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Abstract: The Fiedler vector of a graph is the eigenvector corresponding to the algebraic connectivity, which is the second-smallest eigenvalue (counting multiple eigenvalues separately) of the corresponding Laplacian matrix. We propose a continuous-time distributed control protocol to drive the value of the state variables of a network toward the Fiedler vector, up to a scale factor. Our protocol is unbiased and robust with respect to the initial network state, but the knowledge of the algebraic connectivity is required. By means of the proposed control law, we design a local state feedback that achieves desynchronization on arbitrary undirected connected networks of diffusively coupled harmonic oscillators. We provide numerical simulations to corroborate the theoretical results.

Distributed Fiedler Vector Estimation with Application to Desynchronization of Harmonic Oscillator Networks

Diego Deplano, Mauro Franceschelli, Alessandro Giua, Luca Scardovi

Abstract— **The Fiedler vector of a graph is the eigenvector corresponding to the smallest non-trivial eigenvalue of the corresponding Laplacian matrix, i.e, the algebraic connectivity. We propose and prove the convergence properties of a novel continuous-time distributed control protocol to drive the value of the state variables of a network toward the Fiedler vector, up to a scale factor, assuming known algebraic connectivity. The proposed strategy is unbiased and robust with respect to the initial network state. The proposed strategy does not require initialization of state variables to particular values. By exploiting the proposed control protocol we design a local state feedback that achieves desynchronization on arbitrary undirected connected networks of diffusively coupled harmonic oscillators. We provide numerical simulations to corroborate the theoretical results.**

I. Introduction

The computation of eigenvectors of the graph Laplacian L is a problem of fundamental importance for various applications and it is the cornerstone of spectral graph theory [1]. Among all eigenvectors, the *Fiedler vector* [2] plays a pivotal role: it is the eigenvector corresponding to the second smallest eigenvalue of the Laplacian matrix, also known as the *algebraic connectivity*. To name a few, Fiedler vector is useful in graph partitioning [3], [4], [5] and in the control of algebraic connectivity [6], [7], [8].

Power Iteration (PI) [9] is an established iterative method to compute the leading eigenvalue(s) and associated eigenvector(s) of a matrix. In [6], [8], [10], [11], [12] the Fiedler vector is computed by means of methods based on a distributed implementation of PI. Main drawbacks of [6], which exploits the algorithm proposed in [13], are the centralized initialization step and the high number and size of the messages the nodes need to exchange. In [8] and [11] the decentralization is carried on at each agent by two consensus estimators, which are required to run "fast enough" in order to expect the resulting dynamics to converge: a formal proof is not provided. Similar approaches are used to compute eigenvalues and the algebraic connectivity [14]. Another class of algorithms forces the nodes to oscillate at eigenfrequencies and deduce spectral information through Fast Forurier Transform (FFT). In [15] the Fiedler vector is computed by running at every node the wave equation and computing the eigenvector components through an FFT. This algorithm is proved to be orders of magnitude faster than PI-based algorithms. An FFT approach for distributedly computing the eigenvalues is given in [16].

On one hand, the main limitation of PI-based approaches consists on the distributed normalization of the vectors at each step, which severely affects their convergence speed and requires a centralized initialization step. On the other hand, FFT-based approaches suffer from a rather poor accuracy and robustness issues.

The **first main contribution** of this paper is to propose and prove the convergence properties of a novel continuous-time distributed control protocol to drive a MAS toward the Fiedler vector of its graph Laplacian L. The proposed protocol relies neither on a distributed PI nor a FFT approach, thus guaranteeing robustness to initial conditions, high convergence speed and accuracy. However, it requires the knowledge of the algebraic connectivity, which is a reasonable assumption for static networks (as in the case of our main application) since various distributed algorithms have been proposed to distributedly estimate all the eigenvalues of undirected graph Laplacian [17], [16], [18], [19].

The **second main contribution** is to exploit the zero mean property of the Fiedler vector, to employ the proposed protocol as a local feedback law to desynchronize a network of coupled harmonic oscillators by driving it toward a state proportional to the Fiedler vector. While synchronization has been formally and easily defined [20], [21], [22] as the condition maximizing the order-parameter (magnitude of the centroid of the oscillators), the opposite definition of *desynchronization* is more ambiguous [23], [24], [25], [26]. In this work, we define desynchronization as the condition zeroing the order-parameter, which is dual to the classical definition of synchronization given in [20].

