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Special Issue
Generalized Continuum Models and Higher-Order Partial Differential Equations

Edited by
Dr. Sergei Khakalo and Dr. Emilio Barchiesi

MDPI

https://doi.org/10.3390/math11041024

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Strong Ellipticity and Infinitesimal Stability within Nth-Order Gradient Elasticity

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Abstract: We formulate a series of strong ellipticity inequalities for equilibrium equations of the gradient elasticity up to the Nth order. Within this model of a continuum, there exists a deformation energy introduced as an objective function of deformation gradients up to the Nth order. As a result, the equilibrium equations constitute a system of $2N$-order nonlinear partial differential equations (PDEs). Using these inequalities for a boundary-value problem with the Dirichlet boundary conditions, we prove the positive definiteness of the second variation of the functional of the total energy. In other words, we establish sufficient conditions for infinitesimal instability. Here, we restrict ourselves to a particular class of deformations which includes affine deformations.

Keywords: strong ellipticity; strain gradient elasticity; infinitesimal stability; gradient elasticity of the Nth order; Dirichlet boundary conditions

MSC: 74A20; 74B20; 74G60; 35J48; 35J66

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Citation: Eremeyev, V.A. Strong Ellipticity and Infinitesimal Stability within Nth-Order Gradient Elasticity. *Mathematics* **2023**, *11*, 1024. https://doi.org/10.3390/math11041024

Academic Editors: Sergei Khakalo and Emilio Barchiesi

Received: 2 February 2023
Revised: 13 February 2023
Accepted: 15 February 2023
Published: 17 February 2023

1. Introduction

Among various generalized models of continua, the strain gradient elasticity can be treated as a straightforward extension of the classic elasticity. Although invented together with the classic elasticity, see [1–3] for historical developments in the field, the model did not find essential applications. Nowadays, the situation has completely changed. Indeed, a significant extension of the application of continuum and structural mechanics to various scales including micro- and nanometre scales and to the modelling of new materials has resulted in the extensive use of the strain gradient elasticity for the description of material behaviour at small scales [4–7] as well as in the mechanics of composite materials with a high contrast in the properties of their components [8–11]; see also [12,13] and the references therein. The key idea of the strain gradient elasticity approach is based on the consideration of higher-order deformation gradients as arguments of a deformation energy density. Therefore, one can classify these models according to a maximal order of the considered deformation gradient. As a result, the classic elasticity [14–16] could be treated as a model of the first order, where the models by Toupin [17,18] and by Mindlin [19,20] can be considered as a strain gradient elasticity of the second order. In addition to these models, in the literature, a third-order strain gradient elasticity was used [13,21–23]. A general Nth-order gradient elasticity was discussed in [24]; see also [12,13]. From the physical point of view, the increase of the maximal order of deformation gradient serves as a better description of so-called long-range interactions between material particles in solids and fluids. From the mathematical point of view, the Nth-order gradient elasticity results in a system of linear or nonlinear partial differential equations (PDEs) of the $2N$th order. Their properties such as the existence and regularity of solutions could be studied using the general theory of PDEs [25–28].

Considering systems of PDEs, one can distinguish elliptic, parabolic, hyperbolic or more general types of equations. Here, we consider strongly elliptic systems of PDEs using the definition given by Vishik [29]; see also [30]. Note that ellipticity could be considered as a natural property of statics. The strong ellipticity (SE) condition is closely related to the infinitesimal stability of solutions. For example, in nonlinear elasticity, it was shown that the SE condition implies the infinitesimal stability of an affine deformation for the Dirichlet boundary conditions, whereas infinitesimal stability results in a weak form of the SE conditions, called the Hadamard inequality [14–16]. In the case of higher-order models, the relations between the SE and infinitesimal instability is less straightforward, see [31–33] for the case of second- and third-order models.

The aim of this paper was to discuss the relations between the SE conditions and the infinitesimal stability for the general case, i.e., for the strain gradient elasticity of the Nth order. Here, we restricted ourselves to the first boundary-value problem, i.e., a problem with the Dirichlet boundary conditions assumed on the whole boundary. From the physical point of view, the infinitesimal instability for the first boundary-value problem could be treated as a certain material instability; see [14–16] for the classic elasticity. Indeed, in this case, an infinitesimal instability relates only to the material response as no external loadings are presumed.

The paper is organized as follows. First, in Section 2, we introduce the governing equations of the Nth-order gradient elasticity including the SE conditions. In Section 3, we discuss a hierarchical series of constitutive equations of mth-order strain gradient materials, $m = 1, \ldots, N$, and formulate the corresponding SE conditions called the SE$_m$ conditions. Finally, in the following sections, we establish the relations between the SE conditions and the stability. In Section 4, we introduce the second variation of the total energy. In Section 5, we discuss affine deformations, i.e., deformations with a constant deformation gradient. Finally, in Section 6, we show that the SE$_m$ conditions are sufficient for the infinitesimal stability of an affine deformation. On the other hand, similar to the classic elasticity, stability implies only a weak form of the SE$_N$ condition, which plays the role of the Hadamard inequality for the Nth-order strain gradient elasticity.

2. Governing Equations

Let B be an elastic solid body occupying in a reference placement $\varkappa$ a volume $V \subset \mathbb{R}^3$ with a smooth enough boundary $S = \partial V$. A deformation of B is introduced as a smooth invertible mapping from reference placement $\varkappa$ into a current placement χ as follows:

$$\mathbf{x} = \mathbf{x}(\mathbf{X}), \tag{1}$$

where $\mathbf{x}$ and $\mathbf{X}$ are position vectors in χ and $\varkappa$, respectively.

