq j} n_{ik} n_{jk} + \sum_{i \neq j} n_i n_j + 2 \sum_{ik} n_k n_{ik} + (D-1) \left(\sum_{i \neq j} n_{ij} + 2 \sum_i n_i \right) \right]. \tag{30}$$

These results are in accordance with (23) and can be compared with the spectrum of the standard Snyder oscillator [22–24], for which $E_n \sim \omega \left(\sum_i n_i + \frac{D}{2} \right) + o(\beta^2 m \omega)$. It turns out that the vacuum energy is different in the two cases, while the leading order correction, although different, are of the same order of magnitude. Notice also that the higher-order corrections to the vacuum energy (which vanish in our calculations) depend on the specific operator ordering chosen.

However, in this context, it seems more interesting to choose a different Hamiltonian, invariant only under the D -dimensional rotation group. One can still adopt the same expression for the kinetic term, but assume different frequencies ω and Ω for the vector and tensor degrees of freedom in the interaction term, namely

$$H = \sum_i \left(\frac{\hat{p}_i^2}{2m} + \frac{m\omega^2}{2} \hat{x}_i^2 \right) + \sum_{ij} \left(\frac{\hat{p}_{ij}^2}{4M} + \frac{M\Omega^2}{4} \hat{x}_{ij}^2 \right). \tag{31}$$

Using the realization (11) for \hat{x}_i and \hat{x}_{ij} , the spectrum of the free Hamiltonian gives

$$E_{\{n_i, n_{ij}\}} = \omega \left(\sum_i n_i + \frac{D}{2} \right) + \frac{\Omega}{2} \left(\sum_{i \neq j} n_{ij} + \frac{D(D-1)}{2} \right), \tag{32}$$

while

$$V = \frac{M\omega^2}{8} \sum_{ijk} \left(\beta^2 x_{ij} p_j x_{ik} p_k + \beta^{-2} x_j p_{ij} x_k p_{ik} - x_i p_j p_{ik} x_{jk} - p_i x_j x_{ik} p_{jk} \right) + \frac{M\Omega^2}{4} \left(\sum_{ij} x_i p_j (x_i p_j - x_j p_i) + \sum_{ijkh} x_{ik} p_{jk} (x_{ih} p_{jh} - x_{jh} p_{ih}) + 2 \sum_{ijk} x_i p_j (x_{ik} p_{jk} - x_{jk} p_{ik}) \right) \tag{33}$$

and therefore

$$\Delta E_{\{n_i, n_{ij}\}} = \frac{\lambda^2 \beta^2 m}{8} \left[\Omega^2 \left(\sum_{i \neq j} n_{ik} n_{jk} + \sum_{i \neq j} n_i n_j \right) + 2\omega^2 \sum_{i \neq j} n_i n_{ij} + (D-1) (\Omega^2 + \omega^2) \sum_i n_i + ((D-2)\Omega^2 + \omega^2) \sum_{ij} n_{ij} \right]. \tag{34}$$

It is reasonable to assume $\Omega \gg \omega$. In this case, one can make the approximation that the tensorial degrees of freedom are in the ground state, and then

$$E_{\{n_i, 0\}} \sim \omega \sum_i n_i + \left(\frac{D\omega}{2} + \frac{D(D-1)\Omega}{4} \right) + \frac{\lambda^2 \beta^2 m}{8} \left(2(D-1)(\omega^2 + \Omega^2) \sum_i n_i + \Omega^2 \sum_{i \neq j} n_i n_j \right). \tag{35}$$

It is evident that the vacuum energy and the order- β^2 corrections are greatly increased with respect to the standard Snyder oscillator, with order of magnitude depending on the ratio Ω/ω .

As we have mentioned, the energy spectrum depends on the realization [24]. For example, let us consider the classical realization, with commutation relations (3), (5). This can be obtained by setting [17]

$$\begin{aligned} \hat{x}_i &= x_i - \frac{\lambda\beta^2}{2} x_{ik} p_k \\ \hat{x}_{ij} &= x_{ij} + \frac{\lambda}{2} (2x_i p_j + x_{ik} p_{jk} - (i \leftrightarrow j)). \end{aligned} \tag{36}$$