The paper is structured as follows. After introducing notation and preliminaries in Section II, a novel local protocol in continuous-time is proposed and employed in Section III to distributedly estimate the Fiedler vector in single-integrator MASs and in Section IV to achieve desynchronization in networks of diffusively coupled harmonic oscillators. In Section V numerical simulations corroborating the theoretical results are provided. Concluding remarks are given in Section VI.

II. Preliminaries

We adopt the following notation. The sets $\mathbb{R}$ and $\mathbb{R}_+$ denote, respectively, the reals and the nonnegative reals. The

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set $\mathbb{S}^1$ denotes the *unit circle*, thus a point $\theta \in \mathbb{S}^1$ is an angle. Given n vectors $x_1, x_2, \dots, x_n \in \mathbb{R}^m$ we indicate with $x = [x_1^T, x_2^T, \dots, x_n^T]^T \in \mathbb{R}^{nm}$ the stacking of the vectors. We denote with I_n and $\mathbf{1}_n$ the identity matrix and a vector of ones of dimension n; subscripts are omitted if the dimension is clear from the context. With $A \otimes B$ we denote the Kronecker product of two matrices A and B of opportune dimensions.

A. Multi Agent Systems

Given a Multi-Agent System (MAS), the pattern of interactions among the agents is encoded by an undirected *graph*. A graph $\mathcal{G} = (V, E)$ consists of a set of nodes $V = \{1, \dots, n\}$ representing the agents, and a set of edges $E \subseteq \{V \times V\}$. An edge $(i, j) \in E$, with $i \neq j$, means that i and j can communicate. To each agent i is associated a set of neighbors $\mathcal{N}_i = \{j \in V : (i, j) \in E\}$, representing the set of agents communicating with agent i. Since the graph is undirected, $(i, j) \in E$ if and only if $(j, i) \in E$. An undirected graph is said to be *connected* if between any pair of nodes $i, j \in V$ there exists a *path*, i.e., a finite sequence of adjacent edges that connects node i to node j. The adjacency matrix $A = \{a_{ij}\}$ of a graph $\mathcal{G}$ is an $n \times n$ matrix with coefficients

$$a_{ij} := \begin{cases} 1 & \text{for } (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

The degree of each node is defined as $d_i = \sum_{i=1}^{n} a_{ij}$. The degree matrix is defined as $D = \mathrm{diag}(d)$, and the Laplacian matrix of the graph is defined as $L = D - A$. Let $\lambda_{L,i}$ and $v_{L,i}$ be the eigenvalues and corresponding eigenvectors of the Laplacian matrix L. The eigenvalues $\lambda_{L,i}$ are real and satisfy $0 = \lambda_{L,1} \leq \lambda_{L,2} \leq \cdots \leq \lambda_{L,n}$ if and only if the graph is connected. The *algebraic connectivity* is denoted as $\lambda_{L,2}$ and its associated eigenvector is known as the *Fiedler vector* $v_{L,2}$ and it satisfies $\mathbf{1}^T v_{L,2} = 0$.

III. Fiedler vector Estimation

Consider a network of n single integrator agents whose topology is represented by an undirected graph $\mathcal{G} = (V, E)$. Each agent is modeled as an autonomous continuous-time system with scalar state $y_i \in \mathbb{R}$ evolving according to

$$\dot{y}_i = u_i, \qquad \forall i \in V. \tag{1}$$

The goal of this section is to design the local control law $u_i \in \mathbb{R}$ such that each agent i estimates the i-th Fiedler vector component of the Laplacian matrix L associated to the in their pattern of interaction given by graph $\mathcal{G}$.

A. Centralized solution

Given a square matrix $M \in \mathbb{R}^{n \times n}$ with eigenvalue spectrum satisfying $0 = \lambda_{M,1} < \lambda_{M,2} \leq \lambda_{M,3} \leq \cdots \leq \lambda_{M,n}$ a system according to $\dot{y} = -My$ is marginally stable and it converges to the eigenvector v_{M_1} associated to the zero eigenvalue $\lambda_{M,1}$. It is straightforward to notice that if one were able to design M such that $v_{M,1} = v_{L,2}$, i.e., the eigenvector associated to the zero eigenvalue of M is exactly the Fiedler vector of the graph Laplacian L, the problem would be solved. Such a matrix can be designed as follows

$$M = L + \alpha \mathbf{1}\mathbf{1}^T - \lambda_{L,2} I, \tag{2}$$

whose construction consists of two conceptual steps.