Within the Nth-order strain gradient elasticity there exists a deformation energy W introduced as a function of deformation gradients up to the Nth order [24]

$$W = W(\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_N), \tag{2}$$

where $\mathbf{F}_1 \equiv \mathbf{F} = \nabla\mathbf{x}$ is the deformation gradient and $\mathbf{F}_{i+1} = \nabla\mathbf{F}_i = \nabla^{i+1}\mathbf{x}$, $\nabla^i = \underbrace{\nabla\nabla\ldots\nabla}_{i \text{ times}}$, $i = 1, \ldots, N-1$ are deformation gradients of a higher order, and ∇ is the 3D nabla operator [14,34]. In what follows, we assume that W is a twice continuously differentiable function.

Applying to (2) the principle of the material frame indifference [16], we get the following form of W

$$W = W(\mathbf{C}, \mathbf{K}_1, \ldots, \mathbf{K}_{N-1}), \tag{3}$$

where $\mathbf{C} = \mathbf{F} \cdot \mathbf{F}^T$ is the Cauchy–Green strain measure, $\mathbf{K}_i = \nabla\mathbf{F}_i \cdot \mathbf{F}^T$, $i = 1, \ldots, N-1$ are other Lagrangian strain measures, and $\cdot$ stands for the dot product [34].

In what follows, we restrict ourselves to the first boundary-value problem, i.e., we assume on S the Dirichlet boundary conditions

$$\mathbf{x}\Big|_S = \mathbf{x}_0, \quad \frac{\partial \mathbf{x}}{\partial n}\Big|_S = \mathbf{x}_1, \quad \frac{\partial^2 \mathbf{x}}{\partial n^2}\Big|_S = \mathbf{x}_2, \quad \ldots, \quad \frac{\partial^N \mathbf{x}}{\partial n^N}\Big|_S = \mathbf{x}_N, \tag{4}$$

where $\mathbf{x}_i$, $i = 1, \ldots, N$ are given on S functions and $\partial/\partial n$ is the normal derivative.

The total energy takes the form

$$E = \iiint\limits_V W\, dV - \iint\limits_S \rho \mathbf{f} \cdot \mathbf{u}\, dS, \tag{5}$$

where $\mathbf{f}$ is a mass force vector, ρ is a mass density in $\varkappa$, and $\mathbf{u} = \mathbf{x} - \mathbf{X}$ is the displacement vector. Note that hereinafter, we assume a dead loading, so $\mathbf{f}$ does not depend on $\mathbf{u}$ and its gradients.

Using the Lagrange variational principle, from the variational equation

$$\delta E = 0, \tag{6}$$

we get the equilibrium equation

$$\nabla \cdot \mathbf{T} + \rho \mathbf{f} = \mathbf{0}, \tag{7}$$

where $\mathbf{T}$ is the total stress tensor of the first Piola–Kirchhoff type given by the formulae

$$\begin{aligned} \mathbf{T} =& \mathbf{P}_1 - \nabla \cdot \mathbf{P}_2 + \nabla \cdot (\nabla \cdot \mathbf{P}_3) - \ldots + (-1)^{N-1} \nabla \cdot (\nabla \cdot \ldots (\nabla \cdot \mathbf{P}_N)) \\ =& \mathbf{P}_1 + \sum_{i=2}^{N} (-1)^{i-1} (\underbrace{\nabla \cdot \ldots (\nabla \cdot}_{i \ \text{times}} \mathbf{P}_i) \ldots)), \end{aligned} \tag{8}$$

$$\mathbf{P}_i = \frac{\partial W}{\partial \mathbf{F}_i}, \quad i = 1, \ldots, N. \tag{9}$$

Here, $\mathbf{P}_i$ is the first Piola–Kirchhoff type hyperstress tensors of the $(i+1)$th order.

Obviously, Equation (7) constitutes a system of PDEs of the $2N$th order, in general. The strong ellipticity condition of (7) takes the form

$$(\underbrace{\mathbf{k} \otimes \mathbf{k} \otimes \ldots \otimes \mathbf{k}}_{N \ \text{times}} \otimes \mathbf{a}) \bullet \frac{\partial^2 W}{\partial \mathbf{F}_N^2} \bullet (\underbrace{\mathbf{k} \otimes \mathbf{k} \otimes \ldots \otimes \mathbf{k}}_{N \ \text{times}} \otimes \mathbf{a}) \geq C |\mathbf{k}|^{2N} |\mathbf{a}|^2, \tag{10}$$

where $\mathbf{k}$ and $\mathbf{a}$ are arbitrary constant vectors, C is a positive constant independent on $\mathbf{k}$ and $\mathbf{a}$, $|\mathbf{k}|^2 = \mathbf{k} \cdot \mathbf{k}$, $|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}$, and $\otimes$ and $\bullet$ are dyadic and full products, respectively. For polyadics, the full product is defined as follows:

$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \ldots \otimes \mathbf{a}_n) \bullet (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \ldots \otimes \mathbf{b}_m)$$
$$= \begin{cases} \sum\limits_{i=1}^{n} (\mathbf{a}_i \cdot \mathbf{b}_i) \mathbf{b}_{n+1} \otimes \ldots \otimes \mathbf{b}_m, & n < m \\ \sum\limits_{i=1}^{m} (\mathbf{a}_{n-m+i} \cdot \mathbf{b}_i) \mathbf{a}_1 \otimes \ldots \otimes \mathbf{a}_{n-m}, & n > m \\ \sum\limits_{i=1}^{n} (\mathbf{a}_i \cdot \mathbf{b}_i), & m = n \end{cases}, \tag{11}$$

where $\mathbf{a}_i$ and $\mathbf{b}_j$ are arbitrary vectors. By linearity, this definition of the full product could be extended for tensors of any order; see [34] for more details.