In this case, the zeroth-order energy (32) is of course unchanged, while the correction terms give rise to a different potential, namely

$$\begin{aligned} V &= \frac{M\omega^2}{8} \sum_{ijk} \beta^2 x_{ij} p_j x_{ik} p_k + \frac{M\Omega^2}{8} \left(4 \sum_{ij} x_i p_j (x_i p_j - x_j p_i) \right. \\ &\quad \left. + \sum_{ijkh} x_{ik} p_{jk} (x_{ih} p_{jh} - x_{jh} p_{ih}) + 4 \sum_{ijk} x_i p_j (x_{ik} p_{jk} - x_{jk} p_{ik}) \right). \end{aligned} \tag{37}$$

A calculation analogous to the previous one gives for the leading order corrections to the energy

$$\begin{aligned} \Delta E_{\{n_i, n_{ij}\}} &= \frac{\lambda^2 \beta^2 m}{8} \left[\Omega^2 \left(\sum_{i \neq j} n_{ik} n_{jk} + 4 \sum_{i \neq j} n_i n_j \right) + \omega^2 \left(\sum_{i \neq j} n_i n_{ij} + \frac{D(D-1)}{4} \right) \right. \\ &\quad \left. + (D-1) \left(4\Omega^2 + \frac{\omega^2}{2} \right) \sum_i n_i + \left((D-2)\Omega^2 + \frac{\omega^2}{2} \right) \sum_{ij} n_{ij} \right]. \end{aligned} \tag{38}$$

Hence, although the structure of the corrections is identical to that obtained for the Weyl realization, the numerical coefficients are rather different. This is a typical feature of noncommutative models, where, for a given Hamiltonian, different realizations lead to nonequivalent physical models [22–25].

5. Conclusions

The extended Snyder model includes tensorial degrees of freedom in addition to the standard vectorial ones, allowing a more satisfying definition of its associated Hopf algebra. We have considered an harmonic oscillator in the context of this model, and calculated its energy spectrum. The result is that the corrections to the spectrum are of the same order $\beta^2 m \omega$ as in the standard Snyder model [22–24]. However, if one allows for different frequencies to be associated with vectorial and tensorial degrees of freedom, the magnitude of the corrections can increase much, depending on the ratio of the two frequencies.

We have assigned to the tensorial degrees of freedom zero physical dimension in natural units, like angular momentum. However, we should mention the possibility of assigning them a noncanonical dimension of length, so that it coincides with the one of the vectorial degrees of freedom. In this case, one can associate a mass m to the tensorial variables identical to that of the vectors. The conclusions about the harmonic oscillator are unaffected, since its properties do not depend on the mass, but the properties of more complex models could depend on this choice. For example, if the tensorial variables very weakly interact with the vectors, they constitute a huge hidden mass whose interaction with ordinary matter can be hardly detectable, and could allow the construction of models for dark matter.

As discussed in the Appendix A, similar results are obtained in the case of a Moyal model in which the object of noncommutativity is promoted to a dynamical variable, as proposed in [21]. Differences arise only in the details of the leading-order corrections to the energy. This fact suggests us the conjecture that all noncommutative models which

include antisymmetric dynamical variables lead to the same structure when applied to the harmonic oscillator problem.

Another effect that was pointed out in [21] is the fact that the uncertainty relations can be modified. This happens also in our case, but is realization dependent. For example, it is clear from (4) that the uncertainty relations for $\Delta x_i \Delta p_j$ in the Weyl realization depend on the expectation values of the tensorial degrees of freedom, while in the classical realization (5) they coincide with those of the standard Snyder model. Also, modifications to the Casimir force between conducting plates could be evaluated on the lines of the calculations performed in [26] for the standard case.

A more thorough discussion of the physical applications of our formalism would require the generalization of our results to the interaction of several harmonic oscillators. For example, in [27] it was shown that in standard quantum mechanics the coupling of two harmonic oscillators can give rise to interesting effects in the context of phase transitions in the Rabi model. It would be interesting to extend this investigation to Snyder quantum mechanics. However, to our knowledge, very few attempts at the description of interacting particles in noncommutative spaces have ever been made. It is usually assumed that the interaction term has the same form as in standard quantum mechanics [28,29] (See however [30] for a different approach in the context of Snyder model). However, it is not clear to us whether this assumption is consistent with the structure of noncommutative spaces, and in particular with the deformed composition law of momenta. Further investigations are necessary to clarify this topic.