1) *Matrix inflation*: add a term $\alpha \mathbf{1}\mathbf{1}^T$ to the Laplacian L with $\alpha > \lambda_{L,2}/n$. The smallest eigenvalue of the resulting matrix is $\lambda_{L,2}$ while the eigenvectors are not changed;

2) *Eigenvalues shifting*: subtract matrix $\lambda_{L,2} I$ to the resulting matrix to ensure that there is a single null eigenvalue associated to the Fiedler vector $v_{L,2}$.

The design of the control law u_i such that the closed-loop matrix satisfy the previous reasoning, thus ensuring the MAS to converge to a scaled Fiedler vector, is given in next theorem.

Theorem 1: Consider a MAS with agents dynamics (1) driven by the control law

$$u_i = \sum_{j \in \mathcal{N}_i} a_{ij} \left(y_j - y_i \right) - \alpha \mathbf{1}^T y + \lambda_{L,2} y_i. \tag{3}$$

If G is connected and $\alpha > \lambda_{L,2}/n$ then the MAS converges to a scaled Fiedler vector of graph G.

Proof: Noticing that the closed-loop matrix given by the local control law (3) is $-M$ with matrix M given in (2), the remaining of proof is straightforward and omitted due to space constraints. ■

B. Distributed solution

The control law (3) given in Theorem 1 is not distributed as it relies on the global information $\mathbf{1}^T y$, which represents the actual average of the states. As previously proposed in earlier work [27], [28], [29], this problem can be overcome by employing a distributed estimator of this quantity. In particular, we consider an *integral dynamic consensus algorithm* designed as in (5), which differs from those presented in [27] and tutorial [29].

Theorem 2: Consider a MAS with agents dynamics (1) driven by the control law

$$u_i = \sum_{j \in \mathcal{N}_i} a_{ij} \left(y_j - y_i \right) - \alpha v_i + \lambda_{L,2} y_i. \tag{4}$$

where v_i is a dynamic average estimation given by

$$\begin{aligned} \dot{v}_i &= \dot{y}_i + \beta(y_i - v_i) + K_I \sum_{j \in \mathcal{N}_i} a_{ij} \left(z_j - z_i \right), \qquad (5) \\ \dot{z}_i &= K_I v_i. \end{aligned}$$

If G is connected and $\alpha > \lambda_{L,2}$, $\beta > 0$, $K_I > 0$, then the state $y(t)$ of the MAS converges to a scaled Fiedler vector of graph G for almost any initial condition.

Proof: Let $w = \begin{bmatrix} y & v & z \end{bmatrix}^T \in \mathbb{R}^{3n}$ be the state of the MAS, which under the action of the feedback (4)-(5) can be written in compact form as

$$\dot{w} = \underbrace{\begin{bmatrix} \lambda_{L,2} I - L & -\alpha I & \mathbf{0} \\ (\lambda_{L,2} + \beta) I - L & -(\alpha + \beta) I & -K_I L \\ \mathbf{0} & K_I I & \mathbf{0} \end{bmatrix}}_{M} w.$$

To compute the eigenvalues of matrix M, we proceed by solving $\det\{M - \lambda I_{3n}\} = 0$. Let us partition matrix $M - \lambda I_{3n} = [A\ B; C\ D]$ into blocks, which are given next

$$A = \begin{bmatrix} (\lambda_{L,2} - \lambda)I - L & -\alpha I \\ (\lambda_{L,2} + \beta)I - L & -(\alpha + \beta + \lambda)I \end{bmatrix},$$

$$B = \begin{bmatrix} \mathbf{0} \\ -K_I L \end{bmatrix}, \quad C = \begin{bmatrix} \mathbf{0} & K_I I \end{bmatrix}, \quad D = -\lambda I.$$

By Shur-complement, the determinant of $M - \lambda I_{3n}$ is equal to $\det\{D\} \cdot \det\{A - BD^{-1}C\}$, where $\det\{D\} = \lambda^n$ and $\det\{A - BD^{-1}C\} = \det\{a_1 L^2 + a_2 L + a_3 I\}$, with

$$a_1 = \frac{K_I^2}{\lambda}, \qquad a_2 = K_I^2 - \lambda_{L,2}\frac{K_I^2}{\lambda} + \beta + \lambda,$$
$$a_3 = \lambda\alpha + \lambda\beta + \lambda^2 - \lambda_{L,2}\beta - \lambda_{L,2}\lambda + \alpha\beta.$$