Using the objective representation (3) of W, we can reformulate (10) as follows

$$(\underbrace{\mathbf{k} \otimes \mathbf{k} \otimes \ldots \otimes \mathbf{k}}_{N \ \text{times}} \otimes \mathbf{F} \cdot \mathbf{a}) \bullet \frac{\partial^2 W}{\partial \mathbf{K}_{N-1}^2} \bullet (\underbrace{\mathbf{k} \otimes \mathbf{k} \otimes \ldots \otimes \mathbf{k}}_{N \ \text{times}} \otimes \mathbf{F} \cdot \mathbf{a}) \geq C |\mathbf{k}|^{2N} |\mathbf{a}|^2. \tag{12}$$

Here, we have used the formula

$$\frac{\partial W}{\partial \mathbf{F}_N} = \frac{\partial W}{\partial \mathbf{K}_{N-1}} \cdot \mathbf{F}^T,$$

see [34] for more details on the calculation of derivatives of scalar- and tensor-valued functions of tensorial arguments. Note that (12) coincides with (10) up to some notations and the replacement $\mathbf{a} \longleftrightarrow \mathbf{F} \cdot \mathbf{a}$.

Inequality (10) can also be written as a certain convexity-type condition with respect to the highest-order deformation gradient

$$\frac{d^2}{d\varepsilon^2} W \left(\mathbf{F}, \mathbf{F}_2, \ldots, \mathbf{F}_{N-1}, \mathbf{F}_N + \varepsilon \underbrace{\mathbf{k} \otimes \mathbf{k} \otimes \ldots \otimes \mathbf{k}}_{N \text{ times}} \otimes \mathbf{a} \right) \Bigg|_{\varepsilon=0} \geq C |\mathbf{k}|^{2N} |\mathbf{a}|^2. \tag{13}$$

One can see that any form of the SE conditions affects only the dependence of W on the highest-order deformation gradient.

3. Series of Gradient Models and Their Ellipticity

In addition to (2), let us consider a series of constitutive equations for strain gradient materials of order m, $1 \leq m < N$. For each material, we introduce a deformation energy density as a reduction of (2)

$$W_m = W_m(\mathbf{F}, \mathbf{F}_2, \ldots, \mathbf{F}_m) \equiv W(\mathbf{F}, \mathbf{F}_2, \ldots, \mathbf{F}_N) \Big|_{\mathbf{F}_{m+1}=\mathbf{0}, \ldots, \mathbf{F}_N=\mathbf{0}}. \tag{14}$$

Hereinafter, $\mathbf{0}$ means a zero tensor of any order.

Moreover, we normalize W_m as follows

$$W_1 = W_1(\mathbf{F}) \equiv W(\mathbf{F}, \mathbf{F}_2, \ldots, \mathbf{F}_N) \Big|_{\mathbf{F}_2=\mathbf{0}, \ldots, \mathbf{F}_N=\mathbf{0}}, \tag{15}$$

$$W_2 = W_2(\mathbf{F}, \mathbf{F}_2) \equiv W(\mathbf{F}, \mathbf{F}_2, \ldots, \mathbf{F}_N) \Big|_{\mathbf{F}_3=\mathbf{0}, \ldots, \mathbf{F}_N=\mathbf{0}} - W_1(\mathbf{F}), \tag{16}$$

$$W_m = W_m(\mathbf{F}, \mathbf{F}_2, \ldots, \mathbf{F}_m) \equiv W(\mathbf{F}, \mathbf{F}_2, \ldots, \mathbf{F}_N) \Big|_{\mathbf{F}_{m+1}=\mathbf{0}, \ldots, \mathbf{F}_N=\mathbf{0}} - W_{m-1}(\mathbf{F}, \mathbf{F}_2, \ldots, \mathbf{F}_{m-1}), \quad m = 2, \ldots, N. \tag{17}$$

As a result, we get

$$W_m(\mathbf{F}, \mathbf{F}_2, \ldots, \mathbf{F}_m) \Big|_{\mathbf{F}_m=\mathbf{0}} = 0, \quad m = 2, \ldots, N, \tag{18}$$

$$W_m = \sum_{i=1}^{m} W_i, \quad \mathbf{P}_m = \sum_{k=m}^{N} \frac{\partial W_k}{\partial \mathbf{F}_m}, \quad m = 1, \ldots, N. \tag{19}$$

In addition, let us assume that

$$\mathbf{P}_m \Big|_{\mathbf{F}_m=\mathbf{0}} = \mathbf{0}, \quad m = 2, \ldots, N. \tag{20}$$

Thus, we assume that hyperstress tensor $\mathbf{P}_m$ vanishes simultaneously with the mth deformation gradient. Let us note that this assumption seems to be natural as $\mathbf{P}_m$ is energetically dual to $\mathbf{F}_m$. For example, (20) is fulfilled if $\varkappa$ is a natural reference placement, i.e., without initial stresses and hyperstresses.

The strong ellipticity conditions related to these constitutive equations are given by

$$(\underbrace{\mathbf{k}\otimes\mathbf{k}\otimes\ldots\otimes\mathbf{k}}_{m \text{ times}}\otimes\mathbf{a})\bullet\frac{\partial^2 W_m}{\partial\mathbf{F}_m^2}\bullet(\underbrace{\mathbf{k}\otimes\mathbf{k}\otimes\ldots\otimes\mathbf{k}}_{m \text{ times}}\otimes\mathbf{a})\geq C_m|\mathbf{k}|^{2m}|\mathbf{a}|^2, \tag{21}$$

where C_m is a positive constant independent on $\mathbf{k}$ and $\mathbf{a}$. In what follows, for brevity, we call Equation (21) the SE condition of the mth order or simply the SE_m condition.