Our investigation can easily be extended to a relativistic setting in analogy with the Moyal case studied in [31]. Perhaps more interesting would be to define a quantum field theory on the extended Snyder background, that could solve some of the problems found in the standard theory, whose formalism has been investigated using various approaches in refs. [14–16,32–35]. Of course, the introduction of the new degrees of freedom would greatly modify the formalism.

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Appendix A

In this appendix we compare our results with those arising from a Moyal oscillator with dynamical noncommutativity [21]. A similar calculation has been performed in [21], but the author employed a different approach, in particular choosing a deformed Hamiltonian, such that the energy spectrum maintains its canonical form.

We do not report here the details of the computation, since they are analogous to those performed in the Snyder case. The commutation relations of the Moyal space are [11]

$$[\hat{x}_i, \hat{x}_j] = i\lambda \theta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}_i, \hat{p}_j] = i\delta_{ij}, \quad (A1)$$

where the object of noncommutativity θ_{ij} is a constant antisymmetric tensor of dimension inverse length square, and λ a dimensionless deformation parameter. In [21] it was proposed to promote θ_{ij} to an independent dynamical variable \hat{x}_{ij} with conjugate momentum \hat{p}_{ij} , in order to maintain rotational covariance.

One has then,

$$\begin{aligned} [\hat{x}_i, \hat{x}_j] &= i\lambda \hat{x}_{ij}, & [\hat{p}_i, \hat{p}_j] &= 0, & [\hat{x}_i, \hat{p}_j] &= i\delta_{ij}, \\ [\hat{x}_{ij}, \hat{p}_{kl}] &= i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), & [\hat{x}_i, \hat{p}_{jk}] &= -i\frac{\lambda}{2}(\delta_{ik}\hat{p}_l - \delta_{il}\hat{p}_k), \\ [\hat{x}_i, \hat{x}_{jk}] &= [\hat{x}_{ij}, \hat{x}_{kl}] = [\hat{p}_i, \hat{x}_{jk}] = [\hat{p}_i, \hat{p}_{jk}] = 0. \end{aligned} \tag{A2}$$

The commutation relations (A2) are similar to those of the extended Snyder model and can be obtained analogously in terms of the extended Heisenberg algebra (6), defining

$$\hat{x}_i = x_i - \frac{\lambda}{2}x_{ij}x_i p_j, \quad \hat{p}_i = p_i, \quad \hat{x}_{ij} = x_{ij}, \quad \hat{p}_{ij} = p_{ij}. \tag{A3}$$

Contrary to ref. [21] we choose the standard Hamiltonian (31) for the extended harmonic oscillator. This will give rise to corrections to the energy spectrum of the canonical extended oscillator.

In fact, substituting (A3) in (31) we obtain an effective Hamiltonian in terms of canonical operators x_i, p_i, x_{ij}, p_{ij} ,

$$H = \sum_i \left[\frac{p_i^2}{2m} + \frac{m\omega^2}{2} \left(x_i^2 - \lambda x_{ij}x_i p_j + \frac{\lambda^2}{4} x_{ij}p_j x_{ik}p_k \right) \right] + \sum_{ij} \left(\frac{p_{ij}^2}{4M} + \frac{M\Omega^2 x_{ij}^2}{4} \right). \tag{A4}$$

As before, this can be separated in a free part H_0 , Equation (25) and an interaction part. The free part has of course the same spectrum as in the extended Snyder model. The leading-order corrections to the energy come instead from the term

$$V = \frac{\lambda^2 m \omega^2}{8} x_{ij} p_j x_{ik} p_k, \tag{A5}$$

which also appears in (37).

We then go through the same steps as in the Snyder case, obtaining

$$\begin{aligned} E_{\{n_i, n_{ij}\}} &= \omega \left(\sum_i n_i + \frac{D}{2} \right) + \frac{\Omega}{2} \left(\sum_{i \neq j} n_{ij} + \frac{D(D-1)}{2} \right) \\ &+ \frac{\lambda^2 m \omega^2}{8} \left(\sum_{i \neq j} n_i n_{ij} + \frac{D-1}{2} \sum_i n_i + \frac{1}{2} \sum_{ij} n_{ij} + \frac{D(D-1)}{4} \right). \end{aligned} \tag{A6}$$

Although the details are of course different, this result is qualitatively similar to the one obtained in the Snyder case. It is likely that analogous results hold for any noncommutative model containing antisymmetric degrees of freedom.

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