Finally, letting $H = a_1 L + a_2 I$, we can write

$$\det\{M - \lambda I_{3n}\} = \lambda^n \cdot \det\{HL + a_3 I\} = 0. \tag{6}$$

Matrix HL is a product of two commuting matrices, i.e., $HL = LH$, thus any eigenvalue of HL is a product of the eigenvalues of H and L. Furthermore, H and L share the same set of eigenvectors because matrix $H = a_1 L + a_2 I$ where a_1 and a_2 are real scalars. Thus, from (6) one can derive the next relationships

$$\lambda(\lambda_{L,i}\lambda_{H,i} + a_3) = 0, \quad \forall i \in V,$$

where the eigenvalues $\lambda_{L,i}$ are real and satisfy $0 = \lambda_{L,1} \leq \lambda_{L,2} \leq \cdots \leq \lambda_{L,n}$ since the graph is assumed to be connected. The eigenvalues of H are $\lambda_{H,i} = a_1\lambda_{L,i} + a_2$, thus, substituting coefficients a_i yields for any $\forall i \in V$

$$\lambda^3 + b_i\lambda^2 + c_i\lambda + d_i = 0, \quad \begin{aligned} b_i &= \alpha + \beta - \lambda_{L,2} + \lambda_{L,i} \\ c_i &= (\alpha - \lambda_{L,2})\beta + (K_I^2 + \beta)\lambda_{L,i}. \\ d_i &= K_I^2\lambda_{L,i}(\lambda_{L,i} - \lambda_{L,2}) \end{aligned}$$

One can notice that M has three eigenvalues for each $\lambda_{L,i}$ with $i \in V$, which can be computed by the above equation. We proceed by ensuring that all eigenvalues of matrix M have negative real part using the Routh criteria.

For $i = \{1, 2\}$ it holds $d_i = 0$ and, by the Routh criteria specialized for second degree polynomials, all solutions are strictly negative if and only if coefficients b_i, c_i are positive, which is verified if the conditions of the theorem hold.

For $i \in \{3, \ldots, n\}$ it holds $d_i > 0$ and, by the Routh criteria specialized for third degree polynomials, all solutions are strictly negative if and only if in addition it holds $b_i c_i - d_i > 0$, proved next

$$b_i c_i - d_i = \beta(\lambda_{L,2} - \lambda_{L,i})^2 + (2\alpha\beta + \beta^2)(\lambda_{L,i} - \lambda_{L,2}) + \\ + (\alpha\beta + K_I^2\lambda_{L,i})(\alpha + \beta) > 0,$$

since $\lambda_{L,i} \geq \lambda_{L,2}$ for $i \in \{3, \ldots, n\}$.

We conclude that, under the conditions of the theorem, matrix M has two zero eigenvalues because $d_i = 0$ for $i \in \{1, 2\}$ while all other eigenvalues have negative real part. Stability of the system can be ensured by proving that the geometric multiplicity of eigenvalue 0 is equal to its algebraic multiplicity, which is two. We prove this fact by showing that two distinct eigenvectors are associated to the zero eigenvalue. Recalling that $w = \begin{bmatrix} y & v & z \end{bmatrix}^T \in \mathbb{R}^{3n}$ is the state of the overall system, we compute $\dot{w} = Mw = \mathbf{0}$,

$$\begin{cases} (\lambda_{L,2}I - L)y = \mathbf{0} \\ \beta y - K_I L z = \mathbf{0} \\ v = \mathbf{0} \end{cases},$$

There are two feasible choices for y. First choice is $y = \mathbf{0}$, leading to $Lz = \mathbf{0}$, i.e., $z = \delta v_{L,2}$, $\forall \delta \in \mathbb{R}$. Second choice is $y = \delta v_{L,2}$, leading to $z = \frac{\beta\delta}{K_I\lambda_{L,2}} v_{L,2} + \sigma\mathbf{1}$, $\forall \sigma \in \mathbb{R}$. Thus, the zero eigenvalue has the two distinct eigenvectors

$$e_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ v_{L,2} \end{bmatrix}, \quad e_2 = \begin{bmatrix} v_{L,2} \\ \mathbf{0} \\ \frac{\beta}{K_I\lambda_{L,2}} v_{L,2} + \frac{\sigma}{\delta}\mathbf{1} \end{bmatrix}.$$

Since two linearly independent eigenvectors are associated to the null eigenvalue which has algebraic multiplicity equal to two, it follows that the system is marginally stable, and

$$\lim_{t \to \infty} y(t) = \alpha v_{L,2}, \qquad \alpha \in \mathbb{R}.$$

Let $S \subset \mathbb{R}^{3n}$ be the space orthogonal to the Fiedler vector $v_{L,2}$. Coefficient α is null if and only if $w(0) \in S$, which is a set of measure zero, thus completing the proof. ■