4. Infinitesimal Stability

Let $\tilde{\mathbf{x}}$ be a known solution of (4) and (7). Following [14,15], we call it stable if the second variation of E is positive, $\delta^2 E > 0$, for any small nonzero kinematically admissible deformations. If for a certain perturbation, $\delta^2 E = 0$, we say that $\tilde{\mathbf{x}}$ relates to a neutral equilibrium. For the derivation of $\delta^2 E$, we use the following standard procedure. Let

$$\mathbf{x} = \tilde{\mathbf{x}} + \varepsilon\mathbf{v} \tag{22}$$

be a perturbed deformation, where ε is a small positive number and $\mathbf{v}$ is a vector of additional displacement (perturbation). As $\mathbf{x}$ satisfies (4) we have the homogeneous Dirichlet boundary conditions for $\mathbf{v}$

$$\mathbf{v}\Big|_S = \mathbf{0}, \quad \frac{\partial\mathbf{v}}{\partial n}\Big|_S = \mathbf{0}, \quad \frac{\partial^2\mathbf{v}}{\partial n^2}\Big|_S = \mathbf{0}, \quad \ldots, \quad \frac{\partial^N\mathbf{v}}{\partial n^N}\Big|_S = \mathbf{0}. \tag{23}$$

Substituting (22) into (5), we get

$$E[\mathbf{x}] = E[\tilde{\mathbf{x}}] + \varepsilon\delta E[\tilde{\mathbf{x}},\mathbf{v}] + \varepsilon^2\delta^2 E[\tilde{\mathbf{x}},\mathbf{v}] + o(\varepsilon^2), \tag{24}$$

where

$$\delta E[\tilde{\mathbf{x}},\mathbf{v}] = \frac{d}{d\varepsilon}E[\tilde{\mathbf{x}}+\varepsilon\mathbf{v}]\Big|_{\varepsilon=0}, \quad \delta^2 E[\tilde{\mathbf{x}},\mathbf{v}] = \frac{d^2}{d\varepsilon^2}E[\tilde{\mathbf{x}}+\varepsilon\mathbf{v}]\Big|_{\varepsilon=0} \tag{25}$$

are the first and second Gateaux differentials, respectively.

Since $\tilde{\mathbf{x}}$ is a stationary point of E, the first variation of E vanishes. Therefore, we come to the formula

$$E[\tilde{\mathbf{x}}+\varepsilon\mathbf{v}] - E[\tilde{\mathbf{x}}] = \varepsilon^2\delta^2 E[\tilde{\mathbf{x}},\mathbf{v}] + o(\varepsilon^2), \tag{26}$$

where $\delta^2 E[\tilde{\mathbf{x}},\mathbf{v}]$ takes the form

$$\begin{aligned}\delta^2 E[\tilde{\mathbf{x}},\mathbf{v}] =& \frac{1}{2}\iiint\limits_V \frac{d^2}{d\varepsilon^2}W(\tilde{\mathbf{F}}+\varepsilon\nabla\mathbf{v},\tilde{\mathbf{F}}_2+\varepsilon\nabla\nabla\mathbf{v},\ldots,\tilde{\mathbf{F}}_N+\varepsilon\nabla^N\mathbf{v})\,dV\Big|_{\varepsilon=0}\\ =& \frac{1}{2}\sum_{i,j=1}^{N}\iiint\limits_V \nabla^j\mathbf{v}\bullet\frac{\partial^2 W}{\partial\mathbf{F}_j\partial\mathbf{F}_i}(\tilde{\mathbf{F}},\tilde{\mathbf{F}}_2,\ldots,\tilde{\mathbf{F}}_N)\bullet\nabla^i\mathbf{v}\,dV,\end{aligned} \tag{27}$$

where $\tilde{\mathbf{F}}_i = \nabla^i\tilde{\mathbf{x}}$.

For a neutral equilibrium, we have that $\delta^2 E[\tilde{\mathbf{x}},\mathbf{v}] \geq 0$ and there exists $\mathbf{v}^*$ such that $\delta^2 E[\tilde{\mathbf{x}},\mathbf{v}^*] = 0$. As in [14], we can show that $\mathbf{v}^*$ satisfies the linearized equilibrium equation

$$\sum_{i,j=1}^{N}(-1)^j\underbrace{\nabla\cdot\ldots(\nabla\cdot}_{j \text{ times}}\left(\frac{\partial^2 W}{\partial\mathbf{F}_j\partial\mathbf{F}_i}\bullet\nabla^i\mathbf{v}\right)\ldots\Big) = \mathbf{0} \tag{28}$$

and boundary conditions (23).

Let us show that the infinitesimal stability, i.e., the inequality

$$\delta^2 E[\tilde{\mathbf{x}},\mathbf{v}] > 0, \quad \forall\,\mathbf{v}\neq\mathbf{0}, \tag{29}$$

implies the weak form of (10), that is,

$$H(\mathbf{k},\mathbf{a};\mathbf{X}) \equiv (\underbrace{\mathbf{k}\otimes\mathbf{k}\otimes\ldots\otimes\mathbf{k}}_{N \text{ times}}\otimes\mathbf{a}) \bullet \frac{\partial^2 W}{\partial \mathbf{F}_N^2}[\mathbf{X}] \bullet (\underbrace{\mathbf{k}\otimes\mathbf{k}\otimes\ldots\otimes\mathbf{k}}_{N \text{ times}}\otimes\mathbf{a}) \geq 0, \tag{30}$$

for any vectors $\mathbf{k}$ and $\mathbf{a}$ and for any point $\mathbf{X} \in V$. Hereinafter, for brevity, we use the notation

$$\frac{\partial^2 W}{\partial \mathbf{F}_j \partial \mathbf{F}_i}[\mathbf{X}] = \frac{\partial^2 W}{\partial \mathbf{F}_j \partial \mathbf{F}_i}(\widetilde{\mathbf{F}}(\mathbf{X}), \widetilde{\mathbf{F}}_2(\mathbf{X}), \ldots, \widetilde{\mathbf{F}}_N(\mathbf{X})).$$

Relation (30) is similar to the Hadamard inequality in nonlinear elasticity [14–16]. In order to prove (30), we use the partition of unity technique [25]. Let us consider a vector-function $\mathbf{v}_\varepsilon$ with finite support, $\operatorname{supp}\mathbf{v}_\varepsilon = V_\varepsilon \equiv \{\mathbf{X} : |\mathbf{X}-\mathbf{P}| \leq \varepsilon\}$, where ε is a small positive number and $\mathbf{P}$ is a position vector of a point in V, $\mathbf{P} \in V$, $\mathbf{P} \notin S$. We introduce $\mathbf{v}_\varepsilon$ as follows

$$\mathbf{v}_\varepsilon = f(y_1)f(y_2)f(y_3), \quad \mathbf{y} = \frac{1}{\varepsilon}(\mathbf{X}-\mathbf{P}), \tag{31}$$

where $\mathbf{a}$ is a constant vector and f is an even function such that $f \in C_0^\infty[-1,1]$ and

$$\int_{-1}^{1} f(y)\,dy = 1, \quad f(\pm 1) = 0, \quad f'(\pm 1) = 0, \quad \ldots, \quad f^{(N)}(\pm 1) = 0.$$

As an example, the bump function could be used as f which is defined by

$$f(y) = \begin{cases} \exp\left(\frac{1}{y^2-1}\right), & |y| \leq 1, \\ 0, \quad |y| > 1. \end{cases}$$

Substituting $\mathbf{v}_\varepsilon$ into (29) and changing the variables $\mathbf{X} \to \mathbf{y}$, so $\nabla_y = \varepsilon\nabla$, $dV_y = \varepsilon^3 dV$, we get

$$\begin{aligned} \delta^2 E[\tilde{\mathbf{x}}, \mathbf{v}_\varepsilon] =& \frac{1}{2}\sum_{i,j=1}^{N} \iiint_{V_\varepsilon} \nabla^j \mathbf{v}_\varepsilon \bullet \frac{\partial^2 W}{\partial \mathbf{F}_j \partial \mathbf{F}_i}[\mathbf{X}] \bullet \nabla^i \mathbf{v}_\varepsilon\, dV \\ =& \frac{1}{2}\sum_{i,j=1}^{N} \iiint_{\widetilde{V}} \varepsilon^{-j}\nabla_y^j \tilde{\mathbf{v}} \bullet \frac{\partial^2 W}{\partial \mathbf{F}_j \partial \mathbf{F}_i}[\varepsilon\mathbf{y}+\mathbf{P}] \bullet \varepsilon^{-i}\nabla_y^i \tilde{\mathbf{v}}\, \varepsilon^3\, dV_y, \end{aligned} \tag{32}$$

where $\tilde{\mathbf{v}} = \mathbf{v}_\varepsilon\big|_{\varepsilon=1}$, $\widetilde{V} = V_\varepsilon\big|_{\varepsilon=1}$, $dV_y = dy_1 dy_2 dy_3$. Thus, we get that

$$\delta^2 E[\tilde{\mathbf{x}}, \mathbf{v}_\varepsilon] = \varepsilon^{3-2N}\left[\widetilde{J}_N + O(\varepsilon)\right], \quad \widetilde{J}_N = \frac{1}{2}\iiint_{\widetilde{V}} \nabla_y^N \tilde{\mathbf{v}} \bullet \frac{\partial^2 W}{\partial \mathbf{F}_j \partial \mathbf{F}_i}[\mathbf{P}] \bullet \nabla_y^i \tilde{\mathbf{v}}\, dV_y. \tag{33}$$

Therefore, we can conclude that the term with higher-order gradients in (27) should be at least non-negative, otherwise (29) is violated. As a result, we get the inequality

$$J_N[\mathbf{v}] = \frac{1}{2}\iiint_{V} \nabla^N \mathbf{v} \bullet \frac{\partial^2 W}{\partial \mathbf{F}_N \partial \mathbf{F}_N}[\mathbf{P}] \bullet \nabla^N \mathbf{v}\, dV \geq 0, \tag{34}$$

for all $\mathbf{v}$ and for any point in V.

Finally, let us demonstrate that (34) implies (30). Let us assume the opposite, i.e., that there exist a point $\mathbf{P}^*$ and a vector $\mathbf{k}^*$ such that $H(\mathbf{k}^*, \mathbf{a}; \mathbf{P}^*) < 0$. As $\frac{\partial^2 W}{\partial \mathbf{F}_j \partial \mathbf{F}_i}[\mathbf{X}]$ is continuous, there is a neighbourhood $V_\varepsilon^* = \{\mathbf{X} : |\mathbf{X}-\mathbf{P}^*| \leq \varepsilon\}$ for a small enough number $\varepsilon > 0$, such that for all $\mathbf{X} \in V_\varepsilon^*$, we have $H(\mathbf{k}^*, \mathbf{a}; \mathbf{X}) < 0$. Now, let us consider $\mathbf{v}$ as an oscillating function with finite support such that

$$\mathbf{v} = \cos[\lambda \mathbf{k}^* \cdot (\mathbf{X} - \mathbf{P}^*)]\varphi(\mathbf{X})\mathbf{a}, \tag{35}$$

where $\varphi \in C_0^\infty$ is a function with finite support, $\mathrm{supp}\varphi \subset V_\varepsilon^*$, $\mathbf{a}$ is a constant vector, and λ is a positive number. For $\mathbf{v}$, we have the formulae

$$\begin{aligned}
\nabla \mathbf{v} =& -\lambda \sin[\lambda \mathbf{k}^* \cdot (\mathbf{X} - \mathbf{P}^*)]\varphi(\mathbf{X})\mathbf{k}^* \otimes \mathbf{a} + O(1),\\
\nabla^2 \mathbf{v} =& -\lambda^2 \cos[\lambda \mathbf{k}^* \cdot (\mathbf{X} - \mathbf{P}^*)]\varphi(\mathbf{X})\mathbf{k}^* \otimes \mathbf{k}^* \otimes \mathbf{a} + O(\lambda),\\
\nabla^3 \mathbf{v} =& \lambda^3 \sin[\lambda \mathbf{k}^* \cdot (\mathbf{X} - \mathbf{P}^*)]\varphi(\mathbf{X})\mathbf{k}^* \otimes \mathbf{k}^* \otimes \mathbf{k}^* \otimes \mathbf{a} + O(\lambda^2),\\
&\dots
\end{aligned}$$