IV. Desynchronization in Harmonic Oscillator Networks

In this section we define and study the desynchronization problem in networks of n coupled identical harmonic oscillators with natural frequency $\omega \in \mathbb{R}$. A harmonic oscillator is a second-order linear system modeling both amplitude $M(t)$ and phase $\theta(t)$ of an oscillator as opposed to the popular Kuramoto model, which models only the phase of an oscillator as a first-order non-linear system. The position $p_i(t) = M_i(t)\cos(\omega t + \theta_i(t)) \in \mathbb{R}$ of the i-th oscillators has the following dynamics, see [21], [22]

$$\ddot{p}_i + \omega^2 p_i = u_i^c + u_i^d, \tag{7}$$

where $u_i^c \in \mathbb{R}$ is the local control feedback to be designed, and $u_i^d \in \mathbb{R}$ accounts for the diffusive coupling between the oscillators and it is defined as

$$u_i^d = \sum_{j \in \mathcal{N}_i} a_{ij}\left(\dot{p}_j - \dot{p}_i\right), \tag{8}$$

where $a_{ij} \in \{0, 1\}$ are the entries of the adjacency matrix of the undirected graph $\mathcal{G}$ describing both the coupling network and the communication network: $a_{ij} = 1$ if oscillator i is coupled and can communicate with oscillator j, and $a_{ij} = 0$ otherwise. Introducing the state vector $x_i = [p_i\ \dot{p}_i]^T \in \mathbb{R}^2$, the state-space representation of a network of oscillators (7) can be written as

$$\begin{aligned} \dot{x}_i &= Ax_i + B(u_i^c + u_i^d), \qquad i \in V, \\ y_i &= Cx_i, \end{aligned} \tag{9}$$

where the state matrices are given by

$$A = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \qquad C = B^T = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The steady state output of each oscillator i is given by

$$y_i^{ss}(t) = M_i \cos(\omega t + \theta_i) = \Re\left\{M_i e^{j\theta_i} \cdot e^{j\omega t}\right\},$$

where $M_i \in \mathbb{R}_+$, $\theta_i \in \mathbb{S}^1$ are the steady state magnitude and the phase of oscillator i, j denotes the imaginary unit and $\Re\{\cdot\}$ denotes the real part of a complex number. Thus, the collective steady-state output dynamics

$$\mathbf{1}^T y^{ss}(t) = \sum_{i=1}^{n} y_i^{ss}(t) = \Re\left\{\sum_{i=1}^{n} M_i e^{j\theta_i} \cdot e^{j\omega t}\right\} \tag{10}$$

is encoded in the centroid[1]

$$Re^{j\phi} = \frac{1}{\sum_{i=1}^{n} M_i} \sum_{i=1}^{n} M_i e^{j\theta_i}, \qquad \sum_{i=1}^{n} M_i > 0, \tag{11}$$

where $R \in \mathbb{R}_+$ represents the phase-coherence of the population of oscillators and $\phi \in \mathbb{S}^1$ indicates the average phase. The goal is to prove that the employment of the local protocol proposed in the previous section, allows to achieve desynchronization in a network of diffusively coupled harmonic oscillators in the sense shown next.

Definition 1 (Desynchronization measure): Consider a network of n identical oscillators (9). The network is said to achieve *desynchronization* if

$$R = 0 \Leftrightarrow \mathbf{1}^T y^{ss}(t) = 0, \tag{12}$$

i.e., the collective steady-state output dynamics (10) is non-null with zero mean or, equivalently, the centroid (11) is at the origin of the Complex Plane. ■

In the light of the above definition, consider a simple yet illustrative example of a network of three oscillators. Fig. 1 depicts the following configurations: (a) all phase differences are null, then oscillators are not desynchronized, regardless of their amplitude; (b) amplitudes are equal and the phase differences are $\theta_1 - \theta_2 = \frac{2\pi}{3}$, $\theta_2 - \theta_3 = \frac{2\pi}{3}$, then the oscillators are desynchronized, this configuration is referred in the literature as a *splay state*; (c) if $M_1 = 2M_2 = 2M_3$ and $\theta_2 = \theta_3 = \theta_1 + \pi$, then the oscillators are not in a splay state but they are desynchronized.

[1]If all oscillators have the same fixed amplitude (such is the case for Kuramoto oscillators) the centroid reduces to $Re^{j\phi} = \frac{1}{n}\sum_{i=1}^{n} e^{j\theta_i}$ and R is known as the *order parameter*.