so $J_N[\mathbf{v}]$ takes the form

$$\begin{aligned}
J_N[\mathbf{v}] =& \frac{1}{2}\iiint\limits_{V_\varepsilon^*} \lambda^{2N}(\underbrace{\mathbf{k}^* \otimes \ldots \otimes \mathbf{k}^*}_{N \text{ times}} \otimes \mathbf{a}) \bullet \frac{\partial^2 W}{\partial \mathbf{F}_N \partial \mathbf{F}_N}[\mathbf{P}^*] \bullet (\underbrace{\mathbf{k}^* \otimes \ldots \otimes \mathbf{k}^*}_{N \text{ times}} \otimes \mathbf{a})\, dV + O(\lambda^{2N-1})\\
=& \lambda^{2N} H(\mathbf{k}^*, \mathbf{a}; \mathbf{P}^*) B^* + O(\lambda^{2N-1}),\\
B^* =& \begin{cases} \iiint\limits_{V_\varepsilon^*} \cos^2[\lambda \mathbf{k}^* \cdot (\mathbf{X} - \mathbf{P}^*)]\varphi^2(\mathbf{X})\, dV, & N = 2k,\\ \iiint\limits_{V_\varepsilon^*} \sin^2[\lambda \mathbf{k}^* \cdot (\mathbf{X} - \mathbf{P}^*)]\varphi^2(\mathbf{X})\, dV, & N = 2k-1. \end{cases}
\end{aligned} \tag{36}$$

Considering a la rge enough λ, we can conclude that $J_N < 0$, which contradicts (34). Thus, the assumption was wrong and we came to (30).

5. Affine Deformations and Linearized Equations

For the first boundary-value problem in nonlinear elasticity of simple materials, it was established that the SE condition implied the infinitesimal stability of affine deformations [14,15]. By an affine deformation, we mean such a deformation that $\mathbf{C}$ or $\mathbf{F}$ is constant. For the Toupin–Mindlin strain gradient elasticity, it was shown in [31,32] that it was not the case, in general. The sufficient conditions for the strain gradient elasticity of the third order were established in [33].

Let us consider the infinitesimal stability of an affine deformation within the Nth-order gradient elasticity. As $\mathbf{F}$ is a constant tensor, we have that all higher-order deformation gradients vanish,

$$\mathbf{F}_2 = \mathbf{0}, \ldots, \mathbf{F}_N = \mathbf{0}.$$

Using (20) we also get that the hyperstresses vanish too,

$$\mathbf{P}_2 = \mathbf{0}, \ldots, \mathbf{P}_N = \mathbf{0}.$$

With (18) and (20), we can prove that the second variation and the equilibrium equations take the simpler form

$$\delta^2 E[\tilde{\mathbf{x}}, \mathbf{v}] = \frac{1}{2}\sum_{i=1}^{N} \iiint\limits_V \nabla^i \mathbf{v} \bullet \mathbf{D}_i \bullet \nabla^i \mathbf{v}\, dV, \tag{37}$$

$$\sum_{i=1}^{N} (-1)^i \underbrace{\nabla \cdot \ldots (\nabla \cdot}_{i \text{ times}} \Big(\mathbf{D}_i \bullet \nabla^i \mathbf{v}\Big) \ldots\Big) = \mathbf{0} \tag{38}$$

$$\mathbf{D}_i = \frac{\partial^2 W}{\partial \mathbf{F}_i \partial \mathbf{F}_i}(\tilde{\mathbf{F}}, \mathbf{0}, \ldots, \mathbf{0}).$$

Note that the tangent moduli tensors $\mathbf{D}_i$ are constant, so (38) is a system of PDEs with constant coefficients. The SE_m conditions take the form

$$(\underbrace{\mathbf{k}\otimes\mathbf{k}\otimes\ldots\otimes\mathbf{k}}_{m\ \text{times}}\otimes\mathbf{a})\bullet\mathbf{D}_m\bullet(\underbrace{\mathbf{k}\otimes\mathbf{k}\otimes\ldots\otimes\mathbf{k}}_{m\ \text{times}}\otimes\mathbf{a})\geq C_m|\mathbf{k}|^{2m}|\mathbf{a}|^2,\quad m=1,\ldots,N. \tag{39}$$

6. Stability of Affine Deformation

Let us show that (39) are the sufficient conditions for infinitesimal stability. In what follows, we use the proof of the Gårding inequality; see e.g., [25]. As $\mathbf{v}$ satisfies (23), we can extend it to the whole space as follows

$$\mathbf{w}(\mathbf{X})=\begin{cases}\mathbf{v}(\mathbf{X}), & \mathbf{X}\in V;\\ \mathbf{0}, & \mathbf{X}\in\mathbb{R}^3\backslash V.\end{cases} \tag{40}$$

Let $\widehat{\mathbf{w}}(\mathbf{k})$ be the Fourier transform of $\mathbf{w}(\mathbf{X})$; therefore, we have formulae

$$\widehat{\mathbf{w}}(\mathbf{k})=\frac{1}{(2\pi)^{3/2}}\iiint\limits_{\mathbb{R}^3}e^{-\mathbf{k}\cdot\mathbf{X}}\mathbf{w}(\mathbf{X})\,dX_1\,dX_2\,dX_3,$$
$$\mathbf{w}(\mathbf{X})=\frac{1}{(2\pi)^{3/2}}\iiint\limits_{\mathbb{R}^3}e^{\mathbf{k}\cdot\mathbf{X}}\widehat{\mathbf{w}}(\mathbf{k})\,dk_1\,dk_2\,dk_3,$$

where i is the imaginary unit, $i^2=-1$, and $dV=dX_1\,dX_2\,dX_3$.

Using the Plancherel theorem [35], we can transform δ^2E as follows

$$\delta^2E=\frac{1}{2}\sum_{i=1}^{N}\iiint\limits_{\mathbb{R}^3}(\underbrace{\mathbf{k}\otimes\ldots\otimes\mathbf{k}}_{i\ \text{times}}\otimes\widehat{\mathbf{w}})\bullet\mathbf{D}_i\bullet(\underbrace{\mathbf{k}\otimes\ldots\otimes\mathbf{k}}_{i\ \text{times}}\otimes\overline{\widehat{\mathbf{w}}})\,dk_1\,dk_2\,dk_3. \tag{41}$$

Here, the overbar denotes the complex conjugate. Let us recall that the Plancherel theorem states that

$$\int\limits_{\mathbb{R}}f(X)h(X)\,dX=\int\limits_{\mathbb{R}}\widehat{f}(k)\overline{\widehat{h}}(k)\,dk.$$

for any two functions $f(X)$, $h(X)\in L_2(\mathbb{R})\cap L_1(\mathbb{R})$, where L_1 and L_2 are Lebesgue spaces [36].

Using the SE_m conditions, we get that

$$\begin{aligned}\delta^2E\geq&\frac{1}{2}\sum_{i=1}^{N}\iiint\limits_{\mathbb{R}^3}C_i(\underbrace{\mathbf{k}\otimes\ldots\otimes\mathbf{k}}_{i\ \text{times}}\otimes\widehat{\mathbf{w}})\bullet(\underbrace{\mathbf{k}\otimes\ldots\otimes\mathbf{k}}_{i\ \text{times}}\otimes\overline{\widehat{\mathbf{w}}})\,dk_1\,dk_2\,dk_3\\ =&\frac{1}{2}\sum_{i=1}^{N}\iiint\limits_{\mathbb{R}^3}C_i\nabla^i\mathbf{w}\bullet\nabla^i\mathbf{w}\,dX_1\,dX_2\,dX_3\\ =&\frac{1}{2}\sum_{i=1}^{N}\iiint\limits_{V}C_i\nabla^i\mathbf{v}\bullet\nabla^i\mathbf{v}\,dV>0,\quad\forall\,\mathbf{v}\neq\mathbf{0}.\end{aligned} \tag{42}$$

Thus, the SE_m conditions are sufficient for infinitesimal stability. Obviously, they are not necessary, in general; see [31,32]. In particular, for the linear Toupin–Mindlin strain gradient elasticity it was shown that for uniqueness, one could assume SE_2 whereas SE_1 could be relaxed [32].

Summarizing the previous results, we can formulate the following theorem.

Theorem 1. *Let W be a twice continuously differentiable functions of deformation gradients up to the Nth order. Then, infinitesimal stability, i.e., the positive definiteness of the second variation of the total energy functional, implies the weak form of the strong ellipticity condition given by* (30).

All SE_m inequalities, $m = 1, \ldots, N$, result in the stability of the affine deformations for the first boundary-value problem.

7. Conclusions

Following the general definition of strong ellipticity [29], we introduced the strong ellipticity (SE) condition within the Nth-order strain gradient elasticity and discussed its relation to the infinitesimal stability of an affine deformation for the first boundary-value problem. Let us note that affine deformations play an important role in the mechanics of materials as they can be used for experimental studies of materials. Among them there are tension/compression tests, pure shear, etc. Unlike the nonlinear elasticity of simple materials [14,15], we demonstrated that the SE condition alone was insufficient for stability. Thus, we formulated a series of SE conditions for the models with a reduced order of deformation gradient starting from the SE condition for a simple material. The latter material was introduced as a reduction of the Nth-order gradient material. In a similar way, we introduced a series of hierarchical models of the mth order, $m = 1, \ldots, N$. One can treat the mth-order model as a gradient regularization of the $(m-1)$th model. Such a regularization keeps the ellipticity even if the previous material loses it; see [37] for more details. These SEm conditions are sufficient for infinitesimal stability but not necessary, in general.

Let us note that in nonlinear elasticity, the Hadamard inequality plays the role of a so-called constitutive inequality [14–16], i.e., an additional condition applied to the form of constitutive equations and to deformations of an elastic material. Thus, in the strain gradient elasticity of the Nth order, a similar inequality as (30) could also be treated as a constitutive inequality.