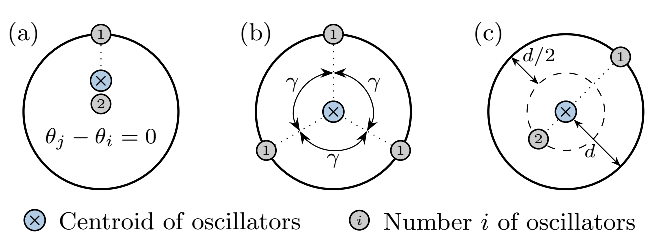

Fig. 1: Steady-state configurations of a network of three oscillators: (a) not desynchronization, (b) desynchronization with same amplitudes, (c) desynchronization with different amplitudes.

A. Main Result

It is known [30], [31] that a network of identical harmonic oscillators (9) under the diffusive coupling (8) achieves synchronization if the interconnection graph is symmetric. The control feedback provided in the next theorem is able to cancel out the synchronization effect of the diffusive coupling while asymptotically steering the network to a non-trivial desynchronized state, according to Definition 1.

Theorem 3 (Desynchronization of harmonic oscillators): Consider a network of n identical harmonic oscillators (9) coupled with the diffusive coupling (8) and driven by the control law

$$u_i = -\beta v_i + \lambda_{L,2} y_i. \tag{13}$$

where v_i is a dynamic average estimation given in (5). If $\mathcal{G}$ is connected and

$$\alpha > \max\left\{\lambda_{L,2}, \frac{\omega^2}{2\lambda_{L,2}}\right\}, \qquad K_I = \sqrt{\frac{2\alpha^2 + \omega^2}{2\lambda_{L,2}}}, \tag{14}$$

then the network achieves desynchronization as in Definition 1 for almost all initial conditions.

Proof: Let $w_i = \begin{bmatrix} x_i & v_i & z_i \end{bmatrix}^T \in \mathbb{R}^4$. The network of coupled harmonic oscillators (9) subject to the diffusive coupling (8) and the feedback control (4) can be written in compact form as

$$\dot{w} = \overbrace{\Big[(I \otimes A^*) - (L \otimes B^*)\Big]}^{M} w \tag{15}$$

$$y = (I \otimes C^*) w, \tag{16}$$

where the operator $\otimes$ denotes the Kronecker product and

$$A^* = \begin{bmatrix} A + \lambda_{L,2} BC & -\alpha B & \mathbf{0} \\ C(A + (\lambda_{L,2} + \alpha) I_2) & -2\alpha & 0 \\ \mathbf{0} & K_I & 0 \end{bmatrix},$$

$$B^* = \begin{bmatrix} BC & \mathbf{0} & \mathbf{0} \\ C & 0 & K_I \\ \mathbf{0} & 0 & 0 \end{bmatrix}, \quad C^* = \begin{bmatrix} C & 0 & 0 \end{bmatrix}.$$

Since L is symmetric, there exists an orthogonal matrix P such that $\Lambda = P^T L P$ is a diagonal matrix. Consider the coordinate change

$$\widetilde{w} = \underbrace{\Big[P \otimes I_4\Big]}_{\widetilde{P}} w \quad \rightarrow \quad \dot{\widetilde{w}} = \underbrace{\Big[\widetilde{P}^T M \widetilde{P}\Big]}_{\widetilde{M}} \widetilde{w}. \tag{17}$$

where, by exploiting the properties of the Kronecker product,

$$\widetilde{M} = \Big[(I \otimes A^*) - (\Lambda \otimes B^*)\Big].$$

Matrices M and $\widetilde{M}$ share the same spectrum. Matrix $\widetilde{M}$ is a block diagonal matrix with blocks $\widetilde{M}_i$ given by

$$\widetilde{M}_i = A^* - \lambda_{L,i} B^* \qquad \forall i \in V,$$

where the eigenvalues $\lambda_{L,i}$ of L are real and satisfy

$0 = \lambda_{L,1} \leq \lambda_{L,2} \leq \cdots \leq \lambda_{L,n}$ since the graph is assumed to be connected. It is known that the eigenvalues of the block diagonal matrix $\widetilde{M}$ are the eigenvalues of the blocks $\widetilde{M}_i$.

By means of the Routh criterion, it can be shown (we omitted the steps due to space constraints) that under condition (14) all the eigenvalues of blocks $\widetilde{M}_i$ are strictly inside the left half of the Gauss plane, except for block $\widetilde{M}_1$, which has a zero eigenvalue, and block $\widetilde{M}_2$, which has a pair of imaginary conjugate eigenvalues. Exploiting the change of variable (17), one can write the state evolution of the system as

$$w(t) = e^{Mt}w(0) = e^{\widetilde{P}\widetilde{M}\widetilde{P}^T t}w(0) = \widetilde{P}e^{\widetilde{M}t}\widetilde{P}^T w(0).$$

and so the output

$$y(t) = (I_n \otimes C^*)w(t) = (I_n \otimes C^*)\widetilde{P}e^{\widetilde{M}t}\widetilde{P}^T w(0).$$

As $t \to \infty$, all blocks $e^{\widetilde{M}_i t}$ for $i = 3, \ldots, n$ tend to zero because of the negative real part of their eigenvalues. Thus, since the columns of P are the eigenvectors of matrix L it follows

$$\begin{aligned}\lim_{t\to\infty} y(t) &= (I_n \otimes C^*)\Big[v_{L,1}v_{L,1}^T \otimes e^{\widetilde{M}_1 t} + v_{L,2}v_{L,2}^T \otimes e^{\widetilde{M}_2 t}\Big]w(0)\\ &= \Big[(v_{L,1}v_{L,1}^T \otimes C^* e^{\widetilde{M}_1 t}) + (v_{L,2}v_{L,2}^T \otimes C^* e^{\widetilde{M}_2 t})\Big]w(0).\end{aligned}$$

Matrix $\widetilde{M}_1$ has only one null eigenvalue (the others have negative real part) with eigenvector $\mathbf{1}_5 \otimes [0\,0\,0\,1]^T$, thus

$$\lim_{t\to\infty} e^{\widetilde{M}_1 t} \propto \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}.$$

Since $\lim_{t\to\infty} C^* e^{\widetilde{M}_1 t} = \mathbf{0}$ it follows

$$\lim_{t\to\infty} y(t) = \Big[v_{L,2}v_{L,2}^T \otimes C^* e^{\widetilde{M}_2 t}\Big]w(0).$$

The collective steady-state output dynamics results in

$$\begin{aligned}\mathbf{1}^T y^{ss}(t) = \lim_{t\to\infty} \mathbf{1}^T y(t) &= \mathbf{1}^T \Big[v_{L,2}v_{L,2}^T \otimes C^* e^{\widetilde{M}_2 t}\Big]w(0)\\ &= \Big[\mathbf{0} \otimes C^* e^{\widetilde{M}_2 t}\Big]w(0) = \mathbf{0}\end{aligned}$$

We proved that the network reaches a steady-state with a zero mean state output dynamics, in which each oscillator has a non-trivial oscillatory behaviour due to the pair of imaginary conjugate eigenvalues of block $\widetilde{M}_2$, and so of M. This completes the proof. ∎

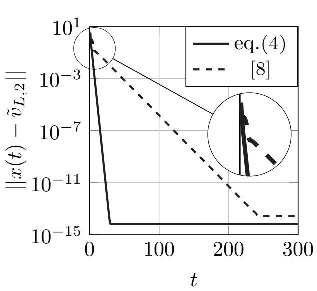

Fig. 2: Fiedler vector estimation error in a network of 5 agents.

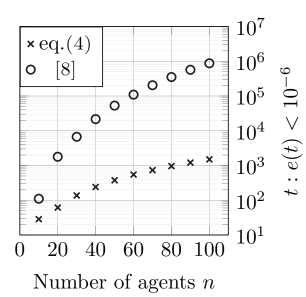

Fig. 3: Convergence time of Fiedler vector estimation in line networks.

V. Numerical Results

Simulations have been carried out by exploiting the 4-th Order Runge–Kutta Method.

A. Fiedler vector estimation

We compare our protocol with the one in [8] because of their similar structure. We keep our notation for any common variable (i.e., α, β and K_I) while we use the notation in [8] for the remaining variables (i.e., k_2, k_3 and K_P).

In the first simulation, a MAS with $n = 5$ agents (1), random topology and local control law (4) is considered. Accordingly with conditions of Theorem 2, we choose the gains for the integral dynamic average consensus estimator as $\beta = 25$, $K_I = 10$, and the gain for the eigenvector estimator as $\alpha = 6$. The additional gains for the algorithm in [8] are chosen as $k_2 = 1$, $k_3 = 20$ and $K_P = 50$. Network topology and free parameters are the same of Example 1 in [8], common variables are initialized to the same values while the others are chosen to nullify the initial error estimation.