Funding: This research was supported by the Strategic Academic Leadership Program "Priority 2030", grant H-496-99_2021-2023.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Maugin, G.A. Generalized Continuum Mechanics: Various Paths. In *Continuum Mechanics Through the Twentieth Century: A Concise Historical Perspective*; Springer: Dordrecht, The Netherlands, 2013; pp. 223–241.
2. Maugin, G.A. *Non-Classical Continuum Mechanics: A Dictionary*; Springer: Singapore, 2017.
3. dell'Isola, F.; Della Corte, A.; Giorgio, I. Higher-gradient continua: The legacy of Piola, Mindlin, Sedov and Toupin and some future research perspectives. *Math. Mech. Solids* **2017**, *22*, 852–872. [CrossRef]
4. Aifantis, E.C. Gradient deformation models at nano, micro, and macro scales. *J. Eng. Mater. Technol.* **1999**, *121*, 189–202. [CrossRef]
5. Aifantis, E. Chapter One–Internal Length Gradient (ILG) Material Mechanics Across Scales and Disciplines. *Adv. Appl. Mech.* **2016**, *49*, 1–110.
6. Cordero, N.M.; Forest, S.; Busso, E.P. Second strain gradient elasticity of nano-objects. *J. Mech. Phys. Solids* **2016**, *97*, 92–124. [CrossRef]
7. Lazar, M.; Agiasofitou, E.; Böhlke, T. Mathematical modeling of the elastic properties of cubic crystals at small scales based on the Toupin–Mindlin anisotropic first strain gradient elasticity. *Contin. Mech. Thermodyn.* **2022**, *34*, 107–136. [CrossRef]
8. Abdoul-Anziz, H.; Seppecher, P. Strain gradient and generalized continua obtained by homogenizing frame lattices. *Math. Mech. Complex Syst.* **2018**, *6*, 213–250. [CrossRef]
9. dell'Isola, F.; Steigmann, D. A two-dimensional gradient-elasticity theory for woven fabrics. *J. Elast.* **2015**, *118*, 113–125. [CrossRef]
10. Rahali, Y.; Giorgio, I.; Ganghoffer, J.F.; dell'Isola, F. Homogenization à la Piola produces second gradient continuum models for linear pantographic lattices. *Int. J. Eng. Sci.* **2015**, *97*, 148–172. [CrossRef]
11. dell'Isola, F.; Steigmann, D.J. *Discrete and Continuum Models for Complex Metamaterials*; Cambridge University Press: Cambridge, UK, 2020.
12. Bertram, A.; Forest, S. (Eds.) *Mechanics of Strain Gradient Materials*; Springer International Publishing: Cham, Switzerland, 2020.
13. Bertram, A. *Compendium on Gradient Materials*; Springer: Cham, Switzerland, 2023.
14. Lurie, A.I. *Non-linear Theory of Elasticity*; North-Holland: Amsterdam, The Netherlands, 1990.
15. Ogden, R.W. *Non-Linear Elastic Deformations*; Dover: Mineola, NY, USA, 1997.
16. Truesdell, C.; Noll, W. *The Non-Linear Field Theories of Mechanics*, 3rd ed.; Springer: Berlin, Germany, 2004.
17. Toupin, R.A. Elastic materials with couple-stresses. *Arch. Ration. Mech. Anal.* **1962**, *11*, 385–414. [CrossRef]

18. Toupin, R.A. Theories of elasticity with couple-stress. *Arch. Ration. Mech. Anal.* **1964**, *17*, 85–112. [CrossRef]
19. Mindlin, R.D. Micro-structure in linear elasticity. *Arch. Ration. Mech. Anal.* **1964**, *16*, 51–78. [CrossRef]
20. Mindlin, R.D.; Eshel, N.N. On first strain-gradient theories in linear elasticity. *Int. J. Solids Struct.* **1968**, *4*, 109–124. [CrossRef]
21. Mindlin, R.D. Second gradient of strain and surface-tension in linear elasticity. *Int. J. Solids Struct.* **1965**, *1*, 417–438. [CrossRef]
22. Reiher, J.C.; Bertram, A. Finite third-order gradient elasticity and thermoelasticity. *J. Elast.* **2018**, *133*, 223–252. [CrossRef]
23. Eremeyev, V.A. Local material symmetry group for first- and second-order strain gradient fluids. *Math. Mech. Solids* **2021**, *26*, 1173–1190. [CrossRef]
24. dell'Isola, F.; Seppecher, P.; Madeo, A. How contact interactions may depend on the shape of Cauchy cuts in *N*th gradient continua: Approach "à la D'Alembert". *ZAMP* **2012**, *63*, 1119–1141. [CrossRef]
25. Fichera, G. *Linear Elliptic Differential Systems and Eigenvalue Problems*; Lecture Notes in Mathematics; Springer: Berlin, Germany, 1965; Volume 8.
26. Volevich, L.R. Solubility of boundary value problems for general elliptic systems (in Russian). *Sb. Math.* **1965**, *68*, 373–416.
27. Agranovich, M. Elliptic Boundary Problems. In *Partial Differential Equations IX: Elliptic Boundary Problems. Encyclopaedia of Mathematical Sciences*; Agranovich, M., Egorov, Y., Shubin, M., Eds.; Springer: Berlin, Germany, 1997; Volume 79, pp. 1–144.
28. Egorov, Y.V.; Shubin, M.A. *Foundations of the Classical Theory of Partial Differential Equations*, 1st ed.; Encyclopaedia of Mathematical Sciences 30; Springer: Berlin/Heidelberg, Germany, 1998; Volume 30.
29. Vishik, M.I. On strongly elliptic systems of differential equations (in Russian). *Sb. Math.* **1951**, *29*, 615–657.
30. Nirenberg, L. Remarks on strongly elliptic partial differential equations. *Commun. Pure Appl. Math.* **1955**, *8*, 648–674. [CrossRef]
31. Eremeyev, V.A. Strong ellipticity conditions and infinitesimal stability within nonlinear strain gradient elasticity. *Mech. Res. Commun.* **2021**, *117*, 103782. [CrossRef]
32. Eremeyev, V.A. On well-posedness of the first boundary-value problem within linear isotropic Toupin–Mindlin strain gradient elasticity and constraints for elastic moduli. *ZAMM-J. Appl. Math. Mech. für Angew. Math. und Mech.* **2023**, p. e202200474.
33. Eremeyev, V.A. On strong ellipticity and infinitesimal stability in third-order nonlinear strain gardient elasticity theory. *Mech. Solids* **2022**, *57*, 1953–1957.
34. Eremeyev, V.A.; Cloud, M.J.; Lebedev, L.P. *Applications of Tensor Analysis in Continuum Mechanics*; World Scientific: Hackensack, NJ, USA, 2018.
35. Yosida, K. *Functional Analysis*, 6th ed.; Springer: Berlin, Germany, 1980.
36. Adams, R.A.; Fournier, J.J.F. *Sobolev Spaces*, 2nd ed.; Pure and Applied Mathematics; Academic Press: Amsterdam, The Netherland, 2003; Volume 140.
37. Eremeyev, V.A.; Reccia, E. Nonlinear strain gradient and micromorphic one-dimensional elastic continua: Comparison through strong ellipticity conditions. *Mech. Res. Commun.* **2022**, *124*, 103909. [CrossRef]