In the second simulation, we consider a MAS with increasing number n of agents, line topology and local control law (4). With the choice of the line topology we are considering the worst case scenario for the dynamic average estimator. According to conditions of Theorem 2, the gains for the integral dynamic average consensus estimator are chosen $\beta = 10$, $K_I = 15$, the gain for the eigenvector estimator is $\alpha = 2\lambda_{L,2}$. The additional gains for the algorithm in [8] are chosen as $k_2 = 1$, $k_3 = 2\lambda_{L,2}$ and $K_P = 25$.

The Figures 2-3 show the results of the two simulations just described. First, Fig 2 shows the error evolution $e(t) = ||x(t) - \widetilde{v}_{L,2}||$, $\widetilde{v}_{L,2} = \lim_{t\to\infty} ||x(t)||v_{L,2}$. Second, Fig. 3 shows, for different values of n, the time required by the two algorithms to reach an error of the order of 10^{-6}. Both simulations reveal that, in the face of the assumption on the knowledge of the algebraic connectivity, the proposed algorithm has a faster convergence rate, making it more scalable with the number of the agents in the network.

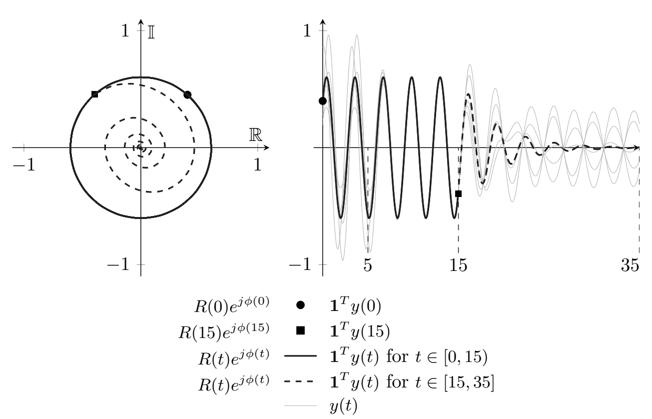

Fig. 4: Evolution of 5 coupled harmonic oscillators. The diffusive coupling (8) is activated at $t = 5$, while the desynchronizing local feedback (13)-(5) is activated at $t = 15$.

B. Desynchronization of harmonic oscillators

Synchronization of networked mechanical oscillator systems have been subject of interest [21], [22], [32], [33]. Here we give a physical example of application of Theorem 3.

Consider the networked mechanical systems consisting of n train wagons of identical mass m with linear dumper b interaction [33] and a mass-spring-damper modeling the interaction with the ground. Assuming a (ideally) null damping in order to guarantee the best comfort to the passengers, and denoting the spring coefficient with k, the model becomes the one in (7)-(8) with natural frequency $\omega = \sqrt{k/m} = 0.1$ and dumping coefficient $b = 1$. The feedback control (13) models the active suspensions between wagons and desynchronization corresponds to the minimum stress on the rails since the sum of the forces becomes as the time passes.

Simulation of a line-topology network with $n = 5$ nodes is shown in Fig. 4 with $\alpha = 1.88$ and $K_I = 2$, according to Theorem 3. The oscillators start evolving without any coupling until at $t = 5$ the diffusive coupling is enabled and synchronization is achieved. Let us denote with $R(t)e^{j\phi(t)}$ the centroid as defined in (10)-(11) given $x(t)$ as the initial condition of the network $x(t)$; it is clear that $\lim_{t\to\infty} R(t) = R$. Thus, as can be seen in Fig. 4, the collective output dynamics has constant module $R(t) = 0.6$ for $t \in [0, 15)$. At $t = 15$ the proposed control feedback is activated and the network is shown to reach desynchronization in the sense of Definition 1, i.e., $R = \lim_{t\to\infty} R(t) = \lim_{t\to\infty} \frac{1}{n}\mathbf{1}^T y(t) = \mathbf{1}^T y^{ss}(t) = 0$.

VI. Conclusions

In this work we proposed a protocol for solving the problem of distributed Fiedler vector estimation in networks of single-integrator agents. Exploiting the zero mean property of the Fiedler vector, we employed the proposed protocol as a local feedback to achieve desynchronization in a network of diffusively coupled harmonic oscillators. The main advantages of the proposed protocol are its robustness to re-initialization and a fast convergence rate. Future works will focus on relaxing the condition on the knowledge of the algebraic connectivity, thus paving the way to address time-varying topologies.